Regular Homotopies of Low-Genus Non-Orientable Surfaces

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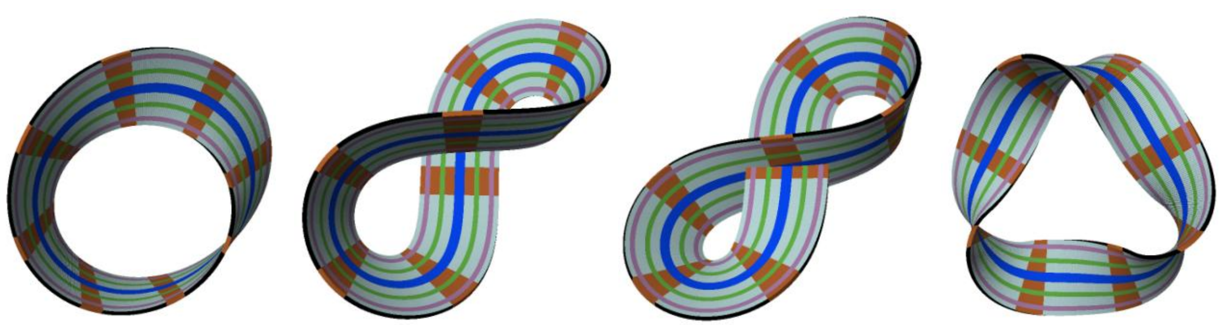
Abstract

The construction of various of Klein bottles that belong to different regular homotopy classes, and which thus cannot be smoothly transformed into one another, is formally introduced. For all cases it is shown how these shapes can be partitioned into two Möbius bands and how the twistedness of these bands defines the homotopy type.

1. Introduction

This paper extends the study of the regular homotopies of tori presented last year in *Tori Story* [19] to the realm of non-orientable surfaces of low genus, in particular to the Projective Plane and to Klein bottles. Topologically, Boy's surface (a compact, smooth, immersive model of the Projective Plane) is a single-sided surface with Euler characteristic $\chi=1$, while Klein bottles have Euler characteristic $\chi=0$. Both surfaces have no boundaries or punctures. Surfaces are in the same regular homotopy class if they can be smoothly transformed into one another without ever experiencing any cuts, or tears, or creases with infinitely sharp curvature; however, a surface is allowed to pass through itself. The paper by Hass and Hughes [10] states (pg.103): Corollary 1.3 (James–Thomas): There are $2^{2-\chi}$ distinct regular homotopy classes of immersions of a marked surface of Euler characteristic χ into $\mathbf{R}^3$. This tells us there should be **two** distinct models of Boy's surface and **four** (decorated) Klein bottle types that cannot be transformed smoothly into one another. However, I have found no publication that shows what these four types might look like. The single-sidedness of these objects also makes it conceptually more difficult to visualize these shapes. But some good discussions and e-mail exchanges with Dan Asimov, Tom Banchoff, Matthias Goerner, Rob Kusner, and John Sullivan helped me clarify the situation. All types of Klein bottles can be built from fusing together two Möbius strips, and the twistedness of these strips plays an important role in the determination of a Klein bottle type. Thus I include this decomposition in the images below and start this exposition with a discussion of these important building blocks.



(a) (b) (c) (d)

Figure 1: *A left-twisting Möbius band (ML). Deforming its sweep path alters its apparent twist sweep from (a) +180° (ccw), to (b) 0°, to (c) –180°, and to (d) –540° (cw).*

2. Möbius Bands

A Möbius band (Fig.1) can be constructed by taking a rectangular domain – for instance a paper strip – and connecting two opposite edges in reversed order, *i.e.*, after executing a 180° flip. For any topological analysis the surface is permitted to pass through itself; thus we can always execute one or more of the

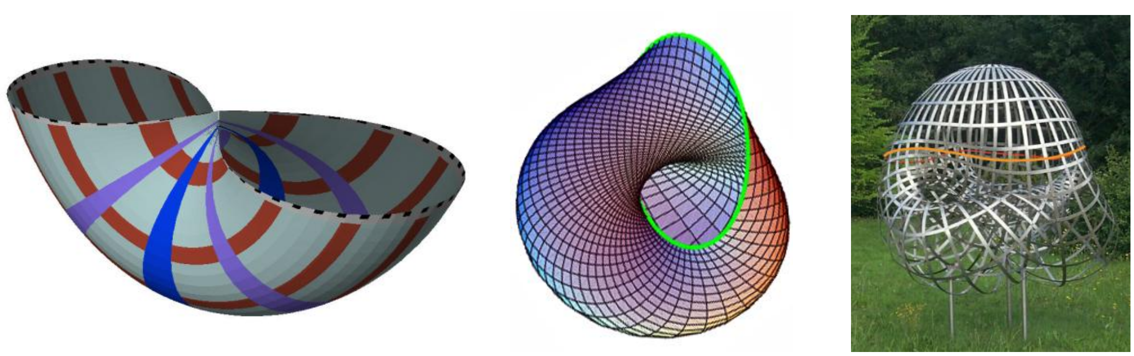
Figure-8 Sweep Cross-over Moves described in *Tori Story* [19]. This allows us to change *twist* (compared to a rotation-minimizing sweep) in increments of ±720°. The same applies to single-sided, non-orientable Möbius bands, and thus there are only two homotopically different Möbius bands, and they differ in their amount of twist by exactly 360°. Thus all the shapes depicted in Figure 1 belong into the same regular homotopy class. Mirroring, on the other hand, would turn the left-turning Möbius band (**ML**) of Figure 1a into a right-turning one (**MR**), and thus cast this surface into a different regular homotopy class.

Several of the stages of different twistedness depicted in Figure 1 have been exploited by various artists: The almost circular shape (Fig.1a) can be found in a wedding band [14]. The shape shown in Figure 1b has been celebrated in a sculpture by Max Bill [2], and the configurations (1c) and (1d) appear in drawings by M.C. Escher [9][8].

(a) (b) (c) (d)

Figure 2: *Artistic Möbius bands: (a) wedding band, (b) sculpture by Bill, (c,d) drawings by Escher.*

If we focus on the edge of the circular Möbius band (Fig.1a) and include a tiny sliver of the band's surface with it, then this edge-band forms a double loop with a total twist of 360°. This double loop can be unfolded into a figure-8 shape without changing the built-in twist as explained in Figure 18 in [20]. This unfolding of the edge will widen the narrow Möbius band into an extended surface that resembles a two-pouch basket (Fig.3a), but which topologically is still a Möbius band. Further deformations can be applied to this edge band, creating different kinds of "baskets" or "goblets." In particular, we can further un-warp the Möbius band edge into a circle. This deformation will change the edge band's twist by ±360°. If we let the twist add up to 720° then we obtain the Sudanese Möbius band [13] depicted in Figure 3b. Alternatively we can let the twist cancel out to zero and then obtain another interesting warped surface – which happens to be equivalent to a Boy surface [3][4] minus a disk (Fig.3c). Some of these shapes will play an important role in our analysis of various Klein bottles.

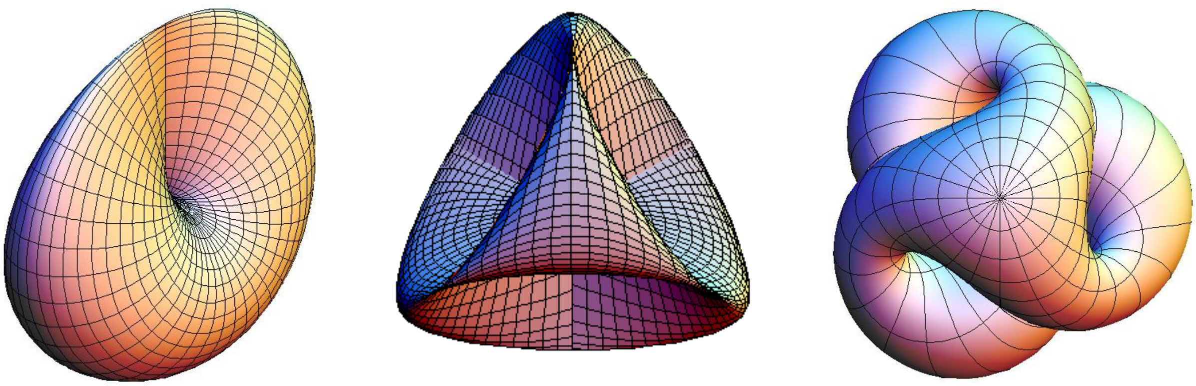


(a) (b) (c)

Figure 3: *Deforming the edge of a Möbius band: (a) into a figure-8 shape with 360° twist; or into a circular loop: (b) with 720° twist, or (c) with zero twist (when [4] is cut at red line).*

3. Boy's Surface and "Boy's Cap"

A Möbius band has just one continuous edge. If a (suitably warped) disk is "glued" to that edge, a single-sided surface without edges and with Euler characteristic $\chi = 1$ results; this is topologically equivalent to the *projective plane*. The simplest compact model of the projective plane in 3D Euclidean space is the *cross cap surface* (Fig.4a). Another model with higher symmetry is *Steiner's Roman surface* [21] (Fig.4b). However, both these models have some singularities, so called *Whitney Umbrellas*, where the surface curvature goes to infinity [23]. It is much harder to find an immersive model without any singularities. Werner Boy found the first such solution [3]. This particularly nice *immersion* has 3-fold rotational symmetry (Fig.4c). This model has a clear sense of chirality, and indeed it appears in two enantiomorphic forms, which we will label **BL** and **BR**. The two versions belong to different regular homotopy classes; and, according to [10], these are the only two regular homotopy classes that we can expect for a surface of Euler characteristic $\chi = 1$.



(a) (b) (c)

Figure 4: *(a) Cross cap surface; (b) Steiner's Roman surface; (c) Boy surface.*

If we puncture a hole into any of the models shown in Figure 4, i.e., remove a topological disk from the surface, we obtain a structure that is equivalent to a Möbius band. This is not easy to visualize, but Indiana University has a web site [5] based on a parameterization by Bryant and Kusner that gives a beautiful visualization of how a triply (right-twisted) Möbius band (**ML**) can be gradually widened to cover almost all of a (left-twisting) Boy surface, except for a small disk missing at the 3-fold symmetric pole (Fig.5). I like to call the final structure a "Boy cap." Smooth and well-formed Klein bottles can also be constructed from pairs of such Boy caps (see Section 13).

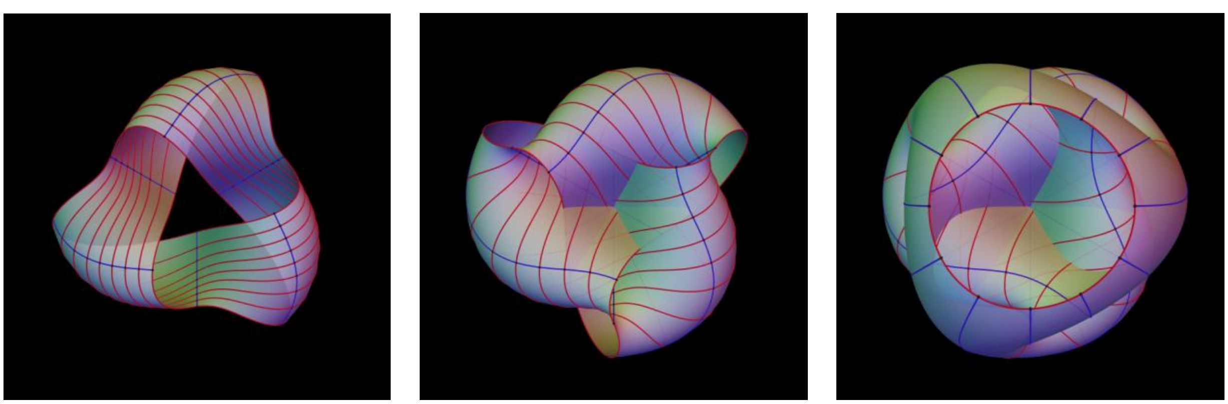
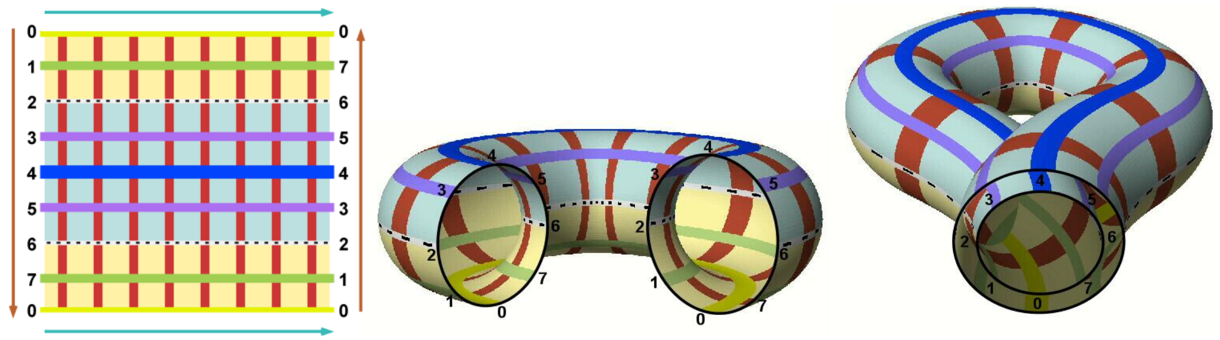


Figure 5: *Transformation of a Möbius band (**M3R=ML**) into a "Boy cap" (**BcL**) – a Boy surface minus a disk – through a gradual widening of the band [5].*

4. The Classical Klein Bottle

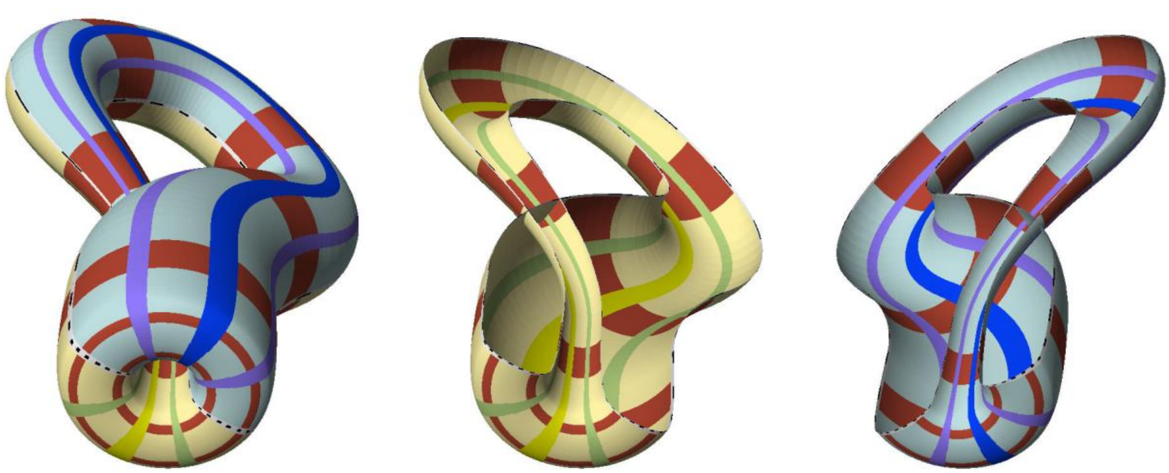
We begin our investigation of the realm of Klein bottles with a formal construction of the well-known "inverted sock" shape. We start with the parameterized, decorated, rectangular domain shown in Figure 6a. As for the case of torus construction, we first merge the two (horizontal) edges marked with parallel cyan arrows to form a generalized cylinder. For now we give this tube simply a round or oval profile (**O**) (Fig.6b). We then close this tube into a **J**-shaped loop (Fig.6c), so that its two ends can be joined with the reversed orientation indicated by the labelling and by the two anti-parallel brown arrows in Figure 6a.

Since we are now dealing with single-sided surfaces, I will use a somewhat different coloring scheme from the one I had used for tori [20]. While I maintain the color **red** for the ***meridian*** bands, I split the domain into two halves with different background colors and give their center bands a fully saturated color, either b**l**ue or ye**ll**ow, to mark the ***parallel*** parameter lines; these two regions will always form the two Möbius bands into which a Klein bottle can be partitioned. Parallel to these central bands I draw lines of less saturated color (o**l**ive, purp**l**e) that will wrap twice around the Klein-bottle loop, thereby executing an even number of 180° flips. Thus a key difference to *Tori Story* is that not all parallels are the same any more. Because of the reverse labelling along the two vertical edges of the rectangular domain, there are only exactly two parallels that meet up with themselves; those form the center-lines of two Möbius bands with ±180° twist. All other parallel parameter lines form double loops with a ±360° twist.



(a) (b) (c)

Figure 6: *Formal construction of a Klein bottle: (a) its rectangular domain; (b) the domain rolled into a tube; (c) the tube bent into a **J**-formation with its two ends lined up in a concentric manner.*



(a) (b) (c)

Figure 7: *The ordinary "inverted sock" Klein bottle resulting from the construction in Figure 6: (a) the complete Klein bottle **KOJ**; (b) the lower half of it is a right-handed Möbius band (**MR**); (c) the upper, left-handed Möbius band (**ML**) shown flipped over.*

For the tube with the **O**-profile, the closing of the loop is most conveniently done by narrowing one end and inserting it sideways into the larger end of the tube – forming some kind of **J**-shape (Fig.6c); this configuration properly lines up all the numbered labels. The two concentric ends are merged by turning the smaller one inside out to yield the classical "inverted sock" Klein bottle, named **KOJ** (Fig.7a).

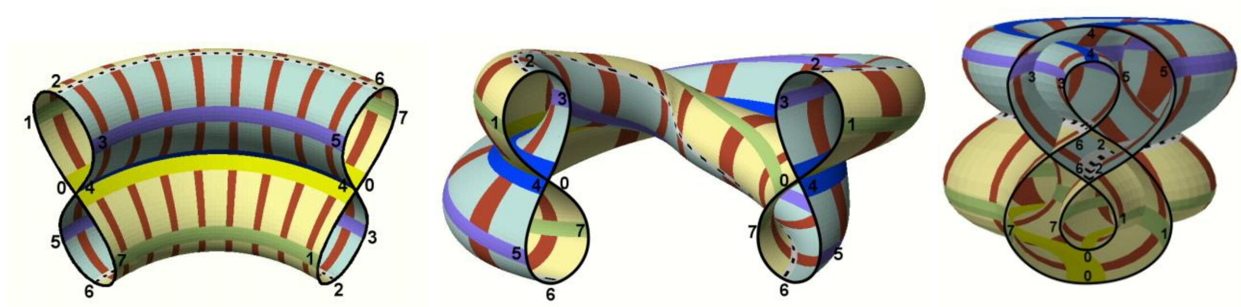
In this case, both "special" parallels form two Möbius bands of opposite handedness. Correspondingly, the **KOJ** Klein bottle can be partitioned into two Möbius bands (yellow and blue) having opposite handedness as shown in Figures 7b and 7c. We can denote this symbolically as:

MR + **ML** = **KOJ**.

The *Tori Story* paper [19] spent much effort to analyze the amount of twist built into the toroidal ring, because tori that differ in their amount of built-in twist by 360° belong to different regular homotopy classes. Interestingly, for the classical Klein bottle twist is a non-issue! By rotating the handle around the symmetry axis of the "inverted-sock" turn-back mouth, any amount of twist can be added to or removed from this **J**-loop [16]. This is a consequence of the fact that in Figure 6a the labeling at the left and right sides of the fundamental rectangular domain can be shifted up or down in a cyclic manner by any arbitrary amount, and there will always be exactly two labels that are lying on the same *parallel* parameter lines. These two parameter lines then form the center-lines of the two Möbius bands of opposite handedness into which this Klein bottle can be decomposed.

5. Figure-8 Klein Bottles

When merging the two parallel edges marked by cyan arrows in Figure 6a to form an initial tube, we are not forced to form a round, circular **O**-profile. Instead we may form a figure-**8** cross section (Fig.8a) or an even more complicated, multiply-rolled generalized cylinder as was discussed for tori [19]. This results in various Klein bottles that may belong to different regular homotopy classes and which thus cannot be smoothly deformed into the classical "inverted-sock" shape.



(a) (b) (c)

Figure 8: *Construction of figure-8 Klein bottles: (a) figure-8 tube; (b) tube twisted through 180° so that the two ends can be merged into a toroidal loop; (c) a new way to line-up the number labels.*

For the tube with the figure-**8** profile, there are a few different ways in which we can fuse the tube ends with the required reversed orientation, and they will result in different Klein bottles. For instance, we can bend the tube into a simple toroidal loop (an **O**-shaped path) and give the figure-**8** cross-section a 180° torsional flip (Fig.8b). This flip can either be clockwise (right-handed: **R**) or counter-clockwise (left-handed: **L**) and this will result in two figure-**8** Klein bottles that belong to two different regular homotopy classes – for the same reason that the Möbius band shown in Figure 1 cannot be smoothly transformed into its own mirror image. We call the resulting two Klein-bottle classes **K8R-O** (Fig.9) and **K8L-O** (Fig.10), respectively. For both classes the two Möbius bands that form the Klein bottle, and which happen to intersect along their center lines, are also shown separately in Figures 9 and 10 (a, b).

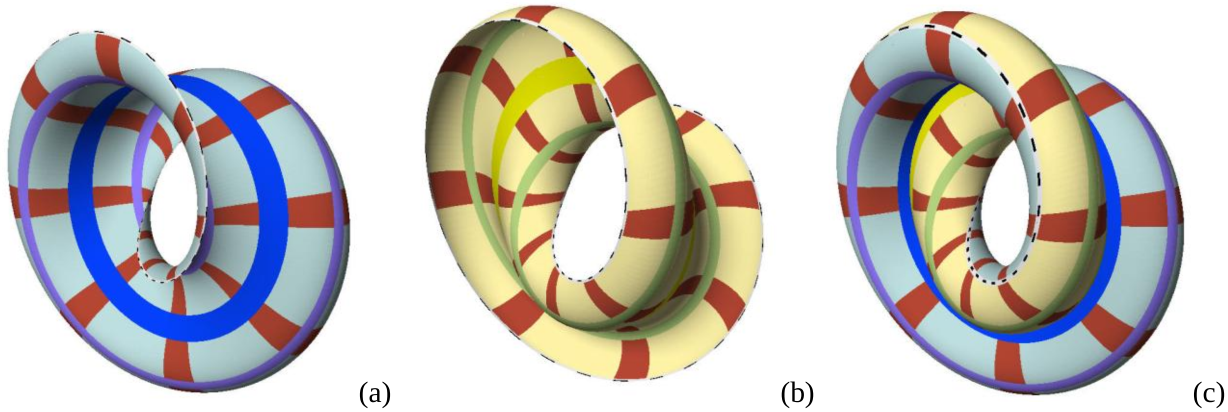


Figure 9: *Two right-handed Möbius bands* ***MR*** *(a,b) form a right-handed Klein bottle* ***K8R-O*** *(c).*

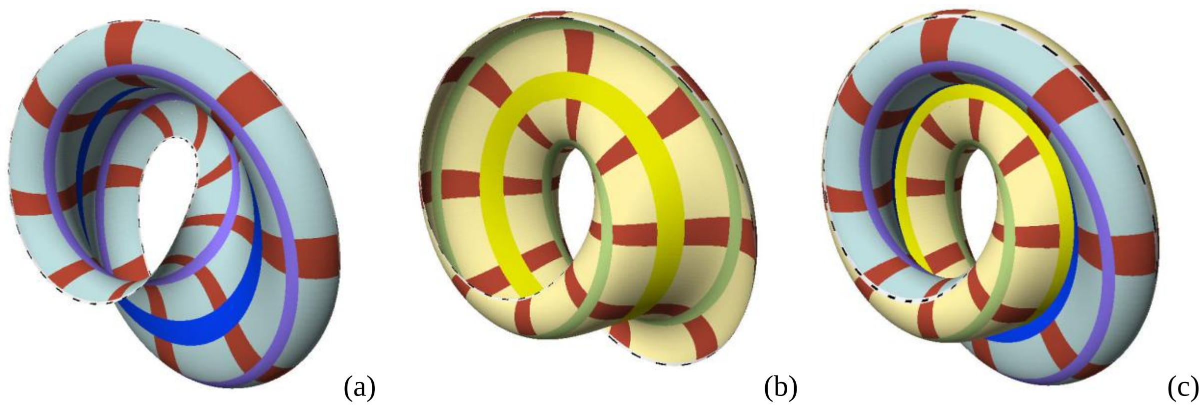


Figure 10: *Two left-handed Möbius bands* ***ML*** *(a,b) form left-handed figure-8 Klein bottle* ***K8L-O*** *(c).*

But there is even a third way in which the **8-**profile tube can be closed into a Klein bottle (Fig.11):

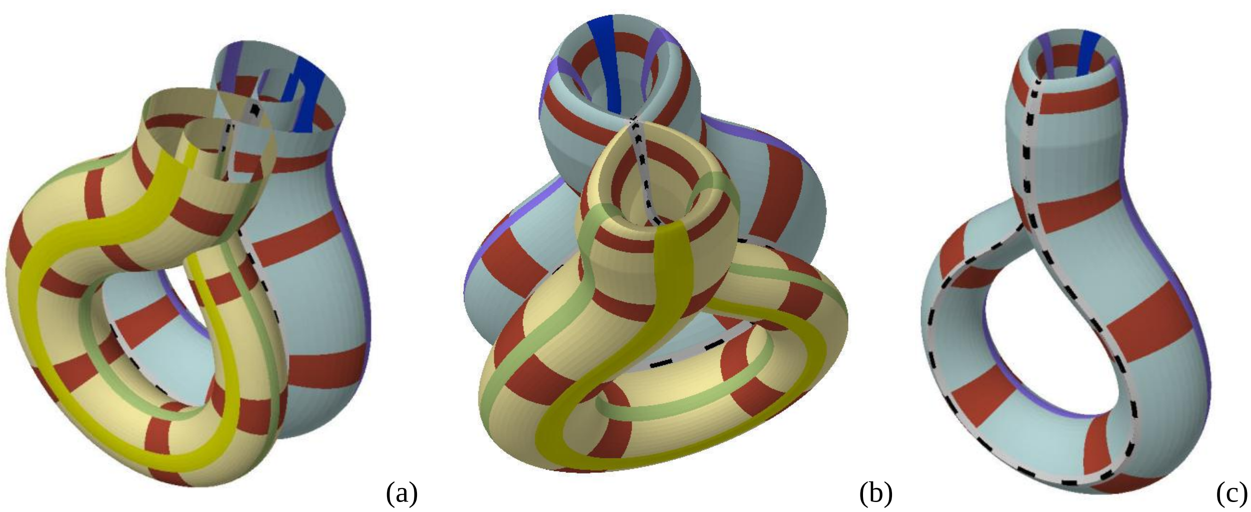


Figure 11: *A Klein-bottle based on an "inverted double sock": (a) without end-caps to show the nesting of the figure-8 tube profiles; (b) completed Klein bottle of type* ***K8L-JJ****; (c) one of its two* ***ML*** *Möbius bands.*

This new type of Klein bottle is constructed by forming the same kind of **J**-shaped sweep path as for the classical Klein bottle and fusing the two nested figure-**8** profiles by turning one of them inside out. To do this smoothly, we use an asymmetrical figure-**8** profile in which one lobe is larger than the other one. As we sweep from one end of the tube to the other one, the larger lobe shrinks and the smaller one grows, so that the end-profiles can be nested as shown in Figures 8c and 11a. Now a nicely rounded, **8**-shaped end cap can smoothly close off this Klein bottle (Fig.11b). I have not seen this particular Klein bottle depicted previously. It is rather special, since it does not just have a single self-intersection line like all the other models, but features two triple points. These triple points occur where the self-intersection line of the figure-**8** cross section passes through the wall of the other tube near the mouth of the Klein bottle. If we split this Klein bottle into its two constituent Möbius bands, they can be separated cleanly along the closed self-intersection loop passing through both triple points. Each component on its own then forms an "inverted sock" Klein bottle shape with a sharp crease line, which itself experiences a 90° twist (Fig.11c).

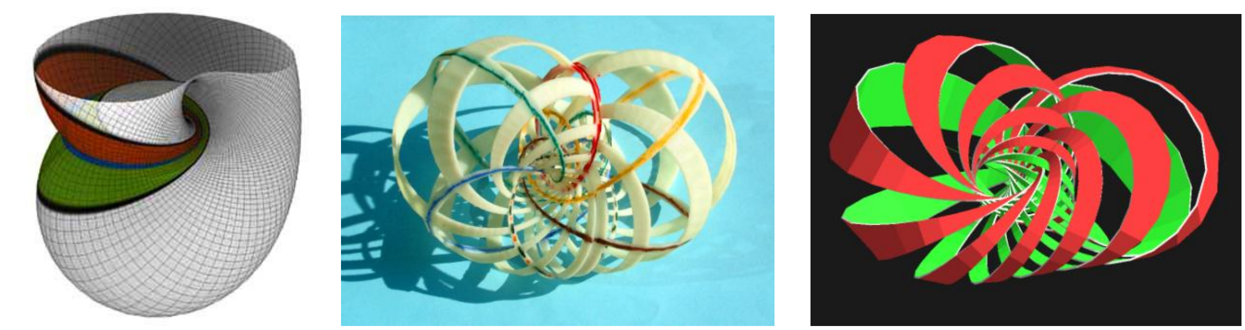
6. Looking for the 4th Type of Klein Bottle

When I first found the Klein bottle shown in Figure 11b I thought it might be the 4th type of bottle that the paper by Hass and Hughes [10] predicted. However, an analysis of the twistedness of its meridians and of its two Möbius bands quickly showed that this shape has C_2 rotational symmetry and is composed of two Möbius bands of the same type. If this shape were different from the three types of Klein bottles already identified, then the mirror image of this new shape would also have to be a separate, different type. But there can only be four different Klein bottles. Thus the two mirror-image versions of this double-sock type must also belong to the classes **K8L** and **K8R**. To distinguish their geometries from **K8L-O** and **K8R-O** we name them **K8L-JJ** and **K8R-JJ**.

So now that we have two attractive representatives for each of these two **K8?** classes, which one should we choose as the most natural one? Can we find a more objective measure for defining the "best" shape? Perhaps the shape with a minimal amount of mean bending energy could be used in this context. This measure, known as Willmore energy [24], and which integrates the square of mean curvature over the whole surface, is scale-independent and thus well-suited for this purpose.

7. Minimum Energy Klein Bottle

Lawson [12] has defined the minimum energy forms for many topological shapes. Among others he has introduced the intriguing shape shown in Figure 12, nicely illustrated by Polthier (Fig.12a) and well explained on his web page [15]. Kusner [11] has conjectured that the Willmore energy [24] of several Klein bottles described by Lawson is indeed minimized by this particular shape.



(a) (b) (c)

Figure 12: *Lawson minimum-energy Klein bottle: (a) Shell with center portions of Möbius bands [15] (top cut off); (b) an FDM model in which the parallel parameter lines have been hand-colored; (c) the set of circular, but twisted meridians.*

Could this be a candidate for the sought-after 4th Klein bottle type? -- No, it is not a candidate. This surface clearly displays a sense of chirality: Along the circular intersection line, we find two Möbius bands of the same twistedness, shown in green and brown in Figure 12a. Because of the limited number of Klein bottles of differing regular homotopy (four), the two mirror versions of the Lawson Klein bottle must again be in the same regular homotopy classes as **K8L-O** and **K8R-O**. To single out these shapes, we denote them as **K8L-Lawson** and **K8R-Lawson**, respectively.

In this case it was not too difficult to convince myself that the two Lawson Klein bottles (**K8?-Lawson**) are in the **K8?-O** classes, and that there is a smooth regular homotopy move that will bring about the conversion between the two geometries. My conceptual model starts from two identical Möbius bands passing through one another at right angles along a circular intersection line. Both bands show rainbow coloring in their longitudinal direction (Fig.13). To form a Klein bottle of type **K8R-O**, one can add figure-8-shaped meridians that cross themselves on this intersection line. Three such meridional bands are shown in Figure 13a, and their color has been matched to the color of the Möbius bands at that cross-over point. Now we shift the two Möbius bands against one another along the intersection circle and let the ends of the two meridional half-loops remain attached to the Möbius bands into which they merge tangentially. This circular shift operation pulls the figure-8 shapes apart and deforms them into warped loops that appear to become more and more twisted as the shifting process proceeds. When the shifting has progressed around half the circle, the meridional bands start to look very much like the bands depicted in Figure 12c. Note, however, that this is not the exact transformation needed to obtain the Lawson Klein bottle. To obtain that exact shape, the shifting would have to be non-linear and result in some compression of the density of the meridional bands along some sector of the intersection circle. Nevertheless, this model made it plausible for me that the shapes are indeed transformable into one another by a relatively simple regular homotopy move.

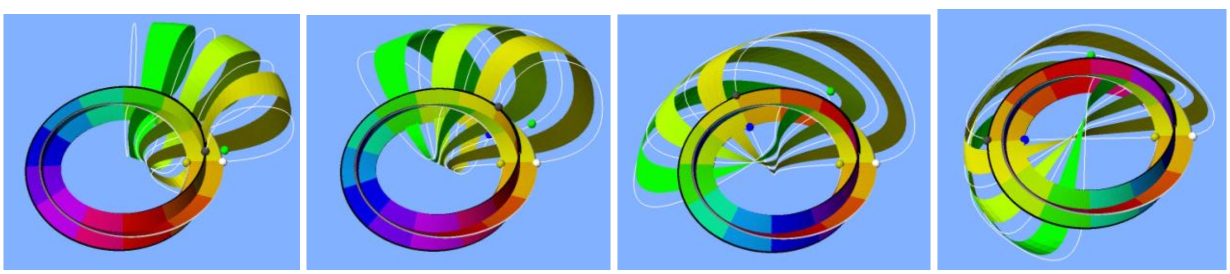


Figure 13: *Conceptual conversion of **K8R-O** into a minimum-energy Lawson Klein bottle.*

Even though the Lawson Klein bottle is an elegant minimum-energy shape, this configuration is still rather difficult to understand. I thus prefer to use **K8?-O** as the representatives for the two figure-8 classes of Klein bottles. But in any case, the search for Klein bottle type #4 needs to continue.

8. Change of Parameterization

In the case of tori, some discrete changes in the parameterization grid can yield tori that belong to different regular homotopy classes. In particular, the introduction of a Dehn twist [7] of 360° in the meridional direction (M-twist) converts a torus of type **TOO** into **TO8**, and a longitudinal (or "equatorial") Dehn twist (E-twist) takes **TOO** into **T8O** [19] [20]. Thus we will try this approach with Klein bottles, too. Let's investigate what kinds of parameter transformations make sense for Klein bottles.

8.1 The Role of Twist in Klein Bottles

Determining the twist of the characteristic bands, and predicting the result of Dehn twist added along characteristic parameter lines is much trickier for Klein bottles than it is for tori. Yet understanding the effects of parametrization changes is crucial to analyze how the domain of all decorated Klein bottles is partitioned into four regular homotopy classes. We will start by looking at the effects of twist along meridians and along longitudes for Klein bottles with circular as well as figure-8 shaped profiles.

Meridional Twist on a Figure-8 Profile

Let's take a Klein bottle of type **K8R-O** (Fig.14a) and cut it along its figure-8 meridian indicated by the black&white double line. Now we focus on a characteristic longitudinal ribbon, e.g., the central portion of one of the two Möbius bands (which follows the intersection line). We hold one of its ends in place while we move its other end once around the whole figure-8 cut line until it comes back to its starting position. This corresponds to adding 360° of meridional Dehn twist; the cut is along the black double line in Figure14b. Now let's analyze how the twistedness of the modified ribbon has been affected. Since the figure-8 loop in the meridional plane has a turning number of zero, the orientation of the moving end of the longitudinal ribbon experiences no net rotation, and thus the twistedness of such a ribbon remains the same. Since the twistedness of meridional ribbons does not change either, the classification of the Klein bottle does not change in response to this addition of M-twist. This also indicates that there must be a regular homotopy move that transforms the initial state into the twisted state, and vice versa.

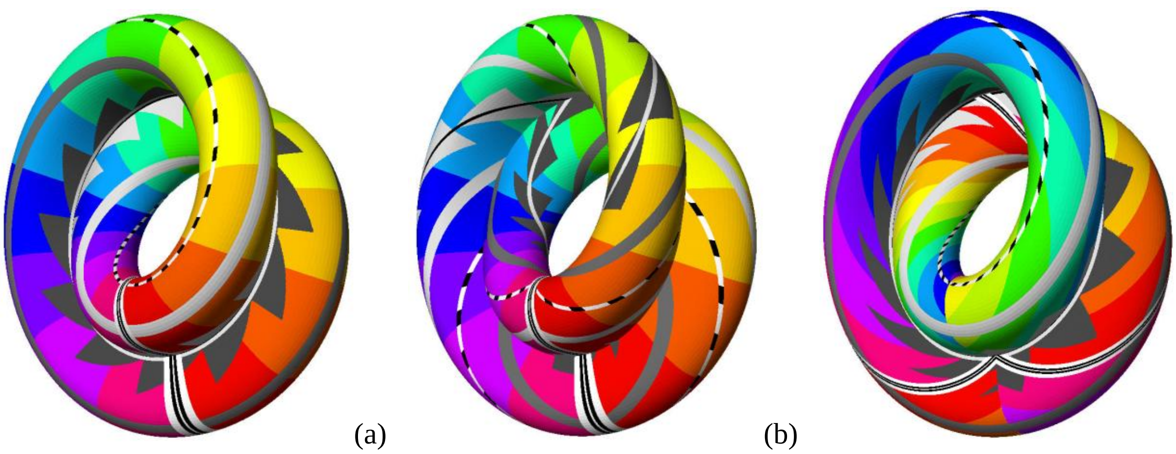

(a) (b) (c)

Figure 14: *(a) Klein bottle of type **K8R-O** with one of its meridional bands enhanced with a black&white double line.; (b) the same shape with a meridional Dehn twist of 360°; (c) the same shape with a longitudinal Dehn twist. (For texture map see Fig.17a.)*

Longitudinal Twist on K8?-O Klein Bottles

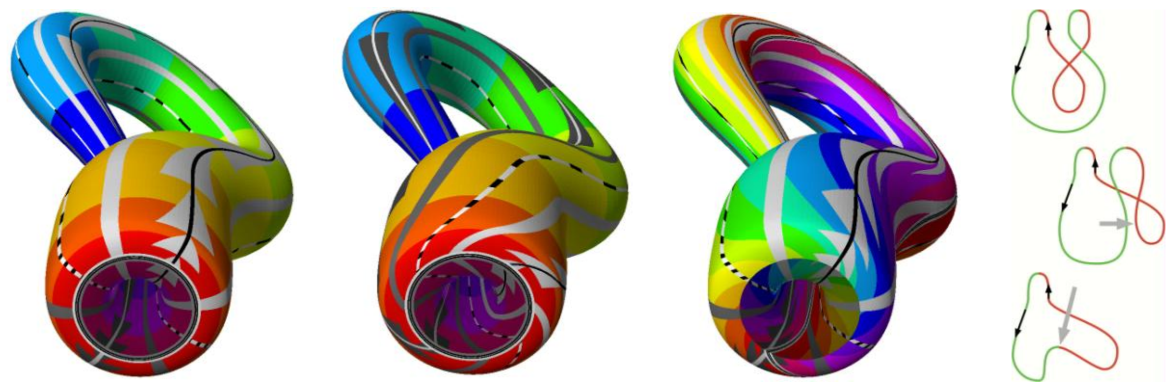
Now we cut the **K8R-O** bottle along a longitudinal line (the line of self-intersection in Figure 14c). A general instance of such a line goes around the loop of the Klein bottle twice before it joins up again with itself. Thus, if we sever a meridional ribbon on this longitudinal cut line, hold one end of it fixed in place, and send the other one traveling once along the whole longitudinal path, then the latter will make two laps around the circular sweep path and thus experience a net change in twistedness of 720°. Since twist is counted modulo 720°, this is again a transformation that does not change the regular homotopy class of the Klein bottle.

Meridional Twist on the KOJ Klein Bottle

Next we take a **KOJ** Klein bottle (Fig.15a) and cut it along a circular meridian, e.g., right at the rim of the mouth. At this location we take the inner branch of the tube and twist it 360° with respect to the other end. This will add a twist of 360° into any longitudinal ribbon crossing the cut (Fig.13b). Since a complete, general longitudinal path crosses this cut-line twice, the net increment in twistedness is again 720°, a value that will not change the classification of the Klein bottle. – This should come as no surprise, since it has already been established that twist in **KOJ** Klein bottles is a mute concept [16], because we can always remove any such twist by rotating the handle around an axis aligned with the cylindrical symmetry axis of the Klein bottle mouth, thereby changing the reference plane that indicates the edges of the two Möbius bands.

Longitudinal Twist on the KOJ Klein Bottle

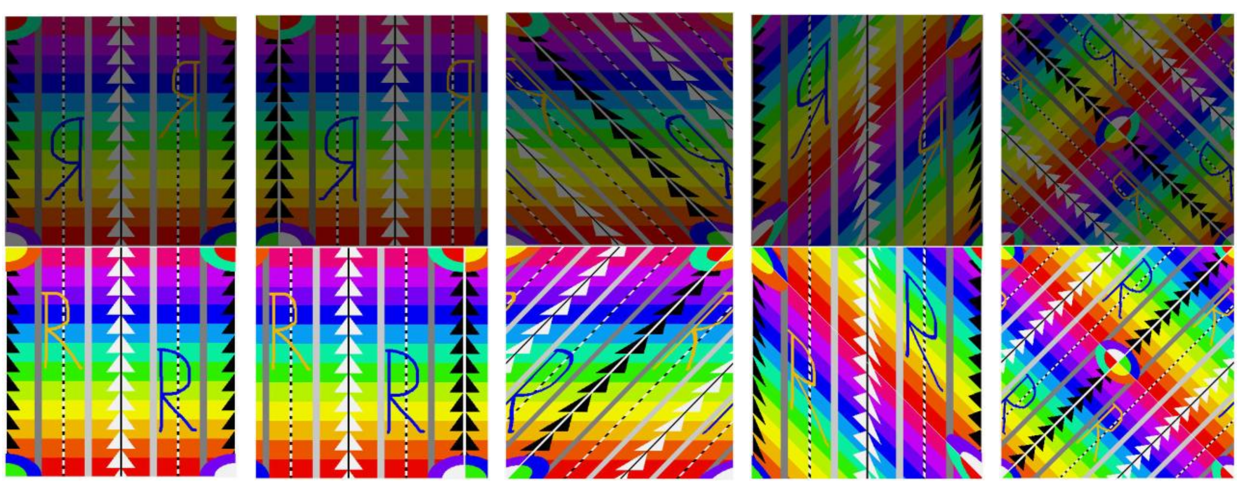
The last case to be analyzed involves a cut along a longitudinal path on the classical **KOJ** Klein bottle. Such a path will in general pass twice through the loopy handle and through the turn-back mouth. If we focus on a **KOJ** bottle with a planar sweep path and then analyze the turning number of the equatorial line, we find that it has a turning number of 1 (Fig.15d). Thus a meridional ribbon cut at one of the crossings with this line, with one end traveling around the longitudinal path, while the other end stays in place will experience the addition of 360° of twist. This amount of change in twistedness does make a change, and thus this twisted Klein bottle (Fig.15d) seems to be no longer in the same regular homotopy class for marked surfaces after the addition of 360° of longitudinal Dehn twist.



(a) (b) (c) (d)

Figure 15: *(a) Classical Klein bottle with one of its meridional rings enhanced; (b) the same shape with a meridional Dehn twist of 360°; (c) the same shape with a longitudinal Dehn twist. (d) Analysis of the turning number of a longitudinal ribbon on a **KOJ** Klein bottle.*

However, Figures 15b and 15c clearly show that something is wrong with a twisted (sheared) texture on a Klein bottle. At some point there is a seam where the tube joins back onto itself with an inside-out reversal. When we look at the effect of introducing twist on the texture map itself (Fig.16) we can see that



(a) (b) (c) (d) (e)

Figure 16: *Dehn twists as seen on the texture map for the Klein bottle: (a) untwisted texture map, (b) with 36° of meridional shift added, (c) with 360° of meridional twist added, (d) with 360° of longitudinal twist, (e) with both meridional and longitudinal twist.*

any kind of twist is simply inconsistent with the Klein bottle surface. Since the rectangular domain of the surface needs to connect to its “backs-side” with features reversed along the x-axis, in Figure 16, we show two repetitions of the texture map: the upper, darker one is the back-side, reversed left-to-right. Any M-shift applied to the texture will move features in opposite directions along the junction line (Fig.16b). An M-twist will result in opposite slope lines that produce kinks in the texture along the junction (Fig.16c); and a longitudinal L-twist (a.k.a. E-twist) will also result in kinks where the now diagonal meridians come together and also exhibit the intersected features in reverse order (Fig.16d). Thus addition of any kind of Dehn twist is not an operation through which we can expect to obtain a novel, yet still legal, Klein bottle that belongs to a different regular homotopy class.

8.2 Other Parameterization Changes

Parameter Swap

We can also rule out the parameter swap that exchanges the roles of meridians and longitudes. On tori this was a permissible and interesting operation. For Klein bottles this operation must also be disallowed, since meridians exhibit twist in integer multiples of 360° and thus form two-sided bands; while there are some longitudes that exhibit 180° of twist and thus form single-sided entities.

Eversion Operation

For tori there was also an evert operation. It is somewhat misleading for this discussion to assume that tori were painted with two different colors, one for the “inside” and the other for the “outside.” One should rather depict these mathematical surfaces as infinitely thin glass surfaces that carry the chosen texture within them so that it can be seen from both sides. In order to know which side we are facing at any particular moment, we give the surface texture some directionality along both coordinate axes. The relationship between the viewing direction and the cross-product of the *meridional* and *longitudinal* direction vectors then determines which side faces the camera. Because of the issues discussed in the previous section, meridians cannot be assigned a consistent direction. Thus no consistent cross-product can be formed, and the surface becomes indeed *non-orientable*. Thus the evert operation is also meaningless for Klein bottles.

Reversing Directionality

However, directionality in the longitudinal direction still makes sense for the **KOJ** bottle, because the Klein bottle mouth is asymmetrical in the direction of travel along the tube first formed in Figure 6b. Thus reversing a texture with directionality in the longitudinal direction is a non-trivial operation on **KOJ**. This seems to be the only parameterization change that has a chance of affecting the regular homotopy class of the Klein bottle subjected to it!

9. Grafting Folds onto Klein Bottles

In our analysis the regular homotopy classes of tori [20] we have seen that we can also obtain all four different representatives by starting with a simple torus and then in turn graft a folded-over surface strip onto one of the three characteristic lines (meridians, parallels, and diagonals). It seems worthwhile trying the same operation with Klein bottles.

Meridional Grafts on KOJ

We can apply a meridional graft anywhere along the tube forming the Klein bottle. In particular we can use the meridian at the rim of the mouth (Fig.17b), which then results in the **collared** structure **KOJ-C** shown in Figure 17c. The extra fold of this graft forces the tube to undergo an extra reversal and thus changes the directionality by which the texture passes through the mouth. While in Figure 17b the arrows seem to flow out of the mouth, in Figure 17c they flow into it. It turns out that we can easily eliminate the additional circular intersection line created by the grafted-on fold by inflating the blue/purple part of the tube; we then simply obtain a Klein bottle in which the thin and the thick arm emerging from the mouth have been switched. Structurally, this makes no difference, since the original **KOJ** could simply be rotated 180° around the mouth axis to coincide with the new shape. However, if we place a texture on the surface that exhibits directionality in the longitudinal direction, then the result is indeed in a different regular homotopy class. – Thus we have found the elusive #4 representative, as conjectured at the end of the previous section! Here is the analysis of the twistedness of the various characteristic ribbons in response to the added meridional graft: The meridional ribbons are not affected. But the longitudinal Möbius bands, which pass through the grafted collar exactly once, experience an extra twist of 360° which reverses their chirality; the left twisting Möbius band becomes right-twisting, and vice versa.

(a) (b) (c) (d)

Figure 17: *(a) A new texture with longitudinal directionality; (b) classical **KOJ** with this texture applied; (c) **KOJ-C** with a collar at the mouth; (d) **KOJ-C** with inflated blue/purple branch.*

Meridional Grafts on K8?-O

We can also graft a fold onto a meridian of a Klein bottle with a figure-8 cross section. Such a fold travels on the outside for half the figure-8 and on the inside for the other half of the profile, but it closes smoothly onto itself. Each Möbius band – a characteristic line of this surface – crosses this fold exactly once and thus changes its twist by 360°, which reverses its twistedness. Other longitudinal lines pass the grafted folds twice and thus do not change their twistedness – as is true for all but the Möbius center lines when a Möbius band is turned into its mirror image. Also, meridional bands are not affected. Thus, adding the figure-8 shaped meridional graft will turn **K8R-O** into **K8L-O** and vice versa.

Let's check what happens when we simply reverse the directionality in the longitudinal direction on a textured Klein bottle of type **K8?-O**. With a 180° flip through 3D space we can negate the effect of such a reversal. But this rotation does not affect the twistedness of the structure. Thus it does not change the regular homotopy class of the Klein bottle.

Longitudinal and Diagonal Grafts

Now let's try a graft along a longitudinal line. First we pick a general longitudinal that runs twice around the Klein bottle loop, so that the grafted fold ends up on the same side that it started on and thus merges nicely with itself. It is hard to visualize such grafts on the Klein bottle itself. Therefore let's look at these grafting operations on the texture map of the Klein bottle.

In Figure 18a a meridional graft line is shown, and we can see that the center lines of the two Möbius bands, indicated with chains of black and white arrows respectively, cross this line exactly once, which causes them to flip their chirality, as already discussed above.

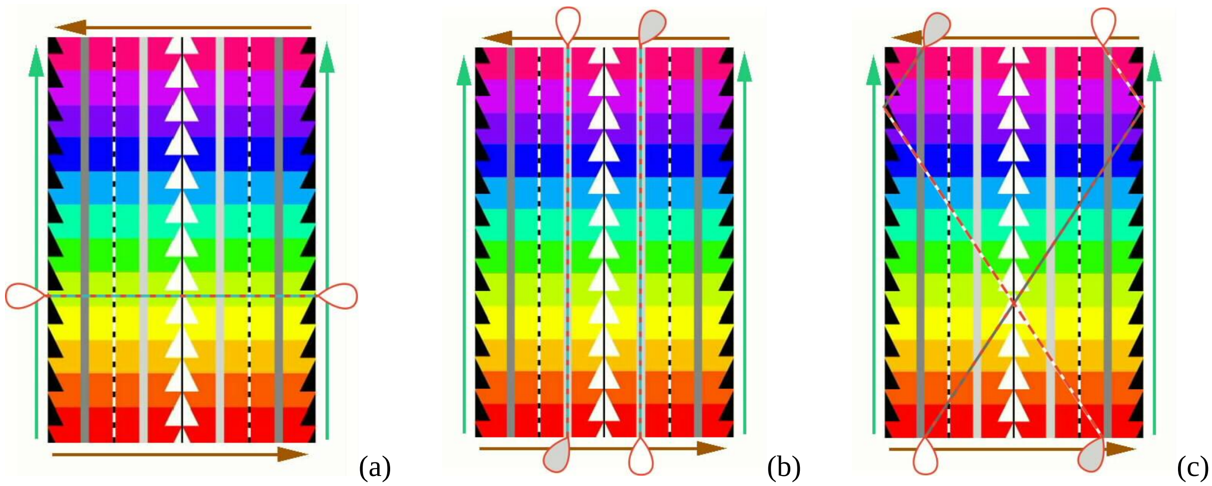


Figure 18: *Graft lines shown on the texture map: (a) meridional graft line, (b) a longitudinal graft line travelling around the loop twice; (c) a single diagonal graft line.*

Next, Figures 18b and 18c show longitudinal and diagonal graft lines in arbitrary positions. Every meridian crosses such a graft line twice and thus its twistedness is not affected. Neighboring longitudes also remain un-affected, and thus the regular homotopy class remains the same.

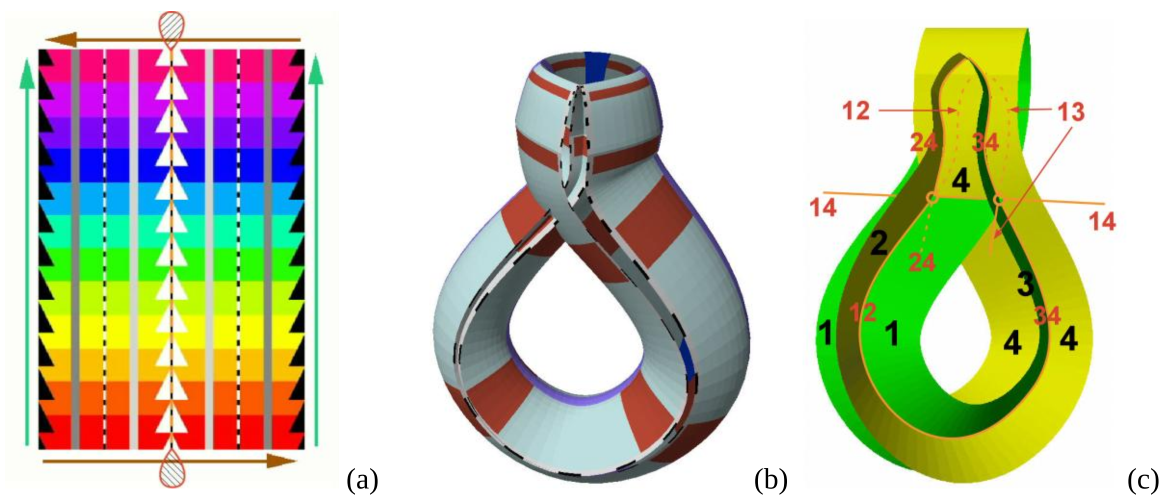


Figure 19: *Klein-bottle-grafts along Möbius center lines: (a) Klein-bottle-graft line on texture map, (b)* ***K8L-JJ*** *cut open along the rim of its left-twisting Möbius band, ready to be grafted with a copy of itself; (c) the boundary zone of the merged Möbius ribbons and an analysis of the resulting graph of intersection lines and triple points.*

Klein-Bottle Grafts on KOJ

However, the longitudinal graft lines should really follow the characteristic lines represented by the centers of the Möbius bands, as shown in Figure 19a. The difficulty is that when we swing the graft loop once around such a Möbius line, it ends up on the other side of the surface and cannot readily close on itself. However, we can use the same trick that **KOJ** uses itself to let the tube close on its own back-face.

In the same way we let the graft loop pass through a Klein bottle mouth inversion and then nicely close on itself. That is what happens in Figure 11. We can look upon Figure 11c as an ordinary **KOJ** Klein bottle with the B&W dashed line marking the center of its right-twisting Möbius band. We now turn this line into a crease and then graft on it another Klein-bottle-graft (of type **ML**) to produce the shape in Figure 11b. Figure 19b shows the blue left-twisting Möbius band (**ML**) of the **KOJ** bottle split open along this crease line; and Figure 19c shows the geometry of the intersection zone after this "KbL-graft." Note that in this process the intersecting surface strips crossing along the graft line have merged into one another at the cross-over at the mouth of the Klein bottle and now form a single double-sided band. Figure 19c further analyzes the resulting graph of intersection lines and triple points. The black labels "1" through "4" mark the dominant vertical portions of this band. The red labels then mark various segments of the graph of intersection lines connecting the two triple points with the pair of ribbon parts involved.

Meridians pass only once through this grafted loop and thereby change their twistedness by ±360°, i.e. going from a circular band to a figure-8 band. This changes the regular homotopy class. The grafted **KOJ** bottle has transformed into type **K8L**.

10. Map of the Regular Homotopy Classes of Klein Bottles

With the insights gained in the previous sections, we are now ready to draw the complete map of the regular homotopy classes for Klein bottles as was done for tori last year [20]. In summary we see that there are now three structural domains, one of which splits into two parts when the surface parameterization is taken into account (Fig.20).

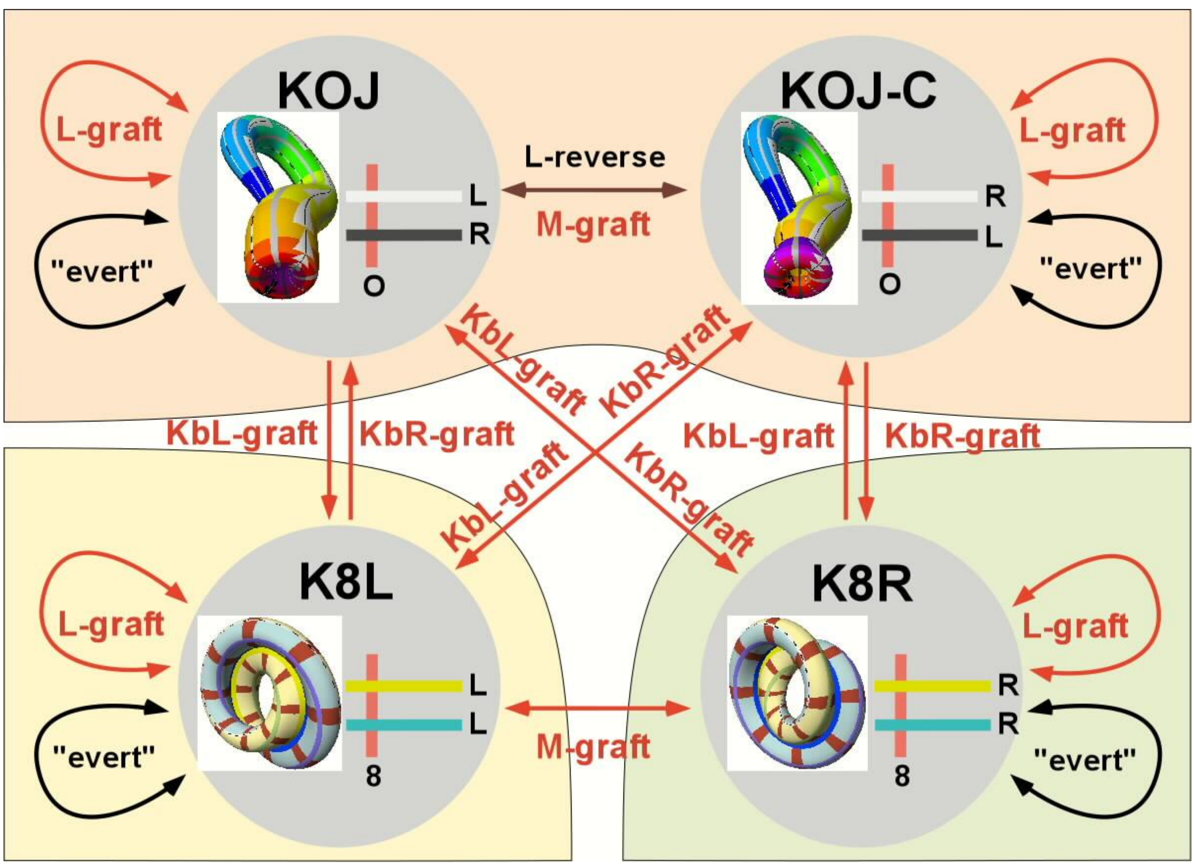


Figure 20: *Complete map of the effects of re-parameterization on the different Klein bottle classes.*

Regular homotopy transformations cannot perform simple mirroring operations; they cannot turn a left-twisting Möbius band into a right-twisting one. Thus we can immediately identify three different classes represented by: **K8L** = **ML**+**ML**, **K8R** = **MR**+**MR**, and by **KOJ** = **ML**+**MR**. These three regular homotopy classes are thus structurally different; they do not depend on any markings of the surface. This exposes yet another difference to the world of tori [19][20], where there are only two structural classes: one formed by the tori **TOO**, **TO8**, and **T8O**, and the other one by **T88** by itself.

For **K8L** and **K8R** we have encountered several interesting geometrical realizations: the twisted figure-8 geometry of **K8L-O** (Fig.10c), the "inverted double sock" configuration **K8L-JJ** (Fig.11b), the Lawson minimum-energy bottle **K8L-Lawson** (Fig.12), and their respective mirror images. As the generic representatives for these classes, I prefer the twisted figure-8 geometry; it is structurally very simple and easy to reconstruct mentally.

For marked Klein bottles we expect to see one more disparate regular homotopy class. It has to be composed of two Möbius bands of the same twistedness; thus structurally it belongs into the same class as **KOJ** and it can only be distinguished from the classical **KOJ** if we place some markings (e.g., a partly directional coordinate grid) on its surface, as discussed in Figure 17.

11. Analyzing and Classifying Complex Bottles

Alan Bennet has constructed some elaborate glass sculptures, which are exhibited at the Science Museum in South Kensington, UK [1]. These models range from an "inverted sock" type with multiply looped (Fig.21a) or helically twisted (Fig.21b) handles to contraptions that exhibit multiple "cross-handles" (Fig.21c) or nest several Klein bottle shapes inside one another (Fig.21d). The latter are topologically no longer simple Klein bottles, since they may be of higher genus or they may form multiple, individual, but interpenetrating surfaces. Other interesting examples of Klein bottles can be found on the home page of Cliff Stoll [22]. All these examples serve as good study object to train one's skills in determining the regular homotopy class of a particular glass sculpture.

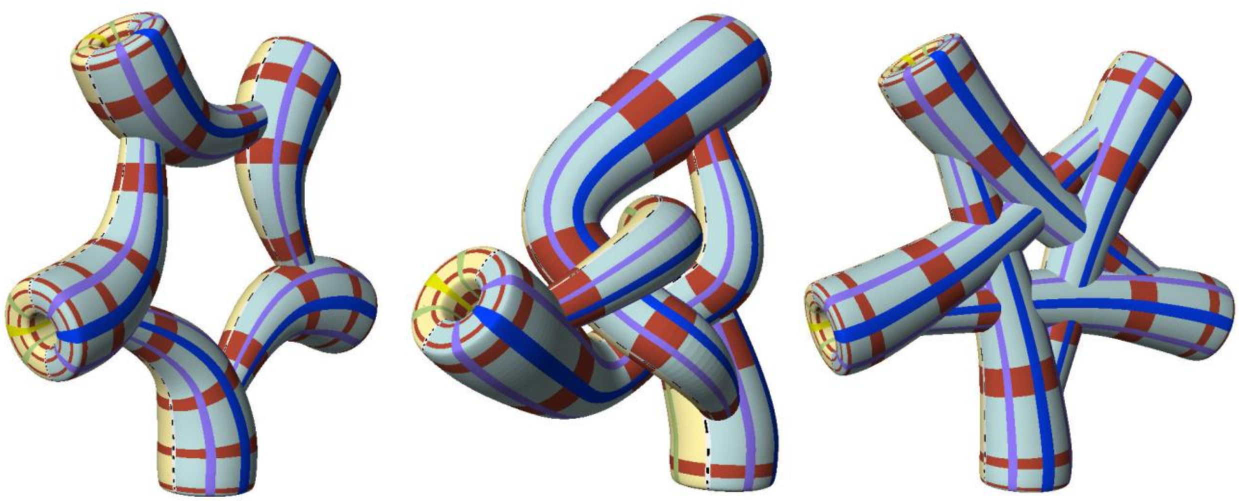
(a) (b) (c) (d)

Figure 21: *Glass work by Alan Bennet exhibited at the Science Museum in South Kensington, UK.*

Figure 20 gives us an idea how we might determine the type of such a fancy, twisted or knotted Klein bottle. For objects with no directional surface decorations, we only need to figure out into which structural class they belong. First we make sure that the surface is indeed single-sided and has Euler characteristic $\chi = V - E + F = 0$; i.e., that it is essentially a single, contorted, self-intersecting "tube," possibly with some "inverted sock" turn-backs – but with no branching. Then we start drawing a set of three "parallel" lanes and wind this "highway" over the surface until it joins itself again. If each of the three lanes joins itself and exhibits a twist that is an even multiple of 180°, then we have found a set of meridians, and based on whether it is twisted or not, we can readily determine whether the surface is in one of the **K8R/L** classes or in **KOJ**. If only the center lane joins itself, and the other two lanes form a double-loop over the surface, then we have found one of the two Möbius bands. The portion of the surface not yet covered must then form the other Möbius band. We need to determine the twist of both of them! To find a representative meridian strip, we start a second "3-lane highway" roughly perpendicular to the first one and look for a way to let it close on itself after only a single intersection with each of the two Möbius bands. The twistedness of these three bands uniquely characterizes the structural regular homotopy class of the shape in question.

12. Klein Knottles

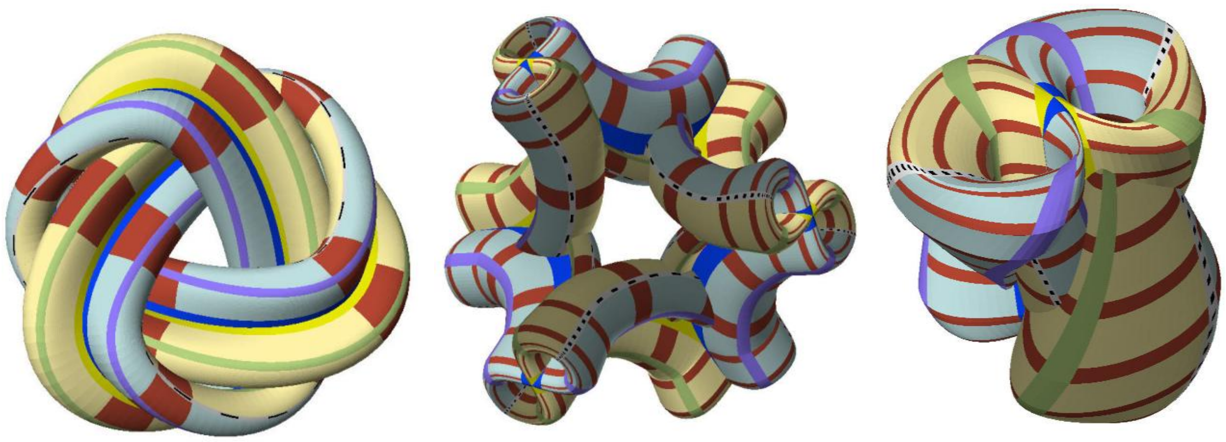
Inspired by the creative bottle shapes shown above, I searched for other intriguing geometries that topologically are proper Klein bottles. Given my investigation of knotted shapes in the past [17] [18], it was natural for me to look for knotted varieties of Klein bottles; I call these geometries "Klein Knottles." A simple way to enhance the visual complexity of a Klein bottle is to put more than one – but typically an odd number – of "inverted sock" turn-backs in series (Fig.22a). All of these surfaces belong in class **KOJ**; pairs of subsequent turn-backs can always be created or eliminated by doing an inversion of the segment in between using the Cheritat eversion move [6], which is also illustrated in Figure 5 in [19]. Any chain of such turn-backs can readily be deformed into a knotted geometry, such as the simple trefoil shown in Figure 22b, since tube branches are allowed to slide through one another. Of course, any intermediate state in such a knotting process may also be the final target of an artistic representation of a Klein bottle, since there is no limit on the number of self-intersections allowed (Fig.22c).



(a) (b) (c)

Figure 22: *Examples of Klein Knottles: (a) ring of five **KOJ**; (b) three **KOJ** forming a trefoil knot; (c) five **KOJ** forming an interpenetrating Knottle.*

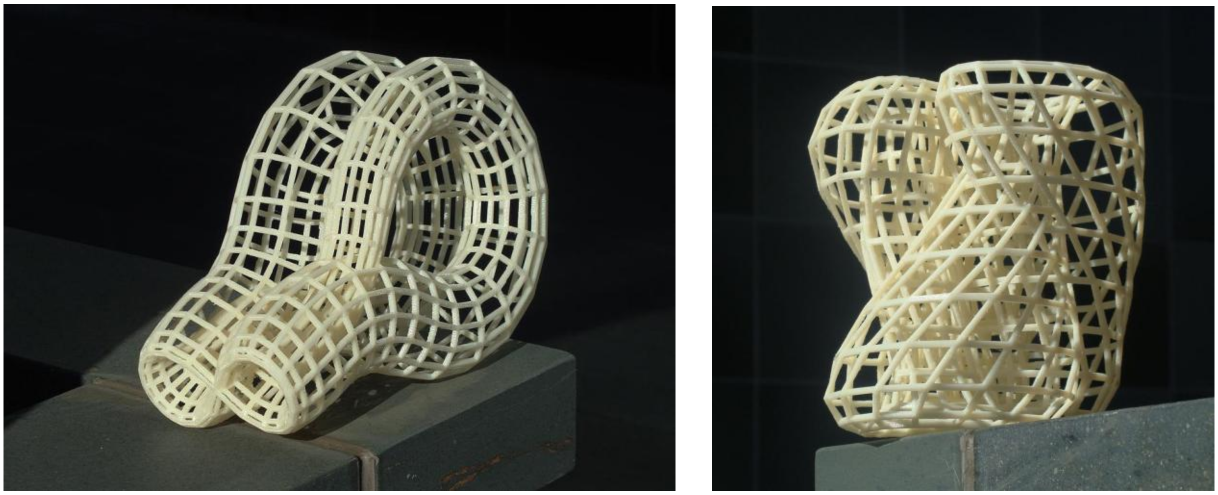
We can also form Knottles with figure-**8** profiles. The toroidal Klein bottles of type **K8R-O** and **K8L-O** can readily be deformed into any knot desired (Fig.23a) – just as long as we make sure that the cross section makes an odd number of 180° flips overall, so that we obtain the inside/outside switch-over needed to make a Klein bottle. We can also form chains with multiple figure-**8** turn-backs; each one of them acts as an inside/outside switchover. However, the figure-8 profile gives us another degree of freedom. It allows us also to accomplish a surface reversal by giving the figure-8 tube a 180° flip – or an odd multiple thereof. These two mechanisms of surface reversals can be combined: We may use an even number of 180° flips together with an odd number of Klein bottle mouths, or, alternatively, an even number of mouths together with an odd number of flips. Figure 23c displays a very "compact" Klein bottle geometry, **K8R-zz**, with two Klein bottle mouths, combined with an overall twist of 180°. This twist is applied in two portions of 90° each, as the figure-8 profile travels once up and down a portion of the *z*-axis. In one of these two passes, the profile also morphs into its twin shape, in which the sizes of the two lobes are exchanged.



(a) (b) (c)

Figure 23: *Klein Knottles with figure-**8** cross sections: (a) **K8OR** wound into a trefoil knot; (b) six **K8J** turn-backs with suitable twist; (c) two **K8J** turn-backs with 90° twist each.*

Figure 24 displays some physical models of these novel Klein bottles shapes. First there is a gridded version of the "inverted double sock" Klein bottle, **K8L-JJ**, already introduced in Figure 11b. This gridded, semi-transparent" model allows a better inspection of how the figure-8 cross section turns inside out at the Klein bottle mouth and then morphs from one asymmetrical configuration to another one, in which the uneven sizes of the two lobes are reversed. Figure 24b allows a similar "transparent look at the very "compact" Klein bottle geometry, **K8R-zz**, with two Klein bottle mouths, combined with an overall twist of 180°, introduced in Figure 23c.



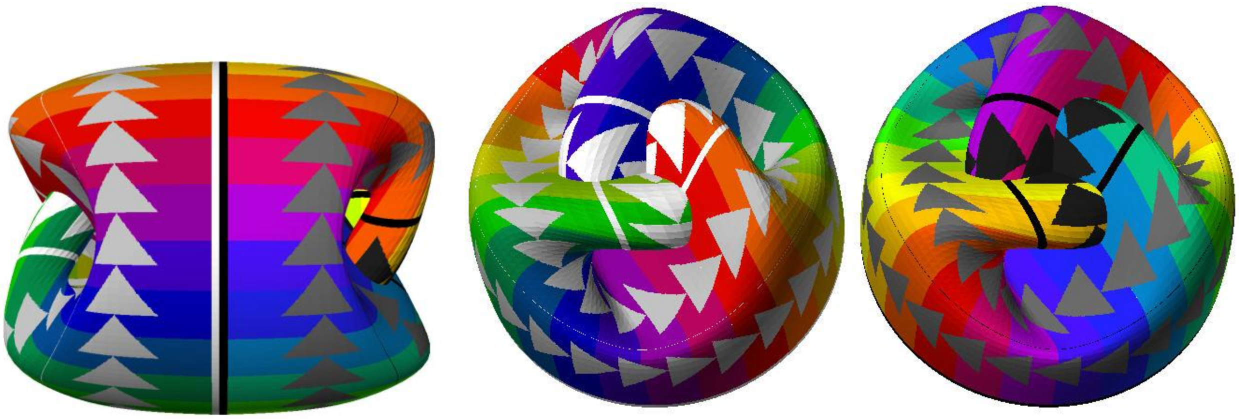
(a) (b)

Figure 24: *Unusual, heavily self-intersecting Klein bottle models: (a) The left-turning **K8L-JJ**; (b) a right-turning Klein bottle with figure-8 profile and a zig-zag sweep path, **K8R-zz**.*

13. Decomposition of Klein Bottles into Two Boy Caps

As mentioned in Sections 3, any model of the projective plane with a disk removed is topologically equivalent to a Möbius band. In particular, when we remove a disk from *Boy's Surface*, we obtain a nice, smooth building block with 3-fold symmetry, which I will call a *Boy cap* (**Bc**) that can be used to form interesting novel Klein bottles with 3-fold rotational symmetry. This special Möbius band also comes in two enantiomorphic forms, denoted **BcL** and **BcR,** respectively. I have taken care of defining a geometry that results in a perfectly circular rim (Fig.25). Two such Boy caps can then be joined at their boundaries (resulting from the removal of a disk) with a rotational degree of freedom. Joining two Boy caps of equal chirality (Fig.25b) results in one of the figure-8 Klein bottles (Fig.25a):

BcL + **BcL** = **K8L** and **BcR** + **BcR** = **K8R**.

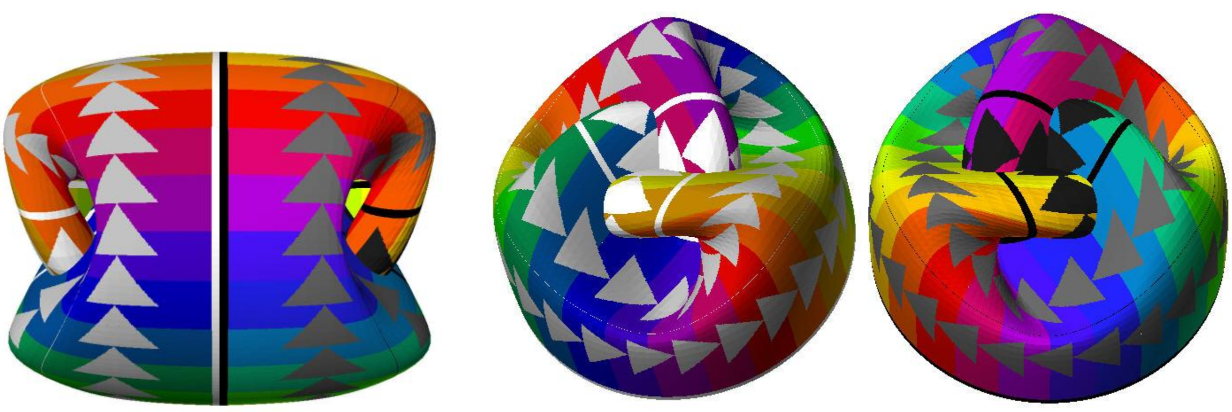


(a) (b)

Figure 25: *(a) Two Boy caps (**BcR**) of the same chirality form a Klein bottle of type **K8R**; (b) the two Boy caps (**BcR**) separated to show the complimentary sense of texture flow at the two ends.*

Joining Boy surfaces of opposite chirality (Fig.26b) yields a Klein bottle of the same structural type as the "inverted sock" model (Fig.26a).

BcL + BcR = KOJ.



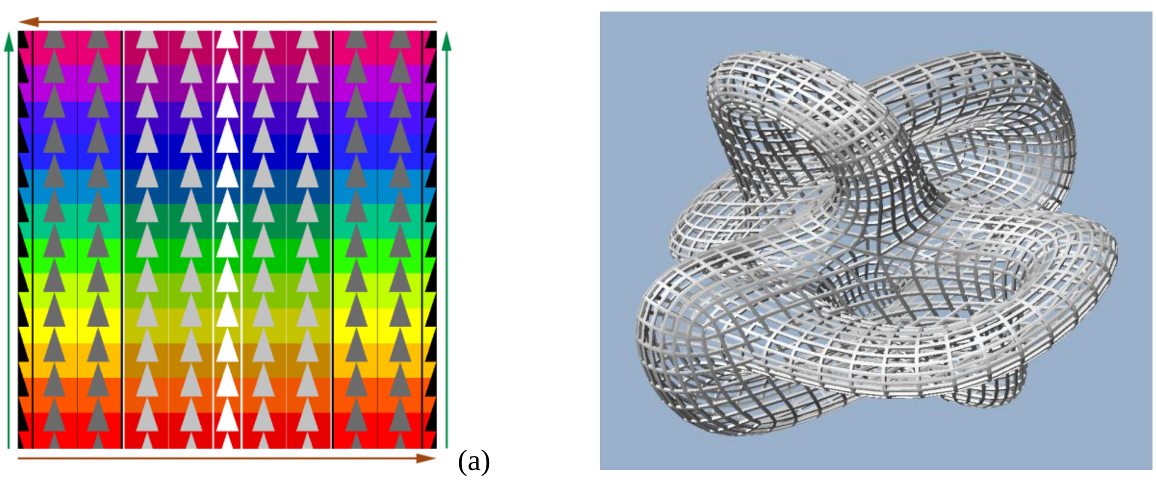
(a) (b)

Figure 26: *(a) Two Boy caps of opposite chirality form a Klein bottle of type **KOJ**; (b) the two Boy caps (**BcL**, **BcR**) separated to show identical direction of texture flow into the triple point.*

Here the surface markings make a difference! We now employ the texture map shown in Figure 28a. If on the Klein bottle shown in Figure 25a the direction of texture flow, as given by the black, white, and gray arrows, is reversed, we will still observe these arrows to flow into one of the two triple points and out of the other one. We can thus simply flip the Klein bottle end-to-end and bring the new texture flow again in alignment with the original one. On the other hand, in the Klein bottle depicted in Figure 26 the texture flows into the triple points at both ends. If the texture flow is reversed it then flows out of the triple points at both ends and there is no smooth transformation that brings these two geometries into alignment. Figures 26 and 27 thus depict two representatives of different regular homotopy classes corresponding to the two classes in the upper half of Figure 20. Figure 28b provides a look into the internal structure of a Klein bottle of type KOJ with an overall symmetry of S_6 composed of two enanthiomorphics Boy caps.

(a) (b)

Figure 27: *(a) Two Boy caps of opposite chirality (**BcL**, **BcR**) form a Klein bottle of type **KOJ**; in this case the texture flow has been reversed with respect to Figure 26; this results in a reversed direction of texture flow at both triple points (b).*



(a) (b)

Figure 28: *(a) Texture map used in the above models. (b) A gridded model of a Klein bottle (**KOJ**) composed from two enanthimorphic Boy caps (**BcL**, **BcR**).*

14. Conclusions

Klein bottles are fascinating geometric objects. Most people are familiar with only the "inverted sock" type (**KOJ**). Many mathematical texts also refer to "the other" Klein bottle – the twisted figure-**8** shape – which actually comes in two different chiral versions (**K8OL**, **K8OR**). Only a few papers or web pages also present Lawson's minimum-energy Klein bottle, which also comes in two chiral forms that belong to the same two homotopy classes. But there is also the possibility of forming turn-backs with a figure-**8** profile and I have not seen this one depicted beforehand (Fig.11).

Of course, there are infinitely many possible geometries that constitute Klein bottles, i.e., single-sided surfaces with Euler characteristic zero. Even with marked surfaces, all of these can be smoothly deformed into one of the four representatives presented in this paper. I have presented two untwisted, marked Klein bottles to distinguish the two versions of the "inverted sock" type, consisting of two Möbius bands of opposite handedness.

Finding the most elegant transformation that will actually accomplish the reduction of an arbitrary Klein bottle into one of these four representatives is a much harder task; it deserves more study. In the meantime the reader may simply enjoy the beautiful glass models created by Alan Bennet and Cliff Stoll, as well as the wild Klein knottles introduced in the last part of the paper.

Acknowledgements

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