Optimal Mixed Spectrum Auction

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Technical Report No. UCB/EECS-2013-129
http://www.eecs.berkeley.edu/Pubs/TechRpts/2013/EECS-2013-129.html

July 9, 2013

Acknowledgement

This work is supported by the NSF grant CNS-0910711 on Economics of the Internet.

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***Abstract*—This work studies the revenue-maximizing auction of a single block of spectrum that can be awarded either for exclusive licensed use by one operator or reserved for unlicensed use. A number of operators bid for exclusive licensed use and a group of non-colluding agents bid to keep the spectrum unlicensed. The revenue of this auction is compared to that of a Vickrey-Clarke-Groves (VCG) auction and that of another auction recently proposed.**

I. Introduction

The commercial success of WiFi demonstrates the value of unlicensed spectrum that can be shared by devices from many vendors. In contrast, cellular spectrum is licensed to individual providers for their exclusive use. Comparing the benefits of these two models, one faces a typical conflict between social welfare and provider profits that may argue in favor of unlicensed spectrum.

In this paper, we do not try to compare the merits of licensed versus unlicensed spectrum. Instead, we limit our study to an exploration of auctions for spectrum that can be used either for an exclusive licensed provider or for a collective of unlicensed users. The value of the spectrum for a licensed provider results from the subscriptions by users. The value for a collective of unlicensed users may be more difficult to quantify but is very real. The collective could consist of content or service providers and equipment vendors for whom the additional access to unlicensed spectrum corresponds to an increase in usage. Thus, even though this collective does not get any exclusivity for spectrum usage, the increase in market size may justify paying for the spectrum. It may very well be that this increase generates enough user welfare and even tax revenue to warrant the government giving away the spectrum for free for unlicensed use. However, political fairness considerations may justify an auction to avoid picking the winner in this conflict; moreover, the revenue that the auction generates can also be used to improve user welfare in some other ways.

The general situation is that of a set of spectrum blocks to be auctioned for either licensed or unlicensed use. A number of bidders are interested in exclusive licensed use of spectrum blocks and a collective of bidders want spectrum blocks for unlicensed use. The problem is to design auctions for this situation and to study their characteristics, such as the revenue they generate and the net utility for the bidders. Here, we explore simple versions of the problem where a single spectrum block is auctioned.

II. Revenue Maximizing Auction

The mixed spectrum auction described in the abstract is a particular case of the following auction. There is a single item to be offered and K groups $\mathcal{G}_1, \dots, \mathcal{G}_K$ of non-colluding agents. The item is given to one of the groups and every agent i of that group derives a valuation V_i. The valuations are independent random variables and V_i has a probability density function $f_i(v)$ that is positive on $[a_i, b_i]$ and zero elsewhere. Let $F_i(v) = P(V_i \leq v)$. We discuss the revenue-maximizing auction among all auctions that are *incentive compatible* and *individually rational*. Incentive compatible means that no bidder has any incentive to lie about his value estimate (honest responses must form a Nash equilibrium in the auction game). Individually rational means that bidders are not forced to participate in the auction (the expected payoff for every user is nonnegative).

Assume that

$$c_i(v) := v - \frac{1 - F_i(v)}{f_i(v)}, a_i \leq v \leq b_i \tag{1}$$

is non-decreasing for every agent. Notice that a sufficient condition is for the distribution to have a non-decreasing hazard rate $f_i(v)/(1 - F_i(v))$. This property holds for the uniform distribution, the exponential distribution, the Gaussian distribution, and many other distributions. (Note: The optimal auction can be derived when $c_i(v)$ is not non-decreasing, but we limit ourselves to this case for now.)

Theorem 1. *The following auction maximizes the revenue. The item is given to the group k with a maximal value of*

$$C_k(\mathbf{V}) := \sum_{i \in \mathcal{G}_k} c_i(V_i),$$

provided that this maximum value exceeds the auctioneer personal value estimate for the object v_0 (it could be zero); if it is not, the seller keeps the item. Also, if the item goes to group k, every agent i of that group pays $x_i(\mathbf{V})$ defined as the minimum non-negative valuation v that makes his group win the auction, i.e., such that

$$c_i(v) + \sum_{j \in \mathcal{G}_k \setminus \{i\}} c_j(V_j) \geq \max\{v_0, C_g(\mathbf{V}), g \neq k\}.$$

Proof:

The proof follows the argument of Myerson [1].

Let $\Pi_k(\mathbf{V})$ be the probability that group k gets the item when the bids are $\mathbf{V}$. Let also $\pi_i(\mathbf{V}) = \Pi_k(\mathbf{V})$ for $i \in \mathcal{G}_k$ and $\pi_i(V_i) = E[\pi_i(\mathbf{V})|V_i]$, the probability that agent i gets the item if he bids V_i. Finally, let $x_i(V_i) = E[x_i(\mathbf{V})|V_i]$ be the expected payment of agent i.

Myerson showed that any mechanism is bayesian incentive compatible if and only if the $\pi_i(V_i)$ are non decreasing in V_i and further the payments are determined from these $\pi_i(V_i)$ uniquely up to an additive constant $S_i(a_i)$. This leads to the following expression for the revenue of the buyer in any incentive compatible mechanism:

$$S_0 = E\left[\sum_i \pi_i(\mathbf{V}) c_i(V)\right] + \sum_i S_i(a_i). \tag{2}$$

Individual rationality further requires that $S_i(a_i) \leq 0$. The allocation rule for the proposed auction indeed possesses the property that $\pi_i(V_i)$ is nondecreasing and it is such that $S_i(a_i) = 0$. Moreover,

$$\pi_i(\mathbf{V}) = \Pi_k(\mathbf{V}), \forall i \in \mathcal{G}_k,$$

because all the agents of the same group get the spectrum together. Thus, we can rewrite (2) as follows:

$$S_0 = \sum_k E[\Pi_k(\mathbf{V}) C_k(\mathbf{V})].$$

Since the auction selects the group with the maximum positive value of $C_k(\mathbf{V})$, it maximizes S_0.

■

III. Examples

A. Uniform Distribution

Consider a valuation V uniformly distributed over the interval $[a, b]$. Then

$$c(v) = v - \frac{1 - F(v)}{f(v)} = v - \frac{1 - \frac{v-a}{b-a}}{\frac{1}{b-a}} = 2v - b.$$

This function is a monotone strictly increasing function and its inverse is

$$c^{-1}(y) = \frac{y + b}{2}.$$

Assume now that agent i has a valuation V_i uniformly distributed in $[a_i, b_i]$ and that the agents are in groups $\mathcal{G}_k$ as before. Then,

$$C_k(\mathbf{V}) = \sum_{i \in \mathcal{G}_k} c_i(V_i) = 2 \sum_{i \in \mathcal{G}_k} V_i - \sum_{i \in \mathcal{G}_k} b_i.$$

The item then goes to the group k with the maximal value of $C_k(\mathbf{V})$, if this value exceeds v_0, and the price of agent i is then the smallest value of v such that

$$2v + 2 \sum_{j \in \mathcal{G}_k \setminus \{i\}} V_j - \sum_{i \in \mathcal{G}_k} b_i$$

exceeds v_0 and $C_g(\mathbf{V})$ for $g \neq k$.

B. Exponential distribution

Consider a valuation V that is exponentially distributed random variable with rate $\lambda > 0$. Then,

$$c(v) = v - \frac{\exp\{-\lambda v\}}{\lambda \exp\{-\lambda v\}} = v - \lambda^{-1}.$$

This function is a monotone strictly increasing function and its inverse is

$$c^{-1}(y) = y + \lambda^{-1}.$$

Assume that agent i has valuation V_i exponentially distributed with rate $\lambda_i > 0$ and that the agents are in groups $\mathcal{G}_k$ as before. Then,

$$C_k(\mathbf{V}) = \sum_{i \in \mathcal{G}_k} V_i - \sum_{i \in \mathcal{G}_k} \lambda_i^{-1}.$$

These expressions enable to determine the winning group and the payments, as before.

C. Normal distribution

Consider a normally distributed random variable X with mean μ and variance σ^2. We know that its cumulative distribution function is equal to $F(x; \mu, \sigma^2) = \psi(\frac{x-\mu}{\sigma})$ where ψ is the cumulative distribution function of the standard normal distribution. We know that $Q(x) := 1 - \psi(x)$, known as the Q-function, is equal to (for details, see [2])

$$\forall x > 0 \quad Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \qquad (3)$$

$$\forall x \leq 0 \quad Q(x) = 1 - Q(-x). \qquad (4)$$

Then Myerson's virtual value of the normal distribution is

$$c(x) = x - \frac{1 - F_X(x)}{f_X(x)} = x - \frac{Q(\frac{x-\mu}{\sigma})}{f_X(x)}$$
$$= x - \left(\sqrt{2\pi\sigma^2} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right)\right) Q\left(\frac{x-\mu}{\sigma}\right).$$

Then

$$\forall x > \mu,$$
$$c(x) = x - \sqrt{\frac{2\sigma^2}{\pi}} \int_0^{\pi/2} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \cot^2\theta\right) d\theta,$$
$$\forall x \leq \mu,$$
$$c(x) = x - \left(\sqrt{2\pi\sigma^2} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right)\right) \left(1 - Q\left(\frac{\mu - x}{\sigma}\right)\right).$$

This function is a monotone strictly increasing function since $c'(x) > 0 \quad \forall x$. However, it is not easy to obtain a closed-form expression for its inverse.

Assume that agent i has valuation V_i normally distributed with mean μ_i and variance σ_i^2 and that the agents are in groups as before. Then

$$C_k(\mathbf{V}) = \sum_{i \in \mathcal{G}_k} c_i(V_i)$$
$$= \sum_{i \in \mathcal{G}_k} V_i - \sum_{i \in \mathcal{G}_k} \left(\sqrt{2\pi\sigma^2} \exp\left(\frac{(V_i - \mu)^2}{2\sigma^2}\right)\right) Q\left(\frac{V_i - \mu}{\sigma}\right).$$

The item then goes to the group k with the maximal value of $C_k(\mathbf{V})$, if this value exceeds v_0, and the price of agent i is then the smallest value of v such that

$$c_i(v) + \sum_{j \in \mathcal{G}_k \setminus \{i\}} c_j(V_j) \geq \max\{v_0, C_g(\mathbf{V}), g \neq k\}.$$

IV. Proposed Mechanisms

In this section, we explain other mechanisms for the licensed vs unlicensed auction. First, we consider a mechanism which have been recently proposed and then the VCG mechanism. We then proceed to compare these mechanisms with the optimal mechanism.

A. Bykowsky Mechanism

The authors of [3] propose a mechanism, which in the case of a single-item is equivalent from an auction theory perspective to a second price auction between groups with a proportional share of the price between the agents within the group with respect to their bids. In other terms, the item is given to the group k with a maximal value of $C_k(\mathbf{V})$. If the item goes to group k, every agent i of that group pays

$$\frac{b_i}{\sum_{j\in\mathcal{G}_k} b_j} \max_{j\neq k} C_j(\mathbf{V})$$

where b_j denotes the bid of agent j.

B. The VCG mechanism

The Vickrey-Clarke-Groves [4], [5], [6] mechanism, or VCG auction, is an incentive compatible auction where each bidder pays the harm he causes to other bidders. In our case, let us recall that there is a single item and K groups $\mathcal{G}_1,\dots,\mathcal{G}_K$ of non-colluding agents. The item is given to one of the groups and every agent i of that group derives a valuation V_i when his group gets the item. The agents of each group k have a joint valuation of $V^k(\mathbf{V}) = \sum_{i\in\mathcal{G}_k} V_i$.

Without agent $i' \in \mathcal{G}_k$, the maximized social welfare corresponds to

$$\max\left\{ \sum_{i\in\mathcal{G}_k\setminus\{i'\}} V_i \,;\, \max\{V^g(\mathbf{V}), g\neq k\} \right\}. \tag{5}$$

With agent $i' \in \mathcal{G}_k$, the welfare of other agents is given by

$$\begin{aligned} &\sum_{i\in\mathcal{G}_k\setminus\{i'\}} V_i \cdot \mathbf{1}_{\{V^k(\mathbf{V})>\max\{V^g(\mathbf{V}),g\neq k\}\}} + \\ &+ \sum_{k'\neq k} V^{k'}(\mathbf{V}) \cdot \mathbf{1}_{\{V^{k'}(\mathbf{V})>\max\{V^g(\mathbf{V}),g\neq k'\}\}}. \end{aligned} \tag{6}$$

The social cost of agent i' winning the object would be equal to the difference between eq. (5) and eq. (6), or equivalently,

$$\begin{aligned} &\left[\max\{V^g(\mathbf{V}), g\neq k\} - \sum_{i\in\mathcal{G}_k\setminus\{i'\}} v_i \right]^+ \\ &\qquad \times \mathbf{1}_{\{V^k(\mathbf{V})>\max\{V^g(\mathbf{V}),g\neq k\}\}}, \end{aligned} \tag{7}$$

which is the payment of agent i'.

If we consider a simple scenario where there are two competing groups, one group with only one bidder L with valuation ℓ and bid b_ℓ and another group of 2 bidders: U_1 with valuation u_1 and bid b_1 and U_2 with valuation u_2 and bid b_2. From eq. (7), the VCG mechanism would be: if L wins, he pays $b_1 + b_2$, otherwise he pays 0; if the group of U_1 and U_2 win, U_1 pays $(b_\ell - b_2)^+$ and U_2 pays $(b_\ell - b_1)^+$, otherwise the U-type bidders pay 0. As we will see in the next section, this simple scenario allows us to evaluate the importance of choosing the right mechanism.

V. Program Description and Results

In this section, we compute the revenues that an auctioneer would get under different mechanisms in two simple cases.

In case 1, there are two competing groups, $\mathcal{G}_1$ and $\mathcal{G}_2$. The group $\mathcal{G}_1$ is composed of only one bidder which can be interpreted as one entity who wants the spectrum to be for its exclusive use or licensed (this bidder is denoted L-type bidder). The group $\mathcal{G}_2$ is composed of two bidders which can be interpreted as two entities who want the spectrum to be unlicensed (these bidders are denoted U-type bidders). The L-type bidder has valuation ℓ and bids b_ℓ and in the group of 2 U-type bidders: U_1 has valuation u_1 and bids b_1 and U_2 has valuation u_2 and bids b_2.

In case 2, there are four groups. The first three groups have a single bidder and the fourth group has five bidders. As before, the single bidders can be thought of as bidders for licensed use of the spectrum, i.e., L-bidders whereas the five bidders in the last group are U-bidders.

In the following we give the program description for Case 1. The program description for Case 2 is similar.

A. Program Description for Case 1

In this subsection, we explain the implementation of the mechanisms described in Section IV for Case 1.

We consider three possible scenarios:

i) Each U-type bidder valuation follows a continuous uniform distribution over the support $[0, 1]$, and the L-type bidder valuation follows a continuous uniform distribution over the support $[0, 2]$.
ii) Each U-type bidder valuation follows an exponential distribution with mean $1/2$, and the L-type bidder valuation follows an exponential distribution with mean 1.
iii) Each U-type bidder valuation follows a normal distribution with mean $1/2$ and variance 0.1, and the L-type bidder valuation follows a normal distribution with mean 1 and variance 0.2.

1) The VCG Mechanisms: To compute the revenue of the auctioneer under VCG Mechanism for the first scenario (the other two scenarios are similar), we consider a Monte Carlo method of 5000 iterations as follows:

i) Initialize the revenue of the auctioneer at zero.
ii) For each iteration
   a) Draw a realization u_1 of the random variable $U_1 \sim \mathcal{U}([0, 1])$, a realization u_2 of the random variable $U_2 \sim \mathcal{U}([0, 1])$ and a realization ℓ of the random variable $L \sim \mathcal{U}([0, 2])$.
   b) If $\ell > u_1 + u_2$, add to the revenue of the auctioneer u_1+u_2; otherwise, add to the revenue of the auctioneer $(\ell - u_2)^+ + (\ell - u_1)^+$.
iii) Divide the total revenue of the auctionner by the number of iterations.

2) Myerson Mechanism: To compute the revenue of the auctioneer under Myerson Mechanism for the first scenario (the other two scenarios are similar), we consider a Monte Carlo method of 5000 iterations as follows:

i) Initialize the revenue of the auctioneer at zero.

ii) Construct functions $c_U(\cdot), c_L(\cdot), c_U^{-1}(\cdot), c_L^{-1}(\cdot)$.

iii) For each iteration

  a) Draw a realization u_1 of the random variable $U_1 \sim \mathcal{U}([0,1])$, a realization u_2 of the random variable $U_2 \sim \mathcal{U}([0,1])$, and a realization ℓ of the random variable $L \sim \mathcal{U}([0,2])$.

  b) If $c_L(\ell) < 0$ and $c_U(u_1) + c_U(u_2) < 0$, add zero to the revenue of the auctioneer

  c) Else, if $c_L(\ell) > 0$ and $c_U(u_1) + c_U(u_2) < 0$, add $c_L^{-1}(0)$ to the revenue of the auctioneer

  d) Else, if $c_L(\ell) < 0$ and $c_U(u_1) + c_U(u_2) > 0$, add $c_U^{-1}(-c_U(u_2)) + c_U^{-1}(-c_U(u_1))$

  e) Otherwise, if $c_L(\ell) > c_U(u_1) + c_U(u_2)$, add $c_L^{-1}(c_U(u_1) + c_U(u_2))$ to the revenue of the auctioneer; otherwise, add to the revenue of the auctioneer $c_U^{-1}(c_L(\ell) - c_U(u_2)) + c_U^{-1}(c_L(\ell) - c_U(u_1))$.

iv) Divide the revenue of the auctionner by the number of iterations.

3) Bykowsky Mechanism: To compute the revenue of the auctioneer under Bykowsky Mechanism, we construct a vector b_ℓ of 100 realizations of the random variable $L \sim \mathcal{U}([0,2])$.

We discretize the domain of possible valuations for U-type bidders in $M+1 = 9$ values $\{v_k\}_{k\in\{0,\ldots,M\}}$, where $v_k = k/M$. We discretize the set of possible bids in $M+1$ values $\{x_k\}_{k\in\{0,\ldots,M\}}$, where $x_k = k/M$.

We compute all the possible combinations of increasing bids for the set of valuations (each of these combinations mimics the function bids vs valuations, and we call them discretized bid vs valuations).

For each discretized bid vs valuation b_1:

- We draw a vector of 100 realizations of the random variable of the U-type bidder (uniform, exponential, normal) and associate the closest v_k and through the discretized bids vs valuations function we get a vector of discretized bids b_1.
- We construct a best response function which takes values of vectors b_ℓ and b_1 and gives the average best response for each v_k.
- We compute the distance between the bid vs valuation b_1 and the best response $\mathrm{BR}(b_1)$

We determine when the best response of the bid coincides with the bid.

With this equilibrium, as in the VCG Mechanism, we compute the revenue of the auctioneer.

B. Results for Cases 1 and 2

In Tables I and IV we show the expected revenue that an auctioneer would get if he run the auction in cases 1 and 2 under the mechanisms previously described (Bykowsky, VCG and Myerson mechanisms). Between Bykowsky mechanism and VCG mechanism, we notice a gain in favor of VCG mechanism (5.45% improvement). Between Bykowsky mechanism and Myerson mechanism the improvement is even higher (29.09% improvement). Between VCG mechanism and Myerson mechanism we also obtain a substantial gain (22.41% improvement) in favor of Myerson mechanism.

In Tables II and V we show the frequency of winning times in cases 1 and 2 under the different mechanisms. VCG mechanism seems to be more fair in the sense that the allocation of the item seems to be closer to the equal repartition between the groups. We notice that Bykowsky mechanism strongly penalizes the unlicensed bidders with respect to the licensed bidders. In Table II, Myerson mechanism appears to be penalizing the unlicensed bidders with respect to the licensed bidders, however, considering only the cases where the auctioneer did not keep the object the unlicensed bidders receive the object a 44.5% of the times compared with the 55.5% of the times for the licensed bidder. In Table V, Bykowsky mechanism penalizes the unlicensed bidders more than Myerson mechanism.

In Tables III, VI and VII, we show the payments of the different bidders for cases 1 and 2. We notice that in Bykowsky mechanism the unlicensed bidders pay much lower than in the other mechanism but this is consequence of the fact that the unlicensed bidders lose in most of the cases.

	Bykowsky	VCG	Myerson
Uniform(0,1)	0.55	0.58	0.71
Improvement wrt. Bykowsky		5.45%	29.09%
Improvement wrt. VCG			22.41%

Table I: 1 L-type bidder vs 2 U-type bidders: Auctioneer revenue under different mechanisms

	Auctioneer	U-type bidders	L1
Bykowsky	-	34,494	65,506
VCG	-	50,061	49,939
Myerson	25,020	33,372	41,608
	-	50,000	50,000

Table II: 1 L-type bidder vs 2 U-type bidders: Frequency of winning times for the uniform distribution

	U1	U2	L1
Bykowsky	0.07199	0.07246	0.40431
VCG	0.08273	0.08302	0.41639
Myerson	0.12548	0.12516	0.45763

Table III: 1 L-type bidder vs 2 U-type bidders: Average payment of each bidder for the uniform distribution

	Bykowsky	VCG	Myerson
Uniform(0,1)	2.51	2.40	2.54
Improvement wrt. Bykowsky		-	1.19%
Improvement wrt. VCG			5.83%

Table IV: 3 L-type bidders vs 5 U-type bidders: Auctioneer revenue under different mechanisms

In the following, we only consider VCG mechanism and Myerson mechanism. There are both technical reasons and practical reasons for not considering Bykowsky mechanism in this part of the analysis. In the technical reasons, we have that the mechanism is not incentive compatible. Moreover,

	Auctioneer	U-type bidders	L1	L2	L3
Bykowsky	-	410	33,126	33,232	33,232
VCG	-	23,851	25,570	25,467	25,112
Myerson	368	5,811	31,185	31,284	31,352
	-	25,000	25,000	25,000	25,000

Table V: 3 L-type bidders vs 5 U-type bidders: Frequency of winning times

	U1	U2	U3	U4	U5
Bykowsky	0.00169	0.00156	0.00165	0.00167	0.00161
VCG	0.06536	0.06497	0.06503	0.06480	0.06476
Myerson	0.02220	0.02213	0.02219	0.02268	0.02227

Table VI: 3 L-type bidders vs 5 U-type bidders: Average payment of each U-type bidder

	L1	L2	L3
Bykowsky	0.83056	0.83409	0.83452
VCG	0.69632	0.69359	0.68397
Myerson	0.80770	0.80753	0.81137

Table VII: 3 L-type bidders vs 5 U-type bidders: Average payment of each L-type bidder

as it was shown in the previous analysis its performance is dominated by VCG and Myerson mechanisms. In the practical side, we notice that it is difficult to consider Bykowsky mechanism since agents do not reveal their true valuation and finding this true valuation (like we did for the three cases presented in Table I) is a combinatorial problem. We consider the improvement in percentage of the auctioneer by choosing Myerson mechanism instead of VCG mechanism as 100*(Revenue under Myerson mechanism - Revenue under VCG mechanism)/Revenue under VCG mechanism.

In Figure 1, the only bidder of $\mathcal{G}_1$ has a valuation uniformly distributed $\sim \mathcal{U}(\mu - \sqrt{3}\mu x, \mu + \sqrt{3}\mu x)$ so that its mean is μ, its standard deviation is μx and thus its coefficient of variation is x. Each of the two unlicensed bidders have valuations uniformly distributed $\sim \mathcal{U}(\frac{\mu}{2} - \frac{\sqrt{6}}{2}\mu x, \frac{\mu}{2} + \frac{\sqrt{6}}{2}\mu x)$, so that each one of them has mean $\frac{\mu}{2}$, standard deviation $\frac{1}{\sqrt{2}}\mu x$, and thus their joint mean is μ, their joint standard deviation is μx and thus their joint coefficient of variation is x. We consider $0 \leq x \leq \frac{1}{\sqrt{6}}$ so that each bidder has always a non-negative valuation for the item.

We notice that the revenue of the auctioneer decreases with the coefficient of variation in both the VCG mechanism and in Myerson mechanism (see Figure 1(a)). However the rate at which the auctioneer revenue decreases in the VCG mechanism is much higher than in Myerson mechanism. This translates in an improvement of the auctioneer revenue by choosing Myerson mechanism which can be as high as 16% (see Figure 1(b)). Another interesting observation from Figure 1(b) is that the improvement of the auctioneer depends on the coefficient of variation but not on the mean (keeping constant the coefficient of variation).

In Figure 2, we consider three different scenarios. In each of these scenarios there is one group with one bidder (that we denote 1L) which competes with: (i) a group with two bidders (that we denote 2U) (ii) a group with three bidders (that we denote 3U) (iii) a group with four bidders (that we denote 4U). In each of these three scenarios the agents distribution is uniform in an interval such that the mean of each group and the coefficient of variation between competing groups is the same, similarly as we did in the scenario of Figure 1.

From Figure 2(a) we notice that the auction described in scenario (i) gives higher revenue than the auction described in scenario (ii), and that the auction described in (ii) gives higher revenue than the auction in scenario (iii). This holds for both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism. A reason for that could be that the auctioneer losses some revenue for the agents to be truthful, and thus with more agents the revenue of the auctioneer decreases. From Figure 2(b), we notice an slightly improvement of the auctioneer revenue, by choosing Myerson mechanism instead of VCG mechanism, with increasing number of U-type bidders.

In Figure 3, we consider one group with only one bidder competing against another group with an increasing number of bidders. In each of these scenarios the agents distribution is uniform in an interval such that the mean of each group ($\mu = 25, 50$, or 100) and the coefficient of variation ($x = 0.2$) between competing groups are the same. We notice a slightly decrease on the auctioneer revenue in both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism as before. From Figure 3(a), we confirm our previous observation that increasing the number of users gives a higher improvement by choosing Myerson.

In Figure 4, Figure 5 and Figure 6, we consider that the valuation follows a normal distribution $\mathcal{N}(\nu, x\nu)$ for different means ν, where x is the coefficient of variation. We consider the coefficient of variation to be between $0 \leq x \leq 0.25$ so that the probability of having positive valuation is 99.99%.

As in the case of the uniform distribution of the valuation, we notice that the revenue of the auctioneer decreases with the coefficient of variation in both the VCG mechanism and in Myerson mechanism (see Figure 4(a)). The rate at which the auctioneer revenue decreases in the VCG mechanism is higher than in Myerson mechanism, which translates in an improvement of the auctioneer revenue by choosing Myerson mechanism which can be as high as 10% (see Figure 4(b)). The improvement of the auctioneer depends on the coefficient of variation but not on the mean (keeping constant the coefficient of variation). From Figure 5(a), we notice that the auction described in scenario (i) gives a higher revenue than the auction described in scenario (ii), and the auction described in (ii) gives a higher revenue than the auction in scenario (iii). This holds for both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism. From Figure 5(b), we notice an improvement of the auctioneer revenue by choosing Myerson mechanism increases with the number of users. We notice a slightly decrease on the auctioneer revenue in both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism as before. From Figure 6(a), we confirm our previous observation that increasing the number of users gives a higher improvement by choosing Myerson with higher improvement than in the uniform case.

In Figure 7, Figure 8 and Figure 9, we consider that the valuation follows an exponential distribution $\exp(\lambda^{-1})$ for different means λ. Notice that the exponential distribution has coefficient of variation always equal to 1. Thus, the x-coordinate of the plots corresponds to the mean.

We notice that the revenue of the auctioneer increases linearly with the mean in both the VCG mechanism and in Myerson mechanism (see Figure 7(a)). The improvement of the auctioneer does not depend on the mean and by choosing Myerson mechanism it is approximately 25% (see Figure 7(b)). As in the previous cases, from Figure 8(a) we notice that the auction described in scenario (i) gives higher revenue than the auction described in scenario (ii), and the auction described in (ii) gives higher revenue than the auction in scenario (iii). This holds for both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism. From Figure 8(b), we notice that the improvement of the auctioneer revenue by choosing Myerson mechanism decreases with the number of users, in contrast to the previous cases. We notice a slightly decrease on the auctioneer revenue in both Myerson mechanism and VCG mechanism, with Myerson mechanism giving higher revenues than VCG mechanism as before. From Figure 9(a), we confirm our previous observation that increasing the number of users gives a smaller improvement by choosing Myerson.

VI. Conclusions

We have studied the auction of a single block of spectrum that can be awarded either for exclusive licensed use by one operator or reserved for unlicensed use. A number of operators bid for exclusive licensed use and a group of non-colluding agents bid to keep the spectrum unlicensed. The revenue of this auction is compared to that of a Vickrey-Clarke-Groves (VCG) auction and that of another auction recently proposed

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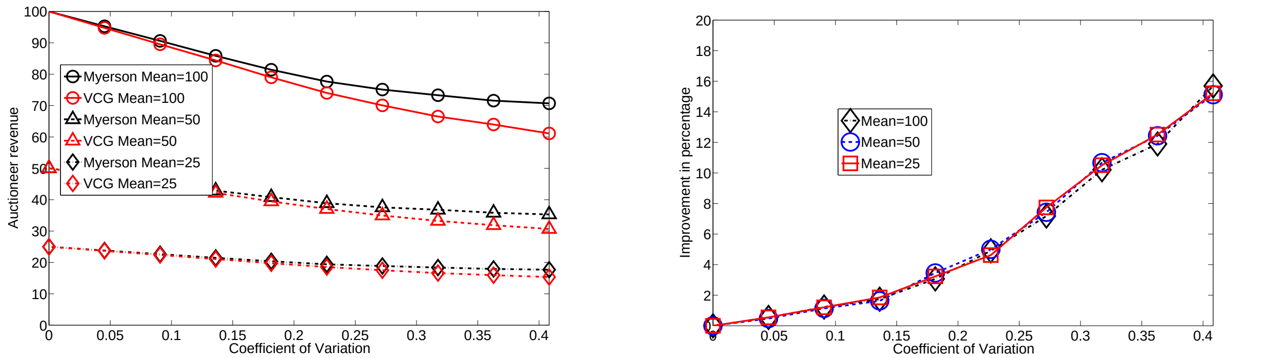
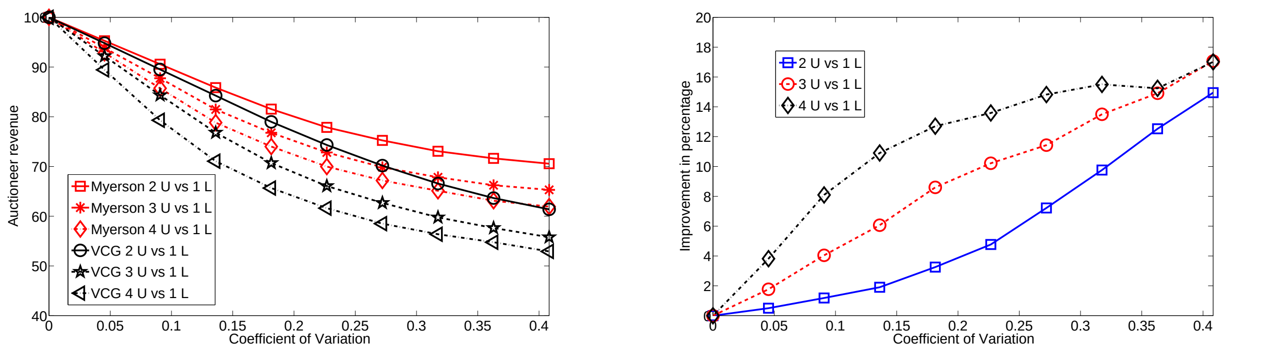


(a) Auctioneer revenue vs coefficient of variation x.

(b) Improvement of the auctioneer revenue (in percentage) vs coefficient of variation x by choosing Myerson instead of VCG mechanism.

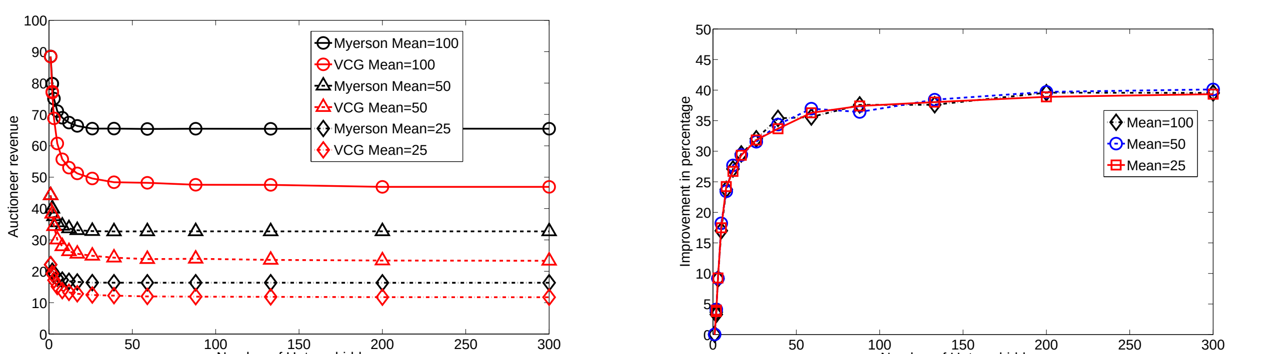
Figure 1: Auctioneer revenue with different means for the uniform distribution: Two U-type bidders with valuations $\sim \mathcal{U}(\frac{\mu}{2} - \frac{\sqrt{6}}{2}\mu x, \frac{\mu}{2} + \frac{\sqrt{6}}{2}\mu x)$ vs one L-type bidder with valuation $\sim \mathcal{U}(\mu - \sqrt{3}\mu x, \mu + \sqrt{3}\mu x)$ where the mean μ takes values 25, 50 and 100 and x is the coefficient of variation, $0 \leq x \leq \frac{1}{\sqrt{6}}$.



(a) Auctioneer revenue vs coefficient of variation x.

(b) Improvement of the auctioneer revenue (in percentage) vs coefficient of variation x by choosing Myerson instead of VCG mechanism.

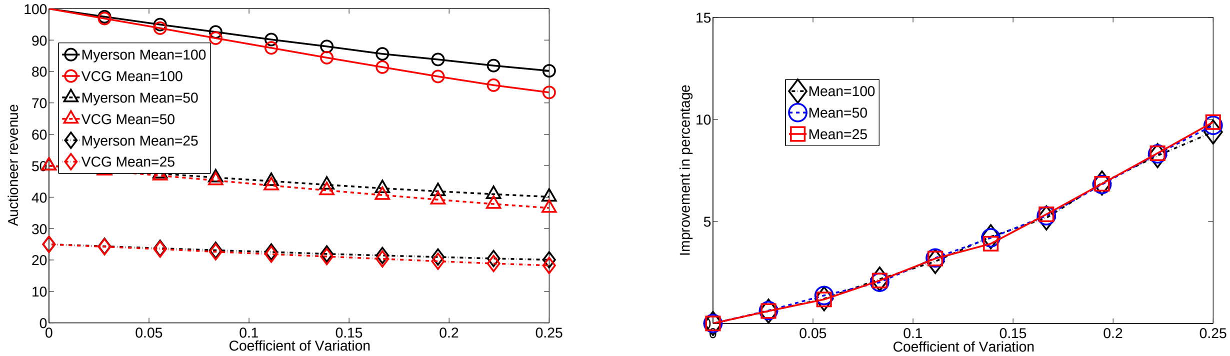
Figure 2: Auctioneer revenue with different number of U-type bidders for the uniform distribution: The U-type bidders bid against a single L-type bidder and the mean of the different groups is the same ($\mu = 100$).



(a) Auctioneer revenue vs number of U-type bidders.

(b) Improvement of auctioneer revenue (in percentage) vs number of U-type bidders by choosing Myerson instead of VCG mechanism.

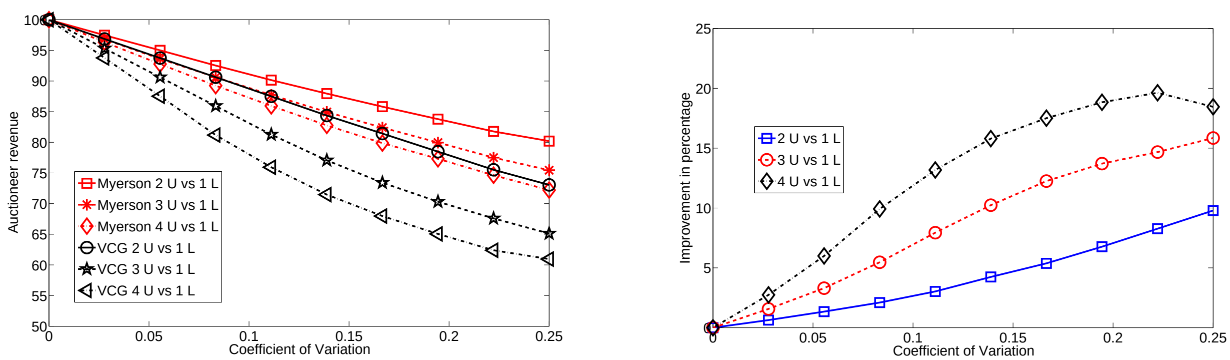
Figure 3: Auctioneer revenue with increasing number of U-type bidders for the uniform distribution: The mean of the different groups is the same ($\mu = 25, 50$ or 100) and the groups bid against a single L-type bidder.



(a) Auctioneer revenue vs coefficient of variation x

(b) Improvement of auctioneer revenue (in percentage) vs coefficient of variation x by choosing Myerson instead of VCG mechanism.

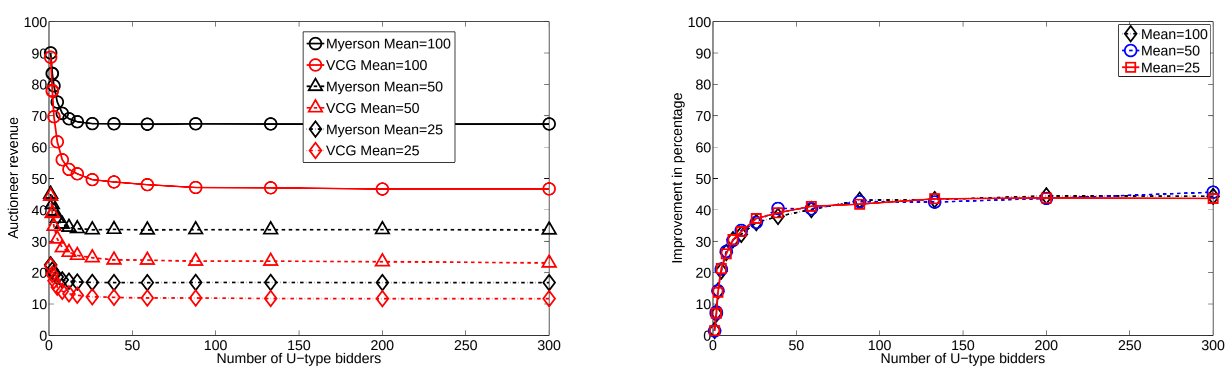
Figure 4: Auctioneer revenue with different means for the normal distribution: Two U-type bidders with valuations $\sim \mathcal{N}(\frac{\mu}{2}, \frac{\mu}{2}x)$ vs one L-type bidder with valuation $\sim \mathcal{N}(\mu, \mu x)$ where the mean μ takes values $25, 50$ and 100 and x is the coefficient of variation, $0 \leq x \leq \frac{1}{4}$.



(a) Auctioneer revenue vs coefficient of variation x.

(b) Improvement of auctioneer revenue (in percentage) vs coefficient of variation x by choosing Myerson instead of VCG mechanism.

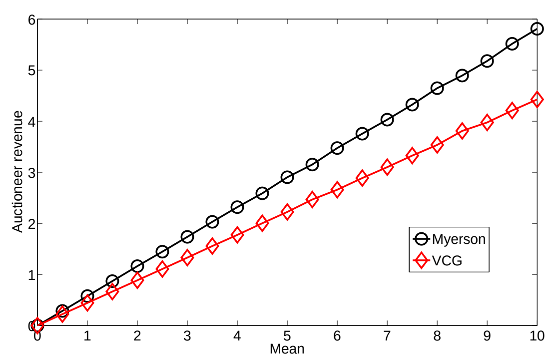
Figure 5: Auctioneer revenue with different number of U-type bidders for the normal distribution: The U-type bidders bid against a single L-type bidder and the mean of the different groups is the same ($\mu = 100$).



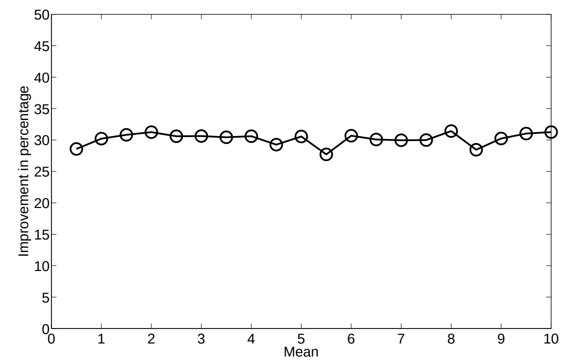
(a) Auctioneer revenue vs number of U-type bidders for different number of U-type bidders.

(b) Improvement of auctioneer revenue (in percentage) vs number of U-type bidders.

Figure 6: Auctioneer revenue with increasing number of U-type bidders for the normal distribution: The mean of the different groups is the same ($\mu = 25, 50$ or 100) and the groups bid against a single L-type bidder.

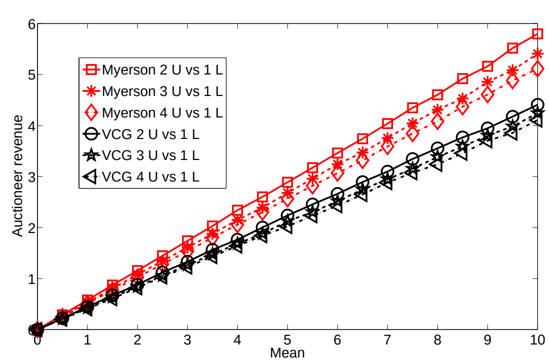


(a) Auctioneer revenue vs mean λ.

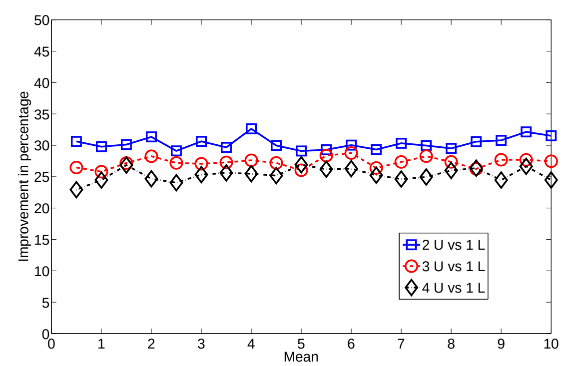


(b) Improvement of auctioneer revenue (in percentage) vs mean λ.

Figure 7: Auctioneer revenue for the exponential distribution: Two U-type bidders with valuations $\sim \exp((\lambda/2)^{-1})$ vs one L-type bidder with valuation $\sim \exp(\lambda^{-1})$.

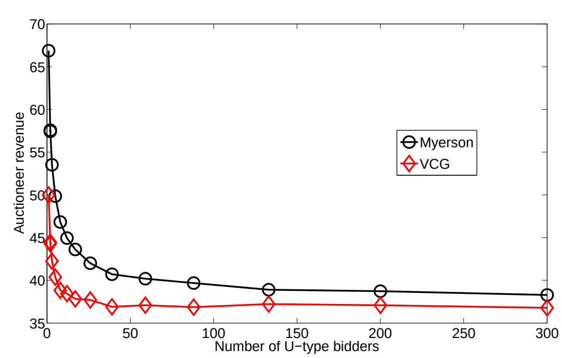


(a) Auctioneer revenue vs mean λ.

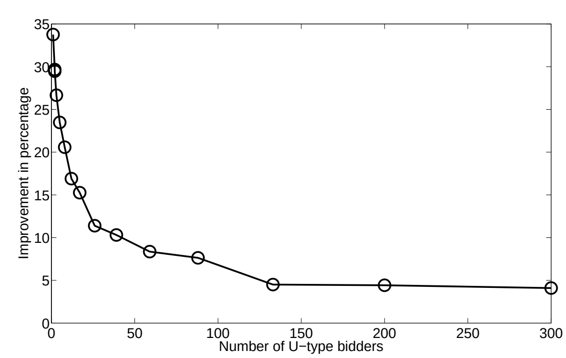


(b) Improvement of auctioneer revenue (in percentage) vs mean λ.

Figure 8: Auctioneer revenue with different number of U-type bidders for the exponential distribution: The U-type bidders bid against a single L-type bidder and the mean of the different groups is the same.



(a) Auctioneer revenue vs number of U-type bidders.



(b) Improvement of auctioneer revenue (in percentage) vs number of U-type bidders.

Figure 9: Auctioneer revenue with increasing number of U-type bidders for the exponential distribution: The U-type bidders bid against a single L-type bidder and the mean of the different groups is the same.