A Succinct Control Theory Derivation and New Experiments with the Square-Root Variant of Fitts' Law for Heterogeneous Targets

Siamak Faridani

Electrical Engineering and Computer Sciences
University of California at Berkeley

Technical Report No. UCB/EECS-2013-39
http://www.eecs.berkeley.edu/Pubs/TechRpts/2013/EECS-2013-39.html

May 1, 2013

A Succinct Control Theory Derivation and New Experiments with the Square-Root Variant of Fitts' Law for Heterogeneous Targets

by

Siamak Faridani

A thesis submitted in partial satisfaction
of the requirements for the degree of

Master of Science
in

Computer Science

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:

Professor Ken Goldberg, Chair
Professor Eric Paulos

Fall 2012

The thesis of Siamak Faridani is approved.

Chair Date

Date

University of California, Berkeley

Fall 2012

A Succinct Control Theory Derivation and New Experiments with the Square-Root Variant of Fitts' Law for Heterogeneous Targets

by

Siamak Faridani

Abstract

A Succinct Control Theory Derivation and New Experiments with the Square-Root Variant of Fitts' Law for Heterogeneous Targets

by

Siamak Faridani

Master of Science in Computer Science

University of California, Berkeley

Professor Ken Goldberg, Chair

In this work, we present a succinct derivation for the Square-Root model based on an explicit model from optimal control theory. Our derivation is intuitive and exact, makes fewer assumptions, and requires far fewer steps than the two-submovement derivation presented in Meyer *et al.* [MAK+88]. To compare the Square-Root model with statistical significance, we performed two extensive experiments with homogeneous and heterogeneous targets that appear in sequence on a plane. Later, we extend the two-parameter model to a more accurate three-parameter that increases the accuracy of the model over the two-parameter formulation. The succinct and intuitive derivation and new experimental results may enhance the appeal of the Square-Root model for design of systems and interfaces with heterogeneous targets.

Professor Ken Goldberg
Thesis Committee Chair

To my parents, Shahram Faridani and Fakhrosadat Moddaress.

Contents

Contributions

This thesis builds on the research that was started almost six years before the author joined UC Berkeley as a graduate student. Mathematical derivation for the two parameter Square-Root Fitts' model is by Ken Goldberg and Ron Alterovitz. The web-based experiment data that is used in this thesis was also collected by Goldberg and Alterovitz.

The contributions of the author in this thesis are as below:

1. An in-lab equivalent of the web-based experiment was designed and conducted on 46 participants

2. Statistical data analysis and hypothesis testing on the data from both in-lab and web-based is performed and a comparison with the Logarithmic variant and Plamondon's Power variant of Fitts's model is presented. The results are shown to be consistent (detailed analysis of this comparison is presented in Chapter 2).

3. Goldberg and Alterovitz provide a two-parameters Square-Root form of the Fitts' law. They assume that in the acceleration of the mouse pointer is constant and its magnitude is proportional to the width (W) of the target: $\ddot{x} = kW$. This assumption results in a two-parameter Square-Root variant of the Fitts law. In this thesis we relax this assumption and assume that the acceleration is proportional to both the width of the target (W) and its distance from the origin (A): $\ddot{x} = kWA^{\beta}$. We show that this assumption leads to a three-parameter Fitt's model ($T = a + b\sqrt{\frac{A^{\lambda}}{W}}$) in which the parameter λ was empirically found to be $\lambda = 2/3$

In addition to the contributions above, in the appendices section we provide a detailed derivation of Meyer's model [MAK$^+$88]. The details of their derivation is missing from their paper hence reproduced here for interested readers.

Chapter 1

Introduction

Human motion models facilitate the design of many systems such as human-computer interfaces, cockpits, and assembly lines. The classic Fitts' Law, characterized by a logarithmic two-parameter relationship between human motion time and the index of difficulty, the ratio of distance over target size, was based on timing measurements between fixed (homogeneous) targets. Alternative two-parameter models, such as the Square-Root model, have also been shown to fit this data. Several researchers argue the Square-Root model provides a better fit in cases such as computer interfaces where targets are heterogenous (have different sizes and relative distances between sequential motions). In this work, we present a succinct derivation for the Square-Root model based on an explicit model from optimal control theory. Our derivation is intuitive and exact, makes fewer assumptions, and requires far fewer steps than the two-submovement derivation presented in Meyer *et al.* [MAK$^+$88].

To compare the Square-Root model with statistical significance, we performed two extensive experiments with homogeneous and heterogeneous targets that appear in sequence on a plane. The first experiment is a controlled laboratory study based on timing data for 46 human participants performing 49 motions each. The second experiment is an uncontrolled study based on timing data for 60,000 motions measured on our Internet site (from an unspecified variety of participants, mouse types, and settings). For both experiments we compute RMS error for each model and perform two-sided paired t-tests on the within subject signed differences in RMS errors for each condition. For *homogeneous* targets of fixed width and position, we did not observe a statistical difference between the models. But for *heterogeneous* movements between circles of varying size and position, the Square-Root model fits the data significantly better than the Logarithmic model. For data from the controlled experiment, the p-value is 2.15×10^{-13} and for data from the uncontrolled experiment, the p-value is 4.74×10^{-46}.

Later, we extend the two-parameter model to a more accurate three-parameter model and show that it is 69% more accurate than the logarithmic variant of the Fitts' law. We empirically show the concavity of the performance function versus the third parameter and use this property to calibrate this new parameter. The appendices section for this work contains a detailed derivation of the Meyer *et al.*, [MAK+88] that is missing from their paper and also a comparison of our model with Plamondon's Power variant of the Fitts' law.

The succinct and intuitive derivation and new experimental results may enhance the appeal of the Square-Root model for design of systems and interfaces with heterogeneous targets. All of our data is available online for future studies[1].

[1] http://www.tele-actor.net/fitts/

Chapter 2

Experiments for Two-Parameter Square-Root Variant of Fitts' Law

Many human-computer interfaces require users to move a mouse or related input device to manually direct a cursor to targeted areas (e.g., menu items, buttons) on a computer screen. Some motions can be performed more efficiently than others. To facilitate efficient human-computer interfaces, designers seek effective models of such motions.

The inherent tradeoff between speed and accuracy of such movements was first quantified by Paul Fitts of Ohio State University in 1954 [Fit54]. Fitts studied *homogeneous* reaching movements, between targets of fixed size and distance, which are common in industrial settings for tasks ranging from installing parts on an assembly line to stamping envelopes in an office. In a series of experiments, Fitts required participants to repetitively move a stylus between two fixed contact plates as quickly as possible for 15 seconds. Fitts set the width W of the plates and the amplitude (distance) A between the plates, as shown in Figure 2.1, which were constant during each experiment and varied between experiments. Fitts measured the movement time T between moving back and forth between these two targets.

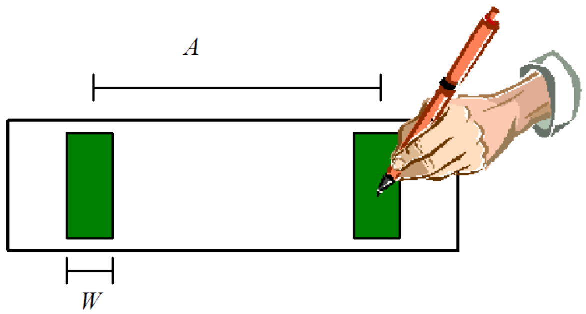


Figure 2.1. In 1954, Paul Fitts measured movement time T for *homogeneous* motions: each participant used a stylus to repetitively tap between two targets of fixed width W separated by a distance A. In this report we also consider *heterogeneous* motions in which A and W change after every tap.

Figure 2.2. Using a Java applet, sequences of rectangular and circular targets are presented, where target distance A and width W can remain constant (homogeneous) or vary (heterogeneous) after every click.

Inspired by Shannon's Information Theory, Fitts empirically fitted a *logarithmic* model to the data he collected from 16 participants. Since then, many researchers have repeated these experiments under varying conditions and proposed alternative models and derivations based on human perception and physiology.

We review related work in the next Section. Our results focus on the Square-Root model studied in depth by Meyer *et al.* in [MAK$^+$88]. In section 2.1.7 we review the Meyer derivation for comparison.

In this chapter, we present a succinct derivation for the Square-Root model based on two assumptions: acceleration is (1) piecewise constant (as predicted by optimal control theory) and (2) proportional to target width: wider targets are perceived as easier to reach and hence correspond to larger accelerations. This derivation is intuitive and exact, makes fewer assumptions, and requires far fewer steps than the two-submovement derivation presented in Meyer *et al.* [MAK+88].

Researchers have argued the Square-Root model provides a better fit for computer interfaces where targets are heterogenous (have different sizes and relative distances between sequential motions) [Kvå80], [MAK+88]. Since Meyer *et al.* report experiments on wrist rotation movements to heterogenous targets for only four participants, we performed extensive new experiments with homogeneous and heterogeneous targets that appear in sequence on a plane. To our knowledge, this is the largest study evaluating Fitts law and the Square-Root model. We analyze the timing data collected in both experiments by computing RMS error for each condition (two-parameter model) and perform two-sided paired t-tests on the within subject signed differences in RMS errors for each condition.

2.1 Related Work

2.1.1 Classic Fitts' Law

In "choice reaction time tasks," a set of stimuli are assigned unique responses, and participants must give the correct response when receiving the stimulus [JF03a]. In 1885, J. Merkel designed an experiment in which the stimulus was a number selected from a set of size N with uniform probability and the participant was required to press a key corresponding to the number [JF03a]. As N increased, so did the reaction time T_R. Merkel found that the reaction time increased by a constant for every doubling of the set of possible numbers.

In 1948, Claude Shannon published the foundational paper on Information Theory, defining the information capacity for a communication channel, C, as:

$$C = B \log_2 \left(\frac{S + N}{N} \right), \tag{2.1}$$

where B is the channel bandwidth, S is signal strength, and N is noise power. Shannon also defined the information I of a symbol based on the probability of receiving the symbol, $I = \log_2 \frac{1}{p}$.

Adopting Shannon's model, Merkel's reaction time can be viewed as proportional to the amount of "information" received by the participant:

$$T_R = a + b \log_2 N$$

where a and b are experimentally determined constants.

In 1953, J. Hyman extended Merkel's work to the case where an element i in the set of possible numbers was selected with non-uniform probability p_i [Hym53]. Hyman found that the average reaction time was consistent with the Shannon's model [JF03a].

In 1954, Fitts hypothesized that the information capacity of the human motor system is specified by its ability to produce consistently one class of movement from among several alternative classes of movements [Fit54]. Fitts then defined the difficulty of a task based based on the minimum amount of "information" required to complete it on average. For the "tapping task", Fitts defined a tap between two targets of distance (amplitude) A with width of W as a movement class. Using Shannon's definition of information as a guideline, Fitts defined the *index of difficulty* (I) to be the "information" transmitted during the task:

$$I = \log_2 \left(\frac{2A}{W} \right).$$

Fitts noted that the choice of the numerator for this index "is arbitrary since the range of possible amplitudes must be inferred," so $2A$ was selected rather than A to ensure that the index is positive in "all practical situations."

Fitts then modeled movement time T as a linear function of the "information" transmitted, producing his classic two-parameter Logarithmic model:

$$T = a + b\log_2\left(\frac{2A}{W}\right). \tag{2.2}$$

2.1.2 Alternative Information Theory Models of Human Motion

In 1960, Welford proposed a revised model based on a Weber fraction where the user must select "a distance from a total distance extending from his starting point to the far edge of the target" [Wel60], [Wel68]. For some constant b, Welford's formulation is given by:

$$T = b\log\left(\frac{A+\frac{1}{2}W}{W}\right) = b\log\left(\frac{A}{W}+0.5\right).$$

I. Scott MacKenzie developed a model that is more closely based on Shannon's Theorem [Mac92]. MacKenzie's communication channel model of Fitts' data considers noise N to be the variation around a specific signal S, so the signal strength equals the movement amplitude ($S = A$) and the noise equals the width ($N = W$). By analogy to Shannon's model (equation 2.1), movement time is given by:

$$T = a + b\log_2\left(\frac{A+W}{W}\right). \tag{2.3}$$

Other researchers, such as Crossman and Welford [Wel68], proposed further modifications to Fitts' Logarithmic model to make it more closely resemble Shannon's communication theorems using methods such as effective target widths to model channel noise.

2.1.3 Alternative Models of Reaching Movements

Alternative models of human reaching movements consider properties of the neuromuscular system, such as minimizing jerk or a sequential impulse model. Crossman and Goodeve [CG83] proposed a movement time model based on a sequence of discrete positional corrective motion impulses, which resulted in a Logarithmic model like Fitts' law. Flash and Hogan developed a mathematical model of voluntary reaching movements based on maximizing the smoothness of trajectories [FH85]. They propose that the human motor system minimizes jerk, the derivative of acceleration. The formula for the time integral of the square of the magnitude of jerk is

$$C = \frac{1}{2}\int_0^{t_f} \sum_{i=1}^{n} \left(\left(\frac{d^3 x_i}{dt^3} \right)^2 \right) dt$$

where n is the dimension of the space, x is the vector coordinate of the pointer as a function of time, and t_f is the time to reach the end point. Minimizing this formula results in 5th order polynomials with 6 unknown parameters for each dimension. One can constrain the position of the start and end points and assume the velocity and acceleration are zero at the start and end of the movement, and then solve for the parameters. The resulting trajectories have smooth position and velocity curves qualitatively similar to experimentally measured data. However, this model differs from Fitts' Logarithmic variant because it does not explicitly consider the tradeoff resulting from varying the size of a target region.

In 1992, Réjean Plamondon proposed an alternative to Fitts' Logarithmic model using a neuromuscular impulse response model [Pla92a], [Pla95a], [Pla95b]. Plamondon's theory for rapid human movements is based on the synergy between the agonist and antagonist neuromuscular systems [Pla95a]. In his model, the agonist and antagonist systems synchronously receive an impulse input $U_0(t - t_0)$ at time t_0 scaled by D_i, where $i = 1$ for the agonist system and $i = 2$ for the antagonist system. Each system independently responds in parallel to the input with impulse response functions $H_i(t)$ to generate output velocities $v_i(t)$ for $i = 1, 2$. Although the two systems may be coupled in reality, Plamondon assumed the output $v(t)$ of the synergy is obtained by subtracting the two parallel outputs. Plamondon proposed defining the impulse response using a log-normal function, a very general formulation based on 7 parameters that can qualitatively predict a variety of velocity profiles including single peaks, double peaks, triple peaks, asymmetric peaks, and multiple peaks with no zero crossing. Reaching movements from one point to another point terminate at a time T when the velocity of motion $v(T)$ equals zero. Solving the velocity equation for the zeros using constraints set by Fitts' experiment, Plamondon modeled movement time

$$T = K \left(\frac{2A}{W} \right)^{\alpha} \tag{2.4}$$

with parameters K and α.

Equation 2.4 defines a power model, an alternative two-parameter formulation based on a fitted log-normal approximation of the velocity profile.

2.1.4 Applications of Fitts' Logarithmic Model

Although Fitts' Logarithmic model was originally developed for industrial pick-and-place tasks [Fit54], it has been applied to a variety of human reaching movements. The first application of Fitts' Logarithmic model to human-computer interaction dates from before the commercialization of modern personal computers. Card, English, and Burr at Xerox PARC studied the relative speed of four input devices: mouse, joystick, step keys, and text keys [CEB78]. They found that Fitts' Logarithmic model accounts for the variation in movement time to select text on a CRT monitor using mice and joysticks. Subsequent studies applied Fitts' Logarithmic model to pen input devices [MSB91]. Fitts' Logarithmic model has also been applied to robotics applications including telemanipulation tasks with remote video viewing [Dra91] and pairs of participants performing tapping motions using a robot manipulator [RPCP04].

Friedlander et al. found that a linear model for movement time fits selection in a non-visual (tactile or auditory) bullseye menu more closely than Fitts' Logarithmic model [FSM98]. Also, Kristensson proposes using context information, such as pattern recognition of likely key presses on a stylus keyboard, to develop input devices that increase the speed of input beyond what would be predicted by Fitts' Logarithmic model [Kri05].

Plamondon and Alimi review studies on speed/accuracy trade-off models and their applications [PA97]. They categorize the experimental procedures used for the speed/accuracy trade-offs into two different categories: spatially constrained movements and temporally constrained movements. For the procedures in the first category, distance (A) and the width (W) are usually given and the time (T) is measured. In the temporal group, movement time is given and the accuracy of reaching the target is being measured. With this definition, Fitts' Law falls into the first category. They classify different studies on the Fitts' Logarithmic model based on different types of movements (tapping, pointing, dragging), limbs and muscles groups (foot, head, hand, etc), experimental conditions (underwater, in flight, etc), device (joystick, mouse, stylus, touchpad, etc), and participants (children, monkeys, adults of different ages, etc). More recently, Hoffmann and Hui study the movement of different arm components like fingers, wrist, forearm and full-arm for reaching a target in an industrial setting. They show for the cases that an operator can use different arm components to reach a target, the limb with the smallest mass moment of inertia should be used to optimally reach the target [HH10].

2.1.5 Extensions to Fitts' Logarithmic Model

Fitts' Logarithmic model, which was originally developed for one-dimensional reaching movements, has been extended to the two-dimensional movements that are common in graphical user interfaces [MB92], [Mac95]. For general two-dimensional targets, both the shape of the target and angle of approach must be considered. For circular targets, the assumptions of the one-dimensional model remain largely intact with target width W being defined by the circle's diameter[1]. For rectangular targets, Card et al. propose the status quo model defines W by the width of the target while ignoring height. This model can result in a negative index of difficulty for near wide targets [Mac95], [CEB78]. MacKenzie et al. proposed two models for rectangular targets [MB92]. The smaller-of model sets W to the smaller of the target width or height. The effective width model sets W by considering an additional parameter: the angle between the start point and the target center. MacKenzie tested the status quo, smaller-of, and effective width models and found that the linear correlation of movement time to Fitts' index of difficulty was significantly greater for both the smaller-of and effective width models compared to the status quo model [MB92].

Gillan et al. examined how Fitts' Logarithmic model can be applied to point-drag movement sequences rather than simply point-click operations. They found that Fitts' Logarithmic model must first be applied as the user points to the left edge of the text object and then applied separately for the dragging distance [GHA+90].

Accot et al. investigated extensions for Fitts' Logarithmic model for trajectory-based interactions, such as navigating through nested menus, drawing curves, or moving in 3D worlds [AZ97]. They developed a "steering law" similar to Fitts' Logarithmic model except the index of difficulty for steering a pointer through a tunnel is defined by the inverse of the width of a tunnel integrated over the length of the

[1] In this work we use the diameter of circular targets for W as suggested by MacKenzie [Mac95].

tunnel. They applied the steering law to participants using 5 input devices (tablet, mouse, trackpoint, touchpad, and trackball), and the linear correlation of movement time to the index of difficulty for steering exceeded 0.98 [AZ99]. Apitz et al. build a crossing-based interface, CrossY [AGZ08]. Unlike point-and-click interfaces, crossing-based interfaces allow participants to trigger an action by crossing a target on the screen instead of clicking on it. Apitz et al. show that a crossing task is as fast as, or faster than a point-and-click task on for the same index of difficulty [AGZ08].

In a related line of research, Wobbrock et al. derive a predictive model for error rates instead of mean times [WCHM08]. Error rate models have practical applications in designing text entry devices and video games [WCHM08].

2.1.6 Input device settings and Fitts' Logarithmic Model

Modern mice and other pointing devices usually offer configurable parameters that adjust the mapping between movement of the device and movement of the cursor on the screen. The most common is mouse speed, a type of "control-display gain" [MC91]. The control-display gain scales the distance d the mouse moves on the table to a distance p in pixels that the cursor moves on the screen. The setting of the gain can have a significant impact on movement time to a target. Thompson et al. experimentally verified that lower gains are better for low amplitude or small target movements while higher gains are better for large amplitude or large target movements [TSB04]. This mixed result makes it difficult to select a single optimal gain for standard computer usage. Blanch et al. introduce semantic pointing, a technique that improves target acquisition by decoupling the visual size of a target from the motor size of the target by dynamically adjusting the control-display gain when the cursor moves over a target [BGB04]. The technique is effective because user movement times are determined primarily by the motor rather than visual space.

Another input device configuration parameter that can be adjusted on many modern personal computers is mouse acceleration. In its most basic form, mouse acceleration includes two parameters, acceleration and threshold [MC91]. When the mouse speed exceeds the threshold, the control-display gain is scaled by the acceleration parameter. Recent operating systems commonly use more complex mouse acceleration models, e.g. multiple thresholds [Mic03]. Mouse speed is defined as the maximum of the x or y axis displacement of the mouse per unit time. When the mouse speed exceeds the first threshold, the operating system doubles the gain. When the speed exceeds the second threshold value, the system quadruples the gain.

Modern input devices like tablets have inspired new research. Hoffmann and Drury study the effect of figure width on Fitts' model and adjust the target width W by considering the width of the target, its proximity to another target and the width of the finger [HD11]. They show that in the case that two keys are adjacent to each other and the width of the finger pad is larger than the clearance between the two key, W can be replaced by "Available W" whose value is $W_{avail} = 2S - W - F$:. In this equation W is the target size, F is the width of the finger pad on the device and S is the target center spacing.

2.1.7 The Square-Root Model

Several researchers have argued that the Square-Root model is superior to the Logarithmic model [Kvå80], [MAK+88]. Using the experiment data for 16 participants from the original Fitts' paper [Fit54], Meyer et al. showed that the Square-Root model fits the original data better than the Logarithmic model [MAK+88].

Meyer et al. provided a complex derivation of the Square-Root model. They do not provide the details of their derivation in the paper but we have presented a detailed derivation of their model in the Appendix section A. The derivation assumes that motion can be partitioned into two submovements, a primary ballistic submovement and a secondary corrective submovement, with near-zero velocity at the transition. The derivation is an approximation based on several strong assumptions: 1) two submovements with a stop between them, 2) submovement endpoints have Gaussian distributions around the center point of the target, and 3) the standard deviation of each Gaussian is linearly related to the average velocity during that submovement, and 4) there are strong numerical bounds on values of A and W for which the approximation holds.

They derive the time T to reach the target as the sum of the average time for the primary submovement T_1 and for the corrective submovement T_2. They estimate T by minimizing its derivative with respect to the submovements and show that when $A/W > 4/z\sqrt{2\pi}$ the value of T can be approximated by

$$T = a + b\sqrt{\frac{A}{W}}. \tag{2.5}$$

where z is the z-score such that 95% of the area under a standard Gaussian distribution $N(0, 1)$ falls inside $(-z, z)$.

In addition to its complexity vis a vis Occam's Razor, there are other drawbacks to this derivation [RG12]. As Meyer et al. note, if the participant reaches the target in a single movement, the derivation collapses to a linear model which fits the data very poorly. The approximation requires numerical bounds on values of A and W. Furthermore, Guiard et al. note that for a fixed positive value of A/W Meyer's model approaches 1 as the number of submovements n approaches infinity [GBMBL01], [RG12]. Meyer et al. evaluated their model with one-dimensional movements using wrist rotation of a dial that can be rotated to different angular targets. In their experiments, 4 participants are presented with 12 target conditions with A/W values ranging from 2.49 to 15.57. This range of A/W does not violate the assumption made for their derivation. However, our experiments presented in Sec. 2.3 suggest that the Square-Root model does not require this assumption and holds even in the range 1.25 to 25.27.

2.2 A Succinct Derivation of the Square-Root Model

In this section we provide a succinct derivation for the Square-Root model[2] that assumes only that acceleration is (1) piecewise constant as predicted by optimal control theory, and (2) proportional to target width: wider targets are perceived by participants as "easier" to reach and hence humans apply larger accelerations as they have a larger margin for error. These are the only two assumptions required.

In control theory, it is well known that the optimal time to reach a target is obtained by "bang-bang" control, where maximal positive acceleration is maintained for the first half of the trajectory and then switched to maximal negative acceleration for the second half [MS82], [JF03a].

To reach a target at distance A, the halfway point (the point reached at the switching time) is defined as $x_{mid} = A/2$. Acceleration as a function of time for bang-bang control is shown in Figure 2.3(a), where the switching time is $s = T/2$.

As shown in Figure 2.3, Acceleration has only two values: full forward or full reverse, hence the term "bang-bang". Velocity is initially zero and then ramps up linearly during the first phase and ramps down during the second. Velocity is thus $\dot{x}(t) = \ddot{x}t$ during the acceleration phase ($t \leq s$) and $\dot{x}(t) = \ddot{x}s - \ddot{x}(t-s)$ during the deceleration phase ($t > s$), where $\ddot{x}$ is the constant magnitude of acceleration.

[2] This derivation is originally by Goldberg and Alterovitz. A journal publication on this derivation along with some of the results that are presented in this thesis is in preparation by Goldberg, Alterovitz and Faridani.

We can integrate this linear velocity with respect to time to get a quadratic function for position $x(t)$. At the switching time s, the position by integration will be $x(s) = \frac{1}{2}\ddot{x}s^2$. By symmetry, position after time $T = 2s$ will be $x(T) = \ddot{x}s^2 = \frac{1}{4}\ddot{x}T^2$. For cursor motion, we set the total distance traveled during movement time T as the amplitude $x(T) = A$. Hence, $A = \frac{1}{4}\ddot{x}T^2$ which implies

$$T = 2\sqrt{\frac{A}{\ddot{x}}}. \tag{2.6}$$

Now, from the second assumption, acceleration magnitude is proportional to the width of the target: $\ddot{x} = kW$ where k is a constant scalar and W is the target width. Substituting into equation 2.6, we get

$$T = 2\sqrt{\frac{A}{kW}}.$$

We now add an initial reaction time a and let $b = 2/\sqrt{k}$. The total movement time is then:

$$T = a + b\sqrt{\frac{A}{W}}. \tag{2.7}$$

This derivation is intuitive, exact, makes fewer assumptions, and requires far fewer steps than the two-submovement derivation presented in Meyer *et al.* [MAK+88].

2.2.1 Asymmetric Acceleration Model

The above derivation does not require a symmetric motion profile. In 1987, C. L. MacKenzie showed empirically that velocity profiles for reaching movements during Fitts' task are often asymmetric [MMD+87a]. We now present a modified derivation based on an asymmetric velocity profile. Let s be the switching time between the acceleration phase and deceleration phase. The peak velocity will occur at the switching time. To complete the reaching movement of amplitude A with $\dot{x}(T) = 0$, the magnitude of constant acceleration $\ddot{x}_a$ before time s may be different from the constant deceleration $\ddot{x}_d$ after s.

MacKenzie showed that normalized time to peak velocity s/T increases roughly linearly as target width W increases and does not depend on amplitude A [MMD+87a]. We approximate the normalized time to peak velocity as linearly proportional to W:

$$\frac{s}{T} = kW$$

where k is a scalar constant. We also assume that initial acceleration $\ddot{x}_a$ for an individual is a fixed maximum acceleration regardless of the task and the deceleration $\ddot{x}_d$ is set so velocity is 0 at time T. The maximum initial acceleration condition implies $|\ddot{x}_a| \geq |\ddot{x}_d|$, which is true according to empirical observations in MacKenzie's results [MMD+87a].

To obtain a relationship between T, A, and W, we first solve for the peak velocity $\dot{x}_{max} = \ddot{x}_a s$. The switching time constraint $s/T = kW$ implies $\dot{x}_{max} = \ddot{x}_a kWT$. Integrating the asymmetric velocity profile in Figure 3.2(a) with respect to time, we get position $x(t)$, shown in Figure 3.2(b).

At time T, position as a function of $\dot{x}_{max}$ is

$$x(T) = \frac{1}{2}\dot{x}_{max}s + \frac{1}{2}\dot{x}_{max}(T - s) = \frac{1}{2}\dot{x}_{max}T. \tag{2.8}$$

Setting $x(T) = A$ and substituting $\dot{x}_{max}$ into equation 3.4 yields:

$$A = \frac{1}{2}\ddot{x}_a kWT^2.$$

Hence,

$$T = \sqrt{\frac{2}{\ddot{x}_a k}\frac{A}{W}}.$$

Letting $b = \sqrt{\frac{2}{\ddot{x}_a k}}$ and adding a fixed initial reaction time a common to all trials for a given participant, we get

$$T = a + b\sqrt{\frac{A}{W}}. \tag{2.9}$$

Equation 3.5 is identical to equation 2.7 except for the definition of the constant term b. Both binary acceleration models were derived based on kinematic assumptions. The former model assumes switching time is fixed relative to T and acceleration is proportional to W while the latter model assumes switching time is proportional to W and initial acceleration is a fixed constant.

2.3 Experiments

To compare the *Square-Root* and *Logarithmic* models we performed two extensive experiments with *homogeneous* and *heterogeneous* targets that appear in sequence on a plane. We created a Java applet with graphical display of targets and recording of completion times. The first experiment is a controlled laboratory study based on timing data for 46 participants performing 49 motions each. The second experiment is an uncontrolled study based on timing data for 60,000 motions measured on our Internet site (from an unspecified variety of participants, mouse types, and settings). We compute unique model parameters for each participant from trajectory data collected by the applet. And the goodness of fit is then measured by calculating the RMS error between the actual timing data and the expected timing data from the trained model. For both experiments we compute RMS error for each model and perform two-sided paired t-tests on the within subject signed differences in RMS errors for each condition.

Uncontrolled (also known as "in the wild") experiments on the web do not provide the consistency of controlled in-lab experiments but can collect data from large numbers of diverse human participants and are gaining acceptance, especially when confirmed by controlled experiments [BJJ+10], [BRMA12]. In their survey, Andreasen et al [ANSS07] systematically compare controlled and uncontrolled (web-based) usability studies and find that synchronous studies (with a live remote human monitor) are more reliable than asynchronous studies (akin to our uncontrolled experiments) but that both "enable collection of use data from a large number of participantsand it would be interesting to perform comparative studies of remote usability testing methods". Kittur et al [KCS08] consider how a system such as Mechanical Turk can be used for user studies and find that the diversity and unknown nature of the user base can be "both a benefit and a drawback." Their results suggest that careful design of the tests to avoid gaming can yield results that are comparable with controlled studies. Both papers acknowledge the inherent tradeoff between quantity and quality of data.

2.3.1 Experiment Conditions

The Java applet asks each participant to finish two *experiments*, defined below, which vary the shape and size of a target region drawn on the screen. Each experiment consists of a sequence of *trials*. In each trial, the user is required to move the cursor to a target region (a green rectangle or circle that appears in the applet window) and click. If the user clicks outside the target, the click is ignored and the trial continues. For each trial, the applet records a single measurement: the time from the appearance of the trial on the screen until the click inside the target region measured in milliseconds (ms).

The two experiments performed were:

1. *Homogeneous Cursor Motion (Fixed Rectangles):* The user repeatedly clicks back and forth between two vertical rectangles of fixed width and amplitude, as shown in Figure 2.2(a). After 11 repetitions, the width and amplitude are

changed. A trial is defined by the task of moving the mouse to the rectangular target and clicking on it. Movement time is the time between successful clicks. Since the purpose of this experiment was to measure movement times for repetitive motions, we discarded the first 3 movement times out each set of 11 repetitions as "warm-up" movements. This experiment includes 24 recorded trials.

2. *Heterogeneous Cursor Motion (Variable Circles):* The user clicks on target circles of varying diameter and location, as shown in Figure 2.2(b). A trial begins when the user clicks on a small "home" circle in the center of the window and ends when the user successfully clicks inside the target circle. After each click on a target circle, the user must return the mouse back to the "home" circle and click on it before starting the next trial. Movement time is the time between clicking on the "home" circle and successfully clicking on the target circle. The target width (circle diameter) and amplitude (distance from target center to home circle center) are varied for every trial. The experiment includes 25 trials.

The distance/amplitude and size/width of the targets for each experiment are shown in Table 2.1. Lengths are measured in units of pixels, so the distance and size of a particular target may appear different on computer systems with different display sizes and resolutions. The order of the trials for each experiment was randomly selected, although the same order was used for each participant.

For compatibility we implemented our applet using Java. It is available online at http://www.tele-actor.net/fitts/. To allow precise measurement of movement times without lag from Internet communications, movement times are measured locally on the participant's computer and sent to our central server after completion of the trials.

2.3.2 Controlled and Uncontrolled User Studies

46 participants responded to ads posted on campus and on Facebook and participated in the controlled study. Upon successfully completing the experiment each participant was given a gift certificate from Amazon.com. All the user experiments were conducted after the human subject certificate was granted (Removed for blind review) We had 17 female (37%) and 29 male (63%) participants in the study. Out of the 46 participants, 4 were left-handed, but still used their right hand to operate the pointing device. Although all of the left-handed participants were given the chance to customize their environment, none of them changed their mouse settings to left-handed; prior studies have shown that this does not disadvantage left-handed users [HCY97]. The average age for our participants was 24.7 (variance = 23.8). The distribution of the ages for participants is shown in Figure 2.5. Our participants play video games for an average of 1.5 hours per week (the population has a high variance of 10.01 hours, suggesting that the majority do not play video games during the week).

Each participant required approximately 15 minutes to complete the Fixed Rectangles experiments. Each participant started with the rectangular object condition and after finishing the first condition is allowed to take a break. Then the participant is presented with the applet for the Variable Circles condition. Applets automatically load 10 trials of each condition, so we collect 250 individual mouse trajectories (25 trials with 10 repetitions) for the Variable Circles condition and 240 mouse trajectories for the Fixed Rectangles conditions. To prevent fatigue, participants were allowed to rest between repetitions. The controlled experiments were performed in front of lab assistants. Lab assistants were instructed to observe participants and redo the experiment if an error happened or the user was not focused.

In addition to the controlled lab experiment we conducted a web-based experiment and collected experimental data online. For Human Subjects' purposes, participants were required to complete an online consent form before starting the first applet. A user entry was created in the server database each time a participant completed the consent form. If one participant runs the applet multiple times, each trial appears as a distinct user.

Web-based studies enable comparisons of large samples in naturalistic settings with greater differing ecological validity conditions than found in a laboratory. However, there are substantial methodological disadvantages with uncontrolled (web-based) studies. We cannot obtain detailed data about the users, some users may perform the experiment multiple times, and we have no control over the user environment nor the input and display devices. But for comparing models of reaching tasks that are intended to generalize over a large number of conditions and devices, the latter may be acceptable uncontrolled web-based experiments hold promise. Furthermore, a major advantage of web-based experiments is in the number of human participants that can be compared and permit a large number of participants. Through the web-based applet were able to collect data from well over a thousand human participants.

As with all experiments with human participants, care must be taken to ensure the integrity of the data. We only consider users who completed all trials for a given experiment. We also verify for each user that no movement time of any trial is unrealistically small (below minimum simple reaction time). However, the movement times of several trials were very large, possibly because the user did not understand the instructions or was distracted from the applet. We consider a movement time an outlier if it is greater than 3 standard deviations from the mean of all users for the given trial. After removing users with outlier movement times, the number of users was 1,640 for the Homogenous cursor motion (Fixed Rectangles) experiment, and 1,561 for the Variable Circles experiment. Our final data set consisted of over 30,000 trajectory measurements for each experiment.

2.3.3 Experimental Results

Controlled User Study Results

As summarized in Table 2.2, we compare the models by computing the Root-Mean-Square (RMS) error between each user's movement times and the predicted movement times using Logarithmic Fitts Model (LOG) and the Square-Root model (SQR) with user-specific parameters. As mentioned in Sec. 2.3 model parameters like a and b are fitted and calculated for each individual participant especially since each participant may use a different pointing device. The RMS error is used to compare models as a measure of the goodness of fit. For further analysis and results of the comparisons between the SQR model, Plamondon's Power Model (2.4) and MacKenzie's Model (2.3) please see the appendix section. When Logarithmic and Square-Root models are compared, for 64.5% of the users the Square-Root model had a lower RMS error while the RMS error for the Logarithmic model was only the lowest for 35.5% of the users. A two-sided paired t-tests is performed on the difference in means of signed differences of RMS errors between the two models. We found that the difference between Logarithmic and Square-Root models was statistically significant for the Variable Circles ($p = 2.15 \times 10^{-13}$). In all of the experiments because of the large number of experimental data a conservative significance threshold of $p = 0.005$ is used for the experimental results. Results of these comparisons are presented in Table 2.2.

Uncontrolled (Web-Based) Study

We compared the RMS errors for Logarithmic model with the RMS errors for the Square-Root model using a two-sided paired t-test with a significance level of $p = 0.005$. The difference between the two models was not statistically significant for Fixed Rectangles ($p = 0.11$) but was statistically significant for the Variable Circles condition ($p = 4.74 \times 10^{-46}$). This agrees with the results from the controlled experiment.

2.3.4 Discussion

In the appendices B and for the heterogeneous Variable Circles condition we first compare Plamondon's Power model (PWR) with the Logarithmic model, in this case the Power model better fits the data for 64.85 percent of the users and the Logarithmic model performed better in 35.15% of the cases. We then compared all three models together. When Power, Square-Root and Logarithmic models are compared together, in only 5.76 percent of the cases the Power model outperforms the other two, and the Square-Root model is more accurate in 61.82 percent of the cases. On the other hand, the Logarithmic model only drops from 35.15% of the cases in the first comparison (comparison between Power model and Logarithmic model) to only 32.42% of the cases when all three models are compared. And when the Square-Root model is compared only with the Logarithmic model, Logarithmic model better fit the data in 35.45% of the cases and the Square-Root model fit 64.55% of the cases. This demonstrates that while the Square-Root model outperforms both models, there are movement cases for which the Logarithmic model constantly outperforms the other two. In other words, the Square-Root model fits better for the movements that the Power model has outperformed the Logarithmic model than in the cases that the Logarithmic model has outperformed the Power model.

In the heterogeneous Variable Circles experiments, the user must perceive and locate each target and plan a new motion. Unlike the Logarithmic model which models the information capacity of the reaching movement, the Square-Root model separates simple reaction and motion planning time (the a parameter) from the time of the actual reaching movement ($b\sqrt{A/W}$) as described in its derivation above. We believe this separation gives the Square-Root model an accurate advantage in modeling reaching movements that require both cognitive and physical movement components.

The empirically fitted values for parameter a in the Square-Root model are consistent with its meaning in the kinematics derivation. Mean simple reaction time for humans, which is shorter than mean recognition and choice reaction times, is generally 150-220ms for a visual stimulus, depending on the strength of stimulus, age of the participant, state of attention, and other physical and mental factors [Wel80]. An additional unknown time must be added to the simple reaction time for motion planning. Expectation may cause the planning required by the brain to be different for differ between repetitive and novel reaching movements; pre-motor cortex potentials begin up to 800ms prior to movement [DSK69]. The mean total reaction times a for the Square-Root model for the uncontrolled study, where a varies from 247.5ms to 461.2ms, exceed simple reaction time and provide support for the physical kinematics derivation of the Square-Root formulation. It is not surprising that, on average, movement time should be higher for heterogeneous Variable Circles cursor motions because of the increased perception and planning requirements of a changing target. Under the single channel theory of stimulus response, a shorter perceptual processing step or response processing step will lead to a faster total reaction time [WH00]. Such a difference was evident in the experimental results: the average empirically fitted value for a in the Square-Root model was lower for the Fixed Rectangles condition (247.5ms) than the Variable Circles condition (360.0ms).

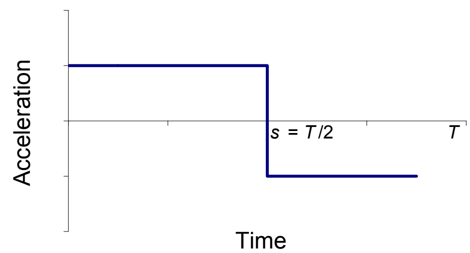
We compare the models by computing the root-mean-square (RMS) error between each user's movement times and the predicted movement times using Fitts' Logarithmic Model, the Power Model, and the Square-Root Model with user-specific parameters for each experiment. Experiment results comparing Fitts' Logarithmic Model and the Square-Root Model are shown in Figure 2.6.

We found that the Square-Root Model performs significantly better than the Power Model for homogeneous cursor motions ($p < 10^{-60}$) and heterogeneous cursor motions ($p < 10^{-50}$) in both experiments.

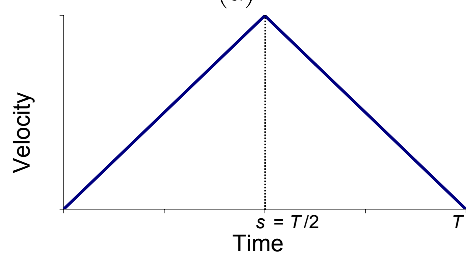
We also explored the relationship between a user's average speed and the model that best fits the user's data for heterogeneous motions. We evaluated a user's speed in an experiment as the the average movement time across all trials in that experiment. For the slowest 10% of users in the variable circle experiment, 46% were best fit by the Square-Root model, 46% were best fit by Fitts' Logarithmic model, and 8% were best fit by the Power model. For the fastest 10% of users in the variable circle experiment, 60% were best fit by the Square-Root model, 34% were best fit by Fitts' Logarithmic model, and 6% were best fit by the Power model.

2.4 Conclusion

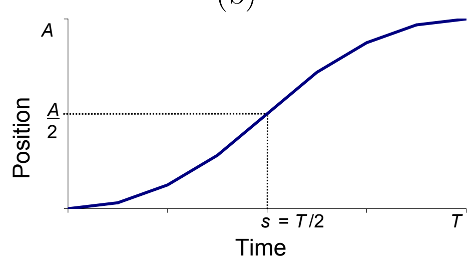
We considered a Square-Root alternative to Fitts' classic model for human reaching movements. We provide a succinct new derivation and perform two user studies to compare the models. In a controlled study with 46 participants using identical mouse hardware, we found that the Square-Root model was statistically equivalent to Fitts' model for motions between fixed-width rectangles. We found that the Square-Root model is significantly better ($p = 2.15 \times 10^{-13}$) for movements between circles of varying size and position (more relevant for screen interfaces). In an uncontrolled web-based remote user study with 1,561 anonymous (and possibly duplicate) participants using a variety of mouse types and settings, results were consistent and much more statistically significant ($p = 4.74 \times 10^{-46}$ for movements between circles of varying size and position). In the uncontrolled experiment we found that Logarithmic and the Square-Root models perform roughly equally well for the slowest 10% of users performing Variable Circles condition, but the latter significantly outperformed the former for the fastest 10% of users. This suggests that the Square-Root model is better for more skilled users. To our knowledge, our uncontrolled web-based study considers more trials than any previous reaching movement study. Web-based experiments have many drawbacks: we cannot record the input device, control-display gains, and mouse acceleration settings for each participant. Also, the participant population is self-selected. However, the web-based study enabled us to obtain movement time measurements in a variety of ecological conditions. Together, the two studies provide strong evidence in favor of the Square-Root model.



(a)

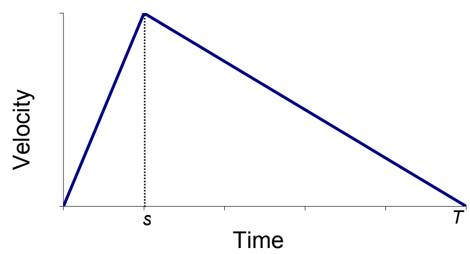


(b)

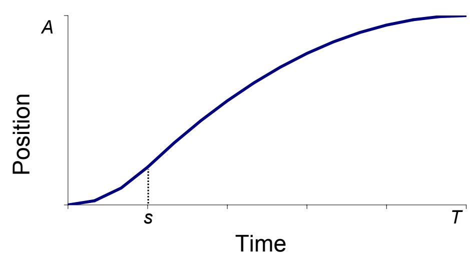


(c)

Figure 2.3. Acceleration vs. Time (a), Velocity vs. Time (b), and Position vs. Time (c) under symmetric optimal control. The “bang-bang” controller maintains the maximal positive acceleration in the first half of the motion and then switches to the maximal negative acceleration until the target is reached (a). The maximal velocity is reached in the middle of the path (b).



(a)



(b)

Figure 2.4. Velocity vs. Time (a) and Position vs. Time (b) for the asymmetric acceleration model. Similar to MacKenzie we assume that the velocity profile is asymmetric and the peak velocity occurs at a switching time s that is not necessarily equal to $T/2$ (a) [MMD+87a].

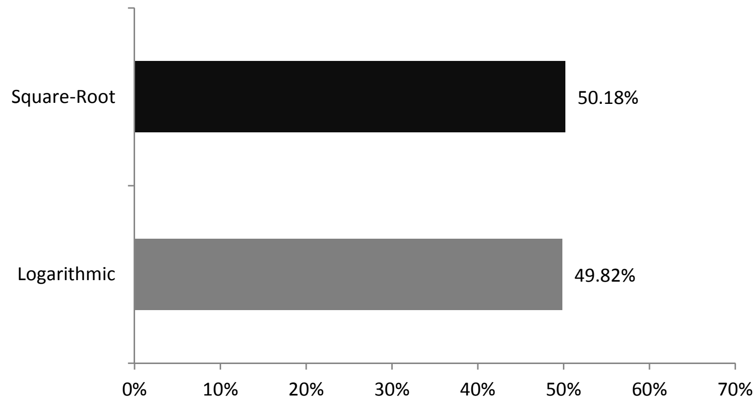
Figure 2.5. Age distribution for participants for the controlled study

Trial	Fixed Rectangles		Variable Circles	
	A	W	A	W
1	370	50	67	20
2	370	50	184	38
3	370	50	280	14
4	370	50	230	29
5	370	50	144	55
6	370	50	249	29
7	370	50	255	14
8	370	50	96	50
9	240	10	225	19
10	240	10	263	12
11	240	10	259	25
12	240	10	229	20
13	240	10	215	31
14	240	10	198	83
15	240	10	301	16
16	240	10	194	66
17	180	70	260	12
18	180	70	296	14
19	180	70	180	44
20	180	70	278	11
21	180	70	283	37
22	180	70	40	32
23	180	70	233	10
24	180	70	191	50
25	-	-	179	18

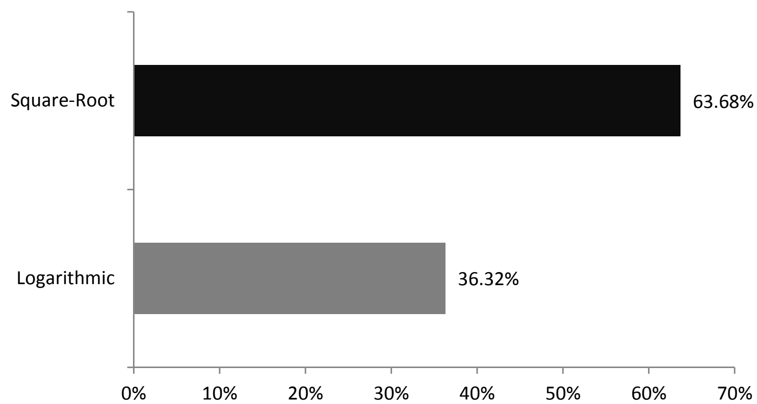
Table 2.1. Target distance/amplitude (A) and size/width (W), in display pixels, for the 24 recorded Fixed Rectangles (Fixed Rectangles) trials and 25 Variable Circles trials.

	Controlled Experiment		Uncontrolled Experiment	
Condition	SQR vs. LOG	SQR vs. PWR	SQR vs. LOG	SQR vs. PWR
Fixed Rectangles	8.93×10^{-3}	1.90×10^{-3}	1.13×10^{-1}	3.03×10^{-58}
Variable Circles	2.15×10^{-13}	1.23×10^{-13}	4.74×10^{-46}	2.10×10^{-64}

Table 2.2. Statistical hypothesis testing results for Fixed Rectangles and Variable Circles conditions in controlled and uncontrolled experiments.



(a) Fixed Rectangles



(b) Variable Circles

Figure 2.6. Comparison of the Fitts' Logarithmic and Square-Root models for the uncontrolled experiments. For each experiment, the bar for each model indicates the percent of users for whom that model best fit the data. For each user in each experiment, we determined the model that best fit the measured movement times by selecting the model that minimized RMS error. A follow up two-sided t-test is performed on the RMS error values for each pair of models. We found that for the Variable Circles case the difference in the value of RMS errors was statistically significant ($p = 4.74 \times 10^{-46}$) while this difference was not statistically significant when we compared Square-Root and Logarithmic models in the he Fixed Rectangles condition ($p = 0.11$). Because of relatively large number of participants a conservative significance level of $p = 0.005$ is used in these comparisons.

Chapter 3

Unimodality of Human Motor Performance in Heterogeneous Cursor Movements Between Circular Objects

In this chapter we report our recent findings on properties of human motor performance. We study the unimodal behavior of the performance of human motor in the case of heterogeneous cursor movements between circular objects. In the heterogenous cursor motions targets vary in distance and width for each trial. We derive a square-root Fitts' law model by using the kinematics and control theories. A third parameter is introduced into the model and the unimodal behavior of performance for the third parameter is shown. This unimodal property enables us to find the optimal performance by calibrating the square-root Fitts' law equation for this new parameter. We compared the performance of the calibrated square-root model with the original logarithmic model and the Plamondon's power law model on human subjects who participated in our study. The root-mean-square (RMS) error between each user's movement times and the predicted movement times are determined for these three cases. The calibrated square-root model outperforms the logarithmic law in 69% of the cases. We also observe that the human motor model transforms from the logarithmic Fitts' law for slower participants to a square-root Fitts' for faster participants. Finally we suggest that this behavior allows us to design adaptive games in which games changes their predicting Fitt's model as the gamer gains skills (Fig 3.1).

Figure 3.1. New developments in multi-touch mobile devices lead to more research on the performance of human motor on circular objects

3.1 Introduction

The starting point of our exposition in this report is the Square-root Fitts' law shown in equation (3.2) first introduced by Meyer et al. [MAK$^+$88] in 1988. They start with the observation reported by Kvålseth [Kvå80] that a power function with an exponent of about 1/2 better predicts the mean movement time between objects than the logarithmic Fitts' law. Meyer et al. [MAK$^+$88] devise equation (3.2) using an optimal dual-submovement model. They also report that the Square-root model (3.2) works better than the logarithmic model (3.1) on Fitts' own data provided in [PM54].

In their model, Meyer et al [MAK+88], split each movement into two submovements: an initial ballistic submovement and an optimal corrective submovement.

$$T = a + b \log_2 \left(\frac{2A}{W} \right) \tag{3.1}$$

$$T = a + b \sqrt{\frac{A}{W}} \tag{3.2}$$

In the previous chapter we derived equation (3.2) by considering that the optimal time to reach a target is attained when the human motor behaves similar to a "bang-bang" controller. In a bang-bang controller the maximal positive acceleration is maintained constant through the first half of the movement and then it switches to maximal negative acceleration [JF03b].

3.2 The Square-Root Model

The three-parameter model presented in this chapter is based on the two-parameter work presented in the previous chapter. Here we assume that the optimal controller hypothesis presented in last chapter holds but the acceleration in human motor is piecewise constant and proportional to both target width W and a power function of the distance distance A^β where the exponent β is not 1 (otherwise the term, A, will be canceled out from the square-root model). Therefore the acceleration $\ddot{x}$:

$$\ddot{x} = kWA^\beta$$

In other words wider targets are perceived to be easier to reach thus correspond to higher accelerations. Additionally farther objects can be reached by higher acceleration while we assume that higher accelerations for closer objects will result in less accuracy in reaching the final target. We later show that the performance of this model as a function of β is unimodal and quasi concave. The concavity of this function let's us calibrate the model for the third parameter by using a 2D optimization method and finding the β for the maximum performance. In this model other parameters, a and b have a physical meaning: a is the initial duration for reaction and planning, and b is related to the maximum attainable acceleration.

3.2.1 Symmetric Acceleration Model

If we consider that $\ddot{x} = kWA^{\beta}$, and assume that the controller switches from $\ddot{x}_{max}$ to $-\ddot{x}_{max}$ at $T = 1/2$ then by following the same procedure as the one presented in the previouse chapter and substituting the equation for acceleration into

$$T = 2\sqrt{\frac{A}{\ddot{x}}}$$

We will get

$$T = 2\sqrt{\frac{A^{1-\beta}}{kW}}$$

We substitute $\gamma = 1 - \beta$ and $b = 2/\sqrt{k}$ and we will have:

$$T = a + b\sqrt{\frac{A^{\gamma}}{W}} \qquad (3.3)$$

3.2.2 Optimal Controller and Asymmetric Velocity Profile

In 1987, C. L. MacKenzie empirically showed that the velocity profiles of each movement from the initial starting point to the target is asymmetric [MMD+87b]. This results in a modification to our optimal controller hypothesis. Here we assume that switching from the highest positive acceleration to the most negative acceleration occurs at time s, not necessarily equal to $T/2$. In other words the velocity profile of

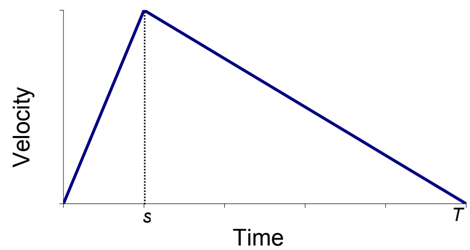


Figure 3.2. Velocity vs. Time for the asymmetric acceleration model.

the movement will be similar to the graph shown in Figure 3.2. The tail in which $T > s$ is where the corrections occur. In this model the maximum attainable positive acceleration before switching time, s, is not necessarily equal to the most negative acceleration right after $T \geq s$.

We also assume that the user stops at the starting point and at the target, therefore $\dot{x}(0) = 0$ and $\dot{x}(T) = 0$. Furthermore, $\ddot{x}_a = \ddot{x}_{max}$ and $\ddot{x}_d = \ddot{x}_{min}$ (a and d stand for acceleration and deceleration phases of the movement, according to empirical observations of MacKenzie [MMD$^+$87b] we have $|\ddot{x}_a| \geq |\ddot{x}_d|$). Similar to the derivation in the first chapter we now assume that the normalized time for the maximum velocity is linearly proportional to

$$\frac{s}{T} = kWA^{\beta}$$

We can now integrate the velocity function presented in Figure 3.2 in order to find the position x. We assume that $\ddot{x}_d$ is such that the velocity at time T is zero. Also $\dot{x}_{max} = \ddot{x}_a s$. By substituting $\frac{s}{T} = kWA^{\beta}$ in the previous equation we will have $\dot{x}_{max} = \ddot{x}_a kWA^{\beta}T$. The position as a function of $\dot{x}_{max}$ is found by integrating the velocity function in Figure 3.2 over time T.

$$x(T) = \frac{1}{2}\dot{x}_{max}s + \frac{1}{2}\dot{x}_{max}(T - s) = \frac{1}{2}\dot{x}_{max}T. \tag{3.4}$$

From the assumptions $x(T) = A$ and by substituting $\dot{x}_{max}$ into equation 3.4

$$D = \frac{1}{2}\ddot{x}_a k W A^{\beta} T^2.$$

Which yields to

$$T = \sqrt{\frac{2}{\ddot{x}_a k}\frac{A^{1-\beta}}{W}}.$$

Denote $b = \sqrt{\frac{2}{\ddot{x}_a k}}$, $\gamma = 1 - \beta$ and by adding a reaction and planing time a we have:

$$T = a + b\sqrt{\frac{A^{\gamma}}{W}} \tag{3.5}$$

Equation 3.5 is identical to equation 3.3 that was derived by using the assumption that $s = 1/2T$.

3.3 Experiment and The Dataset

We developed two graphical Java applets. These two applets are available at http://tele-actor.net/fitts/. Figure 3.3 shows the experiment for the "Variable circle heterogenous cursor motion" where the participant first sees the blue "home" circle in the center; and no green circle. When the participant clicks on the home circle a new green "target" circle with a different diameter and distance appears on the screen. The user must click on the target circle to make it disappear again and then click on the home circle before starting the next trial. The experiment includes 25 trials each of which with a green target with a different distance and size. We use outlier detection techniques in order to make sure that the collected data is reliable for data analysis [RL03].

Experimental data was collected from online users from 2004 to April 2005. Total of 1897 participants provided timing information.

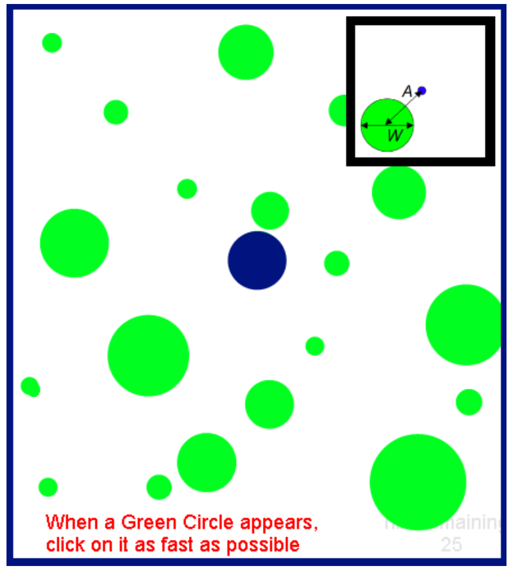


Figure 3.3. The graphical applet for heterogeneous cursor motion for circular objects. Green targets appear and disappear one after another. In this figure 20 consecutive experiments are superimposed on the same diagram.

3.4 Unimodality of Performance

We compared three different Fitts' models. The original Fitts' law [PM54], Plamondon's power law [Pla92b], and our calibrated square-root law. For all of the models a linear regression is used to find parameters a and b. The performance of models were compared by computing the root-mean-square (RMS) error between predicted and actual movement times. Figure 3.4 shows that the maximum performance is reached when we have $\gamma = 2/3$ and the original square root model becomes ($T = a + b\sqrt{\frac{A^{2/3}}{W}}$). The horizontal axis is the value of γ in the square-root formulation and the vertical axis shows the number of cases that our model performed better that the other two in terms of estimating the *mean total time*. By calibrating and modifying the value of γ from 1 to .64 we can increase the number of successful online experiments to 69% in the heterogeneous cursor movement case. As shown in Figure 3.4, performance as a function of the value of γ is quasi concave thus we are able to implement a line search in order to find the value of γ that maximizes the performance of the model compared to the other models. This result shows that unlike former studies (reported in [MMD+87b]), we observe a loose nonlinear correlation between the *normalize peak velocity time* (s/T) and the distance (A) and in our case (heterogeneous repetitive movements between circular targets) the following correlation holds, $\frac{s}{T} = kWA^{1/3}$, which means that the *normalize peak velocity time* (s/T) is correlated with A but with a small exponent of 1/3.

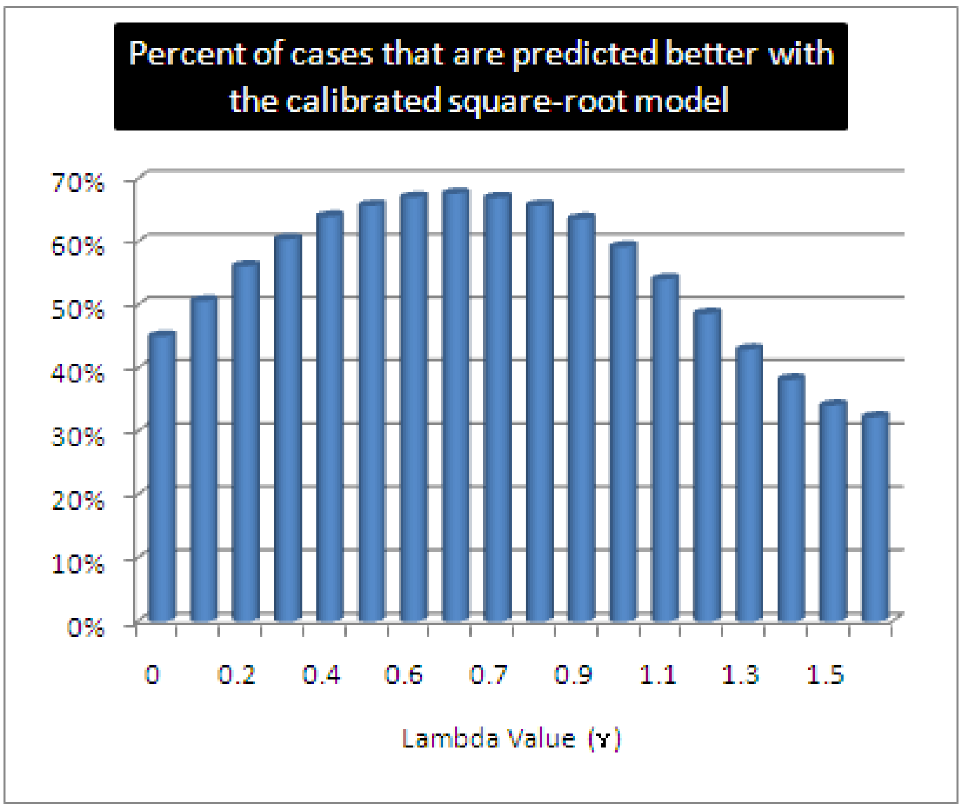


Figure 3.4. Unimodality of the performance as a function of γ helps us find the maximum value for γ with any simple nonlinear optimization method (e.q: steepest decent)

3.5 Discussion

The average speed of the people in this experiment was 861 with the Standard Deviation of 135.5. Figure 3.5 shows a histogram of the average speed for the users.

In order to find out in which cases the Square-root model provides a better solution, we analyzed the results based on the average speed and the device used (mouse, track pad, trackball, track point). In our experiment 10 percent of the slowest people (156 users with average speed of 1050 to 1414 milliseconds) have been analyzed. In this case in 72 cases Logarithmic Fitts' Law provided the best approximation, similarly in 72 other cases our Square-root model outperformed the other two and the Power Law was successful in outperforming the other two in only 12 cases). When fastest 10 percent of the users were considered (157 users with an average speed between 507 to 707), our Square-root model was successful in 60 percent of the cases (94 cases), and the Logarithmic model was successful in 54 cases, and in 9 cases the Power Law outperformed the other two. Additionally, we analyzed 64 cases that have a large RMS value in deferent models (RMS>300) in these cases in 46 cases the Square-root model provided a better approximation, Logarithmic Law was successful in only 15 cases and the Power Law did not outperform those two in any cases. In addition to the worst approximation we analyzed the best approximations as well (RMS<90), in these cases (77 users), our Square-root model provided the best fitting in 46 cases (60 percent of the cases), the original Logarithmic law was successful in 17 cases (22 percent) and the Power Law provided a better approximation in 14 percent of the cases). From the above discussion we can conclude that faster user better follow the optimal controller hypothesis; in other words as people's reaction skills improve (using video games, etc) their motor performance gets closer to the optimal bang-bang model and moves away from the information theoretical model.

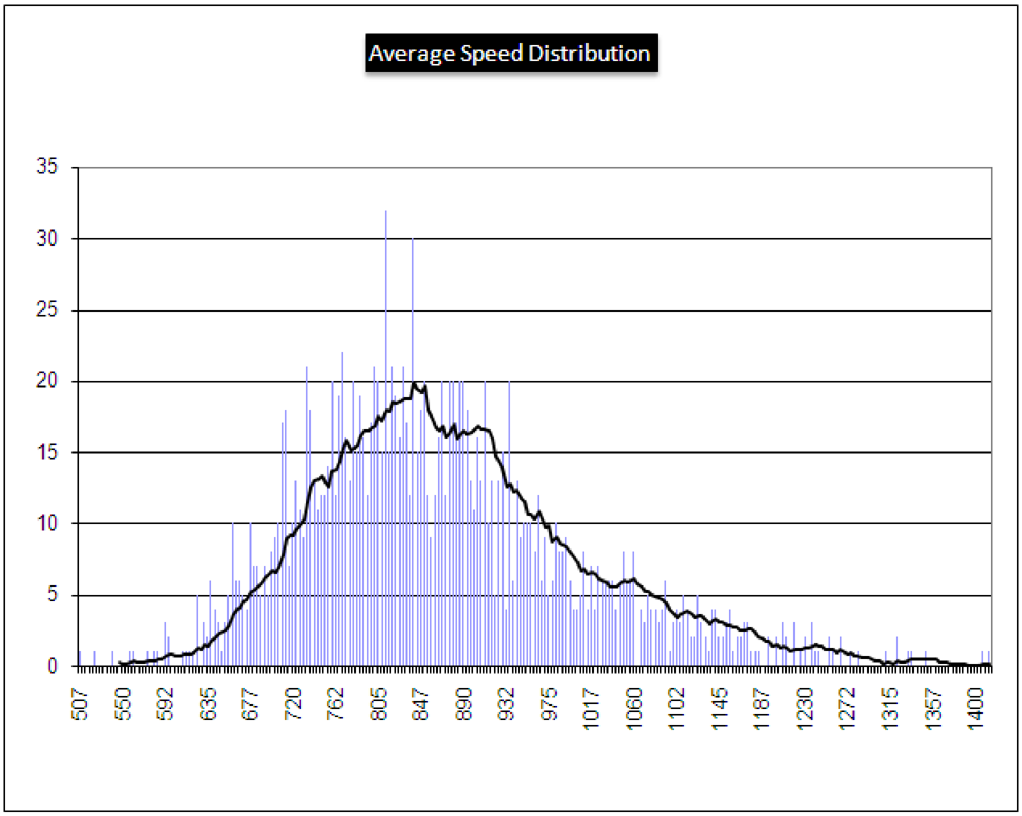


Figure 3.5. The online applet empowered us to collect data from participants with a wide range of average speed that covers the spectrum of different speeds from 507 milliseconds to 1.4 seconds

Chapter 4

Conclusion and Future Work

We provided the experimental results that support the new Square-Root model when models are compared regardless of the value of A/W. To the best of our knowledge this is the largest experiment for evaluating a Fitts' alternative. The three parameter model is presented as a more accurate model to extend the fundamental assumptions of the two-parameter model that $\ddot{x} = kW$. The resulting three parameter was then calibrated by using the empirical data and we obtained $T = a + b\sqrt{\frac{A^{2/3}}{W}}$. We compared this model with the logarithmic and power variations of the Fitts' law and showed that this model has an smaller RMS error in about 60% of the trajectories.

In this thesis we compared models regardless of the magnitude of the index of difficulty (A/W). In the future we would like to compare the Logarithmic and the Square-Root form of the Fitts' law for different values of A/W and understand in what cases the Square-Root generates the smallest RMS error value and in what cases it produces the largest RMS errors. We are hoping that this analysis will shed lights on the fundamental differences between the Logarithmic and Square-Root variants of the Fitts' Law.

Appendix A

Derivation of Meyer's Model:

In this section we discuss the derivation presented by Meyer *et. al.* [MAK$^+$88]. Details of derivation are missing from their paper and that is why a detailed derivation is reproduced here. The terminology used in this section is similar to the one user in Meyer's work and we avoid redefining the terms again. We refer interested readers to that paper for detailed definition for each term.

Meyer et al., [MAK$^+$88] consider two submovements, a primary submovement that takes T_1 and a secondary submovement that takes T_2. As a result the movement will take T.

$$T = T_1 + T_2$$

They assume D_1 to be the mean traveling distance for the primary submovement then:

$$\Rightarrow T_1 = \frac{D_1}{V_1} \xrightarrow[S=KV_1]{assumption} T_1 = \frac{KD_1}{S_1}$$

We replace the above equation in $T = T_1 + T_2$. Additionally Meyer et al., assume that the primary submovement will arrive the pointer to the target therefore if we assume that D is the distance from the home position to the center of the target region, we will have:

$$T = \frac{KD}{S_1} + T_2$$

Now consider Δ the expected length of the secondary corrective submovement. If $|\Delta| \leq \frac{W}{2}$ then we are already in the target region $T_2 = 0$. If on the other hand $|\Delta| > \frac{W}{2}$ then $T_{2\Delta} = \frac{K\Delta}{S_2}$.

Denote C_2 the ration of points that fall in the target region and recall that the z-score is defined as $z = \frac{x-\mu}{\sigma}$ then:

$$Z_{C_2} = \frac{W/2}{S_2} \Rightarrow S_2 = \frac{W}{2Z_{C2}}$$

Which means:

$$P(-Z_{C2} \leq Z \leq Z_{C2}) = C_2$$

Also recall that the expected value of a random variable is $E(x) = \int_{-\inf}^{+\inf} xp(x)dx$

$$T_{2\Delta} = \frac{K\Delta}{S_2}, T_2 = E(T_{2\Delta}$$

$$\Rightarrow T_2 = \int_{-\infty}^{+\infty} \frac{K\Delta}{S_2} p(\Delta)d\Delta$$

In which $p(\Delta)$ is a Gaussian distribution.

$$\Rightarrow T_2 = \int_{-\infty}^{+\infty} \frac{K\Delta}{S_2} n(\Delta|0, S_1)d\Delta$$

Where n is the probability-density function of a normal random variable. Meaning:

$$\Rightarrow n(\Delta|0, S_2) = \frac{1}{S_1\sqrt{2\pi}} e^{-(\frac{\Delta^2}{2S_1^2})}$$

$$T_2 = \int_{-\infty}^{-W/2} \frac{K\Delta}{S_2} n(\Delta|0, S_1) d\Delta + \int_{W/2}^{\infty} \frac{K\Delta}{S_2} n(\Delta|0, S_1) d\Delta$$

We can now replace S_2 with $W/(2Z_{C2})$ and arrive at the following equation for T_2

$$T_2 = \int_{-\infty}^{-W/2} \frac{2KZ_{C2}}{W} n(\Delta|0, S_1) d\Delta + \int_{W/2}^{\infty} \frac{KZ_{C2}}{S_2} n(\Delta|0, S_1) d\Delta$$

$$T_2 = \frac{4KZ_{C2}}{W} \int_{-\infty}^{W/2} \Delta n(\Delta|0, S_1) d\Delta$$

Using integration by part:

$$T_2 = \frac{4KZ_{C2}}{W} \int_{-\infty}^{W/2} \Delta \frac{1}{S_1\sqrt{2\pi}} e^{-(\frac{\Delta^2}{2S_1^2})} d\Delta$$

$$\Rightarrow T_2 = \frac{4KZ_{C2}}{W} \left[-\frac{S_1 e^{-\Delta^2/(2S_1^2)}}{\sqrt{2\pi}} \right]_{W/2}^{\infty}$$

$$\Rightarrow T_2 = \frac{-4KZ_{C2}}{W\sqrt{2\pi}} \left[S_1 e^{-\Delta^2/(2S_1^2)} \right]_{W/2}^{\infty}$$

$$\Rightarrow T_2 = \frac{-4KZ_{C2}}{W\sqrt{2\pi}} \left[0 - S_1 e^{-\frac{W^2/4}{(2S_1^2)}} \right]$$

$$\Rightarrow T_2 = \frac{-4KZ_{C2}}{W\sqrt{2\pi}} S_1 e^{-\frac{W^2/4}{(2S_1^2)}}$$

We now replace S_1 with $W/(2Z_{C1})$. Note that the equation is similar to the one for Z_{C2}

$$\Rightarrow T_2 = \frac{-4KZ_{C2}}{W\sqrt{2\pi}} S_1 e^{-\frac{W^2 4Z_{C1}^2}{(8W^2)}}$$

$$\Rightarrow T_2 = \frac{-4KZ_{C2}}{W\sqrt{2\pi}} S_1 e^{-\frac{Z_{C1}^2}{2}}$$

$$\Rightarrow T_2 = \frac{-4KZ_{C2}S1}{W\sqrt{2\pi}} e^{-\frac{Z_{C1}^2}{2}}$$

Recall $T = T_1 + T_2 = \frac{KD}{S_1} + T_2 \left(S_1 = \frac{W}{2Z_{C1}}\right)$

$$\Rightarrow T = \frac{2KDZ_{C1}}{W} + \frac{2KZ_{C2}}{Z_{C1}\sqrt{2\pi}}.e^{-\frac{Z_{C1}^2}{2}}$$

We are minimizing T with respect to Z_{C1} by taking the derivative and equal it to zero

$$\frac{\partial T}{\partial Z_{C1}} = 0$$

$$\Rightarrow Z_{C1} = \frac{1}{\sqrt{\theta D/W - 1}}$$

Where $\theta = \left[\sqrt{2\pi} e^{Z_{C1}^2/2}\right] / Z_{C2}$

as D/W grows large $\theta \to \frac{\sqrt{2\pi}}{Z_{C2}}$

Meyer et al., assume that D/W is large

$$\Rightarrow T = 2K \frac{2\theta\sqrt{D/W} - \sqrt{W/D}}{\left[\theta\sqrt{\theta - W/D}\right]}$$

We finally get

$$T = 2K\sqrt{D/W}\left[1/\sqrt{\theta} + \theta/\theta\right]$$

$$T = \left[2K\frac{2}{\sqrt{2Z_{C1}}2\pi}\right]\sqrt{D/W}$$

In which $\left[2K\frac{2}{\sqrt{2Z_{C1}}2\pi}\right]$ is B. And by adding a constant reaction time A we arrive at the square-root model (This step is not discussed in Meyer's paper.

$$\Rightarrow T = A + B\sqrt{D/W}$$

Appendix B

Comparison with Plamondon and MacKenzie's Models

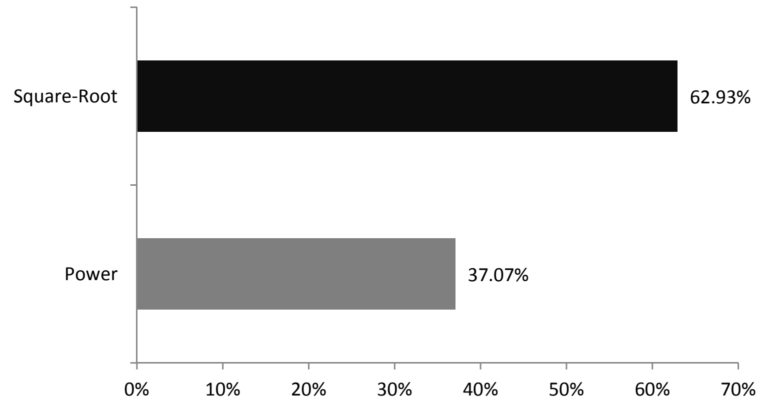
In this section we briefly compare the experimental results from our model with Plamondon's Power Model (2.4) and MacKenzie's Model (2.3).

In addition to the Logarithmic model (3.1), we compared the Square-Root model with the Plamondon's power model (2.4). In the controlled experiments when Square-Root model and Power model are compared, Square-Root model had the lowest RMS error in 63.94% of the users and the Power model performed better for only 36.06% of the users. To determine the statistical significance ($p < 0.005$) of the tree models, we performed a one way within subject ANOVA on RMS values in the three conditions (Logarithmic, Square-Root, and Power models). When ANOVA was used on the Fixed Rectangles condition the p-value was 7.63×10^{-7} and for Variable Circles the p-value was 2.21×10^{-16}. These values suggest that the differences between the RMS errors of these three models are statistically significant in both Fixed Rectangles and Variable Circles conditions. As follow up tests, two-sided paired t-tests were performed on the difference in means of signed differences of RMS errors between the two models. The results of follow up t-tests are summarized in Table 2.2.

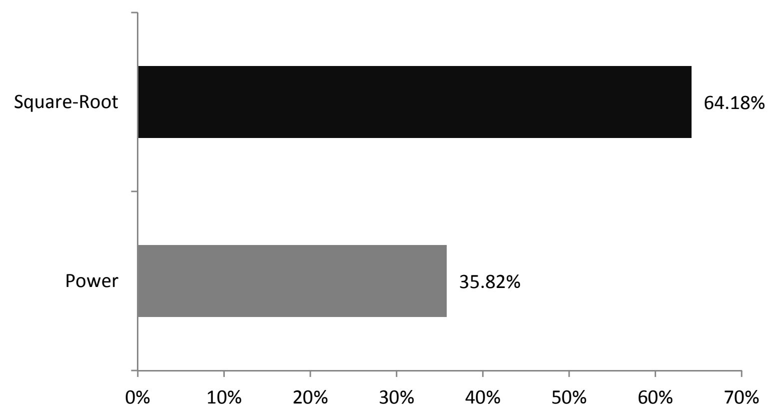
We also used two-way ANOVA analysis to determine whether or not age or gender have been the significant factors for differences between Logarithmic, Square-Root, and Power models. A two-way ANOVA on the dataset with gender and RMS as factors resulted in p-value of 0.45. Similarly another two-way ANOVA with RMS and age groups as factors resulted in p-value of 0.42. These results suggest that gender and age are not contributing to the differences between RMS errors for three models. In the uncontrolled experiments we also found that the Square-Root model performs significantly better than the Power model for Fixed Rectangles ($p < 10^{-60}$) and Variable Circles ($p < 10^{-50}$). Figure B.2 compares the percentages of the cases for which each one of the three models provided the lowest RMS error. Experiment results comparing the Power Model and the Square-Root Model are provided in Figure B.1.

We also explored the relationship between a user's average speed and the model that best fits the user's data for Variable Circles. We evaluated a user's speed in an experiment as the average movement time across all trials in that experiment. For the slowest 10% of users in the Variable Circles condition 46% were best fit by the Square-Root model, 46% were best fit by Logarithmic model, and 8% were best fit by the power model. For the fastest 10% of users in the Variable Circles, 60% were best fit by the Square-Root, 34% were best fit by Logarithmic model, and 6% were best fit by the power model.

To compare our Square-Root model (2.5) with MacKenzie's Model (2.3) we used $\log(A/W+1)$ for the index of difficulty (ID) instead of $ID = \log(2A/W)$ that is used in the Logarithmic Fitts' law (3.1). In that for the Fixed Rectangles experiments, the p-value of the two-sided paired t-tests on the signed differences in RMS error decreased from 8.93×10^{-3} to 2.25×10^{-6}, and for the variable circles experiment the p-value was increased from 2.15×10^{-13} to 1.34×10^{-5} both in favor of our Square-Root model.

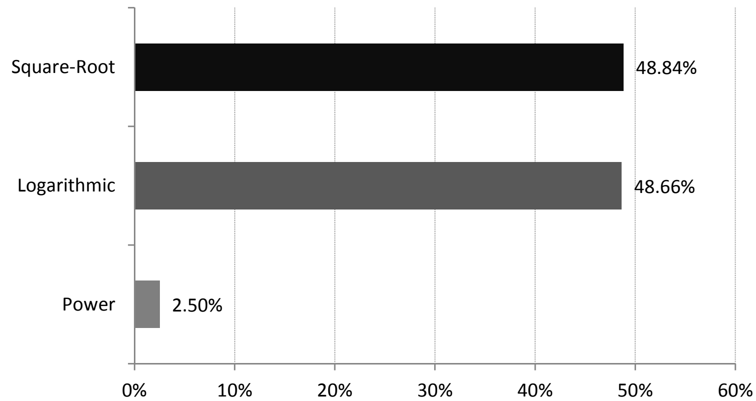


(a) Fixed Rectangles

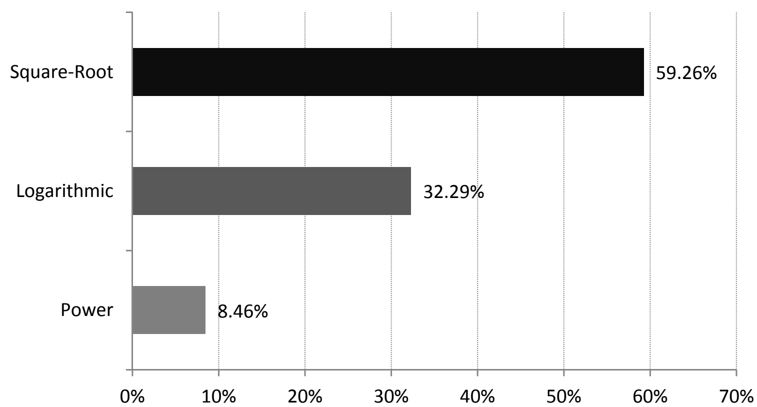


(b) Variable Circles

Figure B.1. Comparison of the Power and Square-Root models for the uncontrolled experiments. The method used to determine the best fit model is similar to the one used in Fig. 2.6. The follow up two-sided paired t-tests on the signed differences in RMS error values revealed that the RMS errors for the Square-Root model was significantly smaller than the ones for the Power model. The p-value for the Fixed Rectangles experiment was smaller than 10^{-60} and was smaller than 10^{-50} for the Variable Circles experiments.



(a) Fixed Rectangles



(b) Variable Circles

Figure B.2. Comparison of the models using the data collected from uncontrolled experiments. Similar to Fig B.1 and Fig 2.6 the models are compared based on the percentage of cases that each model has the smallest RMS error values amongst the other two. A one way within subject ANOVA on the RMS values confirmed that the differences of RMS values among the three models is statistically significant. The follow up t-test revealed that the difference of the RMS values is significant when Logarithmic and Square-Root models are compared in the Variable Circles experiment ($p = 4.74 \times 10^{-46}$) and this difference is not statistically significant in the Fixed Rectangles experiment $p = 1.13 \times 10^{-1}$. These results are consistent with the results from our controlled experiments as shown in Table 2.2.

References

[AGZ08] Georg Apitz, François Guimbretière, and Shumin Zhai. Foundations for designing and evaluating user interfaces based on the crossing paradigm. *ACM Trans. Comput.-Hum. Interact.*, 17(2):9:1–9:42, May 2008.

[ANSS07] M.S. Andreasen, H.V. Nielsen, S.O. Schrøder, and J. Stage. What happened to remote usability testing?: an empirical study of three methods. In *Proceedings of the SIGCHI conference on Human factors in computing systems*, pages 1405–1414. ACM, 2007.

[AZ97] Johnny Accot and Shumin Zhai. Beyond Fitts' law: Models for trajectory-based HCI tasks. In *Proc. ACM CHI '97*, pages 295–302, March 1997.

[AZ99] J. Accot and S. Zhai. Performance evaluation of input devices in trajectory-based tasks: an application of the steering law. In *Proceedings of the SIGCHI conference on Human factors in computing systems: the CHI is the limit*, pages 466–472. ACM, 1999.

[BGB04] Renaud Blanch, Yves Guiard, and Michel BeaudouinLafon. Semantic pointing: Improving target acquisition with control-display ratio adaptation. In *Proc. ACM CHI 2004*, pages 519–526, April 2004.

[BJJ+10] Jeffrey P. Bigham, Chandrika Jayant, Hanjie Ji, Greg Little, Andrew Miller, Robert C. Miller, Robin Miller, Aubrey Tatarowicz, Brandyn White, Samual White, and Tom Yeh. Vizwiz: nearly real-time answers to visual questions. In *Proceedings of the 23nd annual ACM symposium on User interface software and technology*, UIST '10, pages 333–342, New York, NY, USA, 2010. ACM.

[BRMA12] Eytan Bakshy, Itamar Rosenn, Cameron Marlow, and Lada Adamic. The role of social networks in information diffusion. In *Proceedings of the 21st international conference on World Wide Web*, WWW '12, pages 519–528, New York, NY, USA, 2012. ACM.

[CEB78] Stuart K. Card, W. K. English, and B. J. Burr. Evaluation of mouse, rate controlled isometric joystick, step keys, and text keys for text selection on a cathode ray tube. *Ergonomics*, 21:601–614, 1978.

[CG83] E. R. F. W. Crossman and P. J. Goodeve. Feedback control of hand-movement and Fitts' law. *The Quarterly Journal of Experimental Psychology Section A*, 35(2):251–278, May 1983.

[Dra91] David Drascic. Skill acquisition and task performance in teleoperation using monoscopic and stereoscopic video remote viewing. In *Proc. Human Factors Society 35th Annual Meeting*, pages 1367–1371, September 1991.

[DSK69] L. Deecke, P. Scheid, and H. H. Kornhuber. Distribution of readiness potential, premotion positivity, and motor potential of the human cerebral cortex preceding voluntary finger movements. *Experimental Brain Research*, 7:158–168, 1969.

[FH85] Tamar Flash and Neville Hogan. The coordination of arm movements: An experimentally confirmed mathematical model. *The Journal of Neuroscience*, 5:1688–1703, July 1985.

[Fit54] P. M. Fitts. The information capacity of the human motor system in controlling the amplitude of movement. *Journal of Experimental Psychology*, 47:381–391, 1954.

[FSM98] Naomi Friedlander, Kevin Schlueter, and Marilyn Mantei. Bullseye! When Fitts' law doesn't fit. In *Proc. ACM CHI '98*, pages 257–264, April 1998.

[GBMBL01] Y. Guiard, F. Bourgeois, D. Mottet, and M. Beaudouin-Lafon. Beyond the 10-bit barrier: Fitts' law in multi-scale electronic worlds. *Proc. Interaction Homme-Machine / Human-Computer Interaction (IHM-HCI 2001), People and Computers XV - Interactions without frontiers*, pages 573–588, Sept 2001.

[GHA+90] Douglas J. Gillan, Kritina Holden, Susan Adam, Marianne Rudisill, and Laura Magee. How does Fitts' law fit pointing and dragging? In *Proc. ACM CHI '90*, pages 227–234, April 1990.

[HCY97] E.R. Hoffmann, W.Y. Chang, and K.Y. Yim. Computer mouse operation: is the left-handed user disadvantaged? *Applied Ergonomics*, 28(4):245–248, 1997.

[HD11] E.R. Hoffmann and C.G. Drury. Comment on visual layout modulates fittss law: The importance of first and last positions. *Psychonomic bulletin & review*, pages 1–5, 2011.

[HH10] E.R. Hoffmann and M.C. Hui. Movement times of different arm components. *Ergonomics*, 53(8):979–993, 2010.

[Hym53] R. Hyman. Stimulus information as a determinant of reaction time. *Journal of Experimental Psychology*, 45:188–196, 1953.

[JF03a] Richard J. Jagacinski and John M. Flach. *Control Theory for Humans: Quantitative Approaches to Modeling Performance*. Lawrence Erlbum Associates, Mahwah, New Jersey, 1st edition, 2003.

[JF03b] R.J. Jagacinski and J. Flach. *Control theory for humans: Quantitative approaches to modeling performance*. CRC, 2003.

[KCS08] A. Kittur, E.H. Chi, and B. Suh. Crowdsourcing user studies with Mechanical Turk. In *Proceeding of the twenty-sixth annual SIGCHI conference on Human factors in computing systems*, pages 453–456. ACM, 2008.

[Kri05] Per-Ola Kristensson. Breaking the laws of action in the user interface. In *Proc. ACM CHI 2005*, pages 1120–1121, April 2005.

[Kvå80] T.O. Kvålseth. An alternative to Fitts' law. *Bulletin of the psychonomic Society*, 16:371–373, 1980.

[Mac92] I. Scott MacKenzie. Fitts' law as a research and design tool in human-computer interaction. *Human-Computer Interaction*, 7:91–139, 1992.

[Mac95] I. Scott MacKenzie. Movement time prediction in human-computer interfaces. In R. M. Baecker, W. A. S. Buxton, J. Grudin, and S. Greenberg, editors, *Readings in Human-Computer Interaction*, pages 483–493. Kaufmann, Los Altos, CA, 2nd edition, 1995.

[MAK+88] D.E. Meyer, R.A. Abrams, S. Kornblum, C.E. Wright, and JE Keith Smith. Optimality in human motor performance: Ideal control of rapid aimed movements. *Psychological Review*, 95(3):340, 1988.

[MB92] I. S. MacKenzie and W. Buxton. Extending Fitts' law to two-dimensional tasks. In *Proc. ACM CHI '92*, pages 219–226, 1992.

[MC91] M. Moyle and A. Cockburn. Analysing mouse and pen flick gestures. In *Proc. SIGCHI-NZ Symposium On Computer-Human Interaction*, pages 19–24, July 1991.

[Mic03] Microsoft Corporation. SystemParametersInfo function help. Windows System Information Platform SDK, Microsoft Visual Studio .Net 2003, 2003.

[MMD+87a] Christine L. MacKenzie, R.G. Marteniuk, C. Dugas, D. Liske, and B. Eickmeier. Three-dimensional movement trajectories in Fitts' task: Implications for control. *The Quarterly Journal of Experimental Psychology*, 39A:629–647, 1987.

[MMD+87b] Christine L. MacKenzie, R.G. Marteniuk, C. Dugas, D. Liske, and B. Eickmeier. Three-dimensional movement trajectories in Fitts' task: Implications for control. *The Quarterly Journal of Experimental Psychology*, 39A:629–647, 1987.

[MS82] J. Macki and A. Strauss. *Introduction to optimal control theory*. Springer, 1982.

[MSB91] I. S. MacKenzie, A. Sellen, and W. Buxton. A comparison of input devices in elemental pointing and dragging tasks. In *Proc. ACM CHI '91*, pages 161–166, 1991.

[PA97] Réjean Plamondon and Adel M. Alimi. Speed/accuracy trade-offs in target-directed movements. *Behavioral and Brain Sciences*, 20:279–349, 1997.

[Pla92a] Réjean Plamondon. A theory of rapid movements. In G.E. Stelmach and J. Requin, editors, *Tutorials in motor behavior II*, pages 55–69. Elsevier Science, New York, 1992.

[Pla92b] Réjean Plamondon. A theory of rapid movements. In G.E. Stelmach and J. Requin, editors, *Tutorials in motor behavior II*, pages 55–69. Elsevier Science, New York, 1992.

[Pla95a] Réjean Plamondon. A kinematic theory of rapid human movements. Part I. Movement representation and generation. *Biological Cybernetics*, 72:295–307, 1995.

[Pla95b] Réjean Plamondon. A kinematic theory of rapid human movements. Part II. Movement time and control. *Biological Cybernetics*, 72:309–320, 1995.

[PM54] F. PM. The information capacity of the human motor system in controlling the amplitude of movement. *Journal of Erperimental Psychology*, 47(391):1088, 1954.

[RG12] livier Rioulo and Yves Guiard. Power vs. logarithmic model of fitts' law: A mathematical analysis. *"Math. Sci. hum. / Mathematics and Social Sciences"*, 2012. To Appear.

[RL03] P.J. Rousseeuw and A.M. Leroy. *Robust regression and outlier detection.* Wiley-IEEE, 2003.

[RPCP04] Kyle Reed, Michael Peshkin, J. Edward Colgate, and James Patton. Initial studies in human-robot-human interaction: Fitts' law for two people. In *Proc. IEEE Int. Conf. Robotics and Automation (ICRA)*, pages 2333–2338, April 2004.

[TSB04] S.G. Thompson, J. Slocum, and M. Bohan. Gain effects on angle of approach and cursor-positioning time with a mouse in consideration of Fitts' law. In *Proc. Human Factors and Ergonomics Society 48th Annual Meeting*, 2004.

[WCHM08] J.O. Wobbrock, E. Cutrell, S. Harada, and I.S. MacKenzie. An error model for pointing based on fitts' law. In *Proceedings of the twenty-sixth annual SIGCHI conference on Human factors in computing systems*, pages 1613–1622. ACM, 2008.

[Wel60] A. T. Welford. The measurement of sensory-motor performance: Survey and appraisal of twelve years' progress. *Ergonomics*, 3:189–230, 1960.

[Wel68] Alan Traviss Welford. *Fundamentals of Skill.* Methuen, London, 1968.

[Wel80] A. T. Welford. *Reaction times.* Academic Press, New York, 1980.

[WH00] Christopher D. Wickens and Justin Hollands. *Engineering Psychology and Human Performance.* Prentice Hall, Upper Saddle River, New Jersey, 2000.