Further Stochastic Analysis of the k-Server Problem on the Circle

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by Jarett Schwartz

Research Project

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FURTHER STOCHASTIC ANALYSIS OF THE *K*-SERVER PROBLEM ON THE CIRCLE

JARETT SCHWARTZ

Abstract. We consider a stochastic version of the k-server problem, analyzing the cost of the greedy algorithm on the circle. We fully characterize the distribution yielded by this process for $k = 2$. Next, we show new results for larger values of k. Then, we consider other variants of this process. Finally, we confirm the above results in simulations of the process.

1. Preliminaries

The k-server problem [8] is central to the study of online algorithms. In the traditional version of the problem, let $k \geq 2$ be an integer and let $\mathbb{M} = (M, d)$ be a metric space over a set of points M with $|M| > k$ and distance function d. The k-server problem uses an algorithm A to satisfy a sequence of requests $r = \{r_1, r_2, \ldots\}$, where each $r_i \in M$ and r is an infinite stream. For each request r_i, A chooses one of k "servers" to satisfy these requests by moving that server from its current position in M to r_i. The cost $C_i(A)$ for each step of the algorithm is the distance in $\mathbb{M}$ traveled by that server. In the traditional online setting of the k-server problem, the (online) algorithm A is compared against the optimal (offline) algorithm A^* using competitive analysis. The algorithm A is C-competitive if there exists a $c \geq 0$ such that for all request sequences r, $C_i(A) \leq C \cdot C_i(A^*) + c$ for all i. The competitive ratio for A is the infimum over all C for which A is C-competitive. In the deterministic setting, there is a lower bound for the competitive ratio of $\Omega(k)$ and a $(2k-1)$-competitive algorithm [6] is known. In the randomized setting, there is a lower bound for the competitive ratio of $\Omega(\frac{\log k}{\log \log k})$, and a $\operatorname{polylog}(k)$-competitive randomized algorithm [7] was recently shown.

In this report, we will consider a stochastic version of the k-server problem, following the setting of [10] and using their results to simplify the setup. Rather than analyzing an adversary's choice of the requests, as in the traditional setting, each r_i will be chosen identically and independently (i.i.d.) from some distribution over the points in M. While different distributions and metric spaces may yield interesting problems, this report will analyze the particular process generated by the greedy algorithm on requests from the uniform distribution over a circle of circumference k. The positions of requests and servers can take any real value in the interval $[0, k)$. The greedy algorithm sends the server closest to r_i at each step. Ties may be broken uniformly at random when the servers are equidistant from the request in opposite directions on the circle. When the closest servers are at the same point in M, then we choose one such that the ordering of the servers in clockwise order stays fixed. Given this tie-breaking scheme, note that servers will never cross over each other using the greedy algorithm, so their clockwise ordering is invariant.

[10] also defined a discrete version of the stochastic problem, where all requests are chosen from $\ell < \infty$ equidistant positions on the circle. They also showed that the discrete version

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converges to the continuous version as ℓ increases. So, this report only considers the continuous version. The continuous version can be modeled by a Markov chain $(S_n)_{n\geq 0}$ on the state space $\mathbf{\Sigma} = \{\boldsymbol{p} = (p_0, p_1, \dots, p_{k-1}) \in [0,k); p_j \leq p_{j+1} \leq \dots \leq p_0 \leq p_1 \leq \dots p_{j-1} \text{ for some } j\}$. Each p_i represents the position of the i^{th} server on the circle counted in the clockwise direction, and one time step in the Markov chain reflects the change in $\boldsymbol{p}$ after one request. For notational convenience, we will also sometimes use negative indices to label the servers, using the convention $p_{-j} = p_{k-j}$. In turn, each p_{-j} represents the position of the j^{th} server on the circle counted in the counterclockwise direction. More generally, $p_i = p_j$ if $i \equiv j \pmod{k}$. For example, $p_{k+1} = p_1$.

Although it was not assumed in the original description of the process in [10], let the initial positions of the servers S_0 be equidistant around the circle, so $d(p_i, p_{i+1}) = 1$ for all i. This does not change the behavior of S_n as $n \to \infty$, but does simplify some results through symmetry, as the distance between servers, $d(p_i, p_{i+1})$, is equally distributed for all choices of i throughout the process.

From the main theorem in [10], we know that S_n converges exponentially fast to a stationary distribution π and $\dfrac{1}{n}\sum_{i=1}^{n} C_i$ converges to $E_\pi[C]$ as $n \to \infty$, where $E_\pi[C]$ denotes the expected cost of a step in the process after one move from starting positions $\boldsymbol{p}$ chosen from π. See [10] and [9] for more precise convergence bounds. So, in order to analyze the expected average cost of one step in the continuous process, it suffices to analyze the expected cost of one step of the process with initial $\boldsymbol{p}$ chosen from π.

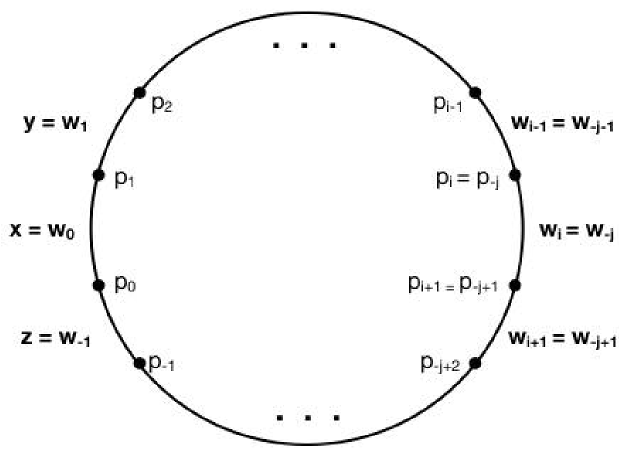

FIGURE 1. k-server on circle labeling

Let X be the random variable equal to $d(p_0, p_1)$ with $\boldsymbol{p}$ chosen from π. As noted above, the distance $d(p_i, p_{i+1})$ is identically distributed for any choice of $i \in 0 \dots k-1$ due to symmetry. Let $f(x)$ for $x \in [0,k]$ be the probability density function of X. Let Y be the random variable equal to $d(p_1, p_2)$ and Z be the random variable equal to $d(p_0, p_{-1})$. Let W_i be the random variable equal to $d(p_i, p_{i+1})$ and W_{-i} be the random variable equal to $d(p_{-i+1}, p_{-i})$ for $i \in [0, k-1]$. Note that $X = W_0$, $Y = W_1$, and $Z = W_{-1}$. Similar to the labeling of p_i, W_i = W_j if $i \equiv j \pmod{k}$. Let $f(w_i, w_j)$ be the joint probability density function of the intervals corresponding to W_i and W_j. For a given set of positions, $\boldsymbol{p}$, the variables x, y, z, w_i refer to the lengths of the intervals in $\boldsymbol{p}$ of the corresponding random variables X, Y, Z, W_i. Refer to Figure 1 for an example of this labeling. In this example, w_i is i segments away from x in the clockwise direction and j segments away in the counterclockwise direction.

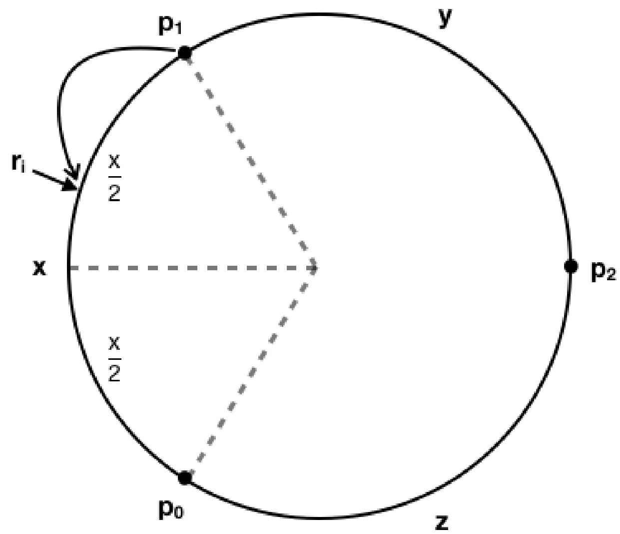

FIGURE 2. Example of a request in segment x satisfied by the greedy algorithm

$E_\pi[C]$ is related to $E[X^2]$ through the following observations. If $X = x$, a request lands in the range $[p_0, p_1]$ with probability $\frac{x}{k}$. The distance that a server moves to satisfy this request is uniformly distributed between $[0, x/2]$, since in the greedy algorithm, a server can move up to half of the length of its neighboring segment. In expectation, the distance moved is $x/4$. See Figure 2 for an example request. In this example, r_i arrives closest to the server located at p_1, so the server at p_1 moves to the position of r_i. Let $c = d(r_i, p_1)$. Then, after this step, $d(p_0, p_1) = x - c$ and $d(p_1, p_2) = y + c$.

Integrating over all choices of length x and summing over all k segments yields the following useful result (also shown in [10]).

Lemma 1.1. $E_\pi[C] = E[X^2]/4$.

Proof.

$$\begin{aligned}
E_\pi[C] &= E[E_\pi[C|\boldsymbol{p}]] \\
&= k\int_0^k \frac{x}{k} f(x)\frac{x}{4}dx \\
&= \frac{1}{4}\int_0^k x^2 f(x)dx \\
&= \frac{E[X^2]}{4}.
\end{aligned}$$

□

So, it suffices to calculate $E[X^2]$ to determine $E_\pi[C]$. In the following sections, $E_k[C]$, $f_k(x)$, and $E_k[X^2]$ will be used to refer to these values given a particular choice of k.

2. Precise Analysis of 2 Servers

In this section a new analysis is provided of the expected cost $E_2[C]$ of the stochastic k-server process on the circle with $k = 2$ servers. This analysis relies on symmetry properties of the distribution $f_2(x)$, so it may be extensible to analogous results for a wider class of process/distribution pairs. It is also shown that $f_2(x)$ does not admit a closed form, a question that was left open in [10].

2.1. **Fabius Function Background.** The definition of the Fabius function is rather ambiguous, as it has been independently derived several times, and the exact definition varies from source to source. In this report, the Fabius function is defined over $[0, 1]$, though it is often extended to $[0, \infty]$. The Fabius function F is the unique function that satisfies both $F(x) = \int_0^{2x} F(s)ds$ and $F(x) = 1 - F(1 - x)$ over $[0, 1]$. The Rvachëv function, $up(x)$, following the setting of [1], is defined over $[-1, 1]$ and takes the value $F(1 + x)$ for $x \in [-1, 0]$ and $F(1 - x)$ for $x \in [0, 1]$.

Several facts about the Fabius function will be helpful in the analysis of the k-server process. When introducing his eponymous function, Fabius showed the following theorem.

Theorem 2.1. *[4] $F(x)$ is nowhere analytic.*

$F(x)$ takes rational values for dyadic x (where $x = 1/2^k$ for some nonnegative k). Calculations of these values can be found in [1] and [5]. One can also derive the moments of $up(x)$ from similar calculations. The following result from [1] will be useful.

Lemma 2.2. *[1] $\int_{-1}^{1} x^2 up(x)dx = 1/9$.*

2.2. **Distribution of $f_2(x)$.** This section will show that the distribution $f_2(x)$ is characterized by the Rvachëv function. Then, the known results on the moments of the $up(x)$ function can be used to directly calculate $E_2[C]$.

First, by analyzing when one step of the process generates $d(p_0, p_1) = x$, it can be shown that $f_2(x)$ satisfies the following recurrence.

Lemma 2.3. *For $x \in [0, 1]$, $f_2(x) = \int_0^{2x} f_2(s)ds$.*

Proof. Let s be the initial value of $d(p_0, p_1)$. Then, consider any step in the process that makes $d(p_0, p_1) = x$. Since π is the stationary distribution, taking one step in the process leaves $f_2(x)$ fixed. Thus, the total density of these steps is equal to $f_2(x)$.

First, consider the case where the request falls in $[p_0, p_1]$. In this case, s becomes $s - c$ for some choice of $c \in [0, s/2]$. So, choices of c that satisfy $x = s - c$ create an interval of size x. Substituting the range $[0, s/2]$ for c gives $s \in [x, 2x]$. Note that the upper bound $x \leq 1$ guarantees that $2x$ is a valid interval length. The probability density for being in a state with interval length s is $f_2(s)$. The request falls in $[p_0, p_1]$ with probability $s/2$. A particular choice of c from the uniform distribution over $[0, s/2]$ has density $2/s$. Integrating over all choices of s, this contributes $\int_x^{2x} f_2(s)(s/2)(2/s)ds = \int_x^{2x} f_2(s)ds$ to the total density of steps that generate $d(p_0, p_1) = x$.

Next, consider the case where the request falls in the complementary interval of length $2-s$. In this case, s becomes $s + c$ for some choice of $c \in [0, (2 - s)/2]$. Choices of c that satisfy $x = s + c$ will yield an interval of size x. Note that $(2 - s)/2 + s = 1 + s/2 \geq 1 \geq x$, so x can always be reached if $s \leq x$. Substituting the range $[0, (2 - s)/2]$ for c gives $s \in [0, x]$. So, this case contributes $\int_0^x f_2(s)ds$ to the total density of steps that generate $d(p_0, p_1) = x$.

Summing both cases, the total probability density $f_2(x)$ equals $\int_0^{2x} f_2(s)ds$. ∎

Lemma 2.4. *For* $0 \le x \le 2$, $f_2(x) = f_2(2-x)$.

Proof. If $d(p_0, p_1) = x$, then $d(p_1, p_0) = 2 - x$. Since each interval is identically distributed, by symmetry, $f_2(x) = f_2(2-x)$. ☐

Lemma 2.5. $\int_0^1 f_2(s)ds = 1/2$.

Proof. This is true via symmetry, but it can also be computed from Lemma 2.4. $\int_0^1 f_2(s)ds = \frac{1}{2}\Big(\int_0^1 f_2(s)ds + \int_1^2 f_2(2-s)ds\Big) = \frac{1}{2}\Big(\int_0^1 f_2(s)ds + \int_1^2 f_2(s)ds\Big) = \frac{1}{2}\int_0^2 f_2(s)ds = \frac{1}{2}$. ☐

Lemmas 2.4 and 2.5 can be combined with Lemma 2.3 in order to get a similar symmetry result on the interval $[0,1]$. The intuition is that the integral $\int_0^{2x} f_2(s)ds$ in the recurrence for $f_2(x)$ can be mirrored over $x = 1$ and defined just on the interval $[0,1]$.

Lemma 2.6. *For* $0 \le x \le 1$, $f_2(x) = 1 - f_2(1-x)$.

Proof. Let $x \in [1/2, 1]$.

$$\begin{aligned}
f_2(x) &= \int_0^{2x} f_2(s)ds && \text{(by Lemma 2.3)} \\
&= \int_0^1 f_2(s)ds + \int_1^{2x} f_2(s)ds \\
&= 1/2 + \int_{2-2x}^1 f_2(s)ds && \text{(by Lemmas 2.4 and 2.5)} \\
&= 1/2 + \left(1/2 - \int_0^{2-2x} f_2(s)ds\right) \\
&= 1 - f_2(1-x) && \text{(by Lemma 2.3)}
\end{aligned}$$

The result follows for $x \in [0, 1/2]$ by substituting $x' = 1 - x$.

☐

Lemmas 2.3 and 2.2 are enough to fully characterize the distribution $f_2(x)$ in terms of the Rvachëv Function $up(x)$.

Theorem 2.7. $f_2(x) = up(x-1)$ *for* $x \in [0,2]$. *Additionally,* $f_2(x)$ *is nowhere analytic.*

Proof. Lemmas 2.3 and 2.2 exactly match the definition of $F(x)$ over $x \in [0,1]$. Hence, on $[0,1]$, $up(x-1) = F(x) = f_2(x)$. On $[1,2]$, $up(x-1) = F(2-x) = f_2(x)$ by Lemma 2.4. $f_2(x)$ is nowhere analytic due to Theorem 2.1. ☐

This theorem proves the conjecture in [10] that $f_2(x)$ does not yield a closed form. It also allows for the calculation of $E_2[C]$ using properties of $up(x)$, as follows.

Corollary 2.8. $E_2[C] = 5/18$.

Proof.

$$
\begin{aligned}
E_2[C] &= E_2[X^2]/4 && \text{(by Lemma 1.1)}\\
&= \frac{1}{4}\int_0^2 x^2 f_2(x)dx\\
&= \frac{1}{4}\int_0^2 x^2 up(x-1)dx && \text{(by Theorem 2.7)}\\
&= \frac{1}{4}\int_{-1}^1 (x+1)^2 up(x)dx\\
&= \frac{1}{4}(E[up(x)^2]+2E[up(x)]+1)\\
&= \frac{1}{36}+0+\frac{1}{4} && \text{(by Lemma 2.2)}\\
&= 5/18.
\end{aligned}
$$

□

These results and the connection to the Fabius function do not easily generalize to $k \geq 3$ servers. For instance, in the smallest case of $k = 3$, the mapping to a pair of intervals (x, y) after one step in the process is the union of intervals of the form $(x-c, y), (x+c, y), (x, y+c), (x, y-c), (x-c, y+c), (x+c, y-c)$. These intervals are more difficult to analyze. In addition, $f_k(x)$ is no longer symmetric around $x = 1$. So, in the next section, the analysis is conducted using only expectations, rather than properties of $f_k(x)$.

3. Analysis of k Servers

3.1. **Alternative calculation of $E_2[X^2]$.** In order to demonstrate the technique for analyzing $E_k[X^2]$, we present an alternative calculation of $E_2[X^2]$ which does not require knowledge of the distribution $f_2(x)$. The following proof will reflect the analysis of the interval length Z in [10].

Proof. (Alternative proof of Corollary 2.8)

Consider the expectation of X^2 after one request r_0. Let $c = \min_i(d(r_0, p_i))$ be the cost of the request. Integrating over c and splitting the analysis into cases corresponding to r_i landing inside or outside of the interval $[p_0, p_1]$ yields:

$$
\begin{aligned}
E_2[X^2] &= \int_0^2 \left(x\int_0^{x/2} (x-c)^2 \frac{1}{x}dc + (2-x)\int_0^{(2-x)/2} (x+c)^2 \frac{1}{2-x}dc \right) f_2(x)dx\\
&= \int_0^2 \left(\int_0^{x/2} (x-c)^2 dc + \int_0^{(2-x)/2} (x+c)^2 dc \right) f_2(x)dx\\
&= \int_0^2 \left(7x^3/24 + (-7x^3/24 + x^2/4 + x/2 + 1/3)dc \right) f_2(x)dx\\
&= \frac{E_2[X^2]}{4} + \frac{E_2[X]}{2} + \frac{1}{3}\\
&= \frac{E_2[X^2]}{4} + \frac{5}{6}.
\end{aligned}
$$

Hence, $E_2[X^2] = 10/9$, and by Lemma 1.1, $E_2[C] = E_2[X^2]/4 = 5/18$.

$\square$

This matches the value calculated for $E_2[X^2]$ in the previous section. Note that the $E_2[X^3]$ term canceled out in this proof. For $k > 3$ the analogous calculation does not lead to this cancellation. However, we can still find the value of $E_k[X^2]$ in terms of $E_k[X^3]$, as we now demonstrate.

3.2. **Moments for $k > 2$.** The following proofs will make extensive use of the fact that X, Y, Z, and all W_i are identically distributed. So, for instance, $E_k[X] = E_k[Y]$ and $E_k[X^aY^b] = E_k[X^bZ^a]$ (by rotational symmetry). However, it is not obvious that $E_k[X^aY^b] = E_k[X^aW_i^b]$ for all i, as the interval corresponding to W_i might not be the same distance on the circle from X as Y is. Nonetheless, this further symmetry can in fact be shown. The following lemmas will establish the equivalence $E_k[X^2W_i] = E_k[X^2W_j]$ for all i, j, but similar techniques work for other choices of a, b.

Lemma 3.1. $E_k[X^2W_i] = E_k[X^2W_{i-1}]/2 + E_k[X^2W_{i+1}]/2$ *for all* $i \geq 2$.

Proof. Consider the value of XW_i after one step in the process. Then,

$$\begin{aligned}
E_k[XW_i] &= E[E_k[XW_i|\boldsymbol{p}]] \\
&= \frac{1}{k}\int_0^k \int_0^k \left(2\int_0^{x/2} (x-c)w_i dc + 2\int_0^{w_i/2} x(w_i - c)dc + \int_0^{y/2} (x+c)w_i dc \right. \\
&+ \int_0^{z/2} (x+c)w_i dc + \int_0^{w_{i-1}/2} x(w_i + c)dc + \int_0^{w_{i+1}/2} x(w_i+c)dc \\
&+ \left. \int_0^{k-x-y/2-z/2-w_i-w_{i-1}/2-w_{i+1}/2} xw_i dc\right) f_k(x, w_i) dw_i dx.
\end{aligned}$$

Each term in the above integral arises from a different location of the request r_1 that defines the single step of the process. Since $i > 2$, the intervals corresponding to X and W_i are not adjacent, so at most one of x or w_i changes after one step. The terms containing $(x - c)$ and $(w_i - c)$ correspond to r_i landing in the intervals corresponding to X and W_i respectively. The terms containing $(x + c)$ correspond to r_i landing in the halves of the intervals corresponding to Z and Y that are closer to p_0 and p_1 respectively. Similarly, the terms containing $(w_i + c)$ correspond to to r_i landing in the halves of the intervals corresponding to W_{i-1}, W_{i+1} that are closer to p_i and p_{i+1} respectively. Finally, the last term corresponds to r_i landing in a interval that does not result in a change in x or w_i.

Expanding the above integral gives:

$$
\begin{aligned}
E_k[XW_i] &= \frac{3E_k[X^2W_i]}{4k} + \frac{3E_k[XW_i]}{4k} + \frac{E_k[W_iXY]}{2k} + \frac{E_k[W_iY^2]}{8k} + \frac{E_k[W_iXZ]}{2k} + \frac{E_k[W_iZ^2]}{8k} \\
&+ \frac{E_k[W_{i-1}^2X]}{8k} + \frac{E_k[W_{i-1}W_iX}{2k} + \frac{E_k[W_{i+1}^2X]}{8k} + \frac{E_k[W_{i+1}W_iX]}{2k} + E_k[W_iX] \\
&- \frac{E_k[W_{i-1}W_iX]}{2k} - \frac{E_k[W_{i+1}W_iX]}{2k} + \frac{E_k[Z^3]}{24k} - \frac{E_k[X^3]}{k} - \frac{E_k[X^2Y]}{2k} - \frac{E_k[X^2Z]}{2k} \\
&- \frac{E_k[W_iXY]}{2k} - \frac{E_k[W_iXZ]}{2k} - \frac{E_k[W_i^2X]}{k} - \frac{E_k[W_iX^2]}{k} \\
&= -\frac{E_k[X^2W_i]}{2k} + \frac{E_k[X^2W_{i-1}]}{4k} + \frac{E_k[X^2W_{i+1}]}{4k} + E_k[W_iX].
\end{aligned}
$$

Rearranging gives: $E_k[X^2W_i] = \dfrac{E_k[X^2W_{i-1}] + E_k[X^2W_{i+1}]}{2}$.

$\square$

Lemma 3.1 relates X^2W_i to its neighbors X^2W_{i+1} and X^2W_{i-1}. Applying this lemma j times leads to a similar relation to neighbors j intervals away.

Corollary 3.2. *For $i \geq 2$ and $j < i$, $E_k[X^2W_{i-j}] = (j+1)E_k[X^2W_i] - jE_k[X^2W_{i+1}]$ and for $i \geq 2$ and $j < k-i$, $E_k[X^2W_{i+j}] = (j+1)E_k[X^2W_i] - jE_k[X^2W_{i-1}]$.*

Proof. Proceed via induction on j. The base case, $j = 1$, is satisfied by Lemma 3.1. Now, assume that the Corollary is true for all $\ell \leq j$.

$$
\begin{aligned}
E_k[X^2W_{i-j-1}] &= 2E_k[X^2W_{i-j}] - E_k[X^2W_{i-j+1}] && \text{(by Lemma 3.1)} \\
&= (2j+2)E_k[X^2W_i] - 2jE_k[X^2W_{i+1}] \\
&+ (j-1)E_k[X^2W_{i+1}] - jE_k[X^2W_i] && \text{(by Inductive Hypothesis)} \\
&= (j+2)E_k[X^2W_i] - (j+1)E_k[X^2W_{i+1}].
\end{aligned}
$$

The corresponding equation for $E_k[X^2W_{i+j}]$ follows from rotational symmetry. $\square$

By applying Corollary 3.2 around the circle until we reach Y in one direction and Z in the other direction, we can show further equivalences.

Lemma 3.3. $E_k[X^2W_i] = E_k[X^2W_j]$ *for all $j > i > 0$.*

Proof. From Corollary 3.2, $iE_k[X^2W_i] - (i-1)E_k[X^2W_{i+1}] = E_k[X^2W_{i-(i-1)}] = E_k[X^2W_1] = E_k[X^2W_{-1}] = E_k[X^2W_{i+(k-i-1)}] = (k-i)E_k[X^2W_i] - (k-i-1)E_k[X^2W_{i+1}$. So, unless $i = (k-i)$, $E_k[X^2W_i] = E_k[X^2W_{i+1}]$. If $i = (k-i)$, then $i = k/2$ and $i - 1 \neq (k-(i-1))$), so $E_k[X^2W_{i-1}] = E_k[X^2W_i]$. By symmetry, $E_k[X^2W_{i-1}] = E_k[X^2W_{k/2-1}] = E_k[X^2W_{k/2+1}] = E_k[X^2W_{i+1}]$. So, $E_k[X^2W_i] = E_k[X^2W_{i+1}]$ for all i.

Then, $E_k[X^2W_i] = E_k[X^2W_{i+1}] = \ldots = E_k[X^2W_{j-1}] = E_k[X^2W_j]$. $\square$

This lemma enables the removal of W_i terms from an expansion of $E_k[X^aW_i]$.

Lemma 3.4. $E_k[X^aY] = (kE_k[X^a] - E_k[X^{a+1}])/(k-1)$

Proof.

$$\begin{aligned}
E_k[X^aY] &= E_k[X^a(k-\sum_{j\neq 1} W_j)] \\
&= kE_k[X^a] - E_k[X^{a+1}] - (k-2)\sum_{j\neq 0,1} E_k[X^aW_j] \\
&= kE_k[X^a] - E_k[X^{a+1}] - (k-2)E_k[X^aY] && \text{(by Lemma 3.3)} \\
&= (kE_k[X^a] - E_k[X^{a+1}])/(k-1).
\end{aligned}$$

□

Then, applying Lemma 3.4 yields an expression for $E_k[X^3]$ in terms of just $E_k[X^2]$.

Theorem 3.5. $E_k[X^3] = \dfrac{3k}{2k+1}E_k[X^2]$.

Proof. Consider the value of $E_k[X^2]$ after one step in the process.

$$\begin{aligned}
E_k[X^2] &= E[E_k[X^2|\boldsymbol{p}]] \\
&= \frac{1}{k}\int_0^k \left(2\int_0^{x/2}(x-c)^2dc + \int_0^{y/2}(x+c)^2dc\right. \\
&\quad \left. + \int_0^{z/2}(x+c)^2dc + \int_0^{k-x-y/2-z/2} x^2dc\right) f_k(x)dx \\
&= \frac{7E_k[X^3]}{12k} + \frac{E_k[X^2Y]}{2k} + \frac{E_k[XY^2]}{4k} + \frac{E_k[Y^3]}{24k} + \frac{E_k[X^2Z]}{2k} + \frac{E_k[XZ^2]}{4k} \\
&\quad + \frac{E_k[Z^3]}{24k} - \frac{E_k[X^3]}{k} - \frac{E_k[X^2Y]}{2k} - \frac{E_k[X^2Z]}{2k} + E_k[X^2] \\
&= E_k[X^2] - \frac{E_k[X^3]}{3k} + \frac{E_k[X^2Y]}{2k} \\
&= E_k[X^2] - \frac{E_k[X^3]}{3k} + \frac{E_k[X^2]}{2(k-1)} - \frac{E_k[X^3]}{(2k)(k-1)} && \text{(by Lemma 3.4)} \\
&= \frac{2k+1}{3k}E_k[X^3].
\end{aligned}$$

□

This technique can be extended to find expressions for $E_k[X^a]$ for $a > 3$ in terms of $E_k[X^2]$, but it requires further analysis to simplify $E_k[X^aY^b]$ (if a, b are both even, longer calculations ensue). It may be possible to generate a more general expression for all a, and use these moment bounds to characterize the function $f_k(x)$.

4. Open Problems and Future Directions

4.1. **Open Problems.** We have addressed several of the open problems posed in [10], but some still remain. While this report fully characterized the distribution of $f_2(x)$, there remain many interesting questions for $k > 2$. Not only is the distribution of $f_k(x)$ unknown for all $k > 2$, but it is not even known if $E_k[C]$ is bounded as $k \to \infty$. Fully determining the distribution of $f_k(x)$ seems like a difficult challenge. On the other hand, showing that $E_k[C]$ is bounded or exactly calculating $E_k[C]$ for all k seems more tractable, based on the new results on the analysis of moments in Section 3.2.

This report only focused on the greedy algorithm on requests from the uniform distribution over the circle, but generalizing the setup may allow for similar results that apply to other processes. When analyzing the stochastic k-server on the circle, the greedy algorithm is only optimal (minimizing cost) for $k = 2$. Fully characterizing the optimal policy for $k > 2$ is still open, so the performance of other algorithms is of interest. Choosing distributions other than uniform for the placement of each r_i could also lead to the analysis of other processes. In the next section, we define a more general framework that allows us to model these related processes.

4.2. **Future Directions.** One proposed way to generalize the stochastic k-server process is to allow for a nonuniform distribution of requests [10]. It is possible to generalize further by considering distributions $D_1, D_2(x), D_3(i)$ and the following steps. Note that D_2 takes parameter x from the length of the interval chosen by D_1 and D_3 takes parameter i from the index of the interval chosen by D_1.

Start with k intervals $w_0, w_1, \dots w_{k-1}$ of size 1.

(1) Select an interval from D_1 with support over all intervals. Let this interval be w_i, with length x.
(2) Set w_i's length to $x - c$, where c is chosen from $D_2(x)$ with support over $[0, x]$
(3) Pick an interval from $D_3(i)$ with support over all k intervals, and increase its length by c

We call the process determined by $D_1, D_2(x), D_3(i)$ a "wealth redistribution" process because the sum of the lengths of all k intervals remains fixed at k at every step. The wealth redistribution process can model the stochastic k-server process as follows: Let D_1 be weighted by the interval lengths, $D_2(x)$ be uniform over $[0, x/2]$, and $D_3(i)$ be uniformly random over w_i's two neighbors on the circle. Note that the greedy algorithm is modeled by the selection of lengths in $D_2(x)$ and the geometry of the circle comes from the selection of neighbors in $D_3(i)$. Changing each D_i can lead to models of very different processes, independent of the geometry exhibited by stochastic k-server on the circle.

For example, it is possible to capture a process studied in economics and physics and analyzed in [2] and [3]. Consider k people each starting with 1 dollar. At each time step, each person gives 1 cent to another person uniformly at random. In the limit, this leads to an exponential distribution on the size of each interval. This process can be modeled by the following wealth redistribution process: Let D_1 be uniform over all intervals, $D_2(x) = 0.01$ be a constant, and $D_3(i)$ be uniform over all intervals. This shows that wealth redistribution processes may be of general interest for applications in various fields, even with these choices of simple distributions D_i.

It is also possible to slightly modify the D_i that model the greedy algorithm for the k-server problem on the circle to define a new process with different properties. As an example, in the next section, we modify $D_2(x)$ and analyze the ensuing process using techniques from Section 2.

4.3. **Modifying k-server process by changing $D_2(x)$.** Choose $\alpha \in (0, 1)$. Then, consider a wealth redistribution process defined as follows: D_1 is weighted by interval lengths, $D_2(x)$ is uniformly random over $[0, x\alpha]$, and $D_3(i)$ is uniformly random over the two neighbors of w_i. Setting $\alpha = 1/2$ yields the process that models the greedy algorithm for the k-server problem described in Section 1. Note that $\alpha = 0$ results in a degenerate case where no intervals change and $\alpha = 1$ in the limit results in a single interval of length k that does a random walk around the intervals of the circle.

Now, consider the two remaining cases: $\alpha < 1/2$ and $\alpha > 1/2$. If $\alpha < 1/2$, then the resulting process models the greedy algorithm for the k-server problem over the circle where the requests always land in the outer 2α fraction of an interval. On the other hand, $\alpha > 1/2$ does not directly model a variant of the stochastic k-server process. Instead, this process can be thought of as selecting an interval with the same weighting as the k-server process and reallocating up to an α fraction of the length of that interval to a random neighbor. Though this models a qualitatively different process, it can be analyzed via the techniques in Section 2.

Let $g_2(x)$ denote the probability density of $d(p_0, p_1)$ for this modified process with $\alpha > 1/2$. Then, the following result follows from an argument similar to the proof of Lemma 2.3.

Theorem 4.1. *For $\alpha > 1/2$ and $2 - 2\alpha \leq x \leq 2\alpha$, $g_2(x) = \dfrac{1}{2\alpha}$.*

Proof. Let s be the initial value of $d(p_0, p_1)$. Then, consider any step in the process that makes $d(p_0, p_1) = x$. Since π is the stationary distribution, taking one step in the process leaves $g_2(x)$ fixed. Thus, the total density of these steps is equal to $g_2(x)$.

First, consider the case where the request falls in $[p_0, p_1]$. In this case, s becomes $s - c$ for some choice of $c \in [0, s\alpha]$. So, choices of c that satisfy $x = s - c$ create an interval of size x. Substituting the range $[0, s\alpha]$ for c gives $s \in [x, x/(1-\alpha)]$. For any $x \geq 2 - 2\alpha$, the possible range of s is $[x, 2]$, since $x/(1-\alpha) \geq 2$. The probability density for being in a state with interval length s is $g_2(s)$. The request falls in $[p_0, p_1]$ with probability $s/2$. A particular choice of c from the uniform distribution over $[0, s\alpha]$ has density $\frac{1}{s\alpha}$. Integrating over all choices of s, this contributes $\int_x^2 g_2(s)(s/2)\frac{1}{s\alpha} ds = \int_x^2 \frac{g_2(s)}{2\alpha} ds$ to the total density of steps that generate $d(p_0, p_1) = x$.

Next, consider the case where the request falls in the complementary interval of length $2 - s$. In this case, s becomes $s + c$ for some choice of $c \in [0, \alpha(2-s)]$. Choices of c that satisfy $x = s + c$ will yield an interval of size x. Note that $(2-s)\alpha + s \geq (2-s)/2 + s = 1 + s/2 \geq 1 \geq x$, so x can always be reached if $s \leq x$. Substituting the range $[0, (2-s)/2]$ for c gives $s \in [0, x]$. So, this case contributes $\int_0^x \frac{g_2(s)}{2\alpha} ds$ to the total density of steps that generate $d(p_0, p_1) = x$.

Summing both cases, the total probability density $g_2(x)$ equals $\int_0^2 \frac{g_2(s)}{2\alpha} ds = \frac{1}{2\alpha}$. ☐

Confirmation of Theorem 4.1 and the shape of $g_2(x)$ for $\alpha \in (0, 1)$ are shown in Figure 5 in the following section. Since we were able to derive some properties of $g_2(x)$, it seems possible that a general framework could analyze the larger set of processes defined by different choices of D_i.

5. Simulations

5.1. **Distribution of $f_k(x)$.** In Section 2.2 we fully characterized the distribution of $f_2(x)$. The results in Section 3 only explored relationships between the moments of $f_k(x)$ for $k > 2$. However, we can approximate the distribution of $f_k(x)$ via simulation as shown in Figure 3.

$f_2(x)$ matches $up(x-1)$ as expected, but $f_k(x)$ for $k > 2$ does not exhibit the same symmetry properties. For instance, $f_k(x)$ is not symmetric around $x = 1$ and does not take a maximum value at $f_k(1)$. $f_{50}(x)$ closely approximates the stationary distribution, as it does not deviate much from simulations of larger choices of k. $f_k(x)$ appears to decrease very quickly for $x > 1$. One strategy to show that $E_k[C]$ is bounded as $k \to \infty$ may be to show that $f_k(x)$ is bounded above by an exponentially decreasing function. Further work would still be necessary to find the precise value of $\lim_{k\to\infty} E_k[C]$.

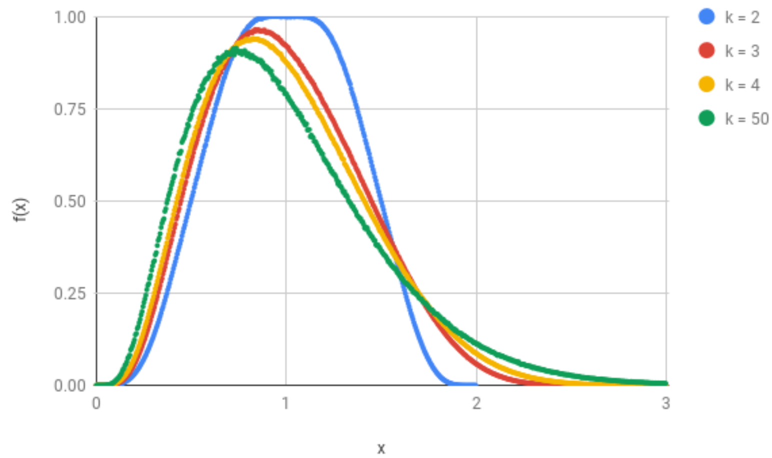

FIGURE 3. Graph of $f_k(x)$ for various choices of k.

5.2. $\frac{E_k[X^2]}{E_k[X^a]}$ **for varying** a**.** Section 3.2 showed a relationship between $E_k[X^3]$ and $E_k[X^2]$. Though it could also be obtained via further calculation, Figure 4 shows the relationship between $E_k[X^a]$ and $E_k[X^2]$ for $a > 3$.

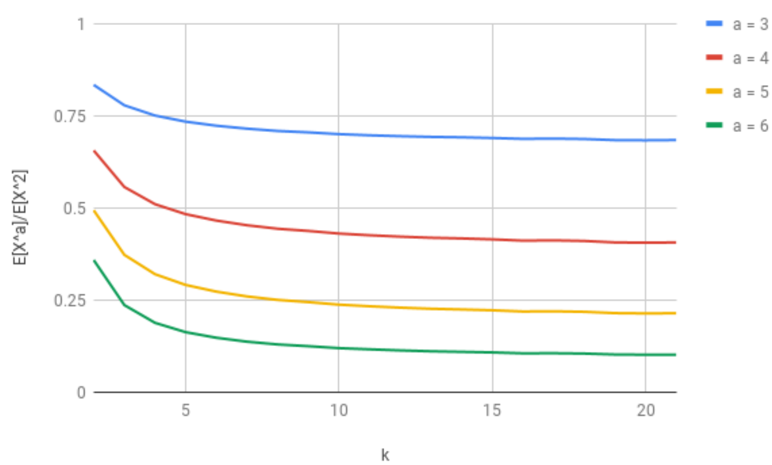

FIGURE 4. $\frac{E_k[X^2]}{E_k[X^a]}$ for choices of a

As expected by the limit of the formula obtained in Theorem 3.5, $\frac{E_k[X^2]}{E_k[X^3]}$ approaches $\frac{2}{3}$ as k increases. Similarly, each $\frac{E_k[X^2]}{E_k[X^a]}$ approaches a nonzero value, suggesting that they may also be defined by a rational function with a finite limit. One could run a regression on rational functions to generate a conjecture for a general formula for $\frac{E_k[X^2]}{E_k[X^a]}$.

5.3. **Distribution of $g_2(x)$ for different α.** As mentioned in Section 4.2, one possible modification to the k-server process is to change $D_2(x)$ from $\alpha = 1/2$ to a value $\alpha \in (0, 1)$, as shown in Figure 5.

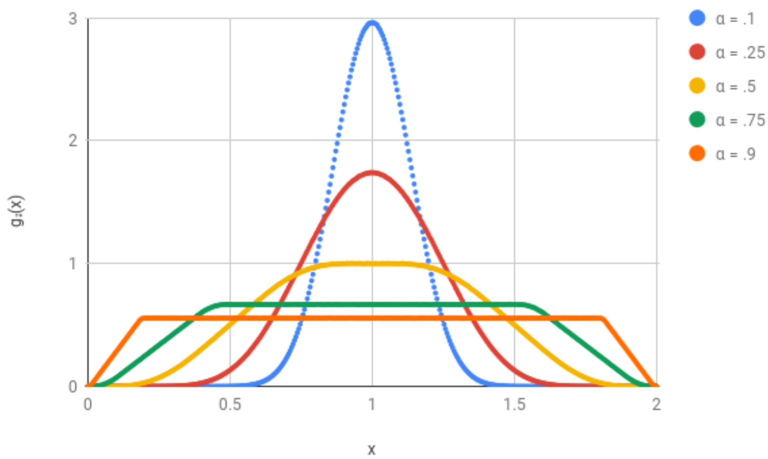

FIGURE 5. Graph of $g_2(x)$ for various choices of α

As predicted by Theorem 4.1, when $\alpha > 1/2$, there is a range for which $g_2(x)$ is constant. For instance, for $\alpha = 3/4$, $g_2(x) = 2/3$ for $x \in [1/2, 3/2]$. When $\alpha < 1/2$, there is still a unique maximum value, but it is greater than the maximum of $f_2(1) = 1$. Finding a formula for these new maxima might help characterize the distribution for $\alpha < 1/2$.

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