Sculpture Designs Based on Borromean Soap Films

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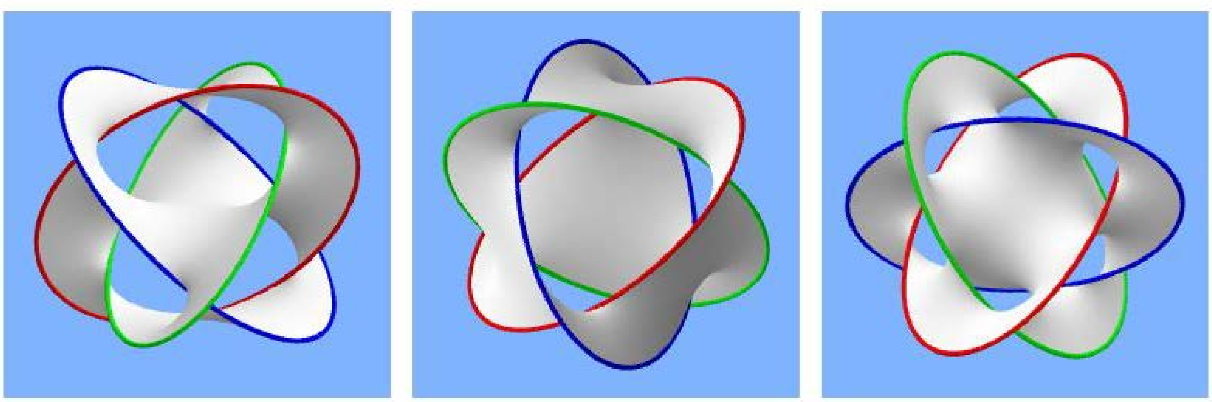
Abstract

The Borromean rings are used as the border curves of aesthetically pleasing 2-manifolds, approximating minimal-area surfaces. Two- and three-level deep recursive configurations are then constructed and the border curves of the various levels are gracefully joined to obtain a single 2-manifold covering all the nested levels. The results are intricate, but highly symmetrical, abstract geometrical sculptures.

Soap Films on Borromean Structures

I have long been fascinated by complex 2-manifolds bordered by intricate smooth border curves, as can be found in sculptures by Eva Hild [5] or Charles Perry [8]. Even on very simple arrangements of border curves, rather intriguing minimal surfaces can be supported. On the highly symmetrical Borromean arrangement of three oval borders, in which no pair of ovals are actually linked, Ken Brakke has identified 15 different possible soap films [1]. Of these, only three are proper 2-manifolds with no self-intersections. The first one is the true orientable Seifert surface [9] with S_3 symmetry (Fig.1a). A second one is a single-sided surface with the symmetry of the orientable tetrahedron (Fig.1b). The third one also is single-sided and has S_3 symmetry, but this one would be unstable for an actual soap film (Fig.1c).



(a) (b) (c)

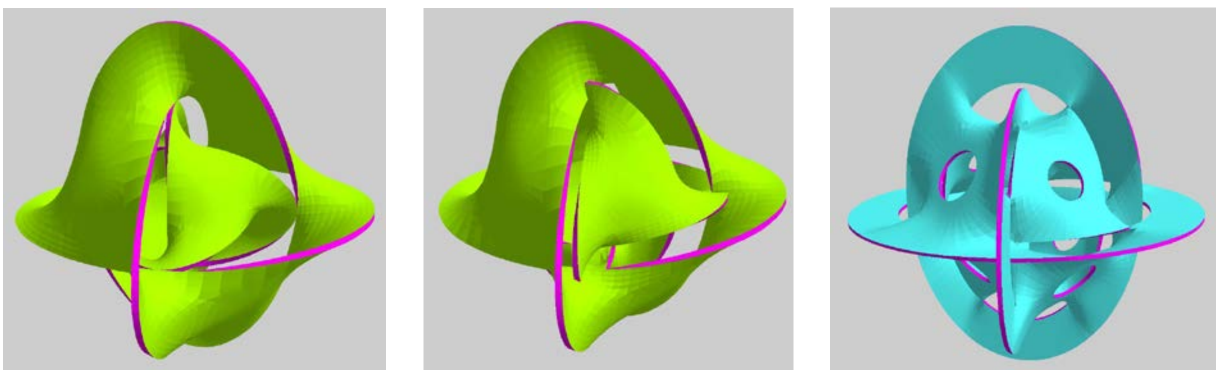
Figure 1: *Intersection-free soap films on the Borromean Rings [1]: (a) two-sided Seifert surface, (b) stable single-sided 2-manifold, (c) unstable non-orientable 2-manifold.*

The second one of the above 2-manifolds is contained in the outermost 20% of a spherical shell and leaves most of the sphere's interior free. Thus, a smaller version of that surface could readily be placed inside; and this process could be repeated recursively. In the work described here, I study how these recursive arrangements can be transformed into single, cohesive 2-manifolds with elegant, smooth border curves. Ideally, the border curves form a cohesive system, where one border curve supports portions of surfaces belonging to different recursive levels, and the spanned surface itself forms a single manifold diving deep into the interior of the sphere, while approximating the shape of a soap film – although I am not shooting for a precise minimal surface.

Two-Level Borromean Structures

To make a 2-level structure, I start with the stable, single-sided soap film surface (Fig.1b) and place a smaller copy of it inside itself so that they just touch (Fig.2a). However, the touching border curves are not a proper edge configuration for borders of a clean 2-manifold surface.

A first solution is to mirror and turn the inner surface by 90 degrees, so that its lobes touch the outer surface with a skew angle of 90°. The inner surface is grown slightly, so that its oval borders also interlink with the borders of the outer surface in a Borromean manner (Fig.2b). The two Borromean soap-films now intersect. To obtain a true, non-intersecting 2-manifold, the surface elements in this neighborhood need to be changed. First, I formed a flat ribbon between the concentric inner and outer ovals lying in the same plane; this ribbon is clearly visible near the apexes of the ovals (Fig.2c). Along the region of lower curvature, the inner oval supports three highly twisted regions that connect to the inner and outer triangular surface regions. In particular, the surface elements that suspend the inner four soap-film octants are very highly twisted. This can be relieved only partially by making the inner ovals smaller and skinnier, and by reducing the size of the opening underneath the six major outer lobes (Fig.2c). These various characteristics made this approach result in a less than optimal sculpture.



(a) (b) (c)

Figure 2: *A 2-level Borromean structure: (a) two nested, scaled Borromean soap films, just touching; (b) the inner surface is scaled, mirrored, rotated by 90°, and interlinked it with the outer borders; (c) a resulting 2-manifold with some extra punctures introduced to enhance transparency.*

A more aesthetically pleasing solution results from changing the border configuration in such a way that inner borders and outer borders form a connected system of loops or knots. A first obvious approach is to shrink the touching inner Borromean soap film, rather than growing it, and then connecting it with six short, broad tabs to the outer one (Fig.3a). However, this approach leads to crescent-shaped border curves with sharp hairpin turns between them. This is not reflecting the spirit of a free-form geometry with features that exhibit minimal bending energy!

Instead, I transformed the connection point between two touching oval borders into a skewed crossing of two smooth curves (Fig.3b). Topologically, this results in a border curve system consisting of six simple interlinked loops. Since the loops are almost circular, we can make this a design feature, and start with a system of border curves that is comprised of six perfect circles (Fig.3c). This results in a very attractive 2-manifold surface with six borders (b = 6). To enhance the transparency of this sculpture and to allow a better look at the inside geometry, I have also cut out 8 small circular holes from the two sets of 3-sided surface regions in the inner and outer soap-films (Fig.3d). This 2-manifold is single-sided and has 6+8 punctures. Disregarding the small circular holes cut for improved visibility, its Euler characteristic is: EC = –10, since it takes (3+5+3) ribbon cuts to turn that surface into a topological "disk." From this, the genus of this surface can now be calculated: genus = 2 – EC – b = 2 + 10 – 6 = 6. The symmetry of the overall geometry is that of the oriented tetrahedron.

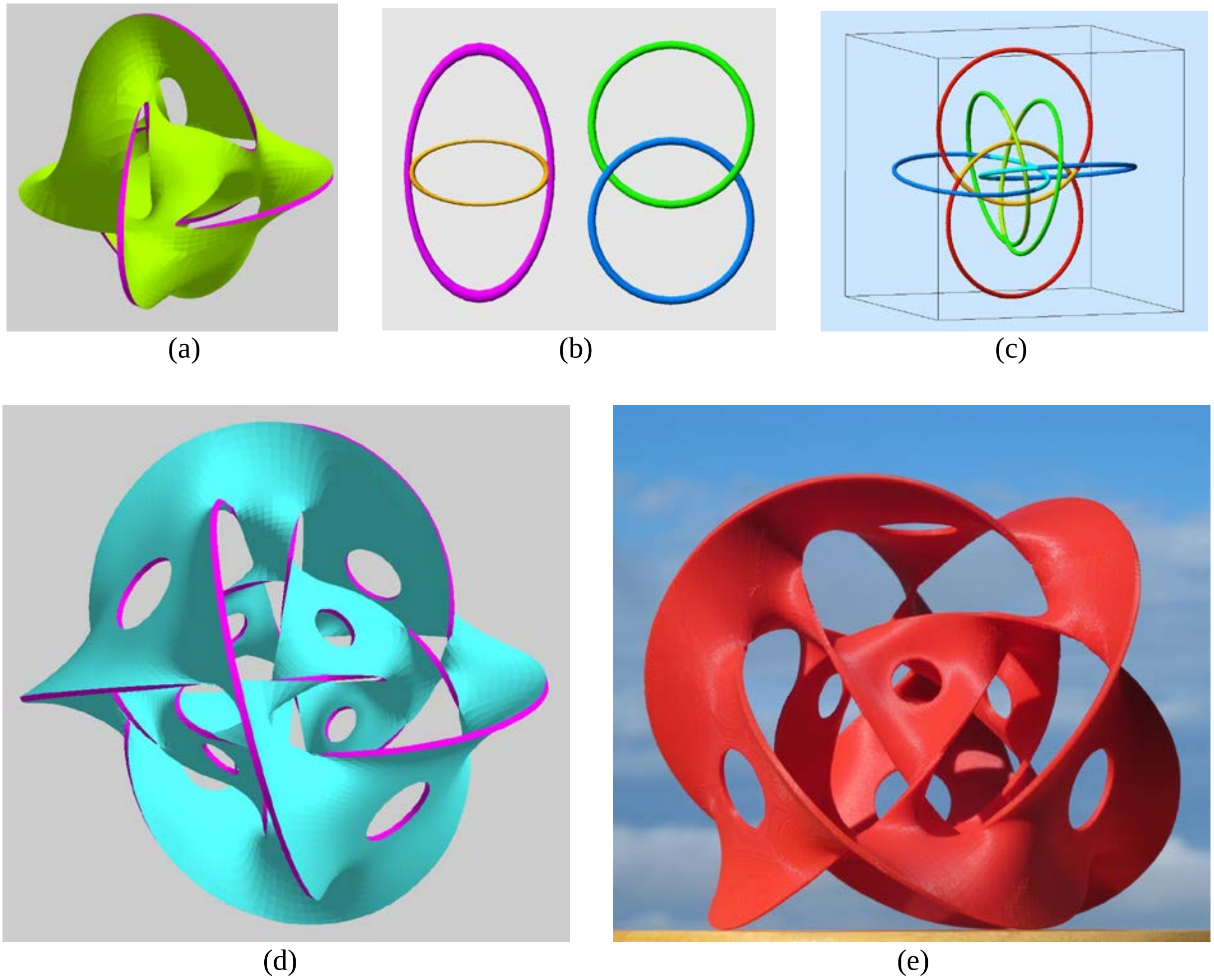


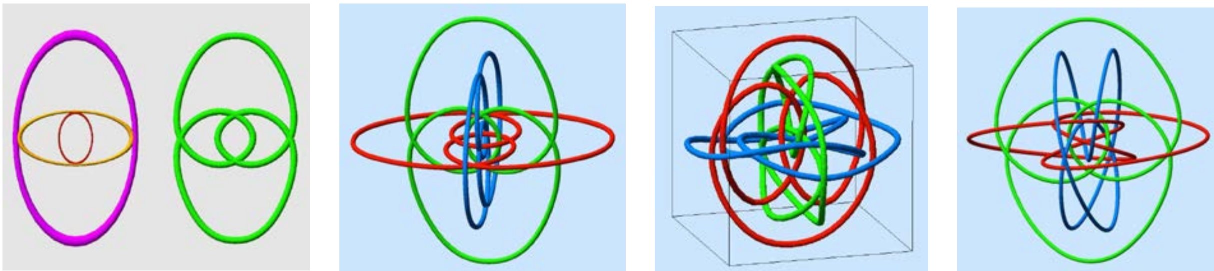
Figure 3: *Other 2-level Borromean constructions: (a) the two surfaces are fused with small tabs; (b) replacing nested scaled ovals with interlinked circles; (c) resulting border curve structure; (d) CAD model of circle-spanned 2-manifold; (e) a high-quality 3D-print.*

Three-Level Structures

Next, I was looking for a nice 2-manifold that connects three nested scale-copies of the Borromean soap-film surface shown in Figure 1b. I followed the approach taken in Figure 3 and integrated the simple oval border curves into a rim structure that spans all three levels. In this case, three nested, scaled ovals can readily be approximated with the single green curve shown in Figure 4a. This green curve can be warped into either a Figure-8 knot or into a (2, 3)-torus-knot, depending on the choice of subsequent over- and under-passes. The linking of three Figure-8 knots (Fig.4c) looked quite busy near the center, and it appeared rather challenging to form an intersection-free 2-manifold in this area. So I started with the other alternative.

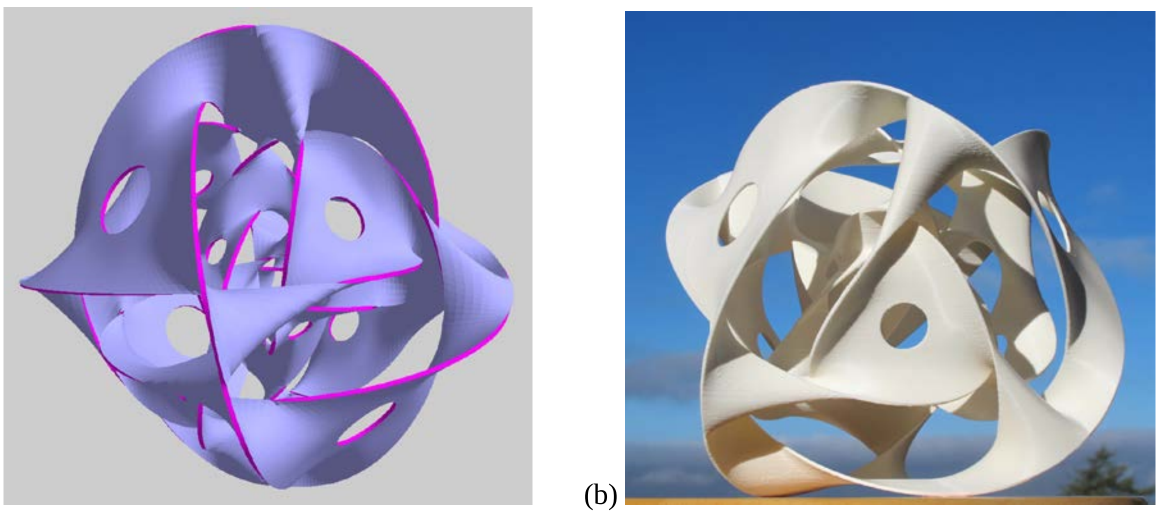
The (2, 3)-torus-knot looks less tangled in the center, and it lead to smoother border curves (Fig.4d), since it involves less undulation in the 3rd dimension than the alternating Figure-8 knot. Figure 5a shows the complete 3-level CAD model, and Figure 5b is a photo of the resulting 3D print.

This sculpture, as well as the one shown in Figure 3e, will be exhibited in the Gallery of Mathematical Art [7] associated with the 2019 Joint Mathematics Meeting in Baltimore.



(a) (b) (c) (d)

Figure 4: *3-level Borromean construction: (a) replacing 3 nested ovals with a single border curve; (b) Borromean configuration of 3 such curves, (c) Borromean tangle of three Figure-8 knots, (d) Borromean linking of three (2, 3)-torus-knots.*



(a) (b)

Figure 5: *3-level Borromean surface based on three (2, 3)-torus-knots: (a) CAD model, (b) 3D-print.*

Construction Hints

Here is a brief outline of how I construct surfaces of this type, explained with the example of designing a 3-level Borromean surface suspended by three Figure-8 knots (Fig.4c). The first step is to define a relatively flat version of the Figure-8 knot. Three of them are then placed at right angle to each other, and the geometrical knot parameters are adjusted to obtain good clearance between any skewed branch crossings (Fig.6a).

I then try to find the best number of sample points along the knot curve: Not too few, because this would result in badly skewed facets connecting adjacent branches. Ideally, I would like to obtain pairs of sample points near all curve crossings where the 2-manifold will flip through a sharp twist to connect to an adjacent patch. Being able to place a patch boundary between two points that mark the closest approach between the two border curves, produces the nicest surface in this neighborhood, when Catmull-Clark subdivision is used for the surface smoothing process [3].

Next, I place disks or annuli in the center of the triangular areas that define the basic Borromean soap-films on each of the three nested levels; they are shown in blue, green, and yellow, respectively in Figure 6b. I adjust their distances from the origin, so that they are in balance with the knot segments that will suspend them. This yields the best start for making the final surface look like the minimal surface of a soap-film. The holes in the annuli give some visibility to the inner portions of the final sculpture. I also adjust the azimuth rotation of these annuli, so that they lead to the nicest, most direct, radial connections with the border curves from which they will be suspended. Figure 6c shows the evolving surface, once a ring of facets has been added around every annulus using NOME [4].

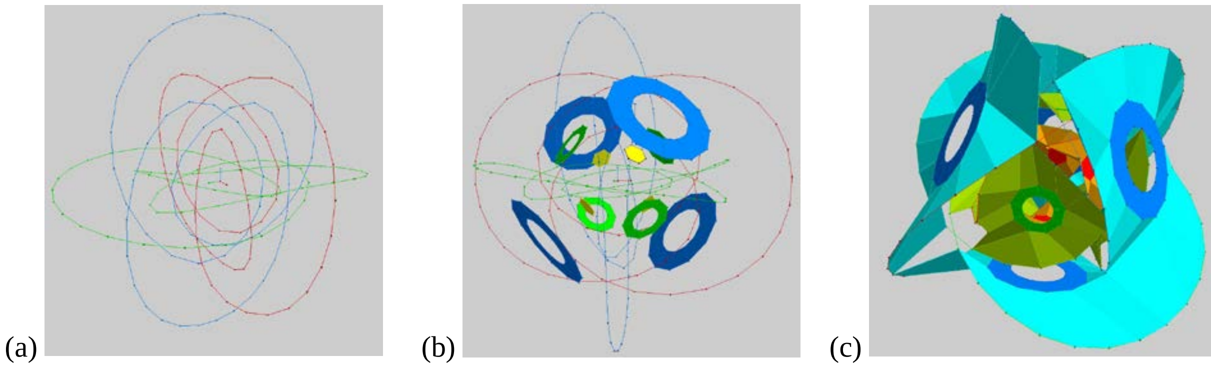


Figure 6: *(a) Three figure8-knots in a Borromean configuration; (b) 12 annuli added to define the centers of the 12 triangular surface regions, (c) rings of facets added around every annulus.*

Now all these surface patches need to be connected to one another. I do this with six sets of facets strung along the six symmetry half-axes (Fig.7a). By themselves, these facets form some twisted columns (Fig.7b). These columns then serve to connect the triangular surface regions to their neighbors in the same level, and they also provide the inter-level connections (Fig.7c).

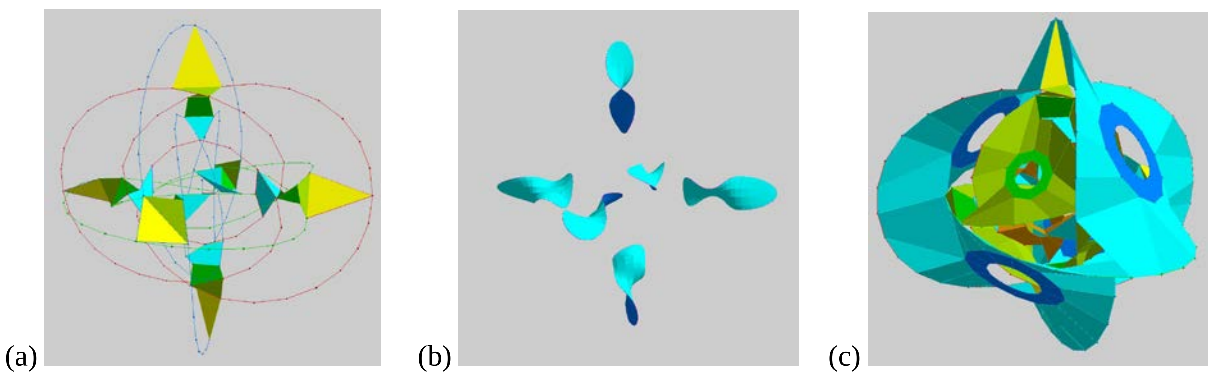


Figure 7: *Interconnecting the three levels: (a) facets strung along the symmetry half-axes, (b) columns formed by these facets; (c) the complete polygonal mesh.*

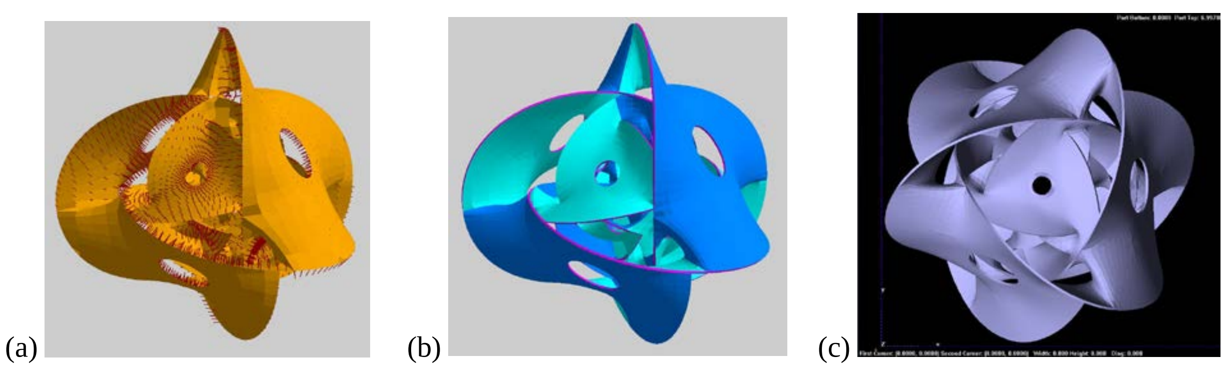


Figure 8: *(a) Surface smoothed by subdivision, (b) and thickened by offsetting; (c) final STL file.*

With all surfaces elements in place, I apply a couple of subdivision steps to smooth the surface (Fig.8a) and then use offsetting to form a physical surface with some thickness (Fig.8b). In this state, I still can fine-tune the parameters defining the exact shape of the Figure-8 knots to make the overall shape as nice as possible – e.g., by getting rid of some unnecessarily sharp twists. This final shape is then saved as an STL file (Fig.8c) and sent to a 3D printer.

Summary and Conclusions

The models described are not true minimal surfaces; they are just rough polyhedral approximations, smoothed with three or four levels of Catmull-Clark subdivision [3]. To obtain closer approximations to minimal surfaces, one could use Ken Brakke's *Surface Evolver* [2]. With the approach taken, the quality of the final results depends somewhat on the way the surface elements are introduced. Ideally, one should start with a rather coarse polyhedral model, in which all facets between neighboring border curves are quadrilaterals that all have roughly the same edge-lengths.

The described approach of nesting Borromean soap films could be continued to a higher level of recursion. However, the network of border curves near the center will get rather busy, and it will be difficult to manually place all the right facets to construct a proper 2-manifold with the desired symmetry. Moreover, there would be a diminishing return, since the visibility into the center is already quite limited in the 3-level structure.

Acknowledgments

I am grateful to Gauthier Dieppedalle and Toby Chen for their efforts towards the construction of an interactive computer-aided design tool (NOME: Non-Orientable Manifold Editor) [4], which makes the modeling of these intricate free-form surfaces possible and enjoyable.

I also would like to express my thanks the staff of Jacobs Institute for Design Innovation [6] for their help with the construction of the various models on their 3D printers.

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Appendix

Below is the text accompanying the two sculptures sent to the 2019 JMM Gallery of Mathematical Art:

Artist Statement:

I have long been fascinated by complex 2-manifolds bordered by intricate smooth curves, as can be found in sculptures by Eva Hild or Charles Perry. Even on very simple arrangements of border curves, rather intriguing minimal surfaces can be supported. On the highly symmetrical Borromean arrangement of three oval borders, in which no pair of ovals are actually linked, Ken Brakke has identified 15 different possible soap films. The simplest one is a stable, a non-orientable minimal surface with the symmetry of the oriented tetrahedron. I am investigating sculptures that result when multiple copies of this structure are recursively nested inside one another.

Sculpture #1

"**Two-Level Borromean Soap Film, Suspended by Six Circles**".
19 x 19 x 19 cm
ABS, 3D-print
2018

I start with the simplest Borromean soap film surface and place a smaller copy inside so that they just touch. To obtain a proper border configuration for a 2-manifold, I transform the six areas where two ovals touch into skewed crossings of two smooth curves. This results in a border curve system consisting of six simple interlinked loops. I force these loops to be perfectly circular, and then construct a soap-film on this border structure.
To enhance the transparency of this sculpture and allow a better look at the inside geometry, I also cut out eight small circular holes from the two sets of 3-sided face patches in the outer and inner levels of the soap film. The 2-manifold is single-sided, of genus 6, and has 6+8 punctures.

Sculpture #2

"**3-Level Borromean Soap Film Bordered by 3 Intertwined Figure-8 Knots**".
22 x 22 x 22 cm
ABS: 3D-print
2018

Here I am placing three of the simple Borromean soap film surfaces inside one another. Again, I want to turn this into a single 2-manifold surface with a cohesive, smooth border curve structure. In this case, when the 12 touch-points of the 9 scaled ovals are turned into skewed crossings of smooth curves, the three ovals that were lying in the same plane readily turn into a single simple knot. This is either a Figure-8 knot or a (2,3)-Torus knot, depending on the choice of subsequent over- and under-passes. The torus knot will lead to smoother border curves, since it requires less undulation in the 3rd dimension than the alternating Figure-8 knot. The resulting single-sided 2-manifold has 3+12 borders and is of genus 9.