Stability of Interlinked Triangle Frames

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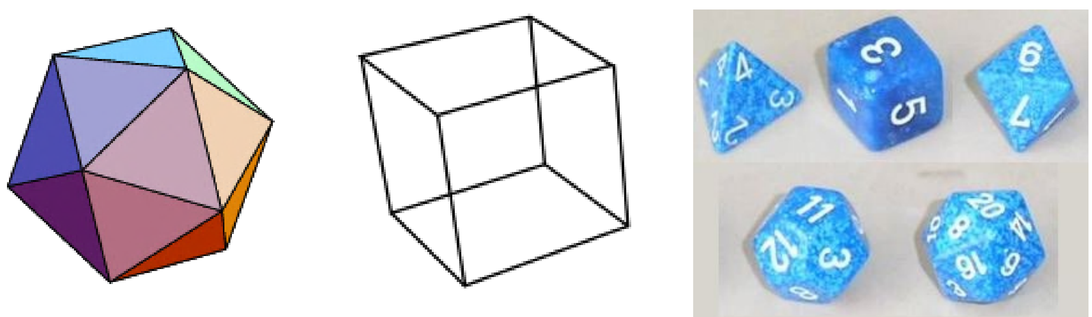
Abstract

Three of the five Platonic solids can be represented with interlinking triangle frames. This report documents the process of improving upon the design of the triangle frames to offer greater overall stability to the composite polyhedral structures.

Introduction

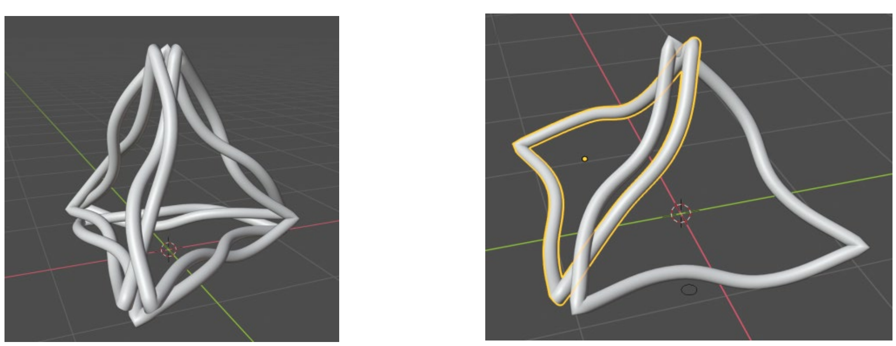
The Platonic solids can be represented in a variety of ways, including solid, closed models (Fig.1a), wireframe models (Fig.1b), and unconventional representations (Fig.1c). Of the five Platonic solids, the tetrahedron, octahedron, and icosahedron are bounded by triangular faces. We can replace these faces with triangular frames that interlink along the shared edges of the polyhedron (Fig.2a).



(a) (b) (c)

Figure 1: *(a) Solid model of a regular icosahedron [1]; (b) wireframe model of a cube [2]; (c) a set of polyhedral dice [3]*

Single-Twist Triangle Frames

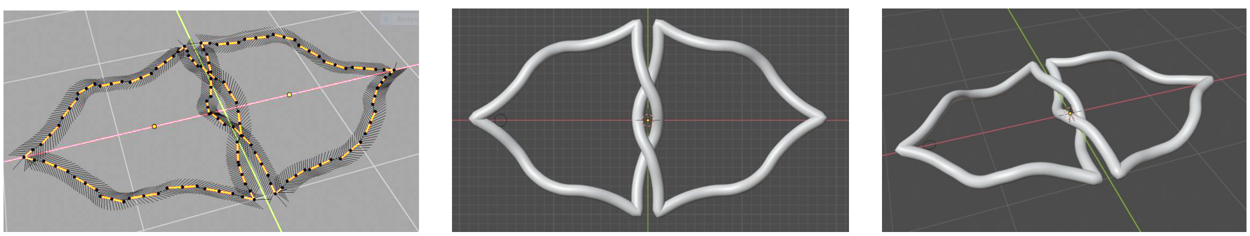


(a) (b)

Figure 2: *(a) Tetrahedral composition of four single-twist triangle frames; (b) frame misalignment caused by an increase in dihedral angle.*

All faces of a Platonic solid are regular polygons. Therefore, the triangle frames we create must be equilateral and be able to interlink with identical frames. One option for designing such a frame is to adjust the geometry of the frame when it is in its final orientation in the desired composition. Figure 2(a) shows a tetrahedral configuration using this method of frame construction. However, this may lead to a problem if we want to use the same triangular frames for different polyhedra with different dihedral angles. There is a possibility that the frames will no longer properly interlink with one another (Fig.2b). Thus, a more challenging goal is to come up with a single design that could construct all 3 Platonic solids bounded by triangles. In this case, two frames must remain interlinked with one another when the dihedral angle is changed. One way of achieving this is by constructing the interlinking frames in a common plane, i.e., with a dihedral angle of 180° between them. It turns out that reducing the dihedral angle from 180° down to 70.53° of the tetrahedron will more easily preserve their interlinking nature.

To design this "universal" triangular frame, we give each edge a helical shape that twists through 360°, so that two such edges can spiral around each other and around the shared original triangle edge, when the two triangles are placed adjacent to one another as shown in Figures 3(b) and 3(c). Each twisted edge is modeled as a Bézier curve with 8 control points, resulting in a total of 21 control points for the complete frame. A circular cross-section of adjustable radius is then swept along the Bézier curve (Fig.3a). For each dihedral angle, there is a maximal radius of the cross section for which the neighboring triangle frames will not intersect one another.

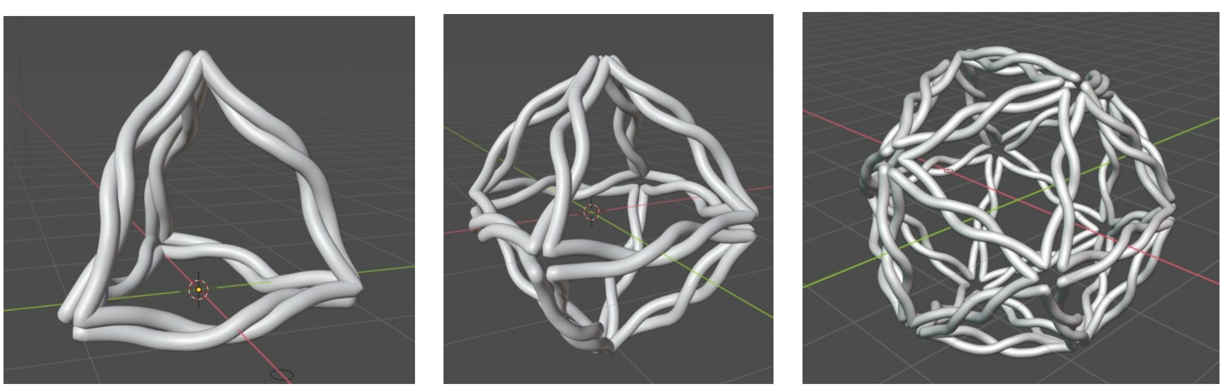


(a) (b) (c)

Figure 3: *Interlinked single-twist triangle frames: (a) Bézier curve; (b) plane view; (c) oblique view.*

Stability of the Composite Polyhedra

For a given cross-section radius, the linking gets tighter as the dihedral angle is reduced. Thus, the composite icosahedron is a much looser assembly than the tetrahedron (Fig.4).

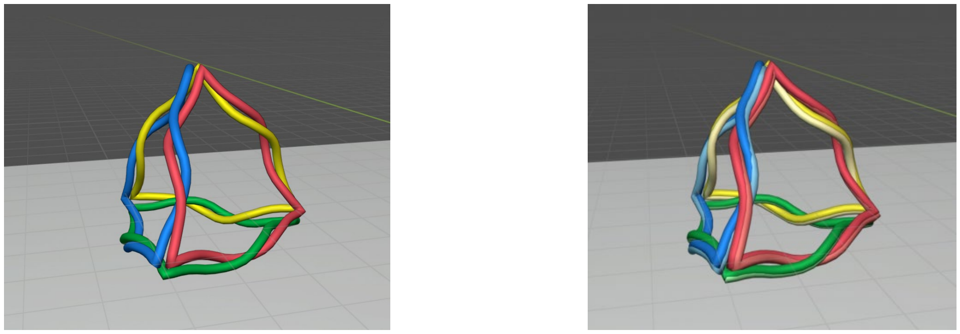


(a) (b) (c)

Figure 4: *Interlinked triangle frames forming: (a) tetrahedron; (b) octahedron; (c) icosahedron.*

To obtain a quantitative measure of rigidity for the various polyhedral assemblies, we use the Blender CAD environment [4]. Here we can investigate the stability of these configurations using Blender's Physics simulation program (Rigid Body Physics) [5]. This program calculates how physical objects would behave when exposed to gravitational forces. We color the frames differently to make it easier to distinguish separate frames. We use four colors for the tetrahedron, four for the octahedron, and five for the icosahedron. The configuration deformed under the influence of gravitational forces will be rendered in lighter shades of color, so that the original assembly (darker colors) and the deformed one (lighter colors) can also be shown in superposition (Figs.5b, 6b,d, and 7a).

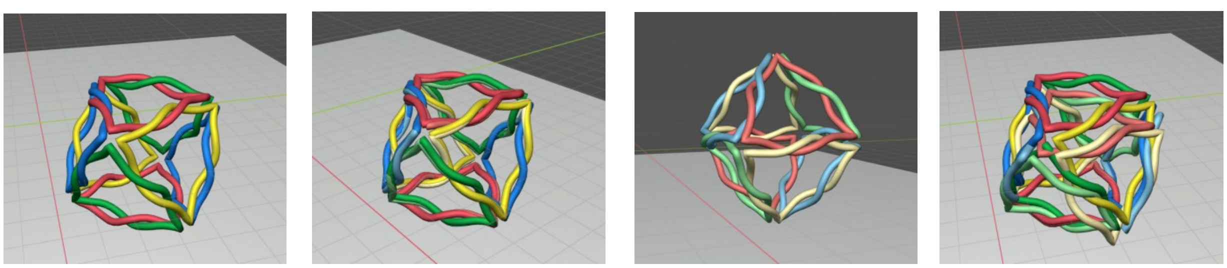
For the tetrahedral assembly, the structure was set up with one "face" atop a flat surface (Fig.5a). After allowing the simulation to run, we observe from the superposed rendering (Fig.5b) that the four triangle frames have barely moved. This indicates that the tetrahedral configuration is rather rigid.



(a) (b)

Figure 5: *Tetrahedral assembly: (a) original configuration; (b) superposition of initial and deformed configurations.*

The octahedral assembly is less rigid (Fig.6). The orientation in which the structures are exposed to gravity affects the degree of deformation. Figure 6(a) depicts the original octahedral configuration with one triangle face on the supporting surface. Figure 6(b) shows some minor, but noticeable deformation when gravity is turned on. When we change the initial orientation, so that the octahedron is balanced on one of its corners (Fig.6c), the assembly experiences more significant deformation. Forces exerted from the support surface will cause a longitudinal displacement between interlinked edges. This will then permit a larger change in some of the dihedral angles after the octahedron has flopped over and has come to rest on one of its faces (Fig.6d).



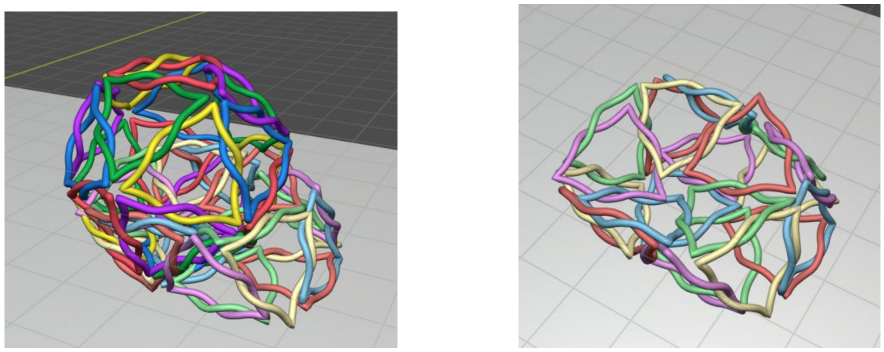
(a) (b) (c) (d)

Figure 6: *Octahedral assembly: (a) original configuration; (b) static deformation due to gravity; (c) different initial set-up; (d) simulation results showing larger deformation.*

This longitudinal shift out of alignment for interlinked triangle frames is even more pronounced in the icosahedral assembly (Fig.7). The top and bottom cupolas of the icosahedron flatten and sag to one side, leading to a final structure that resembles an oblique cylinder.

We observe that the tetrahedron is sufficiently stable, but the icosahedron is unable to maintain its structure well. A tetrahedron has a dihedral angle of 70.53°, an octahedron 109.47°, and an icosahedron 138.19° [6]. From these values and our simulation results, we see that the greater the dihedral angle, the looser the interlinking and the lesser the stability of the overall structure.

The deformation of the structures (Fig.5, 6, 7) are caused by gravitational forces, i.e., by a uniform force in the z-axis direction. It is likely that these structures will deform even more strongly when additional forces in other directions are exerted upon them.



(a) (b)

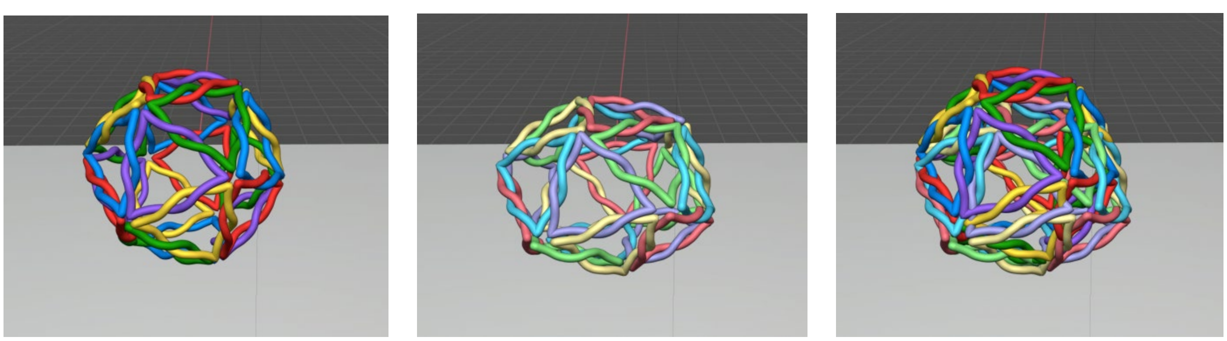
Figure 7: *Icosahedral assembly: (a) superposition of initial and deformed configurations; (b) the deformed configuration by itself.*

Improving Stability

Loose structures, like the one shown in Figure 7, are not truly compatible with our original goal. Looking at the light purple frame in the upper left of Figure 7b, makes it plausible that this frame can easily slip or be maneuvered so that it no longer "shares its edges" with other frames, but is interlinked in a more "free-form" manner. The icosahedral structure might then collapse into a tangle of triangles – way beyond anything resembling an icosahedral shape. Thus, we want to explore some methods to make the various polyhedral assemblies more rigid.

Thickened Triangular Frames

One rather straight-forward approach increases the diameter of the circle swept along the helical path so that it leads to a zero gap-width between the two intertwined sweeps for a particular polyhedron under consideration. This then permits only very minimal changes to the dihedral angle between to neighboring triangle frames and limits the relative frame movements between neighbors. Because the icosahedron is the least stable of the three polyhedra that we are interested in, we will test our stability-improving method on this assembly.



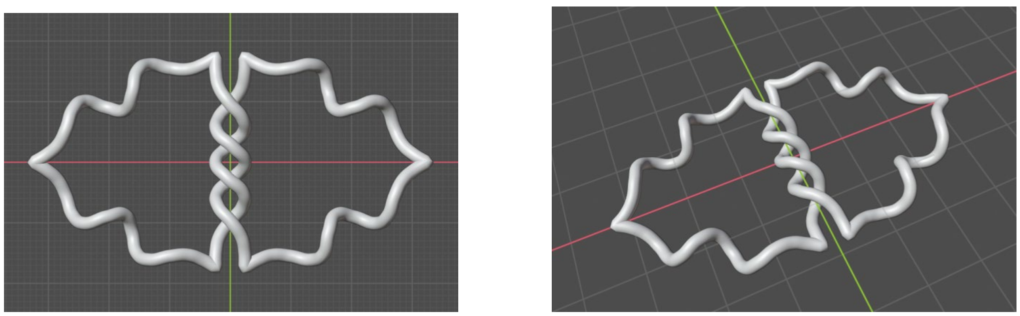
(a) (b) (c)

Figure 8: *Icosahedral assembly: (a) initial configuration with thickened frames; (b) deformation under gravity; (c) superposition of both configurations.*

The original assembly (Fig.7a) had a free gap between the two intertwined helices corresponding to 68% of the sweep-circle diameter. Now we reduce this gap to almost zero. Compared to our previous simulation (Fig.7b), the thickened frame shows a considerable improvement in stability. As shown in Figure 8b, the upper and lower “domes” maintain their shapes better than the results of Figure 7b, and overall, the icosahedron no longer collapses into a skewed prism. We can also apply this same approach to other polyhedral configurations.

Doubly-Twisted Triangular Frames

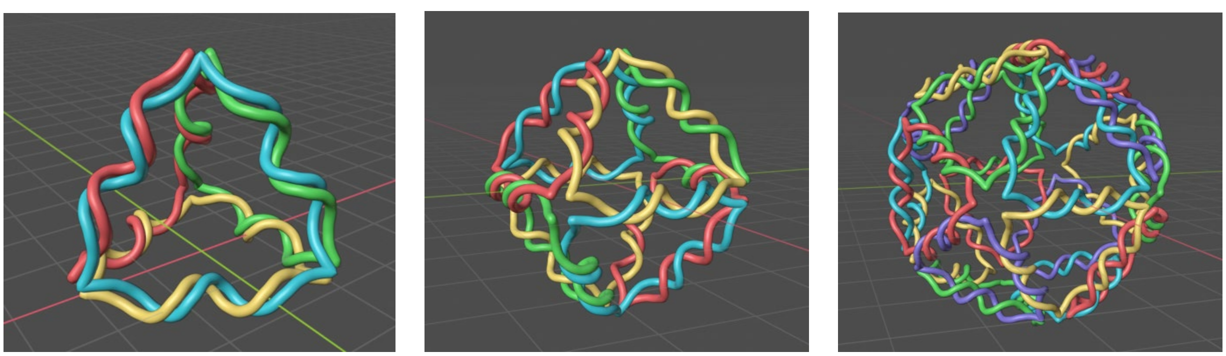
There is also a more structural approach to increase stability. Rather than linking adjacent frames with a single full twist as shown in Figure 3, we can link them with two full helical twists. In designing such doubly-twisted frames, we follow the same basic process that led to the frames in Figure 3. Now the control polygon of the Bézier sweep curve for the triangle frame has a total of 42 control points. The new design (Fig.9) is rather loose; it has a gap between the spirals of 120% of the worm diameter.



(a) (b)

Figure 9: *The new doubly-twisted triangle frames: (a) plane view; (b) oblique view.*

Figure 10 shows the use of these new doubly-twisted frames to construct the three Platonic assemblies.

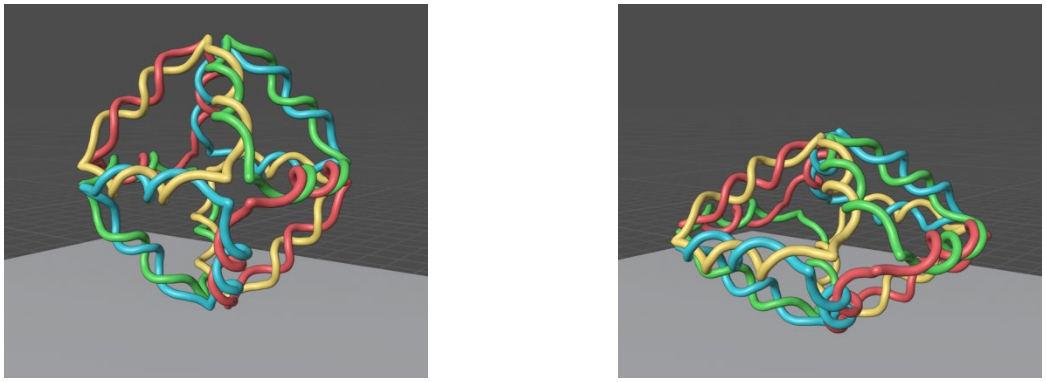


(a) (b) (c)

Figure 10: *Interlinked assemblies of doubly-twisted triangle frames: (a) tetrahedral; (b) octahedral; (c) icosahedral.*

For all three configurations depicted in Figure 10, we have used the same constant frame thickness, which is primarily limited by the tetrahedral configuration. Thus, the tetrahedron (Fig.10a) has the least empty space between the “coils” of adjacent frames. As the dihedral angle between neighboring frames is increased, the gap width between interlinked coils increases, which then allows for more relative movement between them. This then leads to the expectation that the octahedron will be less stable than the tetrahedron, and the icosahedron will be considerably less stable than both of the other two. These conjectures are readily confirmed by simulations conducted within Blender.

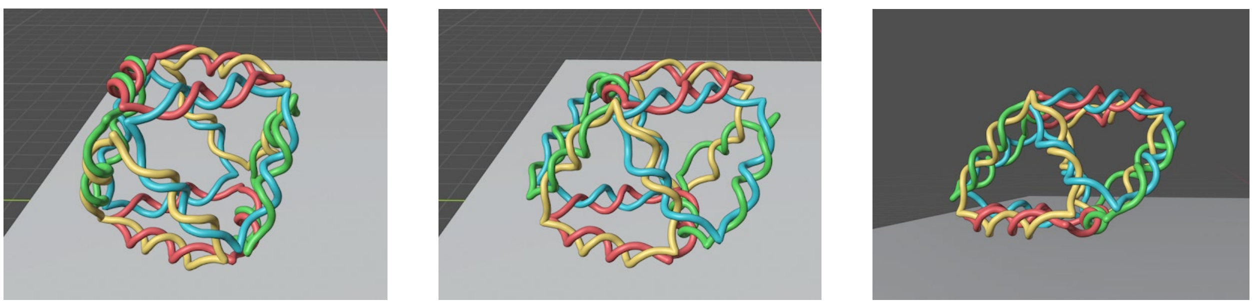
Starting with the octahedron, we first examine its stability when it is balanced on one of its corners (Fig.11a). In our idealistic simulation, the resulting structure remains balanced on its corner (Fig.11b), and the overall deformation can be described as a simple, vertical “squash” of the initial structure.



(a) (b)

Figure 11: *Octahedron simulation: (a) initial configuration; (b) gravity induced deformation.*

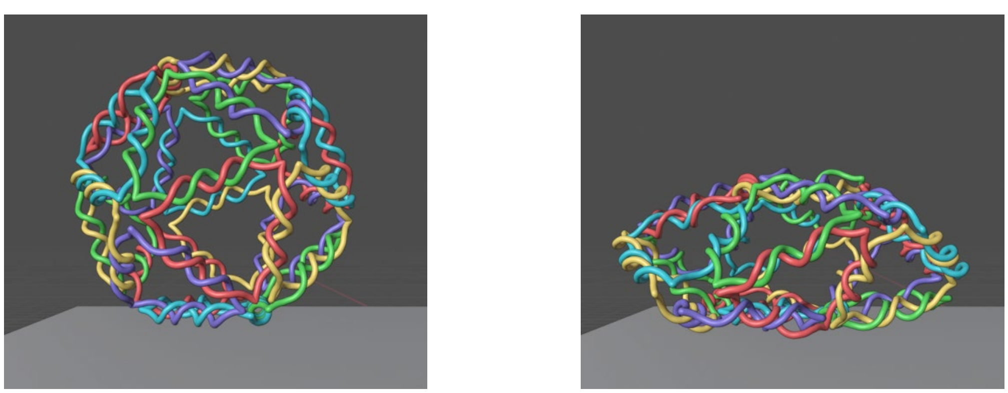
We can also observe how the octahedral configuration behaves when simulated from an initial position in which one "face" of the octahedron is lying atop a flat supporting surface (Fig.12a). The result of this simulation is shown in Figures 12(b) and 12(c). The red frame at the top shifts from its original position, resulting in a "tilt" of the overall structure. Overall, this is almost the same as the "squash" deformation shown in Figure 11(b), after the assembly has been rotated so that the octahedron lies on one of its faces.



(a) (b) (c)

Figure 12: *Second octahedron simulation: (a) initial position; (b) gravity induced deformation; (c) side view of deformation.*

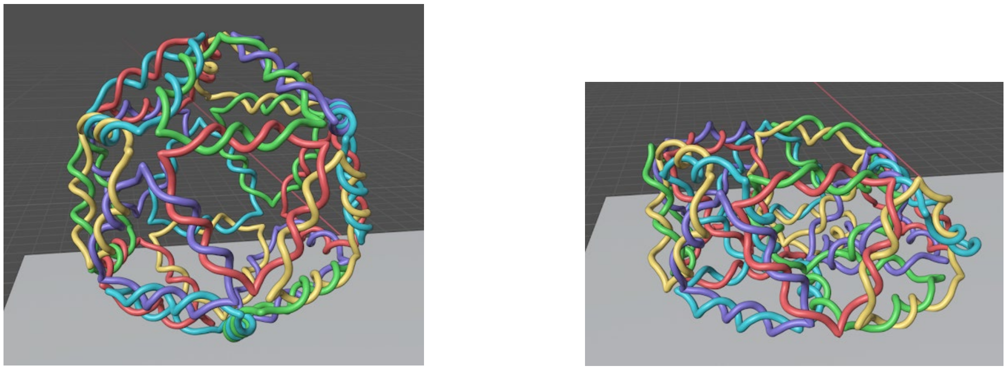
Figures 13 and 14 show equivalent simulations for the icosahedral assembly. Starting with the position where the icosahedron is lying on one of its faces (Fig.13a), the structure collapses to barely half its original height.



(a) (b)

Figure 13: *Icosahedron lying on a face: (a) initial configuration; (b) gravity induced deformation.*

As in the earlier simulations of the single-twist structures, a polyhedral assembly balanced on one of its corners experiences more severe deformation than one lying on one of its faces. For the icosahedron (Fig.14a), the bottom cupola flattens while the top cupola collapses inwards and pulls the assembly to one side (Fig.14b).



(a) (b)

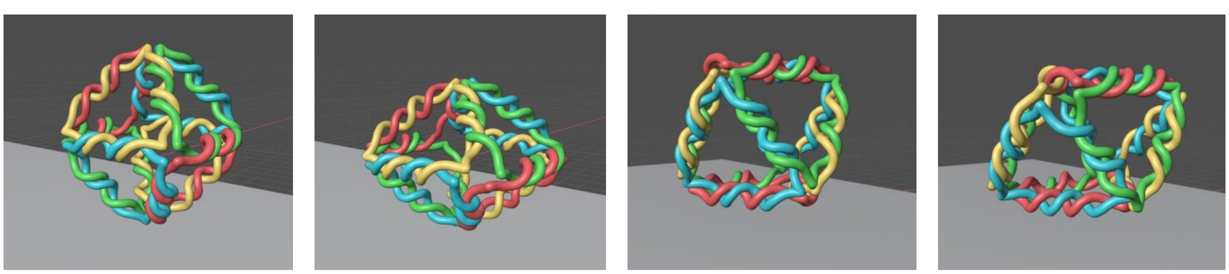
Figure 14: *(a) Icosahedron balanced on a vertex; (b) gravity induced deformation.*

In both the octahedral and icosahedral simulations, we still observe considerable instability, similar to our observations of the single-twisted frame simulations (Figs. 6 and 7). However, there exists one critical difference in behavior between the single-twisted and double-twisted frames. For the single-twisted frames, individual frames are able to rotate about their face-normal axes, which brings the pairs of interlinked edges out of alignment, as discussed earlier. The extra helical twist along each edge in the double-twisted frames disallows this undesired movement. This key feature of the double-twisted frames dramatically increases the topological coherence of our frame structures, and it makes it less likely that the polyhedral structures collapse into wild tangles of irregularly intertwined triangle frames.

Thickened Doubly-Twisted Triangular Frames

Now, let's further improve the stability of our polyhedral structures by applying both of the above methods – using doubly-twisted helices and thickening the frames as much as possible without causing intersections.

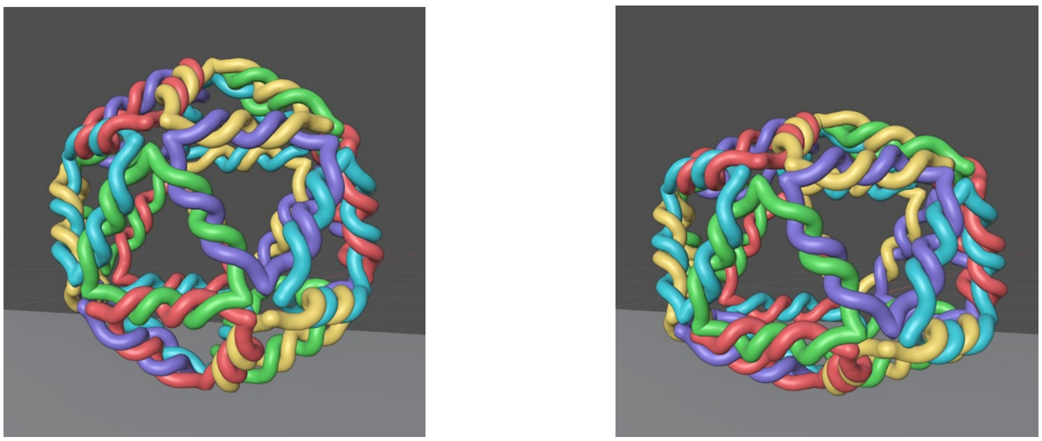
Following the same process as before, we simulate the octahedron's stability in Blender with two different initial orientations (Figs. 15a and 15c). The results are shown in Figures 15(b) and 15(d), respectively. Compared to our simulation results with non-thickened frames (Figs. 11b and 12b), there seems to be only a marginal improvement in stability. The "squash" in Figure 15(b) and the "tilt" in Figure 15(d) are still quite pronounced, but they are less severe than they were in their non-thickened frame counterparts.



(a) (b) (c) (d)

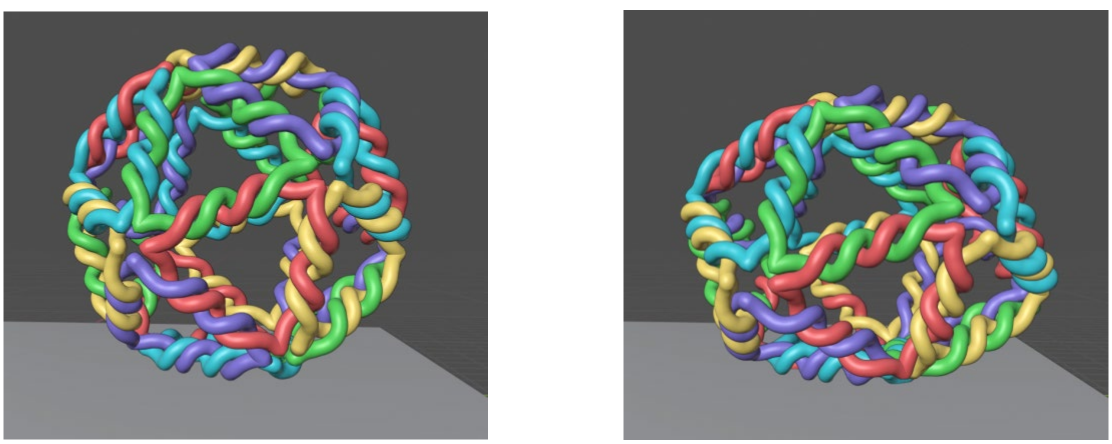
Figure 15: *Octahedron simulation: (a) initial orientation balanced on one "vertex;" (b) simulated result for this orientation; (c) initial orientation lying on one "face;" (d) simulated result for this orientation.*

Figures 16 and 17 show the same two experiments carried out for the icosahedron, balanced on a vertex (Fig.16) and lying on a face (Fig.17). In these cases, we can see significant improvements in stability, when comparing our results to the simulations of the non-thickened frame versions (Figs. 13 and 14). In Figure 16(b), the top and bottom cupolas of the icosahedron no longer flatten nor collapse inwards. Also, the amount of sagging in this structure (Fig.17b) is not nearly as much as that of Figure 13(b), which collapsed to about half the height of the initial structure, whereas our current deformed result is approximately ¾ the height of its initial structure.



(a) (b)

Figure 16: *Icosahedron balanced on a vertex: (a) initial geometry; (b) gravity induced deformation.*



(a) (b)

Figure 17: *Icosahedron lying on a face: (a) initial geometry; (b) gravity induced deformation.*

Now, we can compare the results of the thickened versions of both the single-twisted and double-twisted frame configurations. Based on the simulations of these assemblies, the polyhedra with a larger dihedral angle had more room for improvement. Thus, after increasing the thickness of the frames, the icosahedron displayed a larger degree of improvement than the octahedron.

Summary and Conclusions

We have constructed polyhedral assemblies using triangular frames that interlink along shared edges, representing the tetrahedron, the octahedron, and the icosahedron. If the same frame geometry is used, not all three assemblies can be constructed to be completely rigid. The ones with larger dihedral angles will be looser. Under the influence of gravity, they will lose their nice symmetrical polyhedron shape.

In a real-life environment in which gravity is not the only force acting upon the objects, and where there might be other forces pushing the frames in various directions or "jostling" the object as a whole, the structures will undergo even more severe deformation, and they may collapse into wild tangles of irregularly interlinked triangle frames.

We then studied what design variation would make the assemblies more rigid. One obvious remedy is to make the triangular frames as thick as the dihedral angle associated with a particular assembly would allow. This reduces the relative motions possible between interlinked frames. Another useful remedy is to make adjacent frames interlink with two full helical turns, rather than with just a single one. This tends to keep the frames better in their designed interlinked positions, even when the thickness of the frame is not at the maximum possible before intersection occurs. Of course, both approaches combined will lead to the most rigid and most stable assemblies.

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