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A NOTE ON THE RECIPROCAL ZEROS OF A REAL POLYNOMIAL WITH RESPECT TO THE UNIT CIRCLE

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The purpose of this note is (a) to present a precise formulation of a theorem proposed by Marden[1, 2] which is related to the existence of reciprocal zeros with respect to the unit circle and (b) to indicate a rule which is simpler than Cohn's[3] for determining the number of the reciprocal zeros with respect to the unit circle of a real polynomial.

In an earlier work,[1, 2] Marden has proposed a theorem relating to the number of zeros on the unit circle of a complex (or real) polynomial. This theorem was not precisely formulated as can be readily ascertained from the following example.

$$F(z) = -.5 + 1.65z - .8z^2 - .35z^3 - 1.8z^4 + z^5$$
$$= (z - \tfrac{1}{2})(z - \tfrac{1}{2})(z - 2)\left\{(z + .6)^2 + .8^2\right\} = 0. \qquad (1)$$

If we apply Marden's theorem to Eq. (1) we obtain four roots on the unit circle which is evidently not the case. Hence, we can improve on Marden's theorem and obtain the following modification.

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Theorem: "For a given polynomial $F(z) = a_0 + a_1 z + \dots + a_n z^n$, let the sequence of polynomials $F_{j+1}(z) = \bar{a}_0^{(j)} F_j(z) - a_{n-j}^{(j)} F_j^*(z)$, $j = 0, 1, \dots, n-1$, be constructed. Then, if for some $k < n$, $P_k = \delta_1 \delta_2 \dots \delta_k \neq 0$, for $k = 1, 2, \dots$ but $F_{k+1}(z) \equiv 0$, then $F(z)$ has either $n - k$ zeros on the unit circle $|z| = 1$ or $n - k$ reciprocal zeros with respect to the unit circle or $n - k$ zeros of both reciprocal ones and on the unit circle; it has p zeros in this circle, where p is the number of negative P_j for $j = 1, 2, \dots, k$ and it has $q = k - p$ zeros outside the unit circle." (Note that $\bar{a}_k$ is the conjugate of a_k.)

The above theorem only indicates the necessary condition for the existence of reciprocal zeros with respect to the unit circle or zeros on the unit circle, and to determine the exact number of each, we use the following theorem proposed by Cohn:[3] If $F_{k+1}(z) \equiv 0$, then $F_k(z)$ has in the circle $|z|<|$ as many zeros as the polynomial

$$f_k(z) = [F_k'(z)]^* = z^{n-(k+1)} F_k'(1/z) \tag{2}$$

where

$F_k'(z)$ is the derivative of $F_k(z)$.

While the above procedure is straightforward, it can be considerably simplified for real polynomials. In the following discussion, we will present a simple rule for obtaining the number of reciprocal zeros with respect to the unit circle of a real polynomial. This rule will indirectly determine the number of zeros on the unit circle.

We denote the real polynomial $F_k(z)$ for the sake of generalization as $F_0(z) \equiv F(z)$, and hence we establish our rule for the polynomial $F(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$. The coefficients of the polynomial satisfy the relation

$$|a_n| = |a_0| , \quad |a_{n-1}| = |a_1| , \quad |a_{n-2}| = |a_2| , \ \ldots , \quad . \tag{3}$$

We first form according to Eq. (2) $f(z) = [F'(z)]^*$ which is of degree $n - 1$. It should be noted from Cohn's theorem that $F(z)$ has in the circle $|z|<|$ as many zeros as the polynomial $f(z)$. In determining the zeros of $f(z)$ inside the unit circle we use the following identity for $F(z)$ established by the author:[4]

$$\delta_n = \delta_1^{n-3}\delta_2^{n-4}\delta_3^{n-5} \ldots \delta_{n-3}\delta_{n-2}(A_{n-1} - B_{n-1})^2 F(1)F(-1), \text{ for } n \geqslant 2. \tag{4}$$

In applying the above identity to $f(z)$ we have

$$\delta_{n-1} = \delta_1^{n-3}\delta_2^{n-4} \ldots \delta_{n-3}(A'_{n-2} - B'_{n-2})^2 \; f(1) \, f(-1), \text{ for } n \geqslant 3. \tag{5}$$

If we multiply both sides of the above equation by $\delta_1\delta_2 \ldots \delta_{n-2}$ we establish

$$\text{Sg } P_{n-1} = \text{Sg } [P_1P_2 \ldots P_{n-2}] \cdot \text{Sg}[f(1) \, f(-1)], \text{ for } n \geqslant 3 \ . \tag{6}$$

The above formula indicates that provided $P_k \neq 0$, for $k = 1, \ldots, n - 2$, the sign of P_{n-1} can be readily determined from the sign of P_k, $k = 1, \ldots, n - 2$ and the sign of $f(1) \, f(-1)$. Hence, the calculation of δ_{n-1} is no longer required. Therefore, we can utilize either the sign of $f(1) \, f(-1)$ or the sign of $F(1) \, F(-1)$ in determining the number of real zeros of $f(z)$ or $F(z)$ between $(-1, 1)$ and also the sign of P_{n-1}. The second simplification involves the recognition of the fact that for the real polynomial $f(z)$ or $F(z)$, complex zeros appear only in conjugate pairs. Combining the above simplifications, we will show the rule first for a sixth-order polynomial and then state it for the general case.

Consider a sixth-order polynomial as follows:

$$F(z) = 1 + z + 2z^2 + 3z^3 + 2z^4 + z^5 + z^6. \qquad (7)$$

The above polynomial satisfies Eq. (3) and hence is called a self-inversive polynomial;[4, 5] in which we can apply the procedure discussed above.

$$F'(z) = 1 + 4z + 9z^2 + 8z^3 + 5z^4 + 6z^5 \qquad (8)$$

From Eq. (2) we obtain f(z) and use the following table:[4, 8]

$f(z)$	6	5	8	9	4	1
$f^*(z)$	1	4	9	8	5	6
$f_1(z)$	$\delta_1 = 35$	26	39	46	19	
$f_1^*(z)$	19	46	39	26	35	
$f_2(z)$	$\delta_2 = 864$			1136		

$$\delta_3 = 864^2 - 1136^2 < 0$$

(Note that dotted entries are not needed for determining δ_3 and thus its determination can be avoided.)

The above shortened table indicates that $P_1 > 0$, $P_2 > 0$, $P_3 < 0$; hence at least one zero exists inside the unit circle. Since $F(1) > 0$, $F(-1) > 0$, we can have only two zeros inside the unit circle. These two zeros if real or complex satisfy the condition on $F(1) > 0$ and $F(-1) > 0$. Another two reciprocal zeros are outside the unit circle and the remaining two zeros are on the unit circle. This is the only combination that can exist. Therefore the table form can be considerably simplified. It should be noted that the table test could be termininated, if we merely find that $\delta_1 < 0$. Hence, the proposed rule establishes a minimum and maximum number of the δ_k's or P_k's to be calculated from the table. For the general sixth-order self-inversive polynomial with $F(1)$ and $F(-1)$ having the same sign, the minimum number is δ_1 or P_1 and maximum is δ_4 or P_4. If $F(1)$ and $F(-1)$ have opposite sign, the minimum is two, i.e., P_1 and P_2 and the maximum is four, i.e., up to P_4.

Based on the above example we can readily formulate the following

Rule:

If $n = 4k$ and $F(1)$ and $F(-1)$ have the same sign, then

$$P_{min} = \frac{4k}{2} - 1 \text{ and } P_{max} = n - 2.$$

If $n = 4k - 2$, then

$$P_{min} = \frac{4k - 2}{2} - 1 \text{ and } P_{max} = n - 2.$$

If $n = 4k$ and $F(1)$ and $F(-1)$ have opposite signs, then

$$P_{min} = \frac{4k - 2}{2} - 1 \text{ and } P_{max} = n - 2.$$

Finally, if $n = 4k - 2$, and $F(1)$, $F(-1)$ have opposite signs, then

$$P_{min} = \frac{4k - 2}{2} - 1 \text{ and } P_{max} = n - 2.$$

In the above rule we excluded the cases of $F(z)$ being odd and some of the δ's or $F(-1)$ $F(1)$ vanish. These will be discussed below:

If $F(z)$ is odd then we know that at least a zero exists at $z \pm 1$, which can be factored to yield an even polynomial. The case of $F(1)\ F(-1) = 0$ indicates real zeros on the unit circle which can also be factored. If a certain $\delta_k = 0$ and $f_k \equiv 0$, then the same process applied to $F(z)$ can be applied to $f_k(z)$ to yield the needed information. It should be noted that in some of these cases we can also obtain information on the number of zeros on the unit circle or the number of reciprocal zeros with certain multiplicity. For instance, if $f_1(z)$ has a zero on the unit circle, then $F(z)$ has the same zero with multiplicity two. Finally, from the above rule we readily establish that if $n = 4k - 2$ and $F(1) \quad F(-1) > 0$, then $F(z)$ has either two zeros on the unit circle or two plus $4m$. The same is true if $n = 4k$ and $F(1) \quad F(-1) < 0$.

Furthermore from the preceding discussion we can deduce the following statements:

The necessary and sufficient condition for the zeros of $F(z)$ to lie on the unit circle[3] are: (a) Eq. (3) should be satisfied and (b) all the zeros of $F'(z)$ should lie in, on, or in and on the unit circle. Similarly we can formulate the necessary and sufficient condition for all the zeros of an even $F(z)$ to be reciprocal with respect to the unit circle, by requiring that Eq. 3 be satisfied and that $n/2$ zeros of $f(z) = [F'(z)]^*$ to lie inside the unit circle.

In conclusion, it is hoped that the contents of this note will aid to clarify some of the previous work and to establish a simplified rule for the exact calculation of number reciprocal zeros with respect to the unit circle or zeros on the unit circle of a real polynomial. Although other methods using certain transformations[7] could be used for determining the number of zeros on the unit circle, it is believed that the rule proposed in this note yields simpler and quicker results. The use of the above discussions lies in obtaining the root distribution of any real polynomial in the unit circle and in simplifying the procedures for obtaining the roots of any form of $F(z)$. Such a general form of $F(z)$ could represent the characteristic equation of a linear discrete feedback system or a pulsed network.

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