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THE MINIMAL REALIZATION OF IMPULSE  
RESPONSE MATRICES

by

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# The Minimal Realization of Impulse Response Matrices

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## ABSTRACT

This paper is concerned with the problem of obtaining the minimum realization of a linear nonanticipative system characterized by its impulse response matrix: the problem is to find a linear differential system of least order which is zero-state equivalent to the given one.

For the time-varying case, Kalman's decomposition is used to obtain, in some cases, systems of lower order than Youla's globally reduced systems. In special cases, integrators are time-shared and integrators are saved at the cost of relays; from a mathematical point of view, in such cases, the system's matrices will include  $\delta$  functions.

For the time-invariant case, an explicit way of constructing the minimal order analog computer simulation is given. The method is based on the partial fraction expansion of the transfer function and includes the case of multiple poles. The method is based on careful selection of dyadic expansions; it is an alternative to Kalman's method which is based on the Smith-McMillan form.

## INTRODUCTION

This paper is concerned with the problem of obtaining the minimum realization of a linear nonanticipative system characterized by its impulse response matrix: the problem is to find the linear differential system of least order which is zero-state equivalent to the given one.

For the time-varying case, the key tool is the Kalman<sup>1, 2</sup> decomposition of the impulse response matrix. Our procedure results, in some cases, in a system of lower order than Youla's globally reduced system.<sup>3</sup>

For the time-invariant case, we give an explicit way of constructing the minimal analog computer simulation of the system. Our method is based on the partial fraction expansion and includes the case of multiple poles. This method is an alternative to the one based on the Smith-McMillan canonical form proposed by Kalman.<sup>6</sup>

## I. Time-varying case

### A. Notations

Let  $\underline{W}(t, \tau)$  be an  $r \times p$  impulse response matrix of a nonanticipative system. It is assumed that, for each fixed  $\tau$ ,  $\underline{W}$  is locally integrable with respect to  $t$  and, for each fixed  $t$ ,  $\underline{W}$  is locally integrable with respect to  $\tau$ .

$\underline{W}(t, \tau)$  is said to be realizable<sup>2, 3</sup> if there exists a linear differential system  $S$  of finite dimensional state space, (say  $n$ ), which has a zero-state response to any input applied from  $t_0$  on given by

$$\underline{y}(t) = \int_{t_0}^t \underline{W}(t, \tau) \underline{u}(\tau) dt \quad t \geq t_0. \quad (1)$$

More precisely, let the system  $S$  be characterized by

$$\dot{\underline{x}}(t) = \underline{F}(t) \underline{x}(t) + \underline{G}(t) \underline{u}(t) \quad (2)$$

$$\underline{y}(t) = \underline{H}(t) \underline{x}(t) \quad (3)$$

where  $\underline{F}(\cdot)$ ,  $\underline{G}(\cdot)$  and  $\underline{H}(\cdot)$  are respectively  $n \times n$ ,  $n \times p$  and  $r \times n$  matrices whose elements are real valued functions defined on  $(-\infty, \infty)$ . It is assumed commonly that  $\underline{F}$ ,  $\underline{G}$  and  $\underline{H}$  are square integrable over any finite interval. We shall not impose this requirement on  $\underline{F}$ .

Let  $\underline{\Phi}(t, t_0)$  be the state transition matrix of (2). Then it is well known that  $\underline{W}(t, \tau)$  is realizable by S if and only if

$$\underline{W}(t, \tau) = \underline{H}(t) \underline{\Phi}(t, \tau) \underline{G}(\tau) \quad \text{for all } t \geq \tau. \quad (4)$$

Since the system S is completely characterized by  $\underline{F}$ ,  $\underline{G}$ ,  $\underline{H}$  one uses the locution " $(\underline{F}, \underline{G}, \underline{H})$  realizes  $\underline{W}(t, \tau)$ ."

Under the condition that  $\underline{F}$ ,  $\underline{G}$  and  $\underline{H}$  be locally square integrable, Kalman has given an interesting characterization of realizability:<sup>2</sup>

$\underline{W}(t, \tau)$  is realizable if and only if

$$\underline{W}(t, \tau) = \underline{\psi}(t) \underline{\beta}(\tau) \quad \forall t, \tau \text{ with } t \geq \tau \quad (5)$$

where  $\underline{\psi}(\cdot)$  and  $\underline{\beta}(\cdot)$  are, respectively,  $r \times n$  and  $n \times p$  matrices which are locally square integrable. Note that this characterization is not valid if  $\underline{F}(\cdot)$  is not locally square integrable. The proof is based on the observation that  $(\underline{0}, \underline{\beta}, \underline{\psi})$  realizes  $\underline{W}(t, \tau)$ : thus, under these conditions, it is always possible to simulate any such impulse response matrix using time variable gains and integrators but without feedback.

We shall see below that if we allow the use of switches then, in many instances, there exist simulations which require a smaller number of integrators.

#### B. Reduction algorithm

As a first step in obtaining the minimal realization of  $\underline{W}(t, \tau)$  we show that we can reduce the dimension of the system (2), (3) in some

cases. As we shall see below we shall have to drop the requirement that  $\underline{F}$  be locally integrable. From an engineering point of view, this is of little importance for we shall only need a switch to discharge instantaneously the capacitors of some integrators.

Lemma. Let  $\underline{W}(t, \tau)$  be factored as  $\underline{\psi}(t)\underline{\beta}(\tau)$ , a product of an  $r \times n$  and an  $n \times p$  matrix. If, for some  $t_1$ , the columns of  $\underline{\psi}(\cdot)$  are linearly dependent over  $(t_1, \infty)$  and the rows of  $\underline{\beta}(\cdot)$  are linearly dependent over  $(-\infty, t_1)$  then there is a realization of dimension  $(n-1)$ .

Example 1: let  $r = p = 1$ ,  $\underline{\psi}(t) = (e^{-t}, 1(-t))$  and  $\underline{\beta}(\tau) = \text{col}(e^{-\tau}, 1(\tau-1))$ . Here  $1(t)$  is the Heaviside unit step function. Clearly  $t \geq \tau$  implies that  $\psi_2(t) = 0$  whenever  $\beta_2(\tau) \neq 0$ . Hence

$$\begin{aligned} y(t) &= \int_{-\infty}^t [\psi_1(t)\beta_1(\tau) + \psi_2(t)\beta_2(\tau)] u(\tau) dt \\ &= \int_{-\infty}^t \psi_1(t)\beta_1(\tau) u(\tau) dt = \int_{-\infty}^t e^{-(t-\tau)} u(\tau) dt. \end{aligned}$$

Example 2<sup>(†)</sup>: Let  $r = p = 1$  and  $W(t, \tau) = \psi_1(t)\beta_1(\tau) + \psi_2(t)\beta_2(\tau)$  where

$$\psi_1(t) = \begin{cases} 1 & \text{if } t \in [-2, -1] \\ 0 & \text{elsewhere,} \end{cases} \quad \psi_2(t) = \begin{cases} 1 & \text{if } t \in [3, 4] \\ 0 & \text{elsewhere,} \end{cases}$$

$$\beta_1(\tau) = \begin{cases} 1 & \text{if } \tau \in [-3, -4] \\ 0 & \text{elsewhere,} \end{cases} \quad \beta_2(\tau) = \begin{cases} 1 & \text{if } \tau \in [1, 2] \\ 0 & \text{elsewhere.} \end{cases}$$

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(†) This example is due to P. Varaiya.

Then the differential system

$$\dot{\eta} = -\delta(t)\eta + [l(-t)\beta_1(t) + l(t)\beta_2(t)]u(t)$$

and

$$y = \eta(t)[\psi_1(t) + \psi_2(t)]$$

(where  $\delta(t)$  is the delta "function") is zero-state equivalent to the one characterized by  $W(t,\tau)$ . Note that (a) the matrix  $F(t)$ , which is here  $-\delta(t)$ , is not locally integrable, (b) the globally reduced realization<sup>3</sup> of  $W(t,\tau)$  is of dimension 2, (c) there is no representation of the given  $W(t,\tau)$ , valid for  $t \geq \tau$ , as a product of two real valued functions  $\psi(t)\beta(\tau)$ .

Proof: The linear dependence of the rows of  $\underline{\beta}(\cdot)$  over  $(-\infty, t_1)$  implies that there is a constant column  $n$ -vector  $\underline{b} \neq \underline{0}$  such that

$$\underline{b}'\underline{\beta}(t) = \underline{0} \quad \forall t \text{ in } (-\infty, t_1). \quad (6)$$

The linear dependence of the columns of  $\underline{\psi}(\cdot)$  over  $(t_1, \infty)$  implies that there exists a constant column  $n$ -vector  $\underline{c} \neq \underline{0}$  such that

$$\underline{\psi}(t)\underline{c} = \underline{0} \quad \forall t \text{ in } (t_1, \infty). \quad (7)$$

Case I:  $\underline{b}' \cdot \underline{c} = 0$ . Let us construct an  $n \times n$  real, constant, non-singular matrix  $\underline{M}$  as follows: pick  $(n-1)$  vectors  $\underline{y}^1, \underline{y}^2, \dots, \underline{y}^{n-1}$  to be orthogonal to  $\underline{b}$  and such that  $\{\underline{y}^1, \underline{y}^2, \dots, \underline{y}^{n-1}, \underline{c}\}$  is a set of linear independent vectors; let  $\underline{M}$  be the  $n \times n$  matrix whose columns are  $\underline{y}^1, \underline{y}^2, \dots, \underline{y}^{n-1}, \underline{c}$ . The last row of  $\underline{M}^{-1}$  which is a vector orthogonal to  $\underline{y}^1, \underline{y}^2, \dots, \underline{y}^{n-1}$ , is now vector necessarily collinear with  $\underline{b}'$ ; call it  $\alpha \underline{b}'$ ; note that the scalar  $\alpha \neq 0$ . Now consider

$$\begin{aligned}
\underline{W}(t, \tau) &= \underline{\psi}(t) \underline{\beta}(\tau) = \underline{\psi}(t) \underline{M} \underline{M}^{-1} \underline{\beta}(\tau) \\
&= \underline{\tilde{\psi}}(t) \underline{\tilde{\beta}}(\tau) \qquad t \geq \tau. \qquad (8)
\end{aligned}$$

The  $r \times n$  matrix  $\underline{\tilde{\psi}}(t)$  has  $\underline{\psi}(t)\underline{c}$  as a last column: hence, by (7), the last column of  $\underline{\tilde{\psi}}(t)$  is identically zero for  $t > t_1$ ; the  $n \times p$  matrix  $\underline{\tilde{\beta}}(\tau)$  has  $\alpha \underline{b}'\underline{\beta}(\tau)$  as a last row: hence, by (6), the last row of  $\underline{\tilde{\beta}}(\tau)$  is identically zero for  $\tau < t_1$ . Since nonanticipativeness requires that  $t \geq \tau$ , the last row of  $\underline{\tilde{\beta}}(\tau)$  and the last column of  $\underline{\tilde{\psi}}(\tau)$  may be deleted without affecting the value of the product over  $t \geq \tau$ . We have thus obtained a factorization of  $W(t, \tau)$  in terms of an  $r \times (n-1)$  and an  $(n-1) \times p$  matrix.

Case II:  $\underline{b}'\underline{c} = 0$ , i. e.,  $\underline{b}$  and  $\underline{c}$  are orthogonal. Let  $\underline{N}^{-1}$  be a real, constant, nonsingular  $n \times n$  matrix which has  $\underline{b}'$  as its last row and such that  $\underline{c}$  is the first column of  $\underline{N}$ . This is possible since  $\underline{b}$  is orthogonal to  $\underline{c}$ . Now consider

$$\begin{aligned}
\underline{W}(t, \tau) &= \underline{\psi}(t) \underline{N} \underline{N}^{-1} \underline{\beta}(\tau) \\
&= \underline{\hat{\psi}}(t) \underline{\hat{\beta}}(\tau).
\end{aligned}$$

By (6), the last row of  $\underline{\hat{\beta}}(\cdot)$  is identically zero on  $(-\infty, t_1)$  and by (7) the first column of  $\underline{\hat{\psi}}(\cdot)$  is identically zero on  $(t_1, \infty)$ . Let us consider the implications of these facts in terms of the  $(\underline{0}, \underline{\hat{\beta}}, \underline{\hat{\psi}})$  realization which is shown on Fig. 1. The  $n$ th integrator input is identically zero on  $(-\infty, t_1)$ ; the first integrator output is multiplied by the vector  $\underline{\hat{\psi}}_1(t)$ , (the first column of  $\underline{\hat{\psi}}_1(t)$ ), which is identically zero on  $(t_1, \infty)$ . Thus the first and last integrators of the realization of Fig. 1 may be replaced by a single time-shared integrator. The first and last line of the realization of Fig. 1

are replaced by the one shown on Fig. 2. The differential equation associated with the time-shared integrator is

$$\dot{\eta}_1 = -\delta(t-t_1)\eta_1 + \langle \hat{\underline{\beta}}_1(t)1(t-t_1) + \hat{\underline{\beta}}_n(t), \underline{u}(t) \rangle$$

and the contribution to the output due to  $\eta_1$  is

$$\eta_1(t)[\hat{\underline{\psi}}_1(t) + 1(t-t_1)\hat{\underline{\psi}}_n(t)].$$

This completes the proof of the lemma. As a special case of this lemma we have the following.

Corollary:<sup>3</sup> If there is a nonzero  $\hat{\underline{b}} \in \mathbb{R}^n$  such that

$$\hat{\underline{b}}' \underline{\beta}(\tau) = \underline{0} \quad \forall \tau \in \mathbb{R} \quad (9)$$

or if there is a nonzero  $\hat{\underline{c}} \in \mathbb{R}^n$  such that

$$\underline{\psi}(t)\hat{\underline{c}} = \underline{0} \quad \forall t \in \mathbb{R} \quad (10)$$

then there is a realization of dimension  $(n-1)$ .

Proof: If (9) holds, let  $\underline{M}$  be a real, constant, nonsingular,  $n \times n$  matrix such that  $\underline{M}^{-1}$  has  $\hat{\underline{b}}$  as its last row. By Eq. 8 the factorization of  $\underline{W}(t, \tau)$  is reduced to a product of an  $r \times (n-1)$  matrix  $\tilde{\underline{\psi}}(t)$  and an  $(n-1) \times p$  matrix  $\tilde{\underline{\beta}}(\tau)$ . If Eq. 10 holds, pick  $\underline{M}$  to have  $\hat{\underline{c}}$  as its last column and the same conclusion follows.

### C. Realization of $\underline{W}(t, \tau)$

We start from a factorization of  $\underline{W}(t, \tau)$  as  $\underline{\psi}(t)\underline{\beta}(\tau)$ .

The first step is to check whether there is any vector  $\hat{\underline{b}}$  (or  $\hat{\underline{c}}$ ) in  $R^n$  such that Eq. 9 (or 10) holds. For each such vector we can reduce the realization as shown above. Repeat the procedure until all such vectors are exhausted. For simplicity, we write again the resulting factorization as  $\underline{\psi}(t)\underline{\beta}(\tau)$  where  $\underline{\psi}$  is  $r \times n$  and  $\underline{\beta}$  is  $n \times p$ .

Given any  $t$  in  $(-\infty, \infty)$ , the columns of  $\underline{\psi}(\cdot)$  may be considered as functions mapping  $(t, \infty)$  into  $E^r$ ; as such they span a subspace of  $L_{2r}(t, \infty)$ , the space of square integrable functions mapping  $(t, \infty)$  into  $E^r$ . Call  $n_\psi(t)$  the dimension of this subspace:  $n_\psi(\cdot)$  is an integer-valued function that is monotonically decreasing. Also  $n_\psi(-\infty) = n$ . Similarly consider the rows of  $\underline{\beta}(\cdot)$  as functions mapping  $(-\infty, t)$  into  $E^p$ , call  $n_\beta(t)$  the dimension of the subspace of  $L_{2p}(-\infty, t)$  spanned by the rows of  $\underline{\beta}(\cdot)$ :  $n_\beta(\cdot)$  is an integer-valued function that is monotonically increasing. Note that  $n_\beta(\infty) = n$ . Let for each  $t$  in  $(-\infty, \infty)$

$$\underline{\bar{n}}(t) \triangleq \min \{n_\psi(t), n_\beta(t)\}.$$

Let  $t_1$  be a time such that

$$\underline{\bar{n}} \triangleq \underline{\bar{n}}(t_1) = n_\beta(t_1) = n_\psi(t_1).$$

Call B the subspace of  $R^n$  of all vectors  $\underline{b}$  such that

$$\underline{b}'\underline{\beta}(\tau) = \underline{0} \quad \text{for } \tau < t_1.$$

Call C the subspace of  $R^n$  of all vectors  $\underline{c}$  such that

$$\underline{\psi}(t)\underline{c} = \underline{0} \quad \text{for } t > t_1.$$

Since  $n_{\beta}(t_1) = n_{\psi}(t_1) = \bar{n}$ ,  $\dim B = \dim C$ .

Let us pick an orthonormal set of vectors of  $R^n$  as follows:

- (a) start with an orthonormal basis of  $B \cap C$ ; let  $n_1$  be its dimension;
- (b) pick an orthonormal basis for  $B^\perp \cap C$  and one for  $C^\perp \cap B$ ; let  $k$  be the dimension of each of these bases. From the definition of  $\bar{n}$ , we have  $\bar{n} = n - n_1 - k$ .

Construct a real, constant, nonsingular,  $n \times n$  matrix  $\underline{P}$  whose last  $n_1$  rows are the basis vectors of  $B \cap C$  and whose rows  $n - n_1 - k + 1$ ,  $n - n_1 - k + 2$ ,  $\dots$ ,  $n - n_1$  are the basis vectors of  $C^\perp \cap B$  (see Fig. 3); furthermore the inverse  $\underline{P}^{-1}$  has as its last  $n_1$  columns the basis vectors of  $B \cap C$  and as its first  $k$  columns the basis vectors of  $B^\perp \cap C$ . Now for  $t \geq \tau$ ,

$$\begin{aligned} \underline{W}(t, \tau) &= \underline{\psi}(t) \underline{\beta}(\tau) = \underline{\psi}(t) \underline{P}^{-1} \underline{P} \underline{\beta}(\tau) \\ &= \tilde{\underline{\psi}}(t) \tilde{\underline{\beta}}(\tau). \end{aligned}$$

The effect of the last  $n_1$  columns of  $\underline{P}^{-1}$  and of the last  $n_1$  rows of  $\underline{P}$  is to decrease the dimension of  $\tilde{\underline{\psi}}(t)$  and  $\tilde{\underline{\beta}}(\tau)$  to  $r \times (n - n_1)$  and  $(n - n_1) \times p$ , respectively; indeed, for  $t > t_1$ , the last  $n_1$  columns of the  $r \times n$  matrix  $\underline{\psi}(t) \underline{P}^{-1}$  are zero, and for  $\tau < t_1$  the last  $n_1$  rows of the  $n \times p$  matrix  $\underline{P} \underline{\beta}(\tau)$  are zero; consequently, as long as we consider only the range  $t \geq \tau$ , the product  $(\underline{\psi}(t) \underline{P}^{-1})(\underline{P} \underline{\beta}(\tau))$  is unaffected if we discard the last  $n_1$  columns and the last  $n_1$  rows. From now on we consider  $\tilde{\underline{\psi}}(\cdot)$  and  $\tilde{\underline{\beta}}(\cdot)$  to be  $r \times (n - n_1)$  and  $(n - n_1) \times p$  matrices, respectively. The effect of the first  $k$  rows of  $\underline{P}^{-1}$  is that the first  $k$  columns of  $\tilde{\underline{\psi}}(t)$  are zero for  $t > t_1$ :

$$\tilde{\underline{\psi}}_i(t) = \underline{0} \quad \text{for } t > t_1, \quad i = 1, 2, \dots, k.$$

The effect of the  $k$  rows of  $\underline{P}$  numbered  $n-n_1-k+1, \dots, n-n_1$  is that the last  $k$  rows of  $\tilde{\underline{\beta}}(\tau)$  are zero for  $\tau < t_1$ :

$$\tilde{\underline{\beta}}_i(\tau) = \underline{0} \quad \text{for } \tau < t_1, \quad i = n-n_1-k+1, \dots, n-n_1.$$

Using these facts we have the realization described by the following equation :

$$\dot{\eta}_i(t) = -\delta(t-t_1) \eta_i(t) + \langle \tilde{\underline{\beta}}_i(t) 1(t_1-t) + \tilde{\underline{\beta}}_{n-n_1-i+1}(t), \underline{u}(t) \rangle \quad (11)$$

$$i = 1, 2, \dots, k$$

$$\dot{\eta}_j(t) = \langle \tilde{\underline{\beta}}_j(t), \underline{u}(t) \rangle \quad j = k+1, k+2, \dots, \bar{n} = n-n_1-k. \quad (12)$$

and

$$y(t) = \sum_{i=1}^k \eta_i(t) [\tilde{\underline{\psi}}_i(t) + \tilde{\underline{\psi}}_{n-n_1-i+1}(t) 1(t-t_1)]$$

$$+ \sum_{j=k+1}^{\bar{n}} \eta_j(t) \tilde{\underline{\psi}}_j(t). \quad (13)$$

It is of interest to note that this realization is not minimal in all cases. We may state the following

Assertion: (i) If the subspaces  $B$  and  $C$  have dimension 0, the minimal realization is of dimension  $n$ .

(ii) If  $B^\perp \cap C$  has dimension 0 (and hence so does  $C^\perp \cap B$ )

the minimal realization is of dimension  $n - n_1$ .

(iii) If  $B^\perp \cap C$  is of dimension 1 or more, then the minimal realization is at most of dimension  $\bar{n} = n - n_1 - k$  and at least of dimension  $\bar{n} - k = n - n_1 - 2k$ .

These three statements follow from the realization above.

Case i:  $n_1 = k = 0$ . Let  $t_1$  be an arbitrary point in  $(-\infty, \infty)$ . The columns of  $\underline{\psi}(\cdot)$  and the rows of  $\underline{\beta}(\cdot)$  are linearly independent over  $(t_1, \infty)$  and  $(-\infty, t_1)$  respectively. Consequently, given any  $\underline{y} \in \mathbb{R}^n$ , there is an input  $\underline{u}_{\underline{y}}(\cdot)$  zero for  $t > t_1$  and of the form

$$\sum_{i=1}^n \alpha_i \underline{\beta}_i(t) \quad \text{for } t < t_1, \quad \text{such that}$$

$$\int_{-\infty}^{t_1} \underline{\beta}(\tau) \underline{u}(\tau) dt = \underline{y}.$$

For  $t > t_1$ , the corresponding output is given by

$$\underline{y}(t) = \sum_{i=1}^n \gamma_i \underline{\psi}_i(t) \quad t > t_1.$$

Since the vectors  $\underline{\psi}_i(\cdot)$  are linearly independent over  $(t_1, \infty)$ , the outputs fill an  $n$  dimensional subspace of the output space. It follows from the theory of differential equations that any realization must have a dimension of at least  $n$ . The realization (11), (12) and (13) (for this special case  $k = n_1 = 0$ ) is of dimension  $n$ .

Case ii: Let  $t_1$  be chosen so that  $n_\beta(t_1) = n_\psi(t_1) = n$ . Since here  $k = 0$ ,  $\bar{n} = n - n_1$ . Over  $(-\infty, t_1)$  the  $\bar{n}$  rows of  $\underline{\tilde{\beta}}(\cdot)$  are linearly independent, and over  $(t_1, \infty)$  the  $\bar{n}$  columns of  $\underline{\tilde{\psi}}(\cdot)$  are linearly independent. By the reasoning outlined in Case i, we conclude that any realization must be at least of dimension  $n - n_1$  and we observe that the realization (11), (12) and (13) (for this special case  $k = 0$ ) is of dimension  $\bar{n} = n - n_1$ .

Case iii: ( $n_1 \neq 0$ ,  $k \neq 0$ ). The realization (11), (12) and (13) is of dimension  $\bar{n} = n - n_1 - k$ . This realization is not necessarily minimal in all cases. We know that the first  $\bar{n} = n - n_1 - k$  rows of  $\underline{\tilde{\beta}}(\cdot)$  are linearly independent over  $(-\infty, t_1)$  and that, of the  $(n - n_1)$  columns of  $\underline{\tilde{\psi}}(\cdot)$ ,  $\bar{n} = n - n_1 - k$  of them are linearly independent over  $(t_1, \infty)$ . Clearly, the realization of minimal dimensionality will occur when there is a minimum overlap between the linearly independent rows of  $\underline{\tilde{\beta}}(\cdot)$  and the linearly independent columns of  $\underline{\tilde{\psi}}(\cdot)$ . Consider then the worst case:

$$t < t_1 \quad \left\{ \begin{array}{l} \underline{\tilde{\psi}}_i(\cdot) \quad (i=1, 2, \dots, n-n_1=\bar{n}+k) \text{ are linearly independent,} \\ \underline{\tilde{\beta}}(\cdot) \text{ is identically zero.} \end{array} \right.$$

$$t_1 < t < t_2 < \infty \quad \left\{ \begin{array}{l} \underline{\tilde{\psi}}_i(t) = \underline{0} \text{ for } i = 1, 2, \dots, k, \\ \underline{\tilde{\psi}}_i(\cdot) \quad (i=k+1, k+2, \dots, n-n_1=\bar{n}+k) \text{ are linearly independent,} \\ \underline{\tilde{\beta}}_i(\cdot) \quad (i=1, 2, \dots, \bar{n}=n-n_1-k) \text{ are linearly independent,} \\ \underline{\tilde{\beta}}_i(t) = \underline{0} \quad i=\bar{n}+1, \bar{n}+2, \dots, \bar{n}+k. \end{array} \right.$$

$$t > t_2 \quad \left\{ \begin{array}{l} \underline{\tilde{\psi}}_i(t) = \underline{0} \text{ for } i = 1, 2, \dots, n-n_1, \\ \underline{\tilde{\beta}}_i(\cdot) \quad (i=1, 2, \dots, n-n_1) \text{ are linearly independent.} \end{array} \right.$$

$\underline{W}(t, \tau) = \underline{0}$  whenever  $t \notin (t_1, t_2)$  or  $\tau \notin (t_1, t_2)$ .

When both  $t$  and  $\tau$  are in  $(t_1, t_2)$ ,

$$\underline{W}(t, \tau) = \sum_{i=k+1}^{\bar{n}} \underline{\tilde{\psi}}_i(t) >< \underline{\tilde{\beta}}_i(\tau) \quad t, \tau \in (t_1, t_2).$$

Since  $\underline{W}(t, \tau)$  has a dyadic representation involving only  $(\bar{n} - k)$  dyads, and since these  $\underline{\tilde{\beta}}_i$ 's and these  $\underline{\tilde{\psi}}_i$ 's are linearly independent over  $(t_1, t_2)$ , the realization will require  $\bar{n} - k$  integrators.

### CONCLUSION

We have shown that the global reduction of Youla does not always lead to the minimal realization. It is interesting to note that in some cases, one may time-share an integrator but it is essential to discharge the stored charge at the time of switching. From an engineering point of view it is a significant gain since it replaces an integrator by a few relays.

## II. Time-invariant case

It is well known that an impulse response matrix  $\underline{W}(t)$  is realizable as a linear time-invariant differential system if and only if the Laplace transform of  $\underline{W}(t)$  is a proper rational matrix  $\tilde{W}(s)$ . We start from the partial fraction expansion of  $\tilde{W}(s)$ . Again  $\underline{W}$  and  $\tilde{W}$  are  $r \times p$  matrices. Obviously the problem of finding the minimum realization of  $\underline{W}(t)$  will be solved if we have a method to obtain the minimal representation of each term of the partial fraction expansion;<sup>7, 8</sup> here each distinct pole of  $\tilde{W}(s)$  has associated with it one and only one term of the partial fraction expansion. We shall therefore consider only the minimal realization of a term associated with a single pole.

The description of the procedure for a pole of order  $k$  would require very complicated notations, we shall ease the burden of the reader by describing the method for poles of order 1, 2 and 3. The procedure for the general case will be obvious.

Remark: Since the poles of  $\tilde{W}(s)$  may be complex all subsequent algebra involves in general complex numbers. We use the following notations: let  $\underline{a} = (a_1, a_2, \dots, a_n)$ ,  $\underline{b} = (b_1, b_2, \dots, b_n)$  and  $\underline{x} = (x_1, x_2, \dots, x_n)$  then the complex scalar product is written as  $\langle \underline{a}, \underline{b} \rangle$  and defined by

$$\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^n \bar{a}_i b_i, \text{ where } \bar{a}_i \text{ is the complex conjugate of } a_i.$$

Similarly  $\bar{\underline{a}} = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$ . Finally  $(\underline{a} \rangle \langle \underline{b}) \underline{x} \triangleq \langle \underline{b}, \underline{x} \rangle \underline{a} = \left( \sum_{i=1}^n \bar{b}_i x_i \right) \underline{a}$ .

$\underline{a}^*$  denotes the conjugate transpose of  $\underline{a}$  and  $\underline{K}^*$  denotes the conjugate transpose of the matrix  $\underline{K}$ . It should also be kept in mind that since  $\tilde{W}(s)$  has rational functions with real coefficients as elements, poles occur in complex conjugate pairs. In the final realization, the realizations -- involving complex coefficients -- pertaining to pole  $\lambda$  and to  $\bar{\lambda}$  may be combined in order to obtain a realization involving only real coefficients. The technique is to change variables and use  $\frac{1}{2}(\underline{x} + \bar{\underline{x}})$  and  $\frac{1}{2j}(\underline{x} - \bar{\underline{x}})$  as the new variables. [See Ref. 6, p. 533 for details.]

Remark 2: The numerical evaluation of the partial fraction expansion has been discussed in References 7 and 8.

Pole of the 1st order: As shown by E. G. Gilbert,<sup>4</sup> the minimal representation of a term corresponding to a first order pole, say,

$$\frac{1}{s + \lambda} \underline{K}$$

requires a state space whose dimension  $k \triangleq \text{rank}(\underline{K})$ . In the following we denote by  $R(\underline{K})$  the range of  $\underline{K}$ , i. e., the subspace spanned by its column-vectors. We recall that  $k$  is also the dimension of  $R(\underline{K})$ . We denote by  $R(\underline{K}^*)$  the range of  $\underline{K}^*$ , the conjugate-transpose of  $\underline{K}$ ;  $R(\underline{K}^*)$  is the subspace spanned by the conjugate of the row-vectors of  $\underline{K}$ .

The first step in obtaining a minimal realization of  $\underline{K}/(s + \lambda)$  is to write the  $r \times p$  matrix  $\underline{K}$  as a sum of dyads

$$\underline{K} = \sum_{i=1}^k \hat{\underline{\alpha}}_i > < \underline{\alpha}_i \tag{15}$$

where, for each  $i$ ,  $\underline{\alpha}_i$  is a  $p$ -tuple of complex numbers and  $\hat{\underline{\alpha}}_i$  is an  $r$ -tuple of complex numbers. The set  $\{\underline{\alpha}_i\}_1^k$  is a basis for  $R(\underline{K}^*)$  and  $\{\hat{\underline{\alpha}}_i\}_1^k$  is a basis for  $R(\underline{K})$ . Obviously these bases are related.

We describe two methods for obtaining the representation (15).

Method I. Pick any set of  $k$  linearly independent columns of  $\underline{K}$ ; call them  $\hat{\underline{\alpha}}_1, \hat{\underline{\alpha}}_2, \dots, \hat{\underline{\alpha}}_k$ . Since  $\text{rank}(\underline{K}) = k$ , they constitute a basis for  $R(\underline{K})$ . The related basis  $\{\underline{\alpha}_i\}_1^k$  is obtained by solving the following system of  $k$  linear algebraic equations in  $k$  unknowns:

$$\begin{aligned} \underline{K}^* \hat{\underline{\alpha}}_j &= \left( \sum_{i=1}^k \underline{\alpha}_i \langle \hat{\underline{\alpha}}_i \rangle \right) \hat{\underline{\alpha}}_j \\ &= \sum_{i=1}^k \langle \hat{\underline{\alpha}}_i, \hat{\underline{\alpha}}_j \rangle \underline{\alpha}_i \quad j = 1, 2, \dots, k. \end{aligned} \quad (16a)$$

This system can be solved by the Gauss elimination method, for example. For every choice of  $\{\hat{\underline{\alpha}}_i\}_1^k$ , the solution of the system is unique because the matrix of the system is nonsingular. Indeed its  $(i, j)$  element is  $\langle \hat{\underline{\alpha}}_i, \hat{\underline{\alpha}}_j \rangle$  and the  $\hat{\underline{\alpha}}_i$ 's are linearly independent. In addition the solution  $\underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_k$  spans  $R(\underline{K}^*)$ : indeed  $\underline{K}^*$  is a bijection -- a one-to-one map which is onto -- of  $R(\underline{K})$  on  $R(\underline{K}^*)$  and  $\{\hat{\underline{\alpha}}_j\}_1^k$  is a basis of  $R(\underline{K})$ .

Method II. Pick any set of  $k$  linearly independent rows of  $\underline{K}$ ; call them  $\bar{\underline{\alpha}}_1, \bar{\underline{\alpha}}_2, \dots, \bar{\underline{\alpha}}_k$ . Since  $\text{rank}(\underline{K}^T) = k$ , they constitute a basis for  $R(\underline{K}^T)$ ; equivalently,  $\{\bar{\underline{\alpha}}_i\}_1^k$  is a basis for  $R(\underline{K}^*)$ . The related basis  $\{\hat{\underline{\alpha}}_i\}$  is obtained by solving the following system of  $k$  linear algebraic equations in  $k$  unknowns:

$$\begin{aligned}
\underline{K} \underline{\alpha}_j &= \left( \sum_{i=1}^k \hat{\underline{\alpha}}_i > < \underline{\alpha}_i \right) \underline{\alpha}_j \\
&= \sum_{i=1}^k < \underline{\alpha}_i, \underline{\alpha}_j > \hat{\underline{\alpha}}_i \quad j = 1, 2, \dots, k. \quad (16b)
\end{aligned}$$

This system can be solved by the Gauss elimination method, for example. For every choice of basis  $\{\underline{\alpha}_i\}_1^k$ , the system has a unique solution because the matrix of the system is nonsingular. Indeed, its  $(i, j)$  element is  $< \underline{\alpha}_i, \underline{\alpha}_j >$  and the  $\underline{\alpha}_i$ 's are linearly independent. Furthermore the solution  $\{\hat{\underline{\alpha}}_i\}_1^k$  spans  $R(\underline{K})$ : indeed  $\underline{K}$  is a bijection of  $R(\underline{K}^*)$  on  $R(\underline{K})$  and  $\{\underline{\alpha}_i\}_1^k$  is a basis for  $R(\underline{K}^*)$ .

In the following we shall describe the method for obtaining the minimal representation in terms of the second method just described.

The state representation of  $\underline{K}/(s + \lambda)$  is obtained as follows: to each dyad of Equation 15 is associated one state variable

$$\dot{\underline{x}}_i = -\lambda \underline{x}_i + < \underline{\alpha}_i, \underline{u}(t) > \quad (i = 1, 2, \dots, k). \quad (17)$$

The contribution to the output due to this first order pole is

$$\underline{y}_\lambda(t) = \sum_{i=1}^k \underline{x}_i(t) \hat{\underline{\alpha}}_i. \quad (18)$$

This state representation of the transfer function  $\underline{K}/(s + \lambda)$  is obviously minimal:<sup>1</sup> the system specified by Equations 17 and 18 is completely controllable (since the  $\underline{\alpha}_i$  are linearly independent) and completely

observable (since the  $\hat{\underline{\alpha}}_i$  are linearly independent).

Pole of 2nd order: Let the partial fraction expansion be

$$\frac{1}{s+\lambda} \underline{K}_1 + \frac{1}{(s+\lambda)^2} \underline{K}_2$$

where  $\underline{K}_1$  and  $\underline{K}_2$  are  $r \times p$  matrices with possibly complex elements. The problem is to carefully select the dyadic representation of  $\underline{K}_1$  and  $\underline{K}_2$ :

$$\underline{K}_1 = \sum_1^{k_1} \hat{\underline{\alpha}}_i >< \underline{\alpha}_i, \quad \underline{K}_2 = \sum_1^{k_2} \hat{\underline{\beta}}_i >< \underline{\beta}_i.$$

We know that the  $\underline{\alpha}_i$  must span  $R(\underline{K}_1^*)$  and the  $\underline{\beta}_i$ 's must span  $R(\underline{K}_2^*)$ , however these subspaces may have a proper subspace in common. We therefore proceed as follows:

(i) Let  $B_{12}$  be a basis for the subspace common to  $R(\underline{K}_1^*)$  and  $R(\underline{K}_2^*)$ .

(ii) Let  $B_{12}$  be a set of linearly independent vectors such that

$$B_{12} \cup B_{12} \text{ is a basis for } R(\underline{K}_2^*)$$

(iii) Let  $B_{1\bar{2}}$  be a set of linearly independent vectors such that

$$B_{1\bar{2}} \cup B_{12} \text{ is a basis for } R(\underline{K}_1^*)$$

The  $\underline{\alpha}_i$ 's used to represent  $\underline{K}_1$  are the vectors of  $B_{12}$  and  $B_{1\bar{2}}$ .

The  $\underline{\beta}_i$ 's used to represent  $\underline{K}_2$  are the vectors of  $B_{12}$  and  $B_{1\bar{2}}$ .

Therefore we have three types of block diagrams shown on Fig. 4: the

first one corresponds to a vector,  $\underline{\alpha}_1$ , of the basis  $B_{12}$ , the second one corresponds to a vector,  $\underline{\beta}_2$ , of the basis  $B_{\bar{1}2}$  and the third one corresponds to a vector,  $\underline{\alpha}_3$ , of the basis  $B_{1\bar{2}}$ .

Pole of 3rd order: Let the partial fraction expansion be

$$\frac{1}{s+\lambda} \underline{K}_1 + \frac{1}{(s+\lambda)^2} \underline{K}_2 + \frac{1}{(s+\lambda)^3} \underline{K}_3.$$

Again the problem is to obtain suitable dyadic expansions of  $\underline{K}_1$ ,  $\underline{K}_2$  and  $\underline{K}_3$ :

$$\underline{K}_1 = \sum_1^{k_1} \hat{\alpha}_i \gg \alpha_i, \quad \underline{K}_2 = \sum_1^{k_2} \hat{\beta}_i \gg \beta_i, \quad \underline{K}_3 = \sum_1^{k_3} \hat{\gamma}_i \gg \gamma_i$$

$\{\alpha_i\}_1^{k_1}$   $\left( \{\beta_i\}_1^{k_2}, \{\gamma_i\}_1^{k_3}, \text{ respectively} \right)$  is a basis for  $R(\underline{K}_1^*) \left( R(\underline{K}_2^*), R(\underline{K}_3^*), \text{ respectively} \right)$ .

- (i) Let  $B_{123}$  be a basis for the subspace common to  $R(\underline{K}_1^*)$ ,  $R(\underline{K}_2^*)$  and  $R(\underline{K}_3^*)$ .
- (ii) Let  $B_{\bar{1}23}$  be a set of linearly independent vectors such that  $B_{\bar{1}23} \cup B_{123}$  is a basis for  $R(\underline{K}_2^*) \cap R(\underline{K}_3^*)$ .
- (iii) Let  $B_{1\bar{2}3}$  be a set of linearly independent vectors such that  $B_{1\bar{2}3} \cup B_{123}$  is a basis for  $R(\underline{K}_1^*) \cap R(\underline{K}_3^*)$ .
- (iv) Let  $B_{\bar{1}\bar{2}3}$  be a set of linearly independent vectors such that  $B_{\bar{1}\bar{2}3} \cup B_{\bar{1}23} \cup B_{1\bar{2}3} \cup B_{123} \hat{=} B_3$  is a basis for  $R(\underline{K}_3^*)$ .
- (v) Let  $B_{12\bar{3}}$  be a minimal set of linearly independent vectors which together with some vectors of  $B_3$  forms a basis for  $R(\underline{K}_1^*) \cap R(\underline{K}_2^*)$ .

- (vi) Let  $B_{\bar{1}2\bar{3}}$  be a minimal set of linearly independent vectors which together with some vectors of  $B_3$  and of  $B_{12\bar{3}}$  forms a basis for  $R(\underline{K}_2^*)$ . Let  $B_{2\bar{3}} \triangleq B_{12\bar{3}} \cup B_{\bar{1}2\bar{3}}$ .
- (vii) Let  $B_{1\bar{2}\bar{3}}$  be a minimal set of linearly independent vectors which together with some vectors of  $B_{2\bar{3}}$  and  $B_3$  form a basis for  $R(\underline{K}_1^*)$ .

To each vector in the seven bases defined above corresponds a type of block diagram, some of which are shown on Fig. 5 . The  $\underline{y}_i$ 's used to represent  $\underline{K}_3$  are vectors of  $B_{123}$ ,  $B_{\bar{1}23}$ ,  $B_{1\bar{2}3}$  and  $B_{\bar{1}\bar{2}3}$ ; the  $\underline{\beta}_i$ 's used to represent  $\underline{K}_2$  are the vectors of  $B_{123}$ ,  $B_{\bar{1}23}$  and  $B_{2\bar{3}}$ ; the  $\underline{\alpha}_i$ 's used to represent  $\underline{K}_1$  are the vectors of  $B_{123}$ ,  $B_{1\bar{2}3}$ ,  $B_{12\bar{3}}$  and  $B_{\bar{1}\bar{2}\bar{3}}$ .

The linear independence of the six sets of vectors  $\{\underline{\alpha}_i\}_1^{k_1}$ ,  $\{\hat{\underline{\alpha}}_i\}_1^{k_1}$ ,  $\{\underline{\beta}_i\}_1^{k_2}$ ,  $\{\hat{\underline{\beta}}_i\}_1^{k_2}$ ,  $\{\underline{y}_i\}_1^{k_3}$  and  $\{\hat{\underline{y}}_i\}_1^{k_3}$  implies that the corresponding linear, time-invariant, differential system is completely controllable and completely observable. Hence it is minimal.

For a pole of order  $k$ , the procedure is clear: it requires, however,  $2^k - 1$  bases.

Example: Suppose we are given the matrix transfer function of a system that has two inputs and two outputs. Let the partial fraction expansion of the transfer function be

$$\frac{1}{s+1} \begin{bmatrix} 4 & 7 \\ 5 & 5 \end{bmatrix} + \frac{1}{(s+1)^3} \begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix} \quad (19)$$

The range of  $\underline{K}_3^*$  is spanned by the vector  $\text{col}(1, 3)$ . Therefore we shall use this vector as a row vector of the dyadic representation of  $\underline{K}_1$ . We obtain

$$\underline{K}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [2 \quad 1] + \begin{bmatrix} 2 \\ 1 \end{bmatrix} [1 \quad 3] \quad (20)$$

$$\underline{K}_3 = \begin{bmatrix} 7 \\ 2 \end{bmatrix} [1 \quad 3] \quad (21)$$

The realization of  $\underline{K}_3/(s+1)^3$  requires three integrators, but the first integrator may be used to realize the second dyad of  $\underline{K}_1$ . The realization is illustrated on Fig. 6.

If we were to use the first method, then we would observe that  $R(\underline{K}_3)$  is spanned by  $\text{col}(7, 2)$ . Then we pick  $\text{col}(7, 2)$  and  $\text{col}(1, 0)$  as a basis for  $R(\underline{K}_1)$ . The representations are obtained on the basis of a few elementary calculations:

$$\underline{K}_1 = \begin{bmatrix} 7 \\ 2 \end{bmatrix} [2.5 \quad 2.5] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [-13.5 \quad -10.5] ; \quad \underline{K}_3 = \begin{bmatrix} 7 \\ 2 \end{bmatrix} [1 \quad 3] \quad (22)$$

The corresponding realization is shown on Fig. 7.

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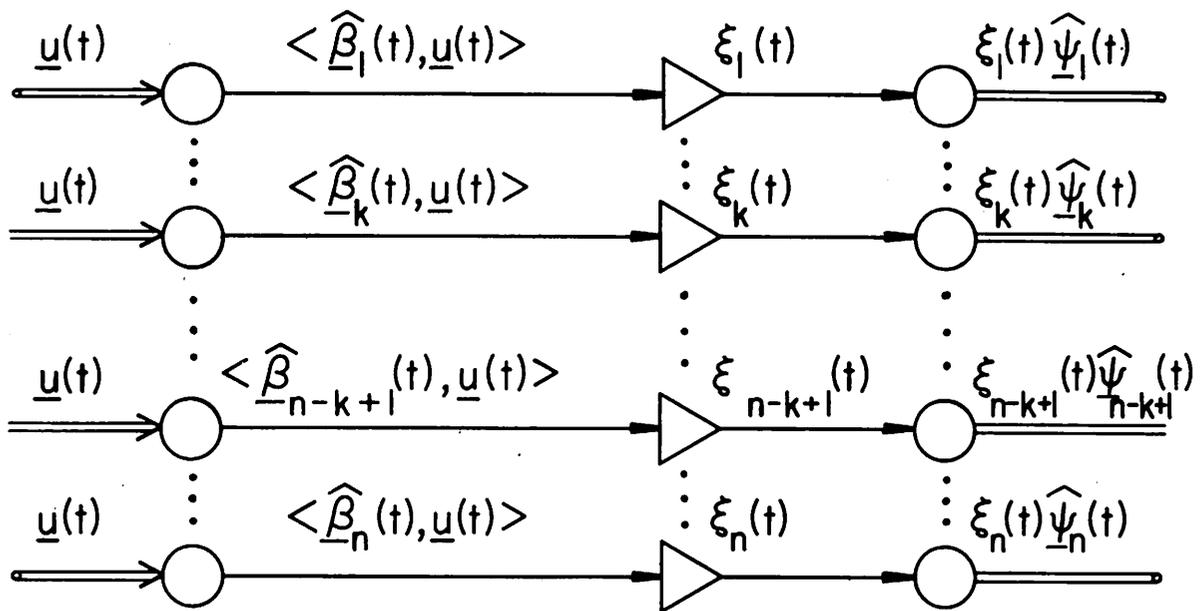


Fig. 1. Analog computer setup to realize the factorization  $\underline{\hat{\psi}}(t) \underline{\hat{\beta}}(t)$ .

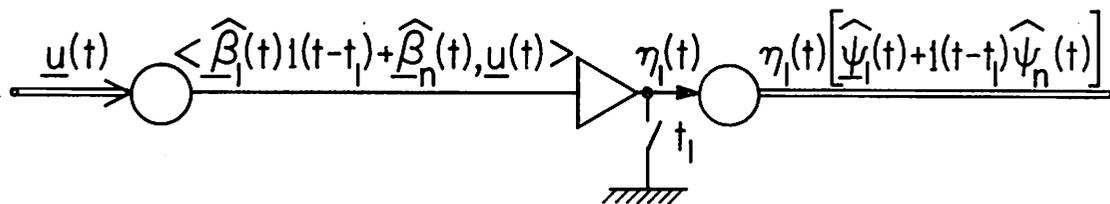


Fig. 2. Realization of the time-shared integrator for the case  $\underline{b}'\underline{c} = 0$ . The switch is closed at time  $t_1$  in order to discharge the integrator capacitor. It is immediately reopened thereafter.

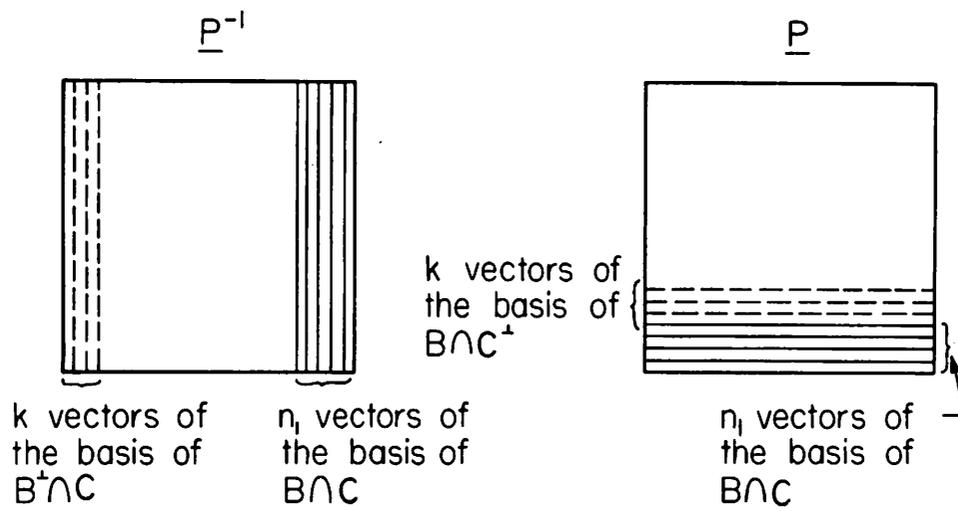


Fig. 3. Illustration showing the choice of columns in  $\underline{P}^{-1}$  and rows in  $\underline{P}$ .

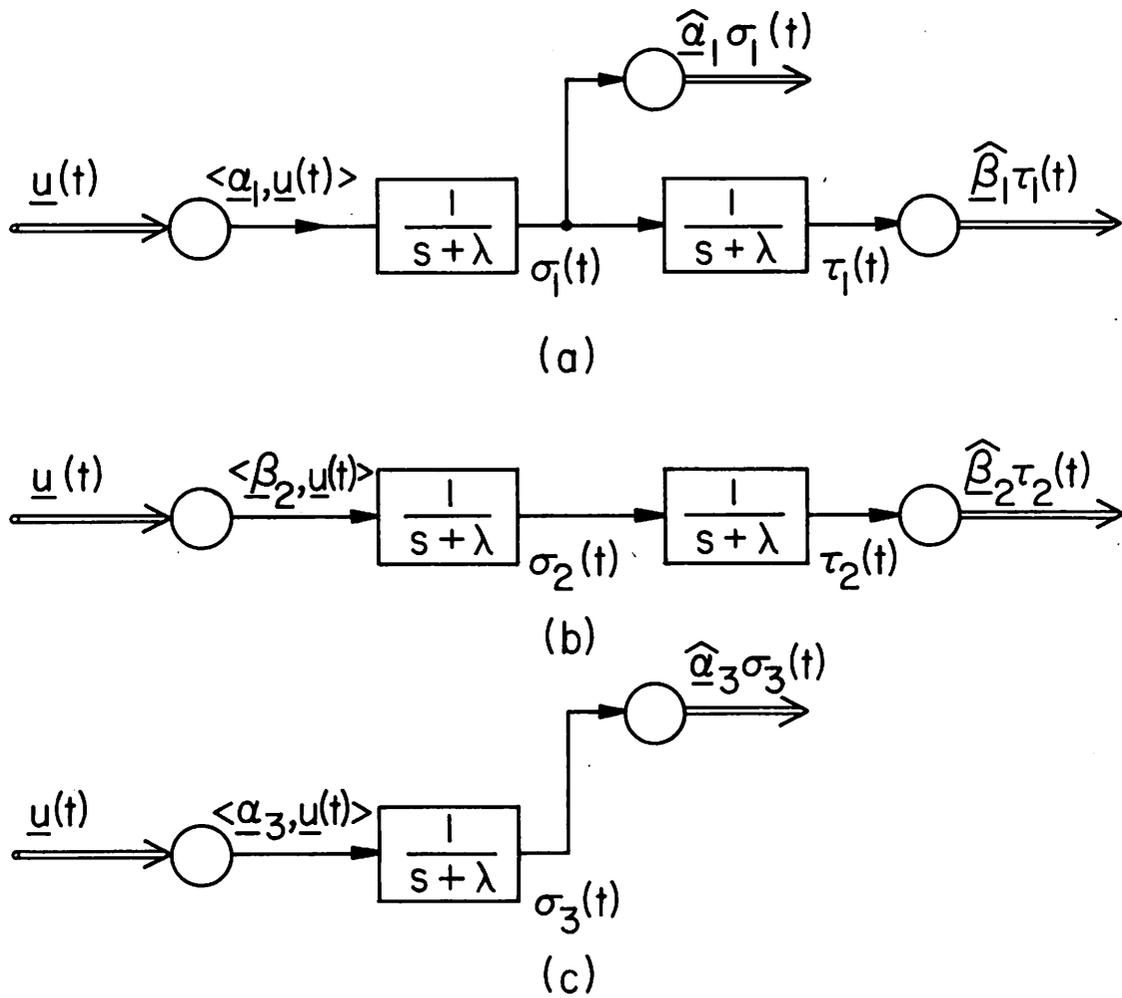


Fig. 4. This figure shows the three possible types of realizations required by a second order pole: the case (a) realizes simultaneously a dyadic term of  $\underline{K}_1$  and a dyadic term of  $\underline{K}_2$ : these two dyads have the same row vector,  $\underline{\alpha}_1$ , which is an element of  $B_{12}$ ; case (b) realizes a dyadic term of  $\underline{K}_2$  whose row vector,  $\underline{\beta}_2$ , is an element of  $B_{12}$ ; case (c) realizes a dyadic term of  $\underline{K}_1$  whose row vector,  $\underline{\alpha}_3$ , is an element of  $B_{12}$ .

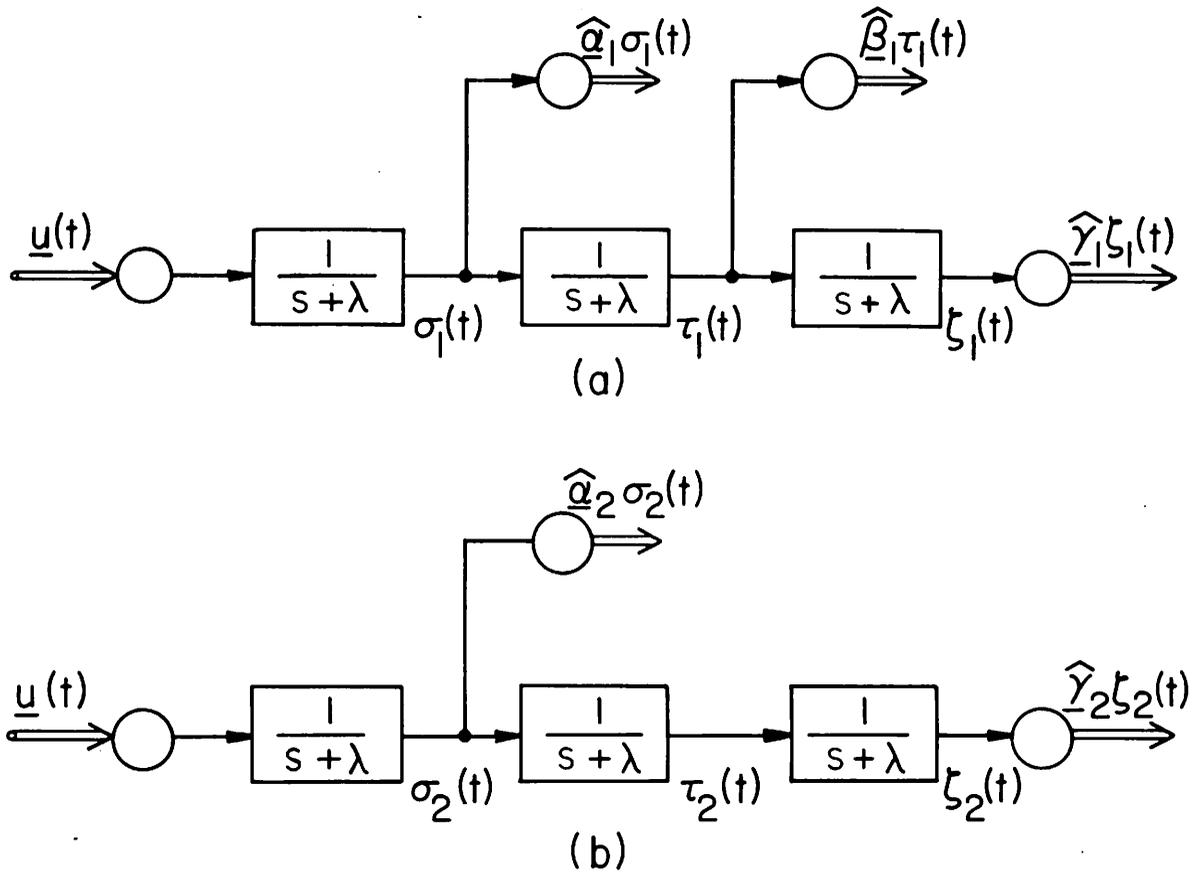


Fig. 5. This figure shows some types of block diagrams required by a third order pole. Case (a) realizes simultaneously a dyadic term of  $\underline{K}_1$ ,  $\underline{K}_2$ ,  $\underline{K}_3$ : each one has the same row vector, a vector of  $B_{123}$ . Case (b) realizes a dyadic term of  $\underline{K}_1$  and  $\underline{K}_3$ : each one has the same row vector, a vector of  $B_{1\bar{2}3}$ .

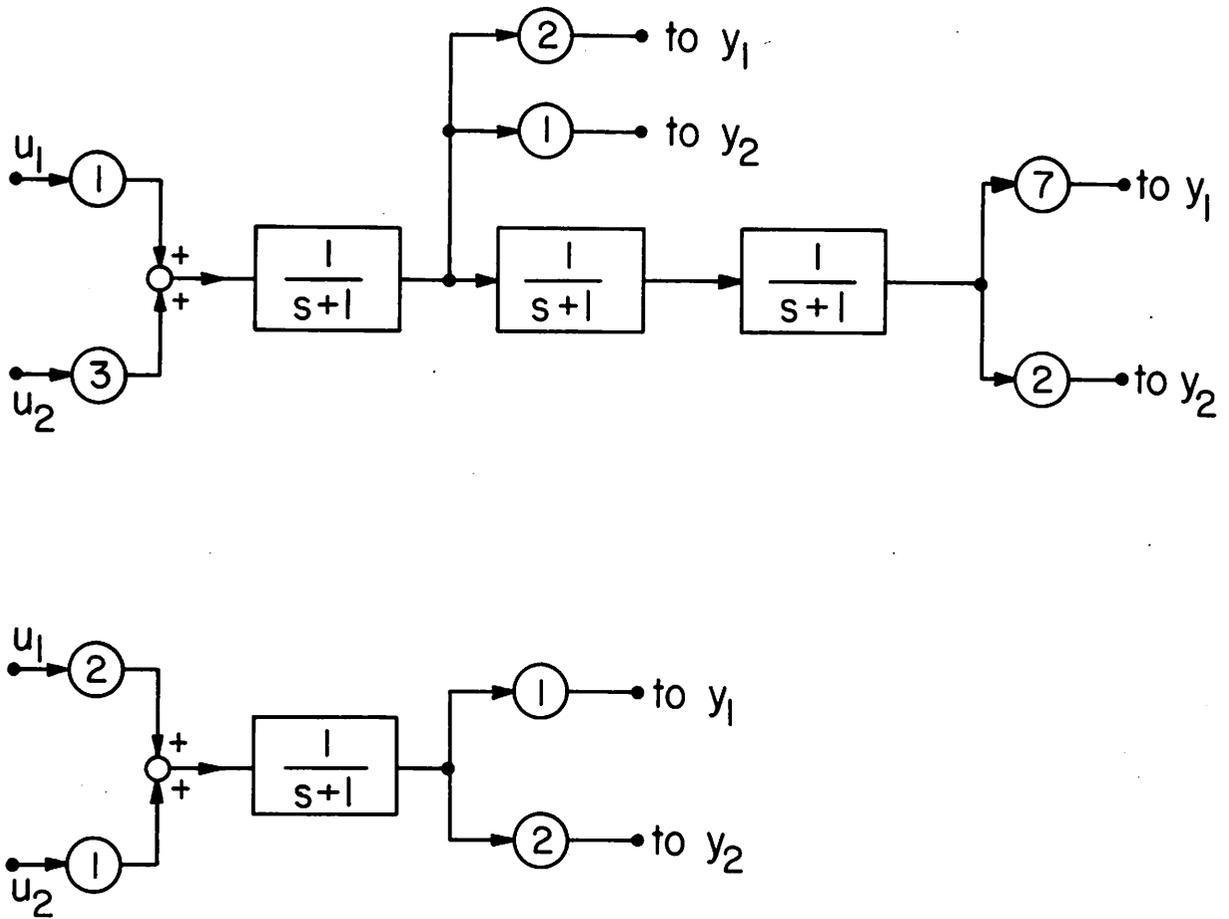


Fig. 6. Realization of the matrix transfer function given by (19). The dyadic expansions used in the realization are given by (20) and (21).

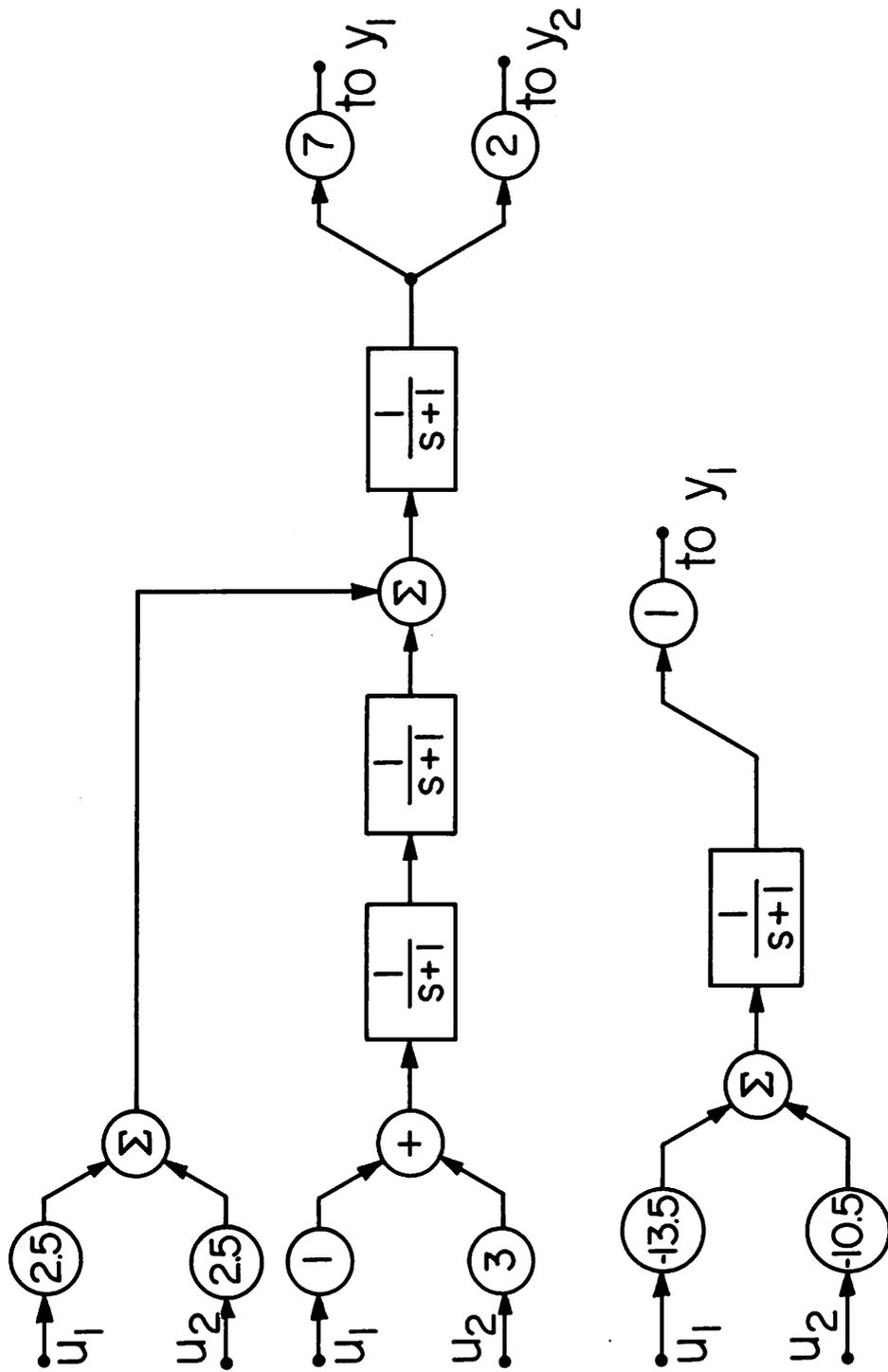


Fig. 7. Realization of the matrix transfer function given by (19) and based on the dyadic expansion (22).