EQUIVALENCE RELATIONS FOR THE CLASSIFICATION AND SOLUTION OF OPTIMAL CONTROL PROBLEMS

by

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INTRODUCTION

This paper is concerned with the use of the concept of equivalence in the study of optimal control problems. The idea of using equivalence relations in the study of problems in system theory is not new, although until recently no apparent attempt had been made to apply this idea to the theory of optimal control. Lately, a number of papers have appeared by Polak [1, 2, 3], Hermes [4], Liu and Leaka [5] in which equivalence relations are defined for optimal control problems and used to obtain theoretical or computational results for broad classes of problems. It is the purpose of this paper to formulate the ideas presented in these papers in a more general form. Actually, because of the type of control problem considered by the authors (the so called "open loop" problem), the definitions of equivalence given in [4] and [5] are not subsumed by the structure developed in this paper. However, it should be clear to the reader that a parallel development for closed loop control problems would unite and generalize the equivalence relations defined in [4] and [5].

It is shown in this paper that equivalence relations of the type defined herein lead to problem classification schemes which are both intuitively appealing and computationally useful. To demonstrate the latter, a new computational procedure for solving optimal control problems is presented and illustrated by examples. Finally, it is hoped that this classification scheme will lead to a greatly improved understanding of the invariant properties of optimal control problems.

THE IDEALIZED PHYSICAL SYSTEM

The mathematical structure for constructing relations between optimal control problems will be based on the idealized regulator system shown in Fig. 1. This system consists of the following elements: a plant, describable by differential or difference equations, a controller, a computer, and a switch (sampler). Let X be the state space of the plant, $T = \{-\infty < t < +\infty\}$, the time axis, and V the space of all possible computer outputs, assumed to be such that $V = W \times T$, where W is a set of quantities whose elements determine the "shape" of the forcing functions produced by the controller. V will be called the control space. When an input $v \in V$ is applied to the controller at time $t = t_0$, it produces a forcing function $u(s; w)$, $0 \leq s \leq \tau_v$, $s = (t - t_0)$, $v = (w, \tau_v)$.

The entire regulator system will be assumed to operate as follows. At time $t = t_0$ the switch closes momentarily, enabling the computer to read the plant state $x(t_0) \in X$, while the time t_0 is supplied by a clock. The computer then produces instantaneously a control v, resulting in a forcing function u which takes the plant state from $x(t_0)$ to a point in a given terminal set $X_f \subset X$.

Clearly, since every feedback control law gives rise to a corresponding open loop control law, this definition of the regulator system does not preclude the control laws implemented by the computer from being feedback laws. However, it will be more convenient for the purpose at hand to consider the system as being open loop.

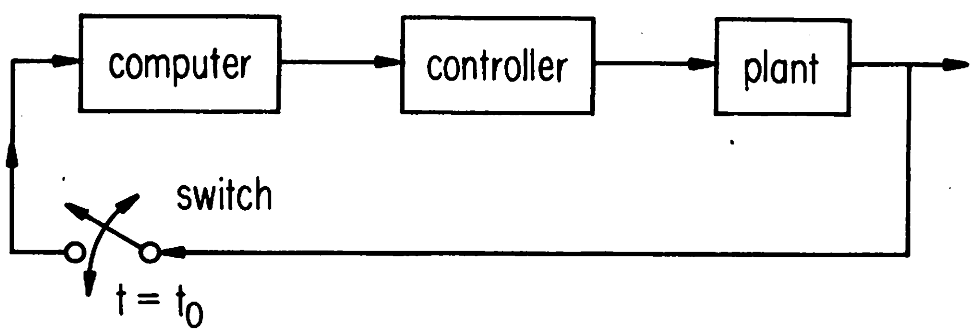

Fig. 1. Idealized regulator system.

AN OPTIMAL CONTROL PROBLEM

An optimal control problem is completely determined by the following seven quantities.

(i) $X^* = X \times T$, the phase space of the system,

(ii) $X_i^* \subset X^*$, the set of initial phases,

(iii) $X_f^* \subset X^*$, the set of terminal phases,

(iv) V, the control space,

(v) $\mathcal{a}: X^* \times V \rightarrow X^*$, the phase transition law of the system, assumed to have the following properties

(a) $\mathcal{a}_v \equiv \mathcal{a}(\cdot, v) : X^* \rightarrow X^*$ is 1-1 and onto for all $v \in V$,

(b) $\mathcal{a}(x_0^*, v_0) = (x_1, t_0 + \tau_0)$, where $x_0^* = (x_0, t_0)$, $v_0 = (w_0, \tau_0)$, i.e., the last component of the image phase is $t_0 + \tau_0$,

(vi) $F_{\mathcal{a}} : X^* \times V \rightarrow R$, a real valued cost functional depending parametrically on the phase transition law.

(vii) $G = \{g : g : X_i^* \rightarrow V, \text{ and } \forall x^* \in X_i^*, \mathcal{a}(x^*, g(x^*)) \in X_f^*\}$, a set of admissible control laws.

The next step is to impose a partial ordering on the set G.

Definition 1: Let g_1, g_2 be any two elements of G. Then $g_1 \leq g_2$ iff $F_{\mathcal{a}}(x^*, g_1(x^*)) \leq F_{\mathcal{a}}(x^*, g_2(x^*))$ for every $x^* \in X_i^*$.
The traditional statement of the optimal control problem can now be enunciated as follows:

Given the seven quantities specified above, find a $g^o \in G$ such that $g^o \leq g$ for every $g \in G$.

Such a g^o will be called an optimal control law. It is quite clear that we could think of an optimal control problem simply as the septuplet $(X^*, X_i^*, X_f^*, V, \mathcal{A}, F_{\mathcal{A}}, G)$, with the task of finding an optimal control $g^o \in G$ always being implied. However, to make the ensuing discussion less cumbersome, we find it convenient to group the first six quantities in the septuplet together as part of the specification of a feasible solution.

Definition 2. Let $\rho = (X^*, X_i^*, X_f^*, V, \mathcal{A}, F_{\mathcal{A}}, g)$, $g \in G$. Then ρ will be called a feasible solution to the optimal control problem specified by $X^*, X_i^*, X_f^*, V, \mathcal{A}, F_{\mathcal{A}}$, and G.

Definition 3. Let $P = \{\rho: \rho = (X^*, X_i^*, X_f^*, V, \mathcal{A}, F_{\mathcal{A}}, g), g \in G\}$. Then, for any $\rho_1, \rho_2 \in P$, we define an order relation between ρ_1 and ρ_2 by

$$\rho_1 \leq \rho_2 \text{ iff } g_1 \leq g_2 .$$

This defines a 1-1, ordered correspondence between feasible solutions, $\rho \in P$, and admissible control laws, $g \in G$.

It is now natural to define an optimal control problem as follows:

Definition 4. An optimal control problem is defined to be a set of feasible solutions, differing only in their control laws, partially ordered according to Definition 3.

The next definition is a logical consequence of the preceding definitions.

Definition 5. A feasible solution, $\rho^o \in P$, is an optimal solution to the optimal control problem P iff $\rho^o \le \rho$ for every $\rho \in P$.

Remark. For any optimal control problem there is always a question of existence of an optimal solution ρ^o. In what follows we shall always assume that an optimal solution exists.

PROPERTIES OF CONTROL LAWS

We now establish some properties of control laws which shall be required later on.

Lemma 1. Consider an optimal control problem P. Let $\rho \in P$ be arbitrary and let g be the corresponding control law. If $X_f^* = \{x_f^*\}$ consists of a single element only, then g is a 1-1 map from X_i^* into V.

Lemma 2. Consider an optimal control problem P. Let $\rho \in P$ be arbitrary and let g be the corresponding control law. If $X_i^* = \{x^* : x^* = (x, t_0), x \in X_i, t_0 \text{ fixed}\}$ and if $X_f^* = \{x^* : x^* = (x_f, t), x_f \text{ fixed}, t_0 \le t < \infty\}$, then g is a 1-1 map from X_i^* into V.

The proofs of both these lemmas follow immediately from the assumed properties of the phase transition law.

EQUIVALENCE RELATIONS FOR OPTIMAL CONTROL PROBLEMS

We now investigate possible ways of defining meaningful equivalence relations on a class of optimal control problems. One such definition, which immediately comes to mind, is to say that two optimal control problems, P_1 and P_2, are equivalent if there exists an isomorphism between the partially ordered sets $\{P_1, \leq\}$ and $\{P_2, \leq\}$, i.e., if there exists a 1-1 correspondence between the solutions of P_1 and P_2 such that if ρ_1^i, ρ_2^i, $i = 1, 2$ are corresponding solutions in P_1 and P_2 respectively, then $\rho_1^1 \leq \rho_1^2$ iff $\rho_2^1 \leq \rho_2^2$. This is clearly an equivalence relation. However, so many widely disparate problems are equivalent under this definition that it makes very little sense. Furthermore, this definition on results in so little structure that it is doubtful that it could lead to any interesting results. In what follows, the authors propose a definition of equivalence which is more satisfying to one's intuition and which at the same time gives a certain amount of useful mathematical structure. This is accomplished by adding to the definition suggested above the condition that the isomorphism be constructed in a certain manner.

EQUIVALENCE

Let $\mathcal{P}$ be a class of optimal control problems and let P_1, P_2 be any two problems in this class, with corresponding subscripts identifying all of the significant quantities of P_1 and P_2. Let R(G) = g_1 $R(G_1)=\bigcup g_1(X_{i1}^*)\subset V_1$, and $R(G_2)=\bigcup g_2(X_{i2}^*)\subset V_2$, with the unions taken over all $g_i \in G_i$, $i = 1, 2$.

Definition 6. We shall say that P_1 is equivalent to P_2, written $P_1 \sim P_2$, iff there exist two maps φ_{12} and ψ_{12} satisfying

(a) $\varphi_{12} : X_1^* \to X_2^*$, 1-1 and onto, and

(i) $\varphi_{12}(X_{i1}^*) = X_{i2}^*$,

(ii) $\varphi_{12}(X_{f1}^*) = X_{f2}^*$;

(b) $\psi_{12} : R(G_1) \to R(G_2)$, 1-1 and onto;

such that the map π_{12} with domain G_1, which is induced by φ_{12}, ψ_{12} according to the relation

$$\pi_{12}(g_1)(x_2^*) = \psi_{12}(g_1(\varphi_{12}^{-1}(x_2^*))), \quad x_2^* \in X_{i2}^*, \quad g_1 \in G_1,$$

(c) maps G_1 onto G_2 in a 1-1 manner, and

(d) induces an isomorphism between the partially ordered sets $\{P_1, \leq\}$ and $\{P_2, \leq\}$. †

Remark. It is trivial to verify that this relation is an equivalence relation.

At first glance, this definition may seem rather complicated and artificial to the reader. However, a little contemplation reveals that it is simply an extension of an intuitive idea of generating an equivalent optimal control problem by making a change of variables on the phase space and/or the control space. This is best illustrated by an example.

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† The map π_{12} induces a correspondence between solutions of P_1 and P_2 by assigning to every solution $\rho_1 \in P_1$ with control law g_1 the solution $\rho_2 \in P_2$ with control law $\pi_{12}(g_1)$.

Example 1. Consider two problems P_1 and P_2 defined as follows. For P_1 the phase transition law is determined by the linear differential equation of the plant and the characteristics of the controller

$$\dot{x}_1 = Ax_1 + bu \tag{1}$$

where $x \in E^n$, A is a constant $n \times n$ matrix, b is a constant n vector, and u is the scalar valued output of the controller, satisfying the condition

$$|u(s;v)| \leq 1, \quad 0 \leq s \leq \tau, \quad \text{for all} \quad v \in V_1.$$

Hence

$$\mathcal{Q}(x_0^*, v) = (e^{A\tau}(x_0 + \int_0^\tau e^{-sA} bu(s;v)\, ds), t_0 + \tau). \tag{2}$$

The final and initial sets of phases are defined by

$$X_{f1}^* = \{(0, t_f)\}, \quad \text{a single point}$$

$$X_{i1}^* = \{(x_1, t_0), \ x_1 \in X_{i1}, \ t_0 \leq t_f \ \text{fixed}\}.$$

The set X_{i1} is the set of all states which can be taken to zero by means of admissible forcing functions u in the time $t_f - t_0$.

The cost functional is defined by

$$F_1 \mathcal{Q}_1(x^*, v) = F(v) = \int_0^\tau |u(s;v)|\, ds, \quad \text{for all} \quad x^* \in X_1^*, \ v \in V_1.$$

For P_2, the state transition law is determined by the time varying vector differential equation

(3) $$\dot{x}_2 = C(t)\, x_2 + d(t)\, u$$

where $x_2 \in E^n$, $C(t) = L^{-1}(t)\, AL(t) - L^{-1}(t)\, \dot{L}(t)$, $d(t) = L^{-1}(t)\, b$, and $L(t)$ is an $n \times n$ matrix, with bounded components, whose derivative $\dot{L}(t)$ exists and has bounded components. In addition $L(t)$ is such that $L(t_0) = I$, the identity matrix, and $|\det L(t)| \geq m > 0$ for all $t \in T$. The final and initial phase sets are defined by

$$X^*_{i2} = X^*_{i1}$$

$$X^*_{f2} = X^*_{f1} \,.$$

The cost functional $F_{2\alpha_2} = F$, defined above, $V_2 = V_1$, and $G_2 = G_1$.

Clearly P_2 has been obtained from P_1 by making the change of variables $x_2(t) = L^{-1}(t)\, x_1(t)$. The reader may verify that these two problems are equivalent with φ_{12} defined by

$$\varphi_{12}(x_1^*) = \varphi_{12}(x_1,\ t_1)) = (L^{-1}(t_1)\, x_1,\ t_1) \quad \text{for all } x_1^* \in X_1^*,$$

and the map ψ_{12} taken as the identity map.

The equivalence relation established above partitions the class $\mathcal{P}$ of optimal control problems into equivalence classes in a very desirable manner. It is reasonably clear that if the equivalence maps, φ_{0i}, ψ_{0i}, connecting a problem P_0 with problems P_i in the same equivalence class are known, then, by solving one problem, one has in fact solved the entire class of equivalent problems. Furthermore, one may also single out and examine sets of solutions in each problem with the same order properties. For example, one may use iterative techniques, such as the steepest descent method, to obtain a sequence of solutions $\{\rho_i^n\}$, in a problem P_1, whose costs, for a given initial phase, converge to the cost of an optimal solution in P_1. If P_2 is a problem equivalent to P_1, then, under the assumptions stated in the lemma below, the image sequence of solutions $\{\rho_2^n\}$ also has the property that, for the image initial phase, the associated sequence of costs converges to the optimal cost.

Let P_1 and P_2 be two equivalent problems under the equivalence maps φ_{12} and ψ_{12} and let g_1^o and g_2^o be optimal laws for P_1 and P_2 respectively with $g_2^o = \pi_{12}(g_1^o)$. Fix $x_1^* \in X_{i1}^*$, and let $x_2^* = \varphi_{12}(x_1^*)$.

<u>Lemma 3</u>. If $F_{\alpha_i}(x_i^*, g_i^o)$, $i = 1, 2$ are cluster points† of the sets $\{F_{\alpha_i}(x_i^*, g_i) = g_i \in G_i\}$, $i = 1, 2$, and $\{\rho_1^n\}$ is any sequence of solutions in P_1, with image sequence $\{\rho_2^n\}$ in P_2, then $F_{\alpha_1}(x_1^*, g_1^n) \downarrow F_{\alpha_1}(x_1^*, g_1^o)$ iff $F_{\alpha_2}(x_2^*, g_2^n) \downarrow F_{\alpha_2}(x_2^*, g_2^o)$.

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† The point x is said to be a cluster point of the set K if every neighborhood of x contains a point of K different from x.

<u>Proof:</u> $\Rightarrow$. The order preserving property of the isomorphism plus the existence of an optimal cost guarantee that $F_{\alpha_2}(x_2^*, g_2^n)$ converges. If $F_{\alpha_2}(x_2^*, g_2^n) \downarrow C > F_{\alpha_2}(x_2^*, g_2^o)$, then there exists a $\hat{g}_2$ with $F_{\alpha_2}(x_2^*, g_2^o) < F_{\alpha_2}(x_2^*, \hat{g}_2) < C$ because $F_{\alpha_2}(x_2^*, g_2^o)$ is a cluster point. Similarly, there exists an N such that $F_{\alpha_1}(x_1^*, g_1^o) < F_{\alpha_1}(x_1^*, g_1^N) < F_{\alpha_1}(x_1^*, \pi_{12}^{-1}(\hat{g}_2))$. This implies that $F_{\alpha_2}(x_2^*, g_2^N) < C$, a contradiction. The implication in the other direction can be proven in a similar manner.

EQUIVALENCE UNDER OPTIMAL CONTROLS

It is reasonably clear that if one is interested in optimal control problem classification schemes depending only on the nature of the optimal solutions, then it is excessive to require that all solutions of one problem have correspondingly ordered images in the other problems belonging to the same equivalence class. We shall therefore confine our attention to the subsets formed by the optimal solutions of the problems under consideration.

Let G be the set of control laws associated with the optimal control problem P. The set $G^o \subset G$ consisting of all the optimal control laws g^o in G will be said to be the set of optimal control laws for the problem P. We now introduce a classification scheme depending on optimal solutions only.

<u>Definition 7.</u> Let P_1 and P_2 be two optimal control problems and let $P_1^o \subset P_1$, $P_2^o \subset P_2$ be nonempty subsets consisting of all their

respective optimal solutions. The problem P_1 will be said to be optimal control equivalent to the problem P_2, written $P_1 \overset{o}{\sim} P_2$, iff $P_1^o \sim P_2^o$, i.e., iff P_1 is equivalent to P_2 when the admissible control law sets G_1, G_2 are reduced to the optimal control law sets G_1^o, G_2^o respectively.

Remark. It is readily seen that the relation $\overset{o}{\sim}$ is symmetric, reflexive and transitive and that it is therefore a true equivalence relation. It will also be observed that condition (d) in Definition 6 is satisfied trivially in the case of optimal control equivalence and hence need not be checked.

By relaxing the conditions under which two problems will be considered equivalent, we have introduced a significantly more useful equivalence relation. To illustrate the nature of optimal control equivalence, we consider the following example.

Example 2.

Problem (a)

Given: $\dot{x}_{a1} = x_{a2}$

$\dot{x}_{a2} = u_a, \quad |u_a| \le 1$

$x_a = x_{ao}$ at $t = 0$

Find: an admissible forcing function $t \rightarrow u_a(t)$ such that $x_{ao} \rightarrow 0$ in minimum time.

Problem (b)

Given: $\dot{x}_{b1} = x_{b2}$

$\dot{x}_{b2} = -2x_{b2} - (x_{b1} - \frac{1}{2}\text{Arctan } x_{b1}) + u_b$

$|u_b| \le 1, \quad -\pi/2 < \text{Arctan } x < \pi/2$

$x_b = x_{bo}$ at $t = 0$

Find: an admissible forcing function $t \rightarrow u_b(t)$ such that $x_{bo} \rightarrow 0$ in minimum time.

It is well known [7] that the optimal forcing functions for problem (a) are "bang-bang" with at most one switching, and Lee and Markus [6] have proved the same to be true for problem (b). If one examines the sets of optimal trajectories in the state plane for these two problems, one is immediately led to the idea that the optimal solutions are "equivalent."

More formally, it is clear that if $X_a^* = X_b^* = E^2 \times T$, $X_{ia}^* = X_{ib}^* = E^2 \times \{0\}$, $X_{fa}^* = X_{fb}^* = \{0\} \times T^+$, where $T^+ = \{t : 0 \le t < \infty\}$, and if $V_a = V_b = \{v : v = (t_1, t_2, \tau), -\infty < t_1 < \infty, -\infty < t_2 > \infty$, $\tau = |t_1| + |t_2|\}$, with $u_j(t; v)$ given by

(4) $$u_j(t;v) = \begin{cases} \operatorname{sgn} t_1 > 0 & t < |t_1| \\ \operatorname{sgn} t_2 > |t_1| \le t < \tau \end{cases}, \quad j = a, b$$

then $g_a^o(X_{ia}^*) = g_b^o(X_{ib}^*) \subset V$. Consequently, $P_a \overset{o}{\sim} P_b$ under the equivalence maps $\psi_{ab} = I$ and $\varphi_{ab} = g_b^{o\,-1} \cdot g_a^o$, where $g_b^{o\,-1}$ exists by virtue of Lemma 2.

The same reasoning may be used to establish that a wide class of minimum time problems with second order nonlinear plants are optimal control equivalent to a "second order integrator" problem (problem (a)). In particular, see Example 4.

Synthesis of Optimal Control Laws

An inherent property of the equivalence relations exhibited so far is that, the equivalence maps may be used to find an optimal solution for any problem in the equivalent class, whenever an optimal solution to one problem is known. This raises the possibility of obtaining a computational method for determining optimal control laws for a whole class of problems by solving the simplest problem in the class. It is shown below that it is indeed possible to solve certain optimal control problems in this manner. However, before demonstrating this, we first introduce a relation between optimal control problems which is still weaker than optimal control equivalence. This relation has the property that it may be used to synthesize optimal control laws in exactly the same way as the other relations.

Definition 8. Let P_1 and P_2 be two optimal control problems and let $P_1^o \subset P_1$, $P_2^o \subset P_2$ be nonempty subsets consisting of all their respective optimal solutions. The problem P_1 will be said to be weak optimal control equivalent to the problem P_2, written $P_1 \overset{w.o.}{\sim} P_2$, iff there exist nonempty subsets $P_{11}^o \subset P_1^o$, $P_{21}^o \subset P_2^o$ such that $P_{11}^o \sim P_{21}^o$.

Remark. The above relation between problems, which, for lack of a better term, we shall call weak optimal control equivalence, is actually not an equivalence relation. It is symmetric and reflexive, but, in general, not transitive.

Remark. It is clear from the definitions that

$$\{P_1 \sim P_2\} > \{P_1 \overset{o}{\sim} P_2\} > \{P_1 \overset{w.o.}{\sim} P_2\} .$$

The reason for introducing the concept of weak-optimal-control-equivalence in the discussion of the synthesis of optimal control laws is that it exhibits all the desirable properties of the other equivalence relations, while possessing two additional advantages. The first advantage is that there are two methods by means of which weak-optimal-control-equivalence is easily established. The first of these methods applies to the class or problems for which the range of optimal control law is known, for example due to the Pontryagin Maximum Principle. The second method applies to the class of optimal control problems for which it is relatively easy to construct isocost sets in the phase space. Second, the authors have found that it is possible to construct "prototype" problems whose optimal solutions can be determined by inspection, and which are weak-optimal-control-equivalent to certain class of optimal control problems. Generally, these "prototype" problems are not optimal-control-equivalent to the problems to which they are weak-optimal-control-equivalent.

The application of equivalence concepts to the synthesis of optimal control laws rests essentially on the following two theorems.

Theorem 1. Let P_1, P_2 be two optimal control problems with identical finite dimensional Euclidean phase spaces, i.e., $X_1^* = X_2^*$. Let $\rho_1^o \in P_1$, $\rho_2^o \in P_2$ be optimal solutions and let g_1^o, g_2^o be the optimal control laws associated with ρ_1^o, ρ_2^o, respectively. If for $k = 1, 2$, (a1) Either the terminal phase sets $X_{fk}^* = \{x_{fk}^*\}$ consist of a single point only, or

(a2) the initial phase sets are contained in hyperplanes $t = t_{0k}$, i.e., $X_{ik}^* = \{x^*, x^* = (x, t_{0k}), x \in X_{ik}, t_{0k} \text{ fixed}\}$, and the terminal phase sets consist of a half line: $X_{fk}^* = \{x^* : x^* = (x_{fk}, t), x_{fk} \text{ fixed}, t_{0k} \leq t < \infty\}$.

(b1) Either $X_{i1}^* \cap X_{f1}^* = X_{i2}^* \cap X_{f2}^* = \phi$, and there exists a map $\psi_{12} = R(g_1^o) \to R(g_2^o)$, 1-1 and onto, or

(b2) $X_{i1}^* \cap X_{f1}^* \neq \phi$ and $X_{12}^* \cap X_{f2}^* = \phi$, and there exists a map $\psi_{12} = R(g_1^o) \to R(g_2^o)$, 1-1 and onto such that $\psi_{12}(g_1^o(X_{i1}^* \cap X_{f1}^*)) = g_2^o(X_{i2}^* \cap X_{f2}^*)$,

then $P_1 \overset{w.o.}{\sim} P_2$.

<u>Proof.</u> Due to the assumptions (a1) and (a2), it is clear that the conditions of either Lemma 1 or Lemma 2 are satisfied, and hence that the optimal control laws g_1^o, g_2^o are both 1-1. Let $g_1^{o^{-1}} : R(g_1^o) \to X_{i1}^*$, $g_2^{o^{-1}} : R(g_2^o) \to X_{12}^*$ be their respective inverses.

Also by assumption, $X_1^* = X_2^*$ and (due to conditions (a1) and (a2) which state that the terminal phase sets are both either a point or a half line) X_{f1}^*, X_{f2}^* can be brought into 1-1 correspondence. Hence there exists an affine map $\hat{\varphi}_{12} = X_1^* \to X_2^*$, 1-1 and onto and such that $\hat{\varphi}_{12} = (X_{f1}^*) = X_{f2}^*$.

Since the map ψ_{12} exists by assumption, it is only necessary to construct a map φ_{12} such that φ_{12}, ψ_{12} are a pair of equivalence maps under which $P_1 \overset{w.o.}{\sim} P_2$. Let $\varphi_{12}: X_1^* \rightarrow X_2^*$ be defined as follows:

$$\varphi_{12}(x_1^*) = \begin{cases} g_2^{o^{-1}} \cdot \psi_{12} \cdot g_1^o(x_1^*) \text{ for all } x_1^* \in X_{i1}^* \\ \hat{\varphi}_{12}(x_1^*) \text{ for all } x_1^* \in X_{i1}^{*c} \end{cases} \tag{5}$$

Due to the nature of ψ_{12}, $\varphi_{12}(X_{i1}^*) = X_{i2}^*$ and due to the assumptions in (b1) and (b2), $\varphi_{12}(X_{f1}^*) = X_{f2}^*$. Clearly φ_{12} is 1-1 and onto from X_1^* onto X_2^*. Furthermore, the image of g_1^o under the induced map π_{12} is

$$\begin{aligned} \pi_{12}(g_1^o) &= \psi_{12} \cdot g_1^o \cdot \varphi_{12}^{-1} \\ &= \psi_{12} \cdot g_1^o \cdot g_1^{o^{-1}} \cdot \psi_{12}^{-1} \cdot g_2^o \text{ on } X_{i2}^* \\ &= g_2^o \text{ on } X_{i2}^* . \end{aligned} \tag{6}$$

Hence the maps φ_{12}, ψ_{12} are a pair of equivalence maps under which $P_1 \overset{w.o.}{\sim} P_2$.

We shall now give an example in which the above theorem is used to show that the problems in a class of minimum time optimal control problems with third order nonlinear plants are each weak optimal control equivalent to a minimum time optimal-control problem with a third order linear plant whose eigenvalues are real. Thus, many methods which have been proposed for the solution of the linear time optimal control problem can be extended to this class of nonlinear time optimal control problems.

Example 3. (Time Optimal Control of a Class of Third Order Nonlinear Systems).

Consider the class $\mathcal{P}$ of problems whose systems can be represented by the black diagram in Fig. 2, where N is a differentiable function with

(7)

$$\text{(a)} \quad N(0) = 0,$$

$$\text{(b)} \quad N'(z) > 0 \text{ for every } z,$$

and

$$\lambda_2 \neq \lambda_3; \lambda_1, \lambda_2, \lambda_3 < 0.$$

One is required in each case, to bring the system from an arbitrary state to the origin in minimum time, subject to the constraint $|u| \leq 1$. The system can also be represented by a block diagram as shown in Fig. 3, and the state equations corresponding to this form are

(8)

$$\dot{x}_1 = \lambda_1 x_1 + N(x_2 + x_3)$$

$$\dot{x}_2 = \lambda_2 x_2 + (\lambda_2 - \lambda_3) u$$

$$\dot{x}_3 = \lambda_3 x_3 - (\lambda_2 - \lambda_3) u.$$

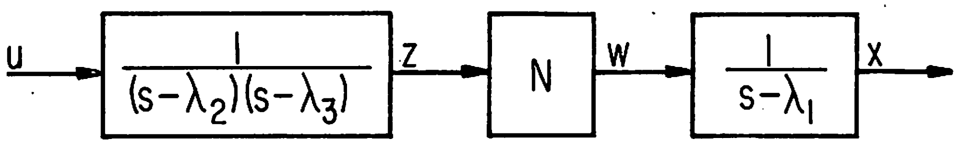

Fig. 2. Block diagram of system of Example 3.

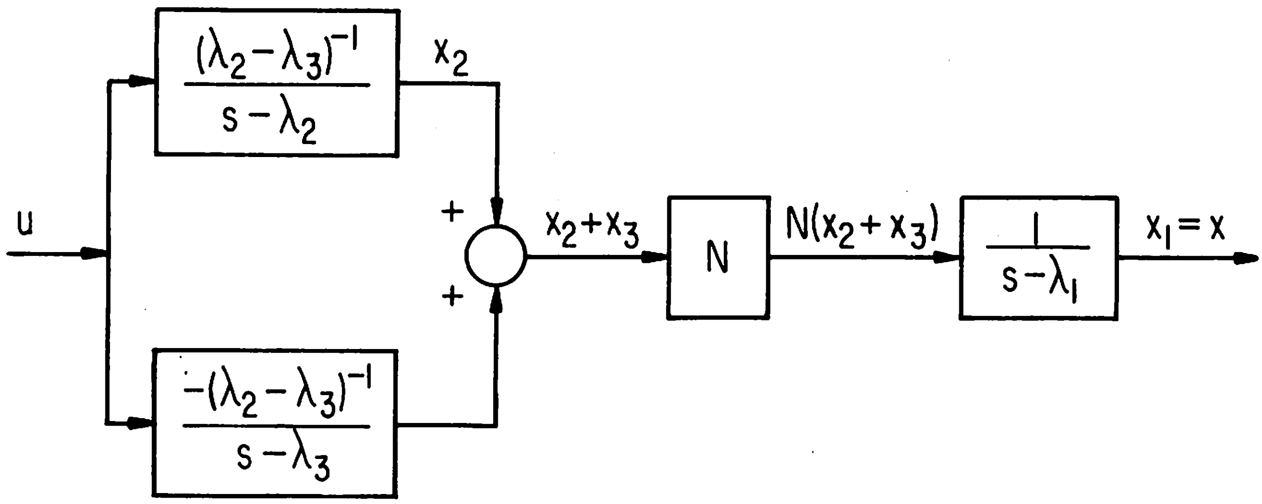

Fig. 3. Modified block diagram of system of Example 3.

Applying Pontryagin's Maximum Principle to this problem, it is easy to show [8] that every optimal forcing function is bang-bang, with at most two switchings. It is equally simple to show that, if the Maximum Principle is also a sufficient condition for optimality, then every bang-bang control with at most two switchings is optimal (equivalently, it is sufficient to show that no two bang-bang controls with at most two switchings bring the same initial state to the origin). It is intuitively obvious, but very difficult to prove, that there are problems in the class $\mathcal{P}$ (with nonlinear plants) for which the range of the optimal control law is the entire set of bang-bang functions with at most two switchings. Therefore, we simply let $\hat{\mathcal{P}}$ be the subclass of $\mathcal{P}$ consisting of problems for which the range of the optimal control law is the entire class of bang-bang functions with at most two switchings.

Now, let P_1 and P_2 be any two problems in $\hat{\mathcal{P}}$. Clearly, we may take ψ_{12} to be the identity map in (b2) of Theorem 1. The rest of (b2) is satisfied since $X_{i1}^* \cap X_{f1}^* = X_{i2}^* \cap X_{f2}^* = \{0\}$, and the time optimal control is the zero control in every case. Condition (a2) is obviously satisfied in this case, and, consequently, $P_1 \overset{w.o.}{\sim} P_2$ by Theorem 1. Indeed, since the optimal control law is unique in every case, the problems in $\hat{\mathcal{P}}$ are all optimal control equivalent, and, therefore, $\hat{\mathcal{P}}$ is an equivalence class. Note that $\hat{\mathcal{P}}$ contains all the problems whose plants are described by linear third order differential equations with real eigenvalues.

Let P_1 be a problem in $\hat{\mathcal{P}}$ with a linear, real eigenvalue plant. By construction, $\varphi_{12} = g_2^{o^{-1}} \cdot g_1^o$, where $g_2^{o^{-1}}$ is the inverse of the optimal control law for P_2 in $\hat{P}$, and $g_2^{o^{-1}}$ can be determined explicitly by solving the plant equation of P_1. Thus, if the optimal control law for P_1 can be determined by some method, then φ_{12} is known explicitly. In fact, φ_{12} can be shown to be a homeomorphism for the class of problems considered. Consequently, knowledge of the optimal control law for the problem with the linear plant determines explicitly the equivalence maps, which relate this problem to all the other problems in $\hat{\mathcal{P}}$ and, moreover, these maps have nice properties.

We shall now examine optimal control problems for which isocost phase sets are relatively easy to construct. Let P be an optimal control problem and let $\rho^o \in P$ be an optimal solution with the associated optimal control law g^o and cost functional $F_{\mathcal{a}}$. The subset of initial phases

$$X_c^* = \{x^* : x^* \in X_i^*,\ F_{\mathcal{a}}(x^*,\ g^o(x^*)) = c,\ c \in R^1\}$$

will be called the c-minicost set. Clearly, X_c^* is the c-isocost set under the optimal control law g^o. The c-minicost sets of a given optimal control law g^o are obviously independent of the particular optimal control law g^o used for their definition. Furthermore, they can often be constructed without the knowledge of an optimal law (see [1], [2], [3]). In such cases the following theorem has been found of value.

<u>Theorem 2.</u> Let P_1, P_2 be two optimal control problems with identical phase control spaces, i.e., $X_1^* = X_2^*$, $V_1 = V_2$, and whose cost functionals have the same form: for k = 1, 2

$$F_{k\mathcal{Q}_k}(x^*, v) = \int_0^{\tau} f(u_k(s;w))\,ds, \quad x^* \in X^*_{ik}, \quad v \in V_k, \quad v = (w, \tau),$$

where u_k is the forcing function produced by the controller of the problem P_k, and f is a scalar valued cost function such that the integral is well defined, and satisfies the condition

$$f(u_1(s;w)) = f(u_2(s;w)), \quad v \in V = V_1 = V_2, \quad 0 \leq s \leq \tau.$$

If there exists a map $\varphi_{12} : X^*_1 \to X^*_2$, 1-1 and onto, such that

(a) $\varphi_{12}(X^*_{i1}) = X^*_{i2}$ (Initial phase sets),

(b) $\varphi_{12}(X^*_{f1}) = X^*_{f2}$ (terminal phase sets),

(c) $\varphi_{12}(X^*_{c1}) = X^*_{c2}$ (c-minicost sets),

(d) for some optimal solution $\rho^o_1 \in P_1$ with associated optimal control law g^o_1, the image control law defined by

$$g^o_2 = g^o_1 \cdot \varphi^{-1}_{12}$$

is a control law in G_2,

then φ_{12}, I (the identify map) are a pair of equivalence maps such that $P_1 \overset{w.o.}{\sim} P_2$, and the control law $g^o_2 = g^o_1 \cdot \varphi^{-1}_{12}$ is an optimal control law.

Proof: We only need to show that the control law g_2^o defined in (d) is optimal, since it is then immediately obvious that the postulated maps φ_{12}, I are indeed a satisfactory pair of equivalence maps. Let x_2^* be an arbitrary point in X_{i2}^*. Hence $x_2^* \epsilon X_{c2}^*$ for some c. It follows from condition (c) that $\varphi_{12}^{-1}(x_2^*) \epsilon X_{c1}^*$. Let $g_1^o(\varphi_{12}^{-1}(x_2^*)) = v$. Then, by definition,

$$g_2^o(x_2^*) = g_1^o(\varphi_{12}^{-1}(x_2^*)) = v,$$

and
$$F_{2\alpha_2}(x_2^*, v) = F_{1\alpha_1}(\varphi_{12}^{-1}(x_2^*), v) = c,$$

i.e., the cost for any initial phase $x^* \epsilon X_{i2}^*$, resulting from the control law g_2^o, is equal to the optimal cost. Hence g_2^o is an optimal control law, and $P_1 \overset{w.o.}{\sim} P_2$.

This theorem was used by one of the authors (see [1], [2], [3]) to construct an optimal control law for minimum time and minimum fuel problems with pulse-width modulation controllers from weak-optimal-control-equivalent problems with pulse-amplitude modulation controllers. The minicost sets were constructed by a method related to dynamic programming.

The second advantage mentioned previously can best be illustrated by an example. It is well known that for a large class of second order nonlinear systems the problem of bringing the system from an arbitrary initial state to the origin in minimum time, with bounded scalar control, has the following unique solution:

Every optimal forcing function is bang-bang with at most one switching, and every bang-bang forcing function with at most one switching is uniquely optimal for the state which it brings to the origin.

One such system was given in Example 2(b). A whole class of such problems is given in the following example.

Example 4. The problems in the class, $\mathcal{P}$, considered here have plants whose state equations take the form

$$\dot{x}_1 = f(x_2) \tag{9}$$
$$\dot{x}_2 = u$$

where f(·) is assumed to be a single valued differentiable function with

(10)

(a) $f(0) = 0$

(b) $f'(z) > 0$ for every z.

One is required in each case to bring the system from an arbitrary initial state to the origin in minimum time, subject to the constraint $|u| \leq 1$.

The reader can easily verify that all the problems in Example 4 have unique optimal solutions of the type mentioned above. Now consider the following problem.

Example 5. (Problem (c))

Given: $$\dot{x}_{c1} = \operatorname{sgn} x_{c2} = \begin{cases} 1, & x_{c2} > 0 \\ 0, & x_{c2} = 0 \\ -1, & x_{c2} < 0 \end{cases}$$

(11)

$$\dot{x}_{c2} = u_c \,, \quad u_c \in \{1, 0, -1\}$$

$$x_c = x_{c0} \text{ at } t = 0$$

Find: an admissible forcing function $t \to u_c(t)$ such that $x_{c0} \to 0$ in minimum time

The possible trajectories for problem (c) are all piecewise linear. We can determine optimal trajectories by inspection, and for almost all initial states there are infinitely many optimal trajectories. Figure 4 illustrates the different types of optimal trajectories for a typical initial state. Among the possible optimal forcing functions for any initial state there is always exactly one which is bang-bang with at most one switching. In fact, this forcing function is given by

(12) $$u(t) = \begin{cases} \operatorname{sgn} t_1, & 0 \le t \le |t_1| \\ \operatorname{sgn} t_2, & |t_1| \le t < |t_1| + |t_2| \end{cases} .$$

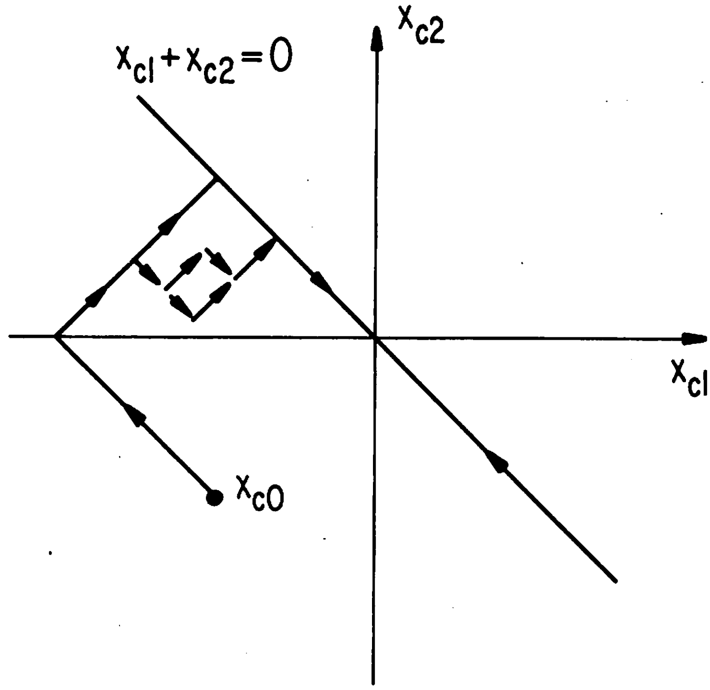

Fig. 4. Optimal trajectories for problem (c) of Example 5. The dotted paths are alternative optimal trajectories for x_{co}.

where

$$\text{(13a)} \qquad t_1 = \begin{cases} -(\frac{1}{2}|x_{c2}| + \frac{1}{2}x_{c1}) - x_{c2}, & \text{if } (x_{c1} + x_{c2}) > 0, \\ (\frac{1}{2}|x_{c2}| - \frac{1}{2}x_{c1}) - x_{c2}, & \text{if } (x_{c1} + x_{c2}) < 0, \\ 0, & \text{if } (x_{c1} + x_{c2}) = 0, \end{cases}$$

$$\text{(13b)} \qquad t_2 = \begin{cases} (\frac{1}{2}|x_{c2}| + \frac{1}{2}x_{c1}), & \text{if } (x_{c1} + x_{c2}) > 0, \\ -(\frac{1}{2}|x_{c2}| - \frac{1}{2}x_{c1}), & \text{if } (x_{c1} + x_{c2}) < 0, \\ -x_{c2}, & \text{if } (x_{c1} + x_{c2}) = 0. \end{cases}$$

Furthermore, the set of forcing functions defined by the above expression for arbitrary initial states is the set of all bang-bang forcing functions with at most one switching.

We may now apply Theorem 1 to show that problem (c) and any one of the other problems mentioned above are weak optimal control equivalent. Clearly, these problems can not be optimal control equivalent since problem (c) has infinitely many optimal solutions while the other problems have unique optimal solutions. However, weak-optimal-control equivalence together with an explicit expression for the optimal control law of problem (c) allow us to determine the optimal control law for any problem of the type specified in Example 4.

Let P_1 be any problem of the type specified in Example 4, and let P_2 be problem (c). Let g_1 be the optimal control law for P_1, and let g_2 be the optimal control law for P_2 given by (13). Then, by Theorem 1, $P_1 \overset{w.o.}{\sim} P_2$ with $\varphi_{12} = g_2^{-1} \cdot g_1$ and $\psi_{12} = I$. Given any $x_1^* \in X_{i1}^*$, we have $g_1(x_1^*) = g_2[\varphi_{12}(x_1^*)]$. Therefore, if we can find $x_2^* = \varphi_{12}(x_1^*)$, we can find $g_1(x_1^*)$. To do this we need to solve

$$\varphi_{12}^{-1}(x_2^*) = g_1^{-1} \cdot g_2(x_2^*) = x_1^*$$

for x_2^*. The functions g_2 and g_1^{-1} are known and, furthermore, for all the problems in Example 4, $\varphi_{12}^{-1} = g_1^{-1} \cdot g_2$ is a homeomorphism and piecewise $C^{(1)}$. Thus, it is possible to solve for x_2^* iteratively. A complete calculation is carried out in the example below.

Example 6. Consider the particle moving in one-dimension according to the equation

(14) $$\frac{d}{dt}(m\dot{y}) = u$$

where y is the position of the particle, u is the applied force, and

(15) $$m = \frac{100}{\sqrt{10^4 - y^2}} \quad \text{for } |\dot{y}| < 100$$

We assume that the force is constrained by $|u| \leq 1$, and the initial velocity satisfies $|y| < 100$. We are required to bring the system to rest at the origin from an arbitrary initial position and an arbitrary initial velocity in the range $|\dot{y}| < 100$ in minimum time.

If we make the substitution $x_1 = y$, $x_2 = p = m\dot{y}$, then the system is described by

$$\dot{x}_1 = f(x_2) = \frac{100\,x_2}{\sqrt{10^4 + x_2^2}} \tag{16}$$

$$\dot{x}_2 = u$$

Inspecting the form of the system equations (16), we see that this problem falls into the class of problems considered in Example 4 et. seq.

Instead of using this form, the authors chose as state variables the quantities $z_1 = y$, $z_2 = \dot{y}$, yielding the equations

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = 10^{-6}\left[10^4 - z_2^2\right]^{1.5} u \tag{17}$$

with the initial phase set $z_i = \{z = -100 < z_2 < 100\}$. Optimal forcing functions for this system are independent of the choice of state variables, and hence we can still find the equivalence map φ_{12}^{-1} from the state space of problem (c) to the set z_i according to

$$\varphi_{12}^{-1} = g_1^{-1} \cdot g_2$$

where g_2 is given by (13), and g_1^{-1} is obtained by integrating (17) backward in time from the origin.

In evaluating g_1, it is sufficient to restrict our attention to the shaded regions of Fig. 5, since the map for the rest of the state space can be obtained by symmetry arguments.

Let (y_1, y_2) be an initial state for problem (c) (P2). Then the map $(z_1, z_2) = \varphi_{12}^{-1}(y_1, y_2)$ is given by

(a) <u>For (y_1, y_2) in Region 1</u>

$$z_1 = \frac{1}{4}\left[\frac{(y_1 - y_2)^2}{1 + \left[1 + \left(\frac{y_1 - y_2}{100}\right)^2\right]^{1/2}} - \frac{(3y_2 - y_1)(y_2 + y_1)}{\left[1 + \left(\frac{y_2}{100}\right)^2\right]^{1/2} + \left[1 + \left(\frac{y_1 - y_2}{200}\right)^2\right]^{1/2}}\right]$$

(18)

$$z_2 = \frac{y_2}{\left[1 + \left(\frac{y_2}{100}\right)^2\right]^{1/2}}$$

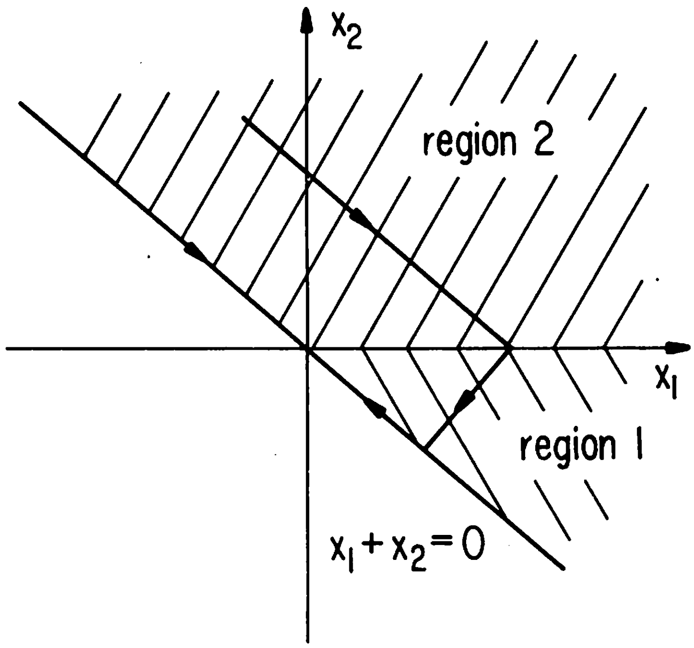

Fig. 5. State space and minimum time switching line for prototype problem of Example 5.

(b) <u>For (y_1, y_2) in Region 2</u>

$$z_1 = \frac{1}{4}\left[\frac{(y_1+y_2)^2}{1+\left[1+\left(\frac{y_1+y_2}{200}\right)^2\right]^{1/2}} - \frac{(3y_2+y_1)(y_2-y_1)}{\left[1+\left(\frac{y_2}{100}\right)^2\right]^{1/2}+\left[1+\left(\frac{y_1+y_2}{200}\right)^2\right]^{1/2}}\right]$$

(19)

$$z_2 = \frac{y_2}{\left[1+\left(\frac{y_2}{100}\right)^2\right]^{1/2}}$$

It is not difficult to verify that φ_{12}^{-1} is indeed a homeomorphism and $C^{(1)}$ everywhere except on the lines $y_2 = 0$ and $y_1 + y_2 = 0$. The equations given here were used in conjunction with an IBM 7094 digital computer to compute $g_1(z)$ according to the formula

$$g_1(z) = g_2(\varphi_{12}(z)).$$

The inversion of φ_{12}^{-1} was accomplished numerically using a modified Newton-Raphson method. Table 1 gives results for various initial states together with the computation time required.

TABLE 1

Computational Results for Example 6

Desired Initial State		Computed Initial State		Optimal Control Law			Comp. Time
ZO(1)	ZO(2)	Z1(1)	Z1(2)	U1	T1	T2	(sec.)
1.0	1.0	1.0059	1.0000	-1	2.2772	1.2272	0.036
10.0	10.0	10.0743	10.0000	-1	17.8314	7.7810	0.036
0.0	50.0	0.0009	50.0000	-1	97.8204	40.0854	0.036
50.0	50.0	50.1494	50.0000	-1	98.4890	40.7545	0.018
50.0	0.0	50.0215	0.0000	-1	7.0770	7.0770	0.036
50.0	-50.0	49.5777	-50.0000	+1	97.1493	39.4143	0.036
90.0	90.0	89.7729	90.0000	-1	337.9153	131.4411	0.084
0.0	95.0	-0.0060	95.0000	-1	489.0511	184.8076	0.234
100.0	95.0	99.6597	95.0000	-1	489.6175	185.3740	0.318
100.0	0.0	100.6797	0.0000	-1	10.0466	10.0466	0.018
100.0	-95.0	100.9541	-95.0000	+1	488.4770	184.2335	0.198
1000.0	95.0	1005.3431	95.0000	-1	494.7471	190.5036	0.048
1000.0	0.0	1004.7053	0.0000	-1	32.0927	32.0927	0.048
1000.0	-95.0	1006.0459	-94.9912	+1	482.8140	178.9100	0.036

CONCLUSION

This paper has attempted to answer the question of whether optimal control problems can be classified in a manner which is both intuitively appealing and computationally useful.

For this purpose, three relations, defined either on all the admissible solution sets, or only on subsets of the optimal solutions sets, were exhibited. The first two of these selections, equivalence and optimal control equivalence, are true equivalence relations. It was shown by means of a number of examples, either worked in this paper or cited from the literature, that the optimal control problems classifications in which they result are highly nontrivial, and that the associated mathematical structure can be quite useful in the construction of algorithms for finding optimal solutions. The third solution, weak-optimal-control equivalence, is reflective and symmetric, but not transitive, and hence it is not a true equivalence relation. Although it is not as useful for classification as the other two relations described, it is by far the most powerful one when applied to the construction of algorithms.

To facilitate the use of equivalence relations in the construction of algorithms, the authors introduce the concept of a prototype problem. This is usually an artificial problem, which can be solved in a very simple manner and which is related to the problem one wishes to solve. It is shown by means of an example how prototypes can be used to obtain algorithms for solving optimal control problems.

Finally, this paper has exhibited what the authors hope will be a new point of view to many who are working in the field of optimal control.

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