Copyright © 1966, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

CONSTRAINED MINIMIZATION UNDER VECTOR-VALUED CRITERIA IN FINITE DIMENSIONAL SPACES

roll 3

roll down 5

by

roll 4

N. O. Da Cunha

1/2

E. Polak

Memorandum No. ERL-M188

roll 1

31 October 1966

ELECTRONICS RESEARCH LABORATORY

roll 3

College of Engineering
University of California, Berkeley
94720

Manuscript submitted: 29 September 1966

The research reported herein was supported wholly by the National Aeronautics and Space Administration under Grant NsG-354, Supplement 3.

INTRODUCTION

Many problems in economics as well as in engineering and in mathematical programming have nonunique solutions, and one is therefore presented with the freedom to seek out optimal solutions. When there is only one criterion of optimality, which is relevant to the problem, we are faced with a straightforward optimization problem. However, when there are several criteria, all of which are important, and whose importance cannot be ordered, the matter becomes considerably more complicated, since we then have a vector-valued criterion optimization problem.

Judging from the literature on the subject, the first formulation of a vector-valued criterion optimization problem is due to the economist V. Pareto in 1896 [1]. Since then, discussions of this problem have kept reappearing in the economics literature (see Kuhn and Tucker [2], Karlin [3], Debreu [4] and, more recently, in the control engineering literature (see Zadeh [5], Chang [6]).

Although the vector-valued criterion formulation of an optimization problem is frequently much closer to reality than a formulation with a scalar valued criterion, very few results have been obtained to date, that shed light on vector-valued optimization problems. The present paper is devoted to developing a broad theory of necessary conditions for characterizing noninferior points, and to determining when a vector-valued criterion problem can be treated as a family of problems with scalar valued criteria.

I. Necessary Conditions for the Basic Problem

Let $f : E^n \to E^p$, $r : E^n \to E^m$ be continuously differentiable functions and let Ω be a subset of E^n. (The function f is the vector-valued performance criterion, while the function r specifies equality constraints.)

1. Basic Problem: Find a point $\hat{x}$ in E^n such that

2. (i) $\hat{x} \in \Omega$ and $r(\hat{x}) = 0$,

3. (ii) for every x in Ω with $r(x) = 0$, the relation $f(x) \leq f(\hat{x})$* (component-wise) implies that $f(x) = f(\hat{x})$.

The solutions $\hat{x}$ of the Basic Problem, i.e., those $\hat{x}$ satisfying (2) and (3), are often referred to as <u>noninferior points</u> [5]. It can easily be shown [7] that they usually constitute an uncountable set of points.

Before we can obtain necessary conditions for a point $\hat{x}$ in E^n to be a solution to the Basic Problem, we must introduce an approximation to the set Ω at $\hat{x}$.

4. <u>Definition:</u> A subset $C(\hat{x}, \Omega)$ of E^n will be called a <u>linearization</u> of the set Ω at $\hat{x}$ if

5. (i) $C(\hat{x}, \Omega)$ is a convex cone,

---

* We use the following notation. For any vector y_1, y_2 in E^p, $y_1 \leqq y_2$ if and only if $y_1^i \leqq y_2^i$ for $i = 1, 2, \cdots, p$; $y_1 \leq y_2$ if and only if $y_1 \neq y_2$ and $y_1 \leqq y_2$; $y_1 < y_2$ if and only if $y_1^i < y_2^i$ for $i = 1, 2, \cdots, p$.

6. (ii) for any finite collection $\{x_1, x_2, \cdots, x_k\}$ of linearly independent vectors in $C(\hat{x}, \Omega)$, there exist a positive scalar ϵ_0 and a continuous map ζ from $\mathrm{co}\{\epsilon x_1, \cdots, \epsilon x_k\}$, the convex hull of $\{\epsilon x_1, \cdots, \epsilon x_k\}$, with $0 < \epsilon \leqq \epsilon_0$, into $\Omega - \{\hat{x}\}$ of the form:

$$\zeta(\delta x) = \delta x + o(\delta x) \text{ for all } \delta x \in \mathrm{co}\{\epsilon x_1, \cdots, \epsilon x_k\},\ 0 < \epsilon \leqq \epsilon_0$$

where the function $o(\cdot)$ is such that $\lim_{\|y\| \to 0} \frac{\|o(y)\|}{\|y\|} = 0$.

An important special case of a linearization is one where the map ζ is the identity map, i.e., $\mathrm{co}\{\epsilon x_1, \cdots, \epsilon x_k\}$ is contained in $\Omega - \{\hat{x}\}$ for $0 < \epsilon \leqq \epsilon_0$. We call this special case a <u>linearization of the</u> first kind.

7. <u>Theorem</u>: Let $\hat{x}$ be a solution to the Basic Problem and let $C(\hat{x}, \Omega)$ be a linearization for Ω at $\hat{x}$. Then, there exist a vector μ in E^p and a vector η in E^m such that

8. (i) $\mu^i \leqq 0$, $i = 1, 2, \cdots, p$,

9. (ii) $(\mu, \eta) \neq 0$,

10. (iii) $\left\langle \mu, \frac{\partial f(\hat{x})}{\partial x} x \right\rangle + \left\langle \eta, \frac{\partial r(\hat{x})}{\partial x} x \right\rangle \leqq 0$ for all $x \in \overline{C(\hat{x}, \Omega)}$,

where $\overline{C(\hat{x}, \Omega)}$ is the closure of $C(\hat{x}, \Omega)$.

Proof: Let

11. $$A(\hat{x}) = \left\{ y \in E^p \,\middle|\, y = \frac{\partial f(\hat{x})}{\partial x} x,\ x \in C(\hat{x}, \Omega) \right\},$$

12. $$\hat{B}(\hat{x}) = \left\{ z \in E^m \,\middle|\, z = \frac{\partial r(\hat{x})}{\partial x}\, x,\ x \in C(\hat{x}, \Omega) \right\},$$

13. $$K(\hat{x}) = \left\{ u \in E^p \times E^m \,\middle|\, u = \left(\frac{\partial f(\hat{x})}{\partial x}\, x,\ \frac{\partial r(\hat{x})}{\partial x}\, x \right),\quad x \in C(\hat{x}, \Omega) \right\}.$$

Since the Jacobian matrices $\frac{\partial f(\hat{x})}{\partial x}$ and $\frac{\partial r(\hat{x})}{\partial x}$ define linear maps, $A(\hat{x})$, $B(\hat{x})$, and $K(\hat{x})$ are convex cones in E^p, E^m, and $E^p \times E^m$, respectively. Clearly, $K(\hat{x}) \subset A(\hat{x}) \times B(\hat{x})$.

Let C and R be the convex cones in E^p and $E^p \times E^m$, respectively, defined by

14. $$C = \{y = (y^1, \cdots, y^p) \in E^p \mid y^i < 0,\ i = 1, 2, \cdots, p\},$$

15. and $$R = \{(y, 0) \in E^p \times E^m \mid y \in C,\ 0 \in E^m\}.$$

Examining (9) and (10), we observe that the claim of the theorem is that the sets $K(\hat{x})$ and R are separated in $E^p \times E^m$. We now construct a proof by contradiction.

Suppose that $K(\hat{x})$ and R are not separated in $E^p \times E^m$. We then find that the following two statements must be true.

16. I. The smallest linear variety containing the union of R and $K(\hat{x})$ is the entire space $E^p \times E^m$, and $R \cap K(\hat{x}) \neq \phi$, the empty set.

17. II. The convex cone $B(\hat{x})$ in E^m, contains the origin as an interior point and since $B(\hat{x})$ is a convex cone, $B(\hat{x}) = E^m$.

This follows from the fact that if 0 is not an interior point of the convex set $B(\hat{x})$, then by the separation theorem,* it can be separated from $B(\hat{x})$ by a hyperplane in E^m, i.e., there exists a nonzero vector η_0 in E^m such that

$$\langle \eta_0, z \rangle \leqq 0 \quad \text{for all} \quad z \in B(\hat{x}) .$$

Clearly, the vector $(0, \eta_0)$ in $E^p \times E^m$ separates R from $A(\hat{x}) \times B(\hat{x})$ and hence from $K(\hat{x})$ contradicting our assumption that they are not separated.

We now proceed to utilize facts I and II. Since the origin in E^m belongs to the nonvoid interior of $B(\hat{x}) = E^m$ (see II), let us construct a simplex Σ in $B(\hat{x})$, with vertices $z_1, z_2, \cdots, z_{m+1}$ such that

18. (i) 0 is in the interior of Σ;

19. (ii) there exists a set of vectors $\{x_1, x_2, \cdots, x_{m+1}\}$ in $C(\hat{x}, \Omega)$ satisfying:

20. (a) $z_i = \dfrac{\partial r(\hat{x})}{\partial x} x_i$ for $i = 1, 2, \cdots, m+1$:

21. (b) $\zeta(x) = x + o(x) \in \{\Omega - \{\hat{x}\}\}$ for all $x \in \text{co}\{x_1, x_2, \cdots, x_{m+1}\}$, where ζ is the map entering the definition of $C(\hat{x}, \Omega)$, see (4).

---

* See [8] p. 118, 2.22. Corollary to the Hahn-Banach Theorem

22. (c) The points $y_i = \frac{\partial f(\hat{x})}{\partial x} x_i$ are in C for $i = 1, 2, \cdots, m+1$.

The existence of such a simplex is easily established. First, we construct any simplex Σ' in $B(\hat{x})$ with vertices $z_1', z_2', \cdots, z_{m+1}'$, which contains the origin in its interior. This is clearly possible since $B(\hat{x}) = E^m$ by (17). Let $x_1', x_2', \cdots, x_{m+1}'$ be any set of points in $C(\hat{x}, \Omega)$ which satisfy (20), i.e., $z_i' = \frac{\partial r(\hat{x})}{\partial x} x_i'$, $i = 1, 2, \cdots, m+1$. If $\frac{\partial f(\hat{x})}{\partial x} x_i' < 0$ for $i = 1, 2, \cdots, m+1$, then (22) is satisfied and we can satisfy (21) by letting $x_i = \epsilon x_i'$, for some $\epsilon > 0$, and still satisfy (18), (20), and (22). But suppose, without loss of generality, that $\frac{\partial f(\hat{x})}{\partial x} x_1' \geqq 0$ and $\frac{\partial f(\hat{x})}{\partial x} x_i' < 0$ for $i = 2, 3, \cdots, m+1$. Since by (16) $K(\hat{x}) \cap R \neq \phi$, there exists a point $u = \left(\frac{\partial f(\hat{x})}{\partial x} \tilde{x}, 0 \right) \in K(\hat{x}) \cap R$, i.e., $\frac{\partial f(\hat{x})}{\partial x} \tilde{x} < 0$ and $\frac{\partial r(\hat{x})}{\partial x} \tilde{x} = 0$. Choose any scalar $\lambda > 0$ such that $\frac{\partial f(\hat{x})}{\partial x} (\lambda x_1' + (1-\lambda) \tilde{x}) < 0$, and let $x_1 = \lambda x_1' + (1-\lambda)\tilde{x}$. Then the simplex Σ with vertices $\epsilon \lambda z_1', \epsilon z_2', \cdots, \epsilon z_{m+1}'$, satisfies conditions (18) and (19) (a), (b), and (c) for the corresponding vectors $x_1, x_2', x_3', \cdots, x_{m+1}'$ and some $\epsilon > 0$.

It is easy to show that (18) implies that the vectors $(z_1 - z_{m+1})$, $(z_2 - z_{m+1}), \cdots, (z_m - z_{m+1})$ are linearly independent. Consequently, since $\frac{\partial r(\hat{x})}{\partial x}$ is a linear map, the vectors $(x_1 - x_{m+1}), (x_2 - x_{m+1}), \cdots,$ $(x_m - x_{m+1})$ are also linearly independent. Let Z be the nonsingular $m \times m$ matrix whose columns are $(z_1 - z_{m+1}), (z_2 - z_{m+1}), \cdots, (z_m - z_{m+1})$ and let X be the $n \times m$ matrix whose columns are $(x_1 - x_{m+1}), (x_2 - x_{m+1}),$ $\cdots, (x_m - x_{m+1})$. Then $z \rightarrow XZ^{-1}(z - z_{m+1}) + x_{m+1}$ is a continuous map from Σ into $co\{x_1, x_2, \cdots, x_{m+1}\}$.

Now, for $0 < \alpha \leqq 1$, let S_α be a sphere in E^m with radius $\alpha\rho$ (where $\rho > 0$), center at the origin, and contained in the interior of the simplex Σ.

Next we define a continuous map G_α from the sphere S_α into E^m by

23. $$G_\alpha(\alpha z) = r(\hat{x} + \zeta(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1}))$$

$$= r(\hat{x} + \alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1} + o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1})),$$

where $||z|| \leqq \rho$, $\alpha z \epsilon S_\alpha$, and ζ is the map associated with the linearization $C(\hat{x},\Omega)$. Since r is continuously differentiable, we can expand the right-hand side of (23) about $\hat{x}$ to obtain:

24. $$G_\alpha(\alpha z) = r(\hat{x}) + \alpha \frac{\partial r(\hat{x})}{\partial x}(XZ^{-1}(z - z_{m+1}) + x_{m+1}) + o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1}).$$

But $r(\hat{x}) = 0$, $\frac{\partial r(\hat{x})}{\partial x} X = Z$, and $\frac{\partial r(\hat{x})}{\partial x} x_{m+1} = z_{m+1}$. Hence, (24) becomes

25. $$G_\alpha(\alpha z) = \alpha z + o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1}).$$

Now, since $\lim_{\alpha \to 0} \frac{||o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1})||}{\alpha} = 0$, there exists for $||z|| = \rho$, an $\bar{\alpha}_0$, $0 < \bar{\alpha}_0 \leqq 1$, such that:

26. $$||o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1})|| < \alpha\rho, \text{ for all } 0 < \alpha \leqq \bar{\alpha}_0 \text{ and } ||z|| = \rho.$$

By assumption, f is differentiable, hence we can expand each component of f about $\hat{x}$ as follows:

27. $f^i(\hat{x} + \zeta(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1})) =$

$$f^i(\hat{x}) + \alpha \frac{\partial f^i(\hat{x})}{\partial x}[XZ^{-1}(z - z_{m+1}) + x_{m+1}] + o(\alpha XZ^{-1}(z - z_{m+1}) + \alpha x_{m+1}).$$

$$i = 1, 2, \cdots, p.$$

Since by construction, (see(22)), $\frac{\partial f^i(\hat{x})}{\partial x} x_j < 0$, for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, m+1$, and the point $XZ^{-1}(z - z_{m+1}) + x_{m+1}$ is in $co\{x_1, x_2, \cdots, x_{m+1}\}$, we have $\frac{\partial f^i(\hat{x})}{\partial x}[XZ^{-1}(z - z_{m+1}) + x_{m+1}] < 0$, with $i = 1, 2, \cdots, p$. Hence there exist $\bar{\alpha}_i$, $i = 1, 2, \cdots, p$, such that

28. $f^i(\hat{x} + \alpha(XZ^{-1}(z - z_{m+1}) + x_{m+1})) < f^i(\hat{x})$ for all $0 < \alpha \leqq \bar{\alpha}_i$, $||z|| = \rho$

and $\quad i = 1, 2, \cdots p.$

Let α^* be the minimum of $\{\bar{\alpha}_0, \bar{\alpha}_1, \cdots, \bar{\alpha}_p\}$. It now follows from Brower's Fixed Point Theorem [9] that there exists a point $\alpha^* z^*$

Now, let $x^* = \hat{x} + \zeta(\alpha^* XZ^{-1}(z^* - z_{m+1}) + \alpha^* x_{m+1})$, then

29. (a) $r(x^*) = 0$ (since $r(x^*) = G_{\alpha^*}(\alpha^* z^*) = 0$),

30. (b) $x^* \epsilon \Omega$, since $(x^* - \hat{x}) \varepsilon \zeta(co\{\alpha^* x_1, \alpha^* x_2, \cdots, \alpha^* x_{m+1}\}) \subset \Omega - \{\hat{x}\}$

by construction.

But (28), (29), and (30) contradict the assumption that $\hat{x}$ is a solution to the Basic Problem. Therefore, the convex cones $K(\hat{x})$ and R are separated in $E^p \times E^m$, i.e., there exists a nonzero vector (μ, η) in $E^p \times E^m$ such that

31. (i) $\left\langle \mu, \frac{\partial f(\hat{x})}{\partial x} x \right\rangle + \left\langle \eta, \frac{\partial r(\hat{x})}{\partial x} x \right\rangle \leqq 0$ for all $x \in C(\hat{x}, \Omega)$,

32. (ii) $\langle \mu, y \rangle + \langle \eta, 0 \rangle \geqq 0$ for all $y \in C$.

But (31) implies that

$$\left\langle \mu, \frac{\partial f(\hat{x})}{\partial x} x \right\rangle + \left\langle \eta, \frac{\partial r(\hat{x})}{\partial x} x \right\rangle \leqq 0 \quad \text{for all} \quad x \in \overline{C(\hat{x}, \Omega)}$$

and (32) and (14) implies that $\mu^i \leqq 0$, $i = 1, 2, \cdots, p$. Q.E.D.

II. Reduction of a Vector-Valued Criterion to a Family of Scalar-Valued Criteria

An examination of (9) and (10) indicates that if we had used the scalar-valued criterion $\langle -\mu, f(x)\rangle$ instead of the vector-valued criterion $f(x)$ in the definition of the Basic Problem (1), with $\mu \in E^p$ specified by Theorem (7) for the vector-valued criterion, we would have obtained from Theorem (7) exactly the same set of necessary conditions. This observation leads us to the following question: can we obtain the solutions to the Basic Problem (1) by solving a family of scalar-valued criterion problems? A partial answer to this question is given below by Theorems (38) and (41).

To simplify our exposition, we lump the constraint set Ω with the set $\{x \;\; E^n | r(x) = 0\}$. We shall therefore consider a subset A of E^n, a continuous mapping f from E^n into E^p and introduce the following definitions.

33. Definition: We shall denote by P the problem of finding a point $\hat{x}$ in A such that for every x in A, the relation $f(x) \leqq f(\hat{x})$ (componentwise) implies that $f(x) = f(\hat{x})$.

34. Definition: Let Λ be the set of all vectors $\lambda = (\lambda^1, \lambda^2, \cdots, \lambda^p)$ in E^p such that $\Sigma_{i=1}^{p} \lambda^i = 1$ and $\lambda^i > 0$, $i = 1, 2, \cdots, p$.

35. Definition: Given any vector λ in E^p, we shall denote by $P(\lambda)$ the problem of finding a point $\bar{x}$ in A such that $\langle \lambda, f(\bar{x})\rangle \leqq \langle \lambda, f(x)\rangle$ for all x in A.

We shall consider the following subsets of E^n:

36. $L = \{x \in A \mid x \text{ solves } P\}$,

37. $M = \{x \in A \mid x \text{ solves } P(\lambda) \text{ for some } \lambda \in \Lambda\}$.

38. Theorem: The set L contains the set M.

Proof: Suppose that $\bar{x} \in M$ and $\bar{x} \notin L$. Then there must exist a point x' in A such that $f(x') \leq f(\bar{x})$. But for any $\lambda \in \Lambda$, this implies that $\langle \lambda, f(x')\rangle < \langle \lambda, f(\bar{x})\rangle$, and hence $\bar{x}$ is not in M, which is a contradiction.

39. Definition: We shall say that a solution $\hat{x}$ of the problem P is regular if there exists a closed convex neighborhood U of $\hat{x}$ such that for any $y \in A \cap U$ the relation $f(\hat{x}) = f(y)$ implies $\hat{x} = y$.

40. Definition: We shall say that the problem P is regular if every solution of P is a regular solution.

It is easy to verify that if f is convex and one of its components is strictly convex then P is regular.

41. Theorem: Suppose that the problem P is regular, that the performance criterion f is convex (component-wise) and that the constraint set A is closed and convex. Then the set L (36) is contained in the closure of

the set M (37).

Proof: We shall show that for every $\hat{x} \epsilon L$, there exists a sequence of points in M which converges to $\hat{x}$.

We begin by constructing a sequence which converges to an arbitrary, but fixed, $\hat{x}$ in L. We shall then show that this sequence is in M.

Let $\hat{x}$ be any point in L. Since we can translate the origins of E^n and E^p, we may suppose, without loss of generality, that $\hat{x} = 0$ and that $f(\hat{x}) = 0$.

Let U be a closed convex neighborhood of $\hat{x}$ satisfying the conditions of definition (39), and let $N \subset U$ be a compact convex neighborhood of $\hat{x}$. For any positive scalar ϵ, $0 < \epsilon \leqq \frac{1}{p}$, (where p is the dimension of the space containing the range of $f(\cdot)$), let

42. $$\Lambda(\epsilon) = \left\{ \lambda = (\lambda^1, \lambda^2, \cdots, \lambda^p) \middle| \sum_{i=1}^{p} \lambda^i = 1, \ \lambda^i \geqq \epsilon, \ i = 1, 2, \cdots, p \right\}.$$

Let g be the real-valued function with domain $A \cap N \times \Lambda(\epsilon)$, defined by:

43. $$g(\lambda, x) = \langle \lambda, f(x) \rangle.$$

Clearly, since f is convex and hence continuous, g is continuous in $A \cap N \times \Lambda(\epsilon)$, furthermore, g is convex in x for fixed λ and linear in λ for fixed x. Since the sets $A \cap N$ and $\Lambda(\epsilon)$ are compact and convex, the sets

44. $$\{x \varepsilon A \cap N \mid g(\bar{\lambda}, x) = \min_{\eta \varepsilon A \cap N} g(\bar{\lambda}, \eta)\},$$

45. $$\{\lambda \varepsilon \Lambda(\epsilon) \mid g(\lambda, \bar{x}) = \max_{\nu \varepsilon \Lambda(\epsilon)} g(\nu, \bar{x})\},$$

are well defined for every $\bar{\lambda} \varepsilon \Lambda(\epsilon)$ and every $\bar{x} \varepsilon A \cap N$, respectively. Obviously, the sets defined in (44) and (45) are convex.

By K. Fan's Theorem [10],* there exist a point $\lambda(\epsilon)$ in $\Lambda(\epsilon)$ and a point $x(\epsilon)$ in $A \cap N$ such that

---

* K. Fan's Theorem:

Let L_1, L_2 be two separated locally convex, topological, linear spaces, and K_1, K_2 be two, compact convex sets in L_1, L_2, respectively. Let g be a real-valued continuous function on $K_1 \times K_2$. If, for any $x_0 \varepsilon K_1$, $y_0 \varepsilon K_2$ the sets

$$\left\{ x \varepsilon K_1 \mid g(x, y_0) = \max_{\nu \varepsilon K_1} g(\nu, y_0) \right\}$$

and

$$\left\{ y \varepsilon K_2 \mid g(x_0, y) = \min_{\eta \varepsilon K_2} g(x_0, \eta) \right\}$$

are convex, then

$$\max_{x \varepsilon K_1} \min_{y \varepsilon K_2} g(x, y) = \min_{y \varepsilon K_2} \max_{x \varepsilon K_1} g(x, y).$$

46. $\langle \lambda(\epsilon), f(x)\rangle \geqq \langle \lambda(\epsilon), f(x(\epsilon))\rangle \geqq \langle \lambda, f(x(\epsilon))\rangle$

for every x in $A \cap N$ and λ in $\Lambda(\epsilon)$.

Since $\hat{x} = 0$ is in $A \cap N$ and $f(\hat{x}) = 0$, we have from (46):

47. $\langle \lambda(\epsilon), f(x(\epsilon))\rangle \leqq 0$.

And from (46) and (47),

48. $\langle \lambda, f(x(\epsilon))\rangle \leqq 0$ for every λ in $\Lambda(\epsilon)$.

Since $A \cap N$ is compact, we can choose a sequence ϵ_n, $n = 1, 2, \cdots$, with $0 < \epsilon_n \leqq 1/p$, converging to zero in such a way that the resulting sequence of points $x(\epsilon_n)$, satisfying (46), converges, i.e.,

49. $\lim_{n \to \infty} x(\epsilon_n) = x^*$, $x^* \varepsilon A \cap N$.

Since $g(\lambda, x)$ is continuous, it follows from (48) and (49) that

$$\langle \lambda, f(x^*)\rangle \leqq 0 \quad \text{for all} \quad \lambda \varepsilon \Lambda,$$

which implies that $f(x^*) \leqq 0$. But $\hat{x}$ is a solution to P, hence $f(x^*) \leqq 0 = f(\hat{x})$ implies that $f(x^*) = f(\hat{x})$. Consequently, since P is regular, $x^* = \hat{x} = 0$. Thus, we have constructed a sequence, $\{x(\epsilon_n)\}$ which converges to $\hat{x}$.

We shall now show that the sequence $\{x(\epsilon_n)\}$ contains a subsequence $\{x(\epsilon_n)\}$ also converging to $\hat{x}$, which is contained in M.

Since $\hat{x}$ is in the interior of N, there exists a positive integer n_0 such that the points $x(\epsilon_n) \epsilon A \cap N$ belong to the interior of N for $n \geqq n_0$.

We will show that for $n \geqq n_0$, $x(\epsilon_n)$ is a solution to $P(\lambda(\epsilon_n))$, i.e., that for $n \geqq n_0$, $x(\epsilon_n) \epsilon M$. By contradiction, suppose that for $n \geqq n_0$, $x(\epsilon_n)$ is not a solution to $P(\lambda(\epsilon_n))$. Then there must be a point x' in A such that

51. $$\langle \lambda(\epsilon_n), f(x') \rangle < \langle \lambda(\epsilon_n), f(x(\epsilon_n)) \rangle .$$

Let $x''(\alpha) = (1-\alpha) x(\epsilon_n) + \alpha x'_1$, $0 < \alpha < 1$; since A is convex, $x''(\alpha)$ is an A for $0 < \alpha < 1$. But for $n \geqq n_0$, $x(\epsilon_n)$ is in the interior of N and hence there exists an α^*, $0 < \alpha^* < 1$ such that $x''(\alpha^*)$ belongs to N.

Now,

52. $$\langle \lambda(\epsilon_n), f(x''(\alpha^*)) \rangle = \langle \lambda(\epsilon_n), f((1-\alpha^*) x(\epsilon_n) + \alpha x') \rangle .$$

But for $\lambda(\epsilon_n) \epsilon \Lambda(\epsilon_n)$, $\langle \lambda(\epsilon_n), f(x) \rangle$ is convex in x. Hence (51) and (52) imply that

53. $$\langle \lambda(\epsilon_n), f(x''(\alpha^*)) \rangle < \langle \lambda(\epsilon_n), f(x(\epsilon_n)) \rangle ,$$

which contradicts (46).

Therefore, for $n \geq n_0$, $x(\epsilon_n)$ is a solution to $P(\lambda(\epsilon_n))$, i.e., $x(\epsilon_n)$ is in M.

Thus, for any given $\hat{x} \epsilon L$ there exists a sequence $\{x(\epsilon_n)\}$ contained in M such that $x(\epsilon_n) \rightarrow \hat{x}$ as $n \rightarrow \infty$. This completes our proof.

III. Applications to Nonlinear Programming

In nonlinear programming the set Ω is usually defined by a set of inequalities. Thus, let q^i, $i = 1, 2, \cdots, s$ be continuously differentiable functions from E^n into E^1. Then Ω is defined by

54. $$\Omega = \{x \in E^n \,|\, q^i(x) \leqq 0, \; i = 1, 2, \cdots, s\} .$$

55. Basic Nonlinear Programming Problem: We shall refer to the particular case of the Basic Problem (1), arising when the constraint set Ω is defined by (54), as the Basic Nonlinear Programming Problem.

At each point x in Ω, the index set of active constraints is defined as

56. $$I(x) = \{i \,|\, q^i(x) = 0, \; i \in \{1, 2, \cdots, s\}\} .$$

Similarly, the index set of inactive constraints is defined as

57. $$\bar{I}(x) = \{i \,|\, q^i(x) < 0, \; i \in \{1, 2, \cdots, s\}\}.$$

Let $\hat{x}$ be a solution to the Basic Nonlinear Programming Problem. In order to bring the additional structure of the Basic Nonlinear Programming Problem into play, it is convenient to begin by allowing the following assumption, which will subsequently be removed.

58. <u>Assumption</u>: There exists a vector z in E^n such that $\frac{\partial q^i(\hat{x})}{\partial x} z < 0$ for every $i \varepsilon I(\hat{x})$.

Under this assumption, the nonvoid set

$$C(\hat{x}, \Omega) = \left\{ x \varepsilon E^n \,\middle|\, \frac{\partial q^i(\hat{x})}{\partial x} x < 0, \ i \varepsilon I(\hat{x}) \right\}$$

is a linearization of the first kind for Ω at $\hat{x}$, and

$$\overline{C(\hat{x}, \Omega)} = \left\{ x \varepsilon E^n \,\middle|\, \frac{\partial q^i(\hat{x})}{\partial x} x \leqq 0, \ i \varepsilon I(\hat{x}) \right\} .$$

By Theorem (7) there exist vectors μ in E^p and η in E^m such that

(i) $\mu^i \leqq 0, \ i = 1, 2, \cdots, p$,

(ii) $(\mu, \eta) \neq 0$,

(iii) $\sum_{i=1}^{p} \mu^i \frac{\partial f^i(\hat{x})}{\partial x} x + \sum_{i=1}^{m} \eta^i \frac{\partial r^i(\hat{x})}{\partial x} x \leqq 0$ for every

$$x \varepsilon \left\{ x \varepsilon E^n \,\middle|\, \frac{\partial q^i(\hat{x})}{\partial x} x \leqq 0, \ i \varepsilon I(\hat{x}) \right\} .$$

And by Farkas' Lemma [11], there exist scalars $\rho^i \leqq 0$, $i \varepsilon I(\hat{x})$ such that

59. $$\sum_{i=1}^{p} \mu^i \frac{\partial f^i(\hat{x})}{\partial x} + \sum_{i=1}^{m} \eta^i \frac{\partial r^i(\hat{x})}{\partial x} + \sum_{i \in I(\hat{x})} \rho^i \frac{\partial q^i(\hat{x})}{\partial x} = 0.$$

Defining $\rho^i = 0$ for $i \in \bar{I}(\hat{x})$, we have just proved:

60. Theorem: Let $\hat{x}$ be a solution to the Basic Nonlinear Programming Problem. If Assumption (58) holds, then there exist scalars μ^i, $i = 1, 2, \cdots, p$, η^j, $j = 1, 2, \cdots, m$ and ρ^k, $k = 1, 2, \cdots, s$ such that

61. (i) $\mu^i \leqq 0$, $i = 1, 2, \cdots, p$,

$\rho^k \leqq 0$, $k = 1, 2, \cdots, s$,

(ii) $(\mu, \eta) \neq 0$,

(iii) $$\sum_{i=1}^{p} \mu^i \frac{\partial f^i(\hat{x})}{\partial x} + \sum_{j=1}^{m} \eta^j \frac{\partial r^j(\hat{x})}{\partial x} + \sum_{k=1}^{s} \rho^k \frac{\partial q^k(\hat{x})}{\partial x} = 0,$$

(iv) $$\sum_{k=1}^{s} \rho^k q^k(\hat{x}) = 0.$$

When the additional Assumption (58) does not hold, we can use the following lemma to obtain somewhat weaker necessary conditions for the Basic Nonlinear Programming Problem, still involving its entire structure.

65. <u>Lemma</u>: Let ν_i, $i=1,2,\cdots,k$ be any k vectors in E^n. If the system

66. $\langle \nu_i, x\rangle < 0$, $i = 1,2,\cdots,k$

has no solution x in E^n, then there exists a nonzero vector $\bar{\rho}$ in E^k, with $\bar{\rho}^i \leqq 0$, $i=1,2,\cdots,k$ such that $\Sigma_{i=1}^{k} \bar{\rho}^i \nu_i = 0$.

<u>Proof</u>: Let $B = \{x \in E^n | x = \sum_{i=1}^{k} \rho^i \nu_i,\ \rho^i \leqq 0,$ not all zero$\}$.

We want to prove that the origin belongs to B. By contradiction, suppose that the origin does not belong to B. Then 0 does not belong to the convex hull of $\{-\nu_1, -\nu_2, \cdots, -\nu_k\}$ since $co\{-\nu_1, -\nu_2, \cdots, -\nu_k\}$ is a subset of B. But $co\{-\nu_1, -\nu_2, \cdots, -\nu_k\}$ is a closed convex set in E^n not containing the origin. Hence, by the strong separation theorem,* there exists a hyperplane in E^n strictly separating the set $co\{-\nu_1, -\nu_2, \cdots, -\nu_k\}$ from the origin, i.e., there exists a nonzero vector $\bar{x}$ in E^n such that

67. $\langle \bar{x}, x\rangle > 0$ for every $x \in co\{-\nu_1, -\nu_2, \cdots, -\nu_k\}$.

Hence,

68. $\langle \bar{x}, \nu_i \rangle < 0$, for $i = 1,2,\cdots,k$,

---

* See Edwards, 8, p. 118, 2.2.3 Corollary to the Hahn-Banach Theorem.

which contradicts the assumption of the Lemma. Therefore $0 \epsilon B$, i.e., there exists a nonzero vector $\bar{\rho}$ in E^k, $\bar{\rho}_i \leqq 0$, $i = 1, 2, \cdots, k$, such that $\Sigma_{i=1}^{k} \bar{\rho}^{\,i} \nu_i = 0$.

Combining Theorem (60), Assumption (58), and Lemma (65), we obtain the following extension of the Fritz-John Theorem [12].

69. Theorem: Let $\hat{x}$ be a solution to the Basic Nonlinear Programming Problem. Then, there exist vectors μ in E^p, η in E^m, and ρ in E^s such that

(i) $\mu^i \leqq 0, \quad i = 1, 2, \cdots, p$,

(ii) $\rho^i \leqq 0, \quad i = 1, 2, \cdots, s$,

(iii) $(\mu, \eta, \rho) \neq 0$,

(iv) $$\sum_{i=1}^{p} \mu^i \frac{\partial f^i(\hat{x})}{\partial x} + \sum_{i=1}^{m} \eta^i \frac{\partial r^i(\hat{x})}{\partial x} + \sum_{i=1}^{k} \rho^i \frac{\partial q^i(\hat{x})}{\partial x} = 0 ,$$

(v) $$\sum_{i=1}^{k} \rho^i q^i(\hat{x}) = 0 .$$

The following corollaries are immediate consequences of Theorem (19):

75. Corollary: If the gradient vectors $\frac{\partial r^1(\hat{x})}{\partial x}, \cdots, \frac{\partial r^m(\hat{x})}{\partial x}$ are linearly independent; then any vectors $\mu \varepsilon E^p$, $\eta \varepsilon E^m$, $\rho \varepsilon E^s$, satisfying the conditions of Theorem (69), also satisfy $(\mu, \rho) \neq 0$.

76. Corollary: If the gradient vectors $\frac{\partial r^1(\hat{r})}{\partial x}, \frac{\partial r^2(\hat{x})}{\partial x}, \cdots, \frac{\partial r^m(\hat{x})}{\partial x}$ together with the gradient vectors $\left\{\frac{\partial q^i(\hat{x})}{\partial x}\right\}$, with $i \varepsilon I(\hat{x})$, are linearly independent, then any vectors $\mu \varepsilon E^p$, $\eta \varepsilon E^m$, $\rho \varepsilon E^s$ satisfying the conditions of Theorem (69), also satisfy $\mu \neq 0$.

77. Corollary: If the set: $\left\{x \varepsilon E^n \middle| \frac{\partial r^j(\hat{x})}{\partial x} x = 0,\ j=1,2,\cdots,m,\ \frac{\partial q^i(\hat{x})}{\partial x} x < 0,\ i \varepsilon I(\hat{x})\right\}$ is nonvoid and the vectors $\frac{\partial r^1(\hat{x})}{\partial x}, \frac{\partial r^2(\hat{x})}{\partial x}, \cdots, \frac{\partial r^m(\hat{x})}{\partial x}$ are linearly independent, then any vectors $\mu \varepsilon E^p$, $\eta \varepsilon E^m$, $\rho \varepsilon E^s$ satisfying the conditions of Theorem (69), also satisfy $\mu \neq 0$.

78. Corollary: If the system

$$\frac{\partial f^i(\hat{x})}{\partial x} x < 0,\ i \varepsilon \{\{1, 2, \cdots, p\} - \{\hat{i}\}\},$$

$$\frac{\partial r^j(\hat{x})}{\partial x} x = 0,\ j = 1, 2, \cdots, m,$$

$$\frac{\partial q^k(\hat{x})}{\partial x} x < 0,\ k \varepsilon I(\hat{x}),$$

has a solution for some $\hat{i}$ $\{1, 2, \cdots, p\}$ and the gradient vectors $\frac{\partial r^1(\hat{x})}{\partial x}, \frac{\partial r^2(\hat{x})}{\partial x}, \cdots, \frac{\partial r^m(\hat{x})}{\partial x}$ are linearly independent, then any vectors $\mu \varepsilon E^p$, $\eta \varepsilon E^m$, $\rho \varepsilon E^s$ satisfying the conditions of Theorem (69), also satisfy $\mu^{\hat{i}} < 0$.

IV. Applications to Optimal Control

79. Definition: Let P be a convex cone in E^s. A subset S of E^s is said to be P-directionally convex if for every z_1, z_2 in S and $0 \leqq \lambda \leqq 1$, there exists a vector $z(\lambda)$ in P such that

$$\lambda z_1 + (1-\lambda) z_2 + z(\lambda) \varepsilon S .$$

80. Remark: It is very easy to show that a subset S of E^s is P-directionally convex if and only if for any finite subset $\{z_1, z_2, \cdots, z_k\}$ of S and any scalars $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ with $\Sigma_{i=1}^k \lambda_i = 1$, $\lambda_i \geqq 0$, $i = 1, 2, \cdots, k$, there exists a vector $z(\lambda_1, \lambda_2, \cdots, \lambda_k)$ in P such that

$$\sum_{i=1}^{k} \lambda_i z_i + z(\lambda_1, \lambda_2, \cdots, \lambda_k) \varepsilon S .$$

On rereading Theorem (7), we observe that it may be rephrased in the following equivalent form.

81. Theorem: Let $\bar{x}$ be any feasible solution to the Basic Problem (1), i.e., $\bar{x}$ Ω and $r(\bar{x}) = 0$, and let $C(\bar{x}, \Omega)$ be a linearization of Ω at $\bar{x}$. If the sets

$$K(\bar{x}) = \left\{ u \varepsilon E^p \times E^m \,\middle|\, u = \left(\frac{\partial f(\bar{x})}{\partial x} x,\ \frac{\partial r(\bar{x})}{\partial x} x \right) ,\ x \varepsilon C(\bar{x}, \Omega) \right\}$$

and

$$R = \left\{ (y, 0) \varepsilon E^p \times E^m \,|\, y^i < 0, \;\; i = 1, 2, \cdots, p \,, \;\; 0 \varepsilon E^m \right\}$$

are not separated, then there exists a vector x^* in Ω, with $r(x^*) = 0$ and $f(x^*) < f(\bar{x})$ (component-wise).

We now make one more observation.

82. Theorem: Let $\Omega' \subset E^n$ be any set with the property that if $x' \varepsilon \Omega'$, then there is a vector x in Ω with $r(x') = r(x)$ and $f(x) \leqq f(x')$. If $\hat{x}$ is a solution to the Basic Problem (1), if $\hat{x} \varepsilon \Omega'$ and if $C(\hat{x}, \Omega')$ is a linearization for Ω' at $\hat{x}$, then there exists a vector μ in E^p and a vector η in E^m such that

83. (i) $\mu^i \leqq 0, \;\; i = 1, 2, \cdots, p \,,$

84. (ii) $(\mu, \eta) \neq 0 \,,$

85. (iii) $\left\langle \mu, \; \frac{\partial f(\hat{x})}{\partial x} x \right\rangle + \left\langle \eta, \; \frac{\partial r(\hat{x})}{\partial x} x \right\rangle \leqq 0$ for all $x \varepsilon \overline{C(\hat{x}, \Omega')}$.

Proof: The theorem claims that the cones

$$K'(\hat{x}) = \left\{ u \varepsilon E^p \times E^m \,|\, u = \left(\frac{\partial f(\hat{x})}{\partial x} x, \; \frac{\partial r(\hat{x})}{\partial x} x \right) \,, \;\; x \varepsilon C(\hat{x}, \Omega') \right\}$$

and

$$R = \left\{ (y, 0) \varepsilon E^p \times E^m \,|\, y^i < 0 \;\; \text{for} \;\; i = 1, 2, \cdots, p, \;\; 0 \varepsilon E^m \right\}$$

must be separated if $\hat{x}$ is a solution to the Basic Problem. Suppose that $K'(\hat{x})$ and R are not separated. Then by Theorem (81) with Ω' taking the place of Ω, there exists a x^* in Ω' such that $r(x^*) = 0$ and $f(x^*) < f(\hat{x})$. But by assumption, there must exist an $\bar{x}$ in Ω such that $r(\bar{x}) = r(x^*) = 0$ and $f(\bar{x}) \leqq f(x^*) < f(\hat{x})$, which contradicts the assumption that $\hat{x}$ is a solution to the Basic Problem.

Now consider a dynamical system described by the difference equation

86. $$x_{i+1} - x_i = f_i(x_i, u_i) \text{ for } i = 0, 1, 2, \cdots, k-1 ,$$

where $x_i \varepsilon E^n$ is the system state at time i, $u_i \varepsilon E^m$ is the system input at time i, and f_i is a function defined in $E^n \times E^m$ with range in E^n.

The <u>Optimal Control Problem</u> is that of finding a control sequence $\hat{U} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_{k-1})$ and a corresponding trajectory $\hat{X} = (\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k)$ determined by (86), such that

87. (i) $\hat{u}_i \varepsilon U_i \subset E^m$, $i = 0, 1, 2, \cdots, k-1$,

88. (ii) $\hat{x}_i \subset X_0 = X_0' \cap X_0''$, with $X_0' = \{x \varepsilon E^n | q_0(x) \leqq 0\}$, and $X_0'' = \{x \varepsilon E^n | q_0(x) = 0\}$, where g_0 maps E^n into E^{ℓ_0} and q_0 maps E^n into E^{m_0},

89. (iii) $\hat{x}_k \varepsilon X_k = X_k' \cap X_k''$, with $X_k' = \{x \varepsilon E^n | q_n(x) \leqq 0\}$ and $X_k'' = \{x \varepsilon E^n | g_k(x) = 0\}$, where g_k maps E^n into E^{ℓ_k} and g maps

E^n into E^{m_k},

90. (iv) $\hat{x}_i \varepsilon X_i = X_i'$, $X_i' = \{x \varepsilon E^n | q_i(x) \leqq 0\}$, $i = 1, 2, \cdots, k-1$ where q_i maps E^n into E^{m_i},

91. (v) for every control sequence $U = (u_0, u_1, \cdots, u_{k-1})$ and corresponding trajectory $X = (x_0, x_1, \cdots, x_k)$, satisfying the conditions (i), (ii), and (iii), the relation $\Sigma_{i=0}^{k-1} c_i(x_i, u_i) \leqq \Sigma_{i=0}^{k-1} c_i(\hat{x}_i, \hat{u}_i)$ implies that $\Sigma_{i=0}^{k-1} c_i(x_i, u_i) = \Sigma_{i=0}^{k-1} c_i(\hat{x}_i, \hat{u}_i)$, where the c_i map E^n into E^p for $i = 0, 1, 2, \cdots, k-1$.

The following assumptions will be made:

92. (i) for $i = 0, 1, 2, \cdots, k-1$ and for every fixed u_i in U_i, the functions $f_i(x_i, u_i)$ and $c_i(x_i, u_i)$ are continuously differentiable in x_i;

93. (ii) let $\bar{R} = \{(y, 0) \varepsilon E^p \times E^m | y \varepsilon E^p,\ y^i \leqq 0,\ i = 1, 2, \cdots, p,\ 0 \varepsilon E^m\}$ and let $\underline{f}_i(x, u) = (c_i(x, u), f_i(x, u))$; then for each x in E^n, the sets $\underline{f}_i(x_i, U_i)$, $i = 0, 1, 2, \cdots, k-1$ are $\bar{R}$ -directionally convex;

94. (iii) the functions $g_0(x)$ and $g_k(x)$ are continuously differentiable and the corresponding Jacobian matrices $\frac{\partial g_0(x)}{\partial x}$, $\frac{\partial g_k(x)}{\partial x}$ are of maximum rank for every x in X_0 and every x in X_k respectively;

95. (iv) for every $x_i \varepsilon X_i'$, $i = 0, 1, 2, \cdots, k$, $\left\{ \frac{\partial q_i^j(x)}{\partial x} \middle| j \varepsilon \{j \,|\, q_i^j(x) = 0,\ j = 1, 2, \cdots, m_i\} \right\}$ is a set of linearly independent vectors.

In order to transcribe the control problem into the form of the Basic Problem, we introduce the following definitions:

96. (i) For $i = 0, 1, 2, \cdots, k-1$, let $\underline{v}_i = (a_i, v_i)$ where $a_i \varepsilon c_i(x_i, U_i)$ and $v_i \varepsilon f_i(x_i, U_i)$, i.e., $\underline{v}_i \varepsilon \underline{f}_i(x_i, U_i)$,

97. (ii) Let $z = (x_0, x_1, \cdots, x_k, \underline{v}_0, \underline{v}_1, \cdots, \underline{v}_{k-1})$,

98. (iii) Let $f(z) = \sum_{i=0}^{k-1} a_i$,

99. (iv) Let $r(z)$ be the function defined by:

$$r(z) = \begin{bmatrix} x_1 - x_0 - v_0 \\ \vdots \\ x_k - x_{k-1} - v_{k-1} \\ g_0(x_0) \\ g_k(x_k) \end{bmatrix},$$

100. (v) Let $\Omega = \{z \mid x_i \varepsilon X_i^{'},\ i = 0, 1, 2, \cdots, k,\ \underline{v}_i \varepsilon \underline{f}_i(x_i, U_i),\ i = 0, 1, \cdots, k-1\}$.

Thus, the Optimal Control Problem is equivalent to the Basic Problem with z, f, r, and Ω given by (97), (98), (99) and (100), respectively.

Let us define the set $\Omega^{'}$ by

101. $\Omega' = \{z \mid x_i \in X_i',\ i = 0,1,2,\cdots,k,\ \underline{v}_i \in \mathrm{co}\, \underline{f}_i(x_i, U_i),\ i = 0,1,2,\cdots,k-1\}$.

We now show that the sets Ω and Ω' defined in (100) and (101), respectively, satisfy the conditions stated in Theorem (82). Let z^* be any point in Ω'. Then for $i = 0,1,2,\cdots,k$, $x_i^* \in X_i'$ and $\underline{v}_i^* = \sum_{j \in J^*} \lambda_i^j \underline{v}_i^j$, where $\sum_{j \in J^*} \lambda_i^j = 1$, $\lambda_i^j \geqq 0$, J^* a finite set and $\underline{v}_i^j \in \underline{f}_i(x_i, U_i)$. But by Assumption (93), the sets $\underline{f}_i(x_i, U_i)$, $i = 0, 1, 2, \cdots, k-1$, are $\bar{R}$-directionally convex and hence there exists a $\tilde{z}$ in Ω such that $\tilde{x}_i = x_i^*$, $\tilde{v}_i = v_i^*$, and $\tilde{a}_i \leqq a_i^*$.

Now let $\hat{z}$ be a solution to the optimal control problem. Then $\hat{z} \in \Omega$ and, since Ω' contains Ω, $\hat{z} \in \Omega'$.

In the appendix we prove that the set

102. $$C(\hat{z}, \Omega') = \Bigg\{ \delta_z = (\delta x_0, \delta x_1, \cdots, \delta x_k, \delta \underline{v}_0, \delta \underline{v}_1, \cdots, \delta \underline{v}_{k-1}) \,\Big|\, \frac{\partial q_i^j(\hat{x}_i)}{\partial x_i} \delta x_i < 0 \text{ for all } j \in \{j \mid q_i^j(\hat{x}_i) = 0\} \text{ and } \delta \underline{v}_i \in \left\{ \frac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x_i} \delta x_i \right\} + RC(\hat{\underline{v}}_i, \mathrm{co}\, \underline{f}_i(\hat{x}_i, U_i)) \Bigg\}^{\dagger}$$

is a linearization for the set Ω' at $\hat{z}$.

---

† <u>Definition:</u> Given a subset A of an Euclidean space, we define the <u>radial cone</u> to A at $\bar{x} \in A$ to be the cone

$$RC(\bar{x}, A) = \{x \mid (\bar{x} + \alpha x) \in A \text{ for all } 0 \leqq \alpha \leqq \epsilon(\bar{x}, x), \text{ where } \epsilon > 0\}$$

It now follows from Theorem (82) that there exists a nonzero vector $\psi = (p^0, \pi)$, $p^0 \varepsilon E^p$, $p^0 \leqq 0$, $\pi = (-p_1, -p_2, \cdots, -p_k, \mu_0, \mu_k)$, $p_i \varepsilon E^n$, $\mu_0 \varepsilon E^{\ell_0}$, $\mu_k \varepsilon E^{\ell_k}$ such that

103. $$p^0 \frac{\partial f(\hat{z})}{\partial z} \delta_z + \pi \frac{\partial r(\hat{z})}{\partial z} \delta z \leqq 0 \quad \text{for all} \quad \delta z \varepsilon \overline{C(\hat{z}, \Omega')}.$$

Substituting (98) and (99) into (103) we obtain

104. $$p^0 \sum_{i=0}^{k-1} \delta a_i - \sum_{i=0}^{k-1} p_{i+1} (\delta x_{i+1} - \delta x_i - \delta v_i)$$
$$+ \mu_0 \frac{\partial g_0(\hat{x}_0)}{\partial x} \delta x_0 + \mu_k \frac{\partial g_k(\hat{x}_k)}{\partial x} \delta x_k \leqq 0$$

for every $\delta_z \varepsilon \overline{C(\hat{z}, \Omega')}$.

Now, by interpreting (104) we obtain the following theorem:

105. <u>Theorem</u>: If the control sequence $\hat{U} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_{k-1})$ and the corresponding trajectory $\hat{X} = (\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k)$ constitute a solution to the Optimal Control Problem, then there exists a vector $p^0 \varepsilon E^p$, $p^0 \leqq 0$, vectors $p_0, p_1, \cdots, p_k$ in E^n, vectors $\lambda_0, \lambda_1, \cdots, \lambda_k$, $\lambda_i \varepsilon E^{n_i}$, $i = 0, 1, \cdots, k$, vectors $\mu_0 \varepsilon E^{\ell_0}$, $\mu_k \varepsilon E^{\ell_k}$ such that

106. (i) $(p^0, p_0, p_1, \cdots, p_k, \mu_0, \mu_k) \neq 0$,

107. (ii) $$p_i - p_{i+1} = p_{i+1} \frac{\partial f_i(\hat{x}_i, \hat{u}_i)}{\partial x} + p^0 \frac{\partial c_i(\hat{x}_i, \hat{u}_i)}{\partial x} + \lambda_i \frac{\partial q_i(\hat{x}_i)}{\partial x},$$
$i = 0, 1, \cdots, k-1$.

108. (iii) $p_0 = -\mu_0 \dfrac{\partial g_0(\hat{x})}{\partial x}$,

109. (iv) $p_k = \mu_k \dfrac{\partial g_k(\hat{x}_k)}{\partial x} + \lambda_k \dfrac{\partial g_k(\hat{x}_k)}{\partial x}$,

110. (v) $\lambda_i q_i(\hat{x}_i) = 0, \ i = 0, 1, \cdots, k$,

111. (vi) the Hamiltonian $H(x, u, p, p^0, i) = \langle p^0, c_i(x, u)\rangle + \langle p, f_i(x, u)\rangle$ satisfies the maximum principle

$$H(\hat{x}_i, \hat{u}_i, p, p^0, i) \geqq H(\hat{x}_i, u_i, p, p^0, i) \text{ for all } u_i \varepsilon U_i, \ i = 0, 1, \cdots, k\text{-}1 .$$

<u>Proof:</u>

(i) This was established in Theorem (82)

(ii) Let $\delta \underline{v}_i = \dfrac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i$. Then (104) becomes:

$$p^0 \frac{\partial c_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i + p_{i+1} \frac{\partial f_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i + p_{i+1} \delta x_i - p_i \delta x_i \leqq 0$$

for every δx_i satisfying $\dfrac{\partial q_i^j(\hat{x}_i)}{\partial x} \delta x_i \leqq 0$, with $q_i^j(\hat{x}_i) = 0$. Applying Farkas' Lemma [11] we obtain (107) and that $\lambda_i q_i(\hat{x}_i) = 0$ for $i = 0, 1, \cdots, k\text{-}1$.

(iii) This is seen to be merely an arbitrary but consistent definition.

(iv) and (v). We select $\delta_z = (0, 0, \cdots, 0, \delta x_k, 0, 0, \cdots, 0)$, with δx_K such that $\frac{\partial q_k^j}{\partial x} \delta x_k \leqq 0$ whenever $q_k^j(\hat{x}_k) = 0$. Again applying Farkas' Lemma, we get (109) and $\lambda_k q_k(\hat{x}_k) = 0$.

(vi) For $i = 0, 1, 2, \cdots, k\text{-}1$, let $\underline{v}_i'$ be an arbitrary point in $\text{co}\,\underline{f}_i(\hat{x}_i, U_i)$, which is convex by construction. Then $\delta \underline{v}_i = \underline{v}_i' - \hat{\underline{v}}_i$ is in $RC(\hat{v}_i, \text{co}\,\underline{f}_i(\hat{x}_i, U_i))$ and, choosing $\delta z = (0, 0, \cdots, 0, \delta \underline{v}_i, 0, \cdots, 0)$, we find that $\delta z \varepsilon C(\hat{z}, \Omega')$, and hence we obtain from (104),

112. $$p^0 \delta a_i + p_{i+1} \delta v_i \leqq 0 .$$

Substituting $\underline{v}_i' - \hat{\underline{v}}_i$ for $\delta \underline{v}_i$ in (112) we obtain

113. $$p^0(a_i' - c_i(\hat{x}_i, \hat{u}_i)) + p_{i+1}(v_i' - f_i(\hat{x}_i, \hat{u}_i)) \leqq 0 .$$

Clearly (113) also holds for every $(a_i', v_i') \varepsilon \underline{f}_i(\hat{x}_i, U_i)$.

Therefore:

$$p^0(c_i(\hat{x}_i, u_i) - c_i(\hat{x}_i, \hat{u}_i)) + p_{i+1}(f_i(\hat{x}_i, u_i) - f_i(\hat{x}_i, \hat{u}_i)) \leqq 0 \text{ for all } u_i \varepsilon U_i,$$

which completes our proof of (111).

Appendix

A1 Given a subset B of a Euclidean space, defined by inequalities, i.e., $B = \{x | q^i(x) \leqq 0,\ i=1,2,\cdots,m\}$, where the q^i are continuously differentiable scalar-valued functions, we define the internal cone to B at $\bar{x}$ B to be the cone

A2 $$IC(\bar{x}, B) = \{x | \frac{\partial q^i(\hat{x})}{\partial x} x < 0 \text{ whenever } q^i(\bar{x}) = 0,\ i \varepsilon \{1,2,\cdots,m\}\}.$$

We now return to the set Ω', which was defined in (101) as

A3 $$\Omega' = \{z = (x_0, x_1, \cdots, x_k, \underline{v}_0, \underline{v}_1, \cdots, \underline{v}_{k-1}) | x_i \varepsilon X_i',\ i = 0,1,2,\cdots,k,$$
$$\underline{v}_j \varepsilon \text{ co } \underline{f}_j(x_j, U_j),\ j=0,1,2,\cdots,k-1\}.$$

We shall prove that the set $C(\hat{z}, \Omega')$ defined in (102), as shown below, is a linearization for the set Ω' at $\hat{z} \varepsilon \Omega'$.

A4 $C(\hat{z}, \Omega') = \{\delta z = (\delta x_0, \cdots, \delta x_k, \delta \underline{v}_0, \cdots, \delta \underline{v}_{k-1}) | \delta x_i \varepsilon IC(\hat{x}_i, X_i')$ for $i = 0, 1, \cdots, k$, and $\delta \underline{v}_i - \frac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i \varepsilon RC(\hat{\underline{v}}_i, \text{ co } \underline{f}_i(\hat{x}_i, U_i))$ for $i = 0, 1, \cdots, k-1\}$.

A5 Lemma: The set $C(\hat{z}, \Omega')$ is a linearization of the set Ω' at $\hat{z}$.

Proof: First of all it is clear that $C(\hat{z}, \Omega')$ is a convex cone. Now, for $j = 1, 2, \cdots, N$, let

A6 $\delta z_j = (\delta x_{0j}, \cdots, \delta x_{kj}, \delta \underline{v}_{0j}, \cdots, \delta \underline{v}_{k-1\,j})$

be N linearly independent vectors in $C(\hat{z}, \Omega')$, and let $S = co\{\bar{\epsilon}\,\delta z_1, \bar{\epsilon}\,\delta z_2, \cdots, \bar{\epsilon}\,\delta z_N\}$ where $\bar{\epsilon}$ is a positive scalar, defined below.

For any δz in S we can uniquely write:

A7 $$\delta z = \bar{\epsilon} \sum_{i=1}^{N} \mu_i(\delta z)\,\delta z_i, \text{ where } \sum_{i=1}^{N} \mu_i(\delta z) = 1, \ \mu_i(\delta z) \geqq 0, \ i = 1, 2, \cdots, N$$

Therefore:

A8 $$\delta x_i = \bar{\epsilon} \sum_{j=1}^{N} \mu_j(\delta z)\,\delta x_{ij}$$

and

A9 $$\delta \underline{v}_i = \bar{\epsilon} \sum_{j=1}^{N} \mu_j(\delta z)\,\delta \underline{v}_{ij}$$

But by definition:

A10 $$\delta \underline{v}_{ij} = \frac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x}\,\delta x_{ij} + \underline{v}_{ij}$$

where $\underline{v}_{ij} \in RC(\hat{\underline{v}}_i, co\, \underline{f}_i(\hat{x}_i, U_i))$

From (A8), (A9) and (A10) ,

A11 $$\delta \underline{v}_i = \frac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i + \bar{\epsilon} \sum_{j=1}^{N} \mu_j(\delta z) \underline{v}_{ij}$$

Now, let's define the positive scalar $\bar{\epsilon}$.

(a) For $j=1, 2, \cdots, N$ and $i=0, 1, \cdots, k$, δx_{ij} belongs to the convex cone $IC(\hat{x}_i, X_i')$. Hence from (A7), $\Sigma_{j=1}^{N} \mu_j(\delta z)\delta x_{ij}$ is also in $IC(\hat{x}_i, X_i')$ for $i=0, 1, \cdots, k$. Therefore there exist positive scalars $\bar{\epsilon}_i$, $i=0, 1, \cdots, k$, possible depending on $\delta z_1, \delta z_2, \cdots, \delta z_N$, such that:

A12 $$\left(\hat{x}_i + \epsilon_i \sum_{j=1}^{N} \mu_j(\delta z)\delta x_{ij}\right) \varepsilon X_i' \text{ for all } 0 \leqq \epsilon_i \leqq \bar{\epsilon}_i$$

(b) Similarly, for $i = 0, 1, \cdots, k\text{-}1$,

A13 $$\sum_{j=1}^{N} \mu_j(\delta z) \underline{v}_{ij} \varepsilon RC(\hat{v}_i, \text{co}\, \underline{f}_i(\hat{x}_i, U_i)),$$

and hence there exist positive scalars $\underline{\epsilon}_i$, possible depending on $\delta z_1, \delta z_2, \cdots, \delta z_N$, such that:

A14 $$\hat{\underline{v}}_i + \epsilon_i \sum_{j=1}^{N} \mu_j(\delta_z) \underline{v}_{ij} \varepsilon\, \text{co}\, \underline{f}_i(\hat{x}_i, U_i) \text{ for all } 0 \leqq \epsilon_i \leqq \underline{\epsilon}_i .$$

We now define $\bar{\epsilon}$ to be minimum of the scalars $\bar{\epsilon}_i$, $i=0,1, \cdots, k$, and $\underline{\epsilon}_j$, $j = 0, 1, \cdots, k-1$.

From (A14), there exists a finite set A_i and scalars λ_α^i such that

A15 $$\bar{\epsilon} \sum_{j=1}^{N} \mu_j(\delta_z) \underset{-}{v}_{ij} = \sum_{\alpha \epsilon A_i} \lambda_\alpha^i \underset{-}{f}_i(\hat{x}_i, u_i^\alpha) - \hat{\underset{-}{v}}_i$$

where $u_i^\alpha \in U_i$, $\alpha \in A_i$, and $\sum_{\alpha \epsilon A_i} \lambda_\alpha^i = 1$, $\lambda_\alpha^i \geq 0$.

Combining (A15) and (A11) we obtain

A16 $$\delta \underset{-}{v}_i = \frac{\partial \underset{-}{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x} \delta x_i + \sum_{\alpha \epsilon A_i} \lambda_\alpha^i \underset{-}{f}_i(\hat{x}_i, u_i^\alpha) - \hat{\underset{-}{v}}_i .$$

We can define a map ζ from S into $\Omega' - \{\hat{z}\}$ by

A17 $$\zeta(\delta z) = (y_0, y_1, \cdots, y_K, \underset{-}{w}_0, \underset{-}{w}_1, \cdots, \underset{-}{w}_{k-1})$$

where

A18 $$y_i(\delta z) = \delta x_i = \bar{\epsilon} \sum_{j=1}^{N} \mu_j(\delta z) \delta x_{ij}, \quad i = 0, 1, \cdots, k, \text{ and}$$

A19 $$\underset{-}{w}_i(\delta z) = \sum_{\alpha \epsilon A_i} \lambda_\alpha^i \underset{-}{f}_i(\hat{x}_i + \delta x_i, u_i^\alpha) - \hat{\underset{-}{v}}_i, \quad i = 0, 1, \cdots, k-1 .$$

From (A12), (A15), (A18) and (A19) it is clear that ζ maps S into $\Omega' - \{\hat{z}\}$.

Expanding (A19) in a Taylor series about $\hat{z}$ we find that:

$$\text{A20} \qquad \underline{w}_i(\delta z) = \frac{\partial \underline{f}_i(\hat{x}_i, \hat{u}_i)}{\partial x}\, \delta x_i + \sum_{\alpha \in A_i} \lambda^i_\alpha\, \underline{f}_i(\hat{x}_i, u^\alpha_i) - \hat{\underline{v}}_i + o_i(\delta z)$$

where $\dfrac{\|o_i(\delta z)\|}{\|\delta z\|} \to 0$ as $\|\delta z\| \to 0$.

Combining (A20), (A13), (A17) and (A18), we see that

$$\zeta(\delta z) = \delta z + o(\delta z), \text{ where } \lim_{\|\delta z\| \to 0} \frac{\|o(\delta z)\|}{\|\delta z\|} = 0$$

Since ζ is obviously continuous, $C(\hat{z}, \Omega')$ is a linearization for Ω' at $\hat{z}$.

Conclusion

This paper has presented a theory of necessary conditions for vector-valued criterion optimization problems, which did not depend on the customary convexity assumptions (see Karlin [3]). When the constraint sets are assumed to be convex and the components of the cost function are also convex, the necessary conditions may also become sufficient [see Karlin p. 218]. Conditions under which a vector-valued criterion problem can be treated as a family of scalar valued criterion problems are very important, as they define the class of problems for which we can effectively compute noninferior points.

Since the conditions presented in this paper are considerably more general then theories hitherto available in the literature, it is hoped that they will open up important classes of optimization problems.

References

1. Pareto, V., Cours d'Economie Politique, Lausanne, Rouge, 1896.
2. H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," Proc. of the Second Berkeley Symposium on Mathematic Statistics and Probability," Univ. of California Press, Berkeley, California, 1951, pp. 481-492.
3. Karlin, S., Mathematical Methods and Theory in Games, Programming and Economics, 1 , Addison-Wesley, Massachusetts, 1959.
4. Debreu, G., Theory of Value, John Wiley, New York, 1959.
5. Zadeh, L. A., "Optimality and Non-Scalar-Valued Performance Criteria," IEEE Transactions on Automatic Control, vol. AC-8, number 1, pp. 59-60; January 1963.
6. Chang, S. S. L., "General Theory of Optimal Processes," J. SIAM Control, Vol. 4, No. 1, 1966, pp. 46-55.
7. Klinger, A., "Vector-valued Performance Criteria," IEEE Transactions on Automatic Control, vol. , pp. 117-118; January 1964.
8. Edwards, R. E., Functional Analysis Theory and Applications, Holt, Rinehart and Winston, New York, 1965.
9. Dieudonne', J., Foundations of Modern Analysis, Academic Press, New York, 1960.
10. Fan, K., "Fixed point and minimax theorems in locally convex topological linear spaces," Proc. Natl. Acad. Sci., U. S. 38(1952), 121-126.

11. Farkas, J., "Über die Theorie der einfachen Ungleichungen," Journal für die reine und angewandte Mathematik, vol. 124, 1901, pp. 1-27.

12. John, F., "Extremum problems with inequalities as side conditions" in Studies and Essays, Courant Anniversary Volume, edited by K. O. Friedrichs, O. E. Neugebauer and J. J. Stoker, Interscience, New York, N. Y., 1948, pp. 187-204.