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A DUAL METHOD FOR OPTIMAL CONTROL PROBLEMS WITH INITIAL AND FINAL BOUNDARY CONSTRAINTS

by

O. Pironneau and E. Polak

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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O. Pironneau and E. Polak

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

This paper presents two new algorithms belonging to the family of dual methods of centers. The first can be used for solving fixed time optimal control problems with inequality constraints on the initial and terminal states. The second one can be used for solving fixed time optimal control problems with inequality constraints on the initial and terminal states and with affine instantaneous inequality constraints on the control. Convergence is established for both algorithms. Qualitative reasoning indicates that the rate of convergence is linear.

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1. Introduction.

The construction of optimal control algorithms is often hampered by two difficulties. The first is due to the fact that the cost function usually has a gradient only in L_∞, while the convergence of the algorithm must be studied in L_2, since control sequences constructed by an optimization algorithm are not likely to converge in L_∞. The second difficulty stems from the fact that "primal type" subproblems, such as those resulting from a direct application of methods of centers or feasible directions, cannot be solved directly and usually require some sort of "dualization". Both of these sources of difficulty are taken into account in the dual method presented in this paper. The algorithm in this paper may be classified as a dual method of centers. It has the very important feature that it is implementable, since both the direction finding procedure and the step size finding procedure are finite, in the sense that they require only a finite number of function evaluations per iteration. Since a closely related algorithm presented in [6] converges linearly on finite dimensional problems, it is reasonably certain that the algorithm presented in this paper also converges linearly on problems in $\mathbb{R}^n$. However, since certain sets, used in the proofs in [6], loose their compactness in general Banach spaces, the proof of rate of convergence given in [6] cannot be extended to general Banach spaces. In spite of this, there are heuristic reasons which lead us to believe that the algorithms presented in this paper does converge linearly at least on a class of optimal control problems with linear dynamics and convex costs.

2. Optimality and Convergence.

Because of the peculiar nature of optimal control problems, which necessitates the simultaneous use of both the L_2 and the L_∞ norms on a space of regulated functions, we need the following abstract structure and accompanying theorems.

Let V be a linear space, let $\|\cdot\|_1$ be a norm on V and let $\langle\cdot,\cdot\rangle_2$ be a scalar product on V, such that $\mathcal{B}_1 = \{V, \|\cdot\|_1\}$ is a Banach space and $\mathcal{B}_2 = \{V, \langle\cdot,\cdot\rangle_2\}$ is a subspace of a Hilbert space. Let $\|\cdot\|_2$ be the norm induced by $\langle\cdot,\cdot\rangle_2$ on $\mathcal{B}_2$ (i.e. $\|z\|_2 = \langle z,z\rangle_2^{1/2}$).

2.1. Assumption:

There exists a $C > 0$ such that $\|z\|_2 \le C\|z\|_1$ for all $z \in V$. □

Now consider the problem

2.2. $$\min\{f^0(z) \mid f^j(z) \le 0,\ j = 1,2,\ \dots,\ m\},$$

where $f^j\colon\ V \to \mathbb{R}^1$ for $j = 0,1,2,\ \dots,\ m$.

2.3. Assumptions:

(i) The functions $f^j(\cdot)$, $j = 0,1,2,\ \dots,\ m$, are Frechet differentiable on $\mathcal{B}_1$, with the Frechet derivative at $\bar{z}$ being denoted by $f_z^j(\bar{z})(\cdot)$, $j = 0,1,2,\ \dots,\ m$.

(ii) The restrictions to $\{z \in \mathcal{B}_2 \mid \|z\|_1$ M of the functions $f^j(\cdot)$, $j = 0,1,2,\ \dots,\ m$, are continuous for any $M \in (0,\infty)$ (i.e. they are continuous in $\|\cdot\|_2$ on $\{z \in V \mid \|z\|_1 < M\}$).

(iii) There exist functions $\nabla f^j\colon\ V \to V$, $j = 0,1,2,\ \dots,\ m$, with the following properties: (a) the f^j are continuous on $\mathcal{B}_1$, (b) the f^j have continuous restrictions on $\{z \in \mathcal{B}_2 \mid \|z\|_1 < M\}$ for any $M \in (0,\infty)$, (c) the ∇f^j satisfy

2.4. $$f^j_z(\bar{z})(h) = \langle \nabla f^j(\bar{z}), h \rangle_2, \; j = 0,1,2, \ldots, m,$$

for any $\bar{z}$, h in $\mathcal{B}_1$. □

2.5. Theorem: Suppose that $\hat{z} \in \mathcal{B}_1$ is a solution of (2.2) (i.e., $f^j(\hat{z}) \leq 0$ for $j = 1,2,\ldots, m$, and $f^0(\hat{z}) = \min\{f^0(z) \mid f^j(z) \leq 0, \; j = 1,2, \ldots, m\}$). Then there exist multipliers $\mu^0 \geq 0, \mu^1 \geq 0, \ldots, \mu^m \geq 0$ such that

2.6 $$\sum_{j=0}^{m} \mu^j \nabla f^j(\hat{z}) = 0 ,$$

2.7 $$\mu^j f^j(\hat{z}) = 0 \text{ for } j = 1,2, \ldots, m ,$$

and

2.8 $$\sum_{j=0}^{m} \mu^j = 1. \; \square$$

Theorem (2.5) is a straight forward generalization of the well known F. John condition of optimality [4]. It can be proved in essentially the same manner as the F. John condition (see the proof of Theorem (3.5.11) in [2]).

2.9. Definition: Let the set of feasible points $\Omega \subset \mathcal{B}_1$ be defined by

2.10 $$\Omega = \{z \in \mathcal{B}_1 \mid f^j(z) \leq 0, \; j = 1,2, \ldots, m\},$$

and let the set of desirable points $\Delta \subset \Omega$ be the set of points $z \in \Omega$ for which there exist multipliers $\mu^j(z)$, $j = 0,1, \ldots, m$, such that

2.11. $$\mu^j(z) \geq 0, \; j = 0,1,2, \ldots, m,$$

2.12. $$\sum_{j=1}^{m} \mu^j(z) = 1,$$

2.13. $$\sum_{j=0}^{m} \mu^j(z)\nabla f^j(z) = 0,$$

2.14. $$\mu^j(z)f^j(z) = 0,\ j = 1,2,\ \ldots,\ m. \qquad \square.$$

Thus, Δ is the set of feasible points which satisfy the optimality condition (2.5). Since in general it is not possible to identify points in Ω which are optimal for (2.2), the best we can hope to achieve is to compute a desirable point.

The algorithm which we shall describe in the next section uses a map $A\colon\ \Omega \to 2^{\Omega}$ and is of the following form.

2.15. Algorithm Model

Step 0: Compute a $z_0 \in \Omega$, and set $i = 0$.

Step 1: Compute a $y \in A(z_i)$.

Step 2: If $f^0(y) < f^0(z_i)$, set $z_{i+1} = y$, set $i = i+1$, and go to Step 1; else, set $\hat{z} = z_i$, and stop. $\square$

The convergence properties of our algorithm are summarized by the following result.

2.16 Theorem: Suppose that (2.3)(ii) is satisfied, that for every $M > 0$ $\Omega_M \triangleq \{z \in \Omega \mid \|z\|_1 < M\}$, and that for every $z \in \Omega$, $z \notin \Delta$, there exist an $\varepsilon(z) > 0$ and a $\delta(z) < 0$ such that for every $M > \|z\|_1$

2.17 $$f^0(z'') - f^0(z') \leq \delta(z)$$

for all $z' \in \{z' \in \Omega_M \mid \|z'-z\|_2 \leq \varepsilon(z)\}$, for all $z'' \in A(z')$.

Suppose that $\{z_i\}$ is a sequence generated by algorithm model (2.15). If $\{z_i\}$ is finite, then its last element $\hat{z}$ is in Δ. If $K \subset \{0,1,2, \dots \}$ is an infinite subset and $z^* \in \Omega$ is such that either (i) $\lim_{i \in K} \|z_i - z^*\|_1 = 0$ or (ii) $\lim_{i \in K} \|z_i - z^*\|_2 = 0$ and $\|z_i\|_1 < M$ for some $M > 0$ and all $i \in K$, then $z^* \in \Delta$. □

We omit a proof of this theorem since it follows directly from Theorem (1.3.10) in [7] and the assumption (2.1).

With the preliminaries out of the way, we can now get down to the task of establishing a specific algorithm for finding points in the set Δ.

3. A Dual Method of Centers.

For the algorithm below to make sense, we need the following additional hypothesis, as is usual in conjunction with methods of centers and methods of feasible directions (see Secs. 4.2 and 4.3 in [7]).

3.1. Assumption: The set $\tilde{\Omega} = \{z \in \mathcal{B}_1 \mid f^j(z) < 0,\ j = 1,2, \ldots, m\}$ is not empty.*

3.2. Algorithm ($\beta \in (0,1)$ is a step size parameter).

Step 0: Compute a $z_0 \in \Omega$, and set $i = 0$.

Step 1: Compute $\mu(z_i) = (\mu^0(z_i), \mu^1(z_i), \ldots, \mu^m(z_i))^T \in \mathbb{R}^{m+1}$ to be a solution of the quadratic programming problem

3.3
$$\phi(z_i) \triangleq \max\{ \sum_{j=1}^{m} \mu^j f^j(z_i) - \frac{1}{2} \| \sum_{j=0}^{m} \mu^j \nabla f^j(z_i) \|_2^2 \mid \sum_{j=0}^{m} \mu^j = 1,\ \mu \geq 0\}.$$

Step 2: If $\phi(z_i) = 0$, set $\hat{z} = z_i$, and stop; else, set

3.4
$$h(z_i) = -\sum_{j=0}^{m} \mu^j(z_i) \nabla f^j(z_i),$$

and go to Step 3.

Step 3: Compute the smallest non-negative integer $k(z_i)$ such that

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*When $\tilde{\Omega}$ is empty, the algorithm below stops at z_0 and hence is useless.

3.5 $$\theta(\beta^{k(z_i)}, h(z_i), z_i) - \frac{1}{2}\beta^{k(z_i)}\phi(z_i) \leq 0,$$

where $\theta\colon \mathbb{R}^1 \times \mathcal{B}_1 \times \mathcal{B}_1 \to \mathbb{R}^1$ is defined by

3.6 $$\theta(\lambda,h,z) = \max\{f^0(z+\lambda h)-f^0(z);\ f^j(z+\lambda h),\ j = 1,2,\ \ldots,\ m\}.$$

Step 4: Set $z_{i+1} = z_i + \beta^{k(z_i)} h(z_i)$, set $i = i+1$, and go to Step 1. □.

The following result is obvious.

3.7 Proposition: Let $\phi\colon \mathcal{B}_1 \to \mathbb{R}^1$ be defined as in (3.3) and let $z \in \Omega$ be arbitrary. Then $\phi(z) \leq 0$, and $(z) = 0$ if and only if $z \in \Delta$. □.

3.8 Lemma: Suppose that $z_i \in \Omega$ is such that $\phi(z_i) \neq 0$, and let $h(z_i)$ be defined as in (3.4). Then

3.9 $$\max\{\langle \nabla f^0(z_i),h(z_i)\rangle_2;\ f^j(z_i) + \langle \nabla f^j(z_i),h(z_i)\rangle_2,\ j = 1,2,\ \ldots,\ m\} \leq \phi(z_i) - \frac{1}{2}\|h(z_i\|_2^2 < 0.$$

Proof: Since (3.3) is a quadratic problem in $\mathbb{R}^n$ and $\mu(z_i)$ is an optimal solution for this problem, it follows from the Kuhn-Tucker optimality conditions (see (3.3) in [2]) that there exist real multipliers $\lambda^0 \geq 0$, $\lambda^1 \geq 0$, ... , $\lambda^m \geq 0$ and a real multiplier ψ, such that

3.10 $$-\frac{\partial}{\partial\mu}\left(\sum_{j=1}^{m} \mu^j(z_i)f^j(z_i) - \frac{1}{2}\Big\| \sum_{j=0}^{m} \mu^j(z_i)\nabla f^j(z_i)\Big\|_2^2\right) + \psi e + \lambda = 0,$$

where $e = (1,1,1,\ \ldots,\ 1)^T \in \mathbb{R}^{m+1}$, $\lambda = (\lambda^0,\lambda^1,\ \ldots,\ \lambda^m)^T \in \mathbb{R}^{m+1}$, and

3.11 $$\lambda^j\mu^j(z_i) = 0 \text{ for } j = 1,2,\ \ldots,\ m.$$

Setting $h(z_i) = -\sum_{j=0}^{m} \mu^j(z_i)\nabla f^j(z_i)$, (3.10) yields,

3.12 $$-\langle \nabla f^0(z_i), h(z_i) \rangle_2 + \psi - \lambda^0 = 0$$

3.13 $$-(f^j(z_i) + \langle \nabla f^j(z_i), h(z_i) \rangle_2) + \psi - \lambda^j = 0,\ j = 1,2, \dots, m.$$

Multiplying (3.13) by $\mu^j(z_i)$, for $j = 1,2, \dots, m$, and (3.12) by $\mu^0(z_i)$, and summing, we obtain,

3.14. $$-\sum_{j=1}^{m} \mu^j(z_i)f^j(z_i) + \|h(z_i)\|_2^2 + \psi = 0.$$

Hence

3.15 $$\psi = \sum_{j=1}^{m} \mu^j(z_i)f^j(z_i) - \|h(z_i)\|_2^2$$

$$= \phi(z_i) - \frac{1}{2}\|h(z_i)\|_2^2 .$$

Inequality (3.9) now follows from (3.15), (3.12), (3.13) and the fact that $\lambda^j \geq 0$ for $j = 0,1,2, \dots, m$. □

3.16 Corollary: Suppose that $z_i \in \Omega$ is such that $\phi(z_i) < 0$, then there exists an integer $k(z_i) \geq 0$ such that (3.5) holds.

Proof: This corollary follows directly from the definition of a Frechet differential, (2.4) and the fact that by (3.9), $\langle \nabla f^0(z_i), h(z_i) \rangle_2 \leq \phi(z_i)$, and $\langle \nabla f^j(z_i), h(z_i) \rangle_2 \leq \phi(z_i)$ for all $j \in \{1,2, \dots, m\}$ such that $f^j(z_i) = 0$. □

3.17. Theorem: Let $\{z_i\}$ be a sequence generated by algorithm (3.2) in the process of searching the set Ω for a point in Δ (see (2.9)), and suppose that assumptions (2.1), (2.3) and (3.1) are satisfied. Then, either $\{z_i\}$ is finite and its last point $\hat{z} \in \Delta$, or $\{z_i\}$ is infinite, in which case any $z^* \in \Omega$ satisfying either $\lim_{i\in K}\|z_i-z^*\|_1 = 0$ or $\lim_{i\in K}\|z_i-z^*\|_2 = 0$, where K is an infinite subset $\{0,1,2, \ldots \}$, also satisfies $z^* \in \Delta$, provided there exists an $M \in (0,\infty)$ such that $\|z_i\|_1 \leq M$ for all $i \in K$.

Proof: We shall show that algorithm (3.2) is of the form of algorithm (2.15) and that it satisfies the assumptions of theorem (2.16). Thus, let $S: \Omega \to 2^V$ be defined by

3.18. $$S(z) = \{ -\sum_{j=0}^{m} \mu^j \nabla f^j(z) \,|\, \mu \geq 0, \sum_{j=0}^{m} \mu^j = 1; \sum_{j=1}^{m} \mu^j f^j(z) - \frac{1}{2}\|\sum_{j=0}^{m} \mu^j \nabla f^j(z)\|_2^2 = \phi(z)\},$$

and let $A: \Omega \to 2^\Omega$ be defined by

3.19 $$A(z) = \{z' = z + \beta^{k(z,h)}h \,|\, h \in S(z)\},$$

where $k(z,h)$ is the smallest non-negative integer which satisfies (3.5) for $z_i = z$, $h(z_i) = h$ and $k(z_i) = k(z,h)$. (Since by (3.6) and (3.5) $f^j(z') \leq \frac{1}{2}\beta^{k(z,h)}\phi(z) \leq 0$ for $j = 1,2, \ldots, m$, it is clear that $A(\cdot)$ maps Ω into 2^Ω). Thus, to complete our proof, we only need to show that (2.17) is satisfied by the maps $f^0(\cdot)$ and $A(\cdot)$, as defined in (3.19). Therefore, suppose that $z^* \in \Omega$ is such that $\phi(z^*) < 0$. Then, because of lemma (3.8) and because $S(z^*)$ is compact (see (A1)), there exists an integer $\overline{k}(z^*)$ such that (see (3.5))

3.20 $$\max_{h\in S(z^*)} \theta(\beta^{\overline{k}(z^*)},h,z^*) - \frac{1}{2}\beta^{\overline{k}(z^*)}\phi(z^*) < 0.$$

Now, let $M > \|z^*\|_1$ be arbitrary. Then since by theorem (A.1), the restriction, S_M, of $S(\cdot)$ to Ω_M is upper semi continuous, it follows from the maximum theorem [([1], page 116)] and our assumptions that the function $\eta_M: \Omega_M \to \mathbb{R}^1$, defined by

3.21 $$\eta_M(z) = \max_{h \in S(z)} \theta(\beta^{\bar{k}(z^*)}, h, z) - \frac{1}{2}\beta^{\bar{k}(z^*)}\phi(z)$$

is upper semi-continuous in both norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. It also follows from the same maximum theorem that $\phi_M: \Omega_M \to \mathbb{R}^1$, defined by $\phi_M(z) = \phi(z)$, is also upper semi continuous in both norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Hence there exists an $\varepsilon(z^*) > 0$ such that

3.22 $$\phi(z) \leq \frac{1}{2}\phi(z^*)$$

and

3.23 $$\eta_M(z) \leq 0$$

for all $z \in \Omega_M$ such that $\|z-z^*\|_2 \leq \varepsilon(z^*)$. Now, for every $z' \in A(z)$, with $z \in \Omega_M$ such that $\|z-z^*\|_2 \leq \varepsilon(z^*)$, we must have (see (3.19), (3.5) and (3.21))

3.24 $$k(z,h) \leq \bar{k}(z^*) \text{ for every } h \in S(z).$$

Hence we obtain that (see (3.6), (3.5), (3.24), (3.22)) for every $h \in S(z)$

3.25 $$f^0(z+\beta^{k(z,h)}h) - f^0(z) \leq \max_{h \in S(z)} \theta(\beta^{k(z,h)}, h, z),$$

3.26 $$\max_{h \in S(z)} \theta(\beta^{k(z,h)}, h, z) \leq \frac{1}{2} \beta^{k(z,h)} \phi(z) \leq \frac{1}{2} \beta^{\bar{k}(z^*)} \phi(z) ,$$

and hence

3.27 $$f^0(z + \beta^{k(z,h)} h(z)) - f^0(z) \leq \frac{1}{4} \beta^{\bar{k}(z^*)} \phi(z^*) \triangleq \delta(z^*) < 0$$

for all $z \in \Omega_M$ such that $\|z-z^*\|_2 \leq \varepsilon(z)$, which completes our proof. □

4. An Application to an Optimal Control Problem.

We shall now show that the algorithm, presented in the preceding section, can be used to solve the following problem,

4.1. $$\min\{\int_{t_0}^{t_f} h^0(x(t),u(t),t)dt \;\middle|\; \frac{d}{dt}x(t) = h(x(t),u(t),t),$$
$$t \in [t_0,t_f];\ g_0(x(t_0)) \leqq 0,\ g_f(x(t_f)) \leqq 0;$$
$$u \in L_\infty^s [t_0,t_f]\},$$

where h^0: $\mathbb{R}^n \times \mathbb{R}^s \times [t_0,t_f] \to \mathbb{R}^1$, h: $\mathbb{R}^n \times \mathbb{R}^s \times [t_0,t_f] \to \mathbb{R}^n$, g_0: $\mathbb{R}^n \to \mathbb{R}^{m'}$, g_f: $\mathbb{R}^n \to \mathbb{R}^{m''}$, and $L_\infty^s [t_0,t_f]$ is the space of equivalence classes of essentially bounded integrable functions from $[t_0,t_f]$ into $\mathbb{R}^s$.

We must begin by transcribing problem (4.1) into the form of problem (2.2). Therefore, let $V = \{(\xi, u) \mid \xi \in \mathbb{R}^n,\ u \in L_\infty^s [t_0,t_f]\}$, let the norm $\|\cdot\|_1$: $V \to \mathbb{R}^1$ be defined by

4.2 $$\|(\xi, u)\|_1^2 = |\xi|^2 + \operatorname*{ess\,sup}_{t\in[t_0,t_f]} |u(t)|^2,$$

where $|\cdot|$ denotes the euclidean norm, and finally, let the scalar product $\langle \cdot,\cdot \rangle_2$ on V be defined by

4.3 $$\langle (\xi, u),(\xi', u') \rangle_2 = \langle \xi, \xi' \rangle + \int_{t_0}^{t_f} \langle u(t),u'(t) \rangle\, dt,$$

where $\langle \cdot,\cdot \rangle$ denotes the euclidean scalar product. Then we see that the space $\mathcal{B}_1 = \{V, \|\cdot\|_1\}$ is a Banach space and the space $\mathcal{B}_2 = \{V, \langle \cdot,\cdot \rangle_2\}$ is a subspace of a Hilbert space. Furthermore, setting $\|\cdot\|_2 = \langle \cdot,\cdot \rangle_2$, it is not difficult to show that there exists a $C \in (0,\infty)$ such that $\|\cdot\|_2 \leq C \|\cdot\|_1$. Next, let $x(t,\xi,u)$, $t \in [t_0,t_f]$, denote the solution of the differential equation

4.4 $$\frac{d}{dt} x = h(x,u,t),\; x(t_0) = \xi\;,\; t \in [t_0,t_f],$$

corresponding to a given $(\xi,u) \in V$. Then we define the functions f^0: $V \to \mathbb{R}^1$, f_1: $V \to \mathbb{R}^{m'}$ and f_2: $V \to \mathbb{R}^{m''}$ as follows:

4.5 $$f^0(\xi,u) = \int_{t_0}^{t_f} h^0(x(t,\xi,u),u(t),t)dt$$

4.6 $$f_1(\xi,u) = g_0(\xi),$$

4.7 $$f_2(\xi,u) = g_f(x(t_f,\xi,u))\;.$$

With the above definitions problem (4.1) can be written as follows, setting $z = (\xi,u)$,

4.8 $$\min\{f^0(z) \mid f_1(z) \leqq 0,\; f_2(z) \leqq 0\},$$

i.e. it can be written in the form (2.2).

4.9. Assumptions.

(i) For every $(\xi,u) \in V$, the solution $x(\cdot,\xi,u)$ of (4.4) exists and is unique.

(ii) The functions h^0 and h are continuously differentiable in x and in u, and h^0, h, $\frac{\partial h^0}{\partial x}$, $\frac{\partial h^0}{\partial u}$, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial u}$ are piecewise continuous in t.

(iii) The functions g_0 and g_1 are continuously differentiable.

(iv) The set $\{z = (\xi,u) \in V \mid f_1(z) < 0,\; f_2(z) < 0\}$ is not empty. □

4.10 Lemma: Suppose that (4.9)(i) - (4.9)(iii) are satisfied. Then the functions f^0, f_1 and f_2, defined in (4.5) - (4.7), are Frechet differentiable on $\mathcal{B}_1$, with their differentials f_z^0, f_{1z}, f_{2z}, defined as follows:

4.11. $$f_z^0(z')(h) = \langle \nabla f^0(z'),h \rangle_2,$$

4.12. $$f_{1z}^i(z')(h) = \langle \nabla f_1^i(z'),h \rangle_2,\ i = 1,2,\ \ldots,\ m',$$

4.13. $$f_{2z}^i(z')(h) = \langle \nabla f_2^i(z'),h \rangle_2,\ i = 1,2,\ \ldots,\ m'',$$

where, for $z' = (\xi',u')$,

4.14 $$\nabla f^0(z') = (-p(t_0,\xi',u'),\ -\frac{\partial h}{\partial u}(x(\cdot,\xi',u'),u'(.),\cdot)^T \times p(\cdot,\xi',u') + \frac{\partial h^0}{\partial u}(x(\cdot,\xi',u'),u'(\cdot),\cdot)^T),$$

with $p(t,\xi',u')$ defined by

4.15 $$\frac{d}{dt}p(t,\xi',u') = -\frac{\partial h}{\partial x}(x(t,\xi',u'),u'(t),t)^T p(t,\xi',u') + \frac{\partial h^0}{\partial x}(x(t,\xi',u'),u'(t),t)^T,\ t \in [t_0,t_f],$$

$$p(t_f,\xi',u') = 0;$$

4.16 $$\nabla f_1^i(z') = (\frac{\partial g_0^i}{\partial x}(\xi')^T,\ 0)\ i = 1,2,\ \ldots,\ m';$$

and

4.17 $$\nabla f_2^i(z') = (-q_i(t_0,\xi',u'),\ -\frac{\partial h}{\partial u}(x(\cdot,\xi',u'),u'(\cdot),\cdot)^T \times q_i(\cdot,\xi',u')),\ i = 1,2,\ \ldots,\ m'';$$

with $q_i(t,\xi',u')$ $i = 1,2,\ \ldots,\ m''$, defined by

4.18 $$\frac{d}{dt} q_i(t,\xi',u') = -\frac{\partial h(x(t,\xi',u'),u'(t),t)^T}{\partial x} q_i(t,\xi'u'),$$

$$t \in [t_0,t_f];\ q_i(t_f,\xi',u') = -\frac{\partial g_f^i}{\partial x}(x(t_f,\xi',u')^T. \quad \square$$

4.19 Corollary: For any $M > 0$, f^0, f_1, f_2, ∇f^0, ∇f_1^i, $i = 1,2, \ldots, m'$, and ∇f_2^i, $i = 1,2, \ldots, m''$ have continuous restrictions on $\{z \in \mathcal{B}_2 \mid \|z\|_1 < M\}$. $\square$

This lemma and the corollary follow directly from Theorem (10.7.1) in [3] and from Theorem A1 in [5]. We therefore omit its proof.

Thus we see that the functions f^0, f_1 and f_2 satisfy the assumptions (2.3). We now show that the set Δ defined in (2.9), with $f^j = f_1^j$, for $j = 1,2, \ldots, m'$, and $f^{j+m'} = f_2^j$ for $j = 1,2, \ldots, m''$, is the set of initial states and controls for which the Pontryagin-Maximum-Principle in differential-form is satisfied.

4.20 Lemma: Let Ω and Δ be defined as in (2.9), with $f^j = f_1^j$, for $j = 1,2, \ldots, m'$, and $f^{j+m'} = f_2^j$, for $j = 1,2 \ldots, m''$, and let $m = m' + m''$. If $(\hat{\xi},\hat{u}) \in \Delta$, then there exists a multiplier function $\hat{\lambda}$: $[t_0,t_f] \to \mathbb{R}^n$, and a scalar $\hat{\lambda}^0 \leq 0$ such that

4.21 $$\frac{d}{dt}\hat{\lambda}(t) = -\frac{\partial h}{\partial x}(x(t,\hat{\xi},\hat{u}),\hat{u}(t),t)^T \hat{\lambda}(t) + \\ + \hat{\lambda}^0 \frac{\partial h^0}{\partial x}(x(t,\hat{\xi},\hat{u}),\hat{u}(t),t),$$

$$t \in [t_0,t_f],$$

4.22 $$\hat{\lambda}(t_0) = \frac{\partial g_0(\hat{\xi})^T}{\partial x}\hat{\nu}_0$$

4.23 $$\hat{\lambda}(t_f) = -\frac{\partial g_f(x(t_f,\hat{\xi},\hat{u}))^T}{\partial x}\hat{\nu}_f,$$

where $\hat{\nu}_0 \geqq 0$, $\hat{\nu}_f \geqq 0$ are such that $(\hat{\nu}_0,\hat{\nu}_f) \neq 0$,

$$4.24 \qquad \langle \hat{\nu}_0, g_0(\hat{\xi}) \rangle = \langle \hat{\nu}_f, g_f(x(t_f,\hat{\xi},\hat{u}) \rangle = 0,$$

and

$$4.25 \qquad \frac{\partial}{\partial u}[\hat{\lambda}^0 h^0(x(t,\hat{\xi},\hat{u}),\hat{u}(t),t) + \langle \hat{\lambda}(t), h(\hat{x}(t,\hat{\xi},\hat{u}),\hat{u}(t),t) \rangle] \equiv 0.$$

(since $(\hat{\xi},\hat{u}) \in \Delta \subset \Omega$, we must have $g(\hat{\xi}) \leqq 0$ and $g_f(x(t_f,\hat{\xi},\hat{u})) \leqq 0$ by definition). □

Proof: Let $(\hat{\xi},\hat{u}) \in \Delta$ and let $\hat{\mu}^j$, $j = 0,1, \ldots, m$, be such that (2.11) - (2.14) are satisfied. Let $\hat{\nu}_0 = (\hat{\mu}^1, \hat{\mu}^2, \ldots, \hat{\mu}^{m'})^T$, let $\hat{\nu}_f = (\hat{\mu}^{m'+1}, \ldots, \hat{\mu}^{m'+m''})$, let $\hat{\lambda}^0 = \hat{\mu}^0$, and let $\hat{\lambda}$: $[t_0,t_f] \to \mathbb{R}^n$ be defined by

$$4.26 \qquad \hat{\lambda}(t) = \hat{\mu}^0 \hat{p}(t) + \sum_{i=1}^{m''} \hat{\mu}^{i+m'} \hat{q}_i(t), \; t \in [t_0,t_f],$$

where $\hat{p}(t) = p(t,\hat{\xi},\hat{u})$ and $\hat{q}_i(t) = q_i(t,\hat{\xi},\hat{u})$, are defined as in (4.15) and (4.18) respectively. Next (2.13), in conjunction with (4.14), (4.16) and (4.17) yields

$$4.27 \qquad -\hat{\mu}^0 \hat{p}(t_0) + \frac{\partial g_0(\hat{\xi})^T}{\partial x} \hat{\nu}_0 - \sum_{i=1}^{m''} \hat{\mu}^{i+m'} \hat{q}_i(t_0) = 0,$$

and also (with $\hat{x}(t) \equiv x(t,\hat{\xi},\hat{u})$),

$$4.28 \qquad \hat{\mu}^0 \left(-\frac{\partial h}{\partial u}(\hat{x}(t),\hat{u}(t),t)^T \hat{p}(t) + \frac{\partial h^0}{\partial u}(\hat{x}(t),\hat{u}(t),t)^T\right) + \sum_{i=1}^{m''} \mu^{i+m'} \frac{\partial h}{\partial u}(\hat{x}(t),\hat{u}(t),t)^T \hat{q}_i(t) = 0, \; t \in [t_0,t_f].$$

Now, making use of (4.26), (4.15) and (4.18), we find that $\hat{\lambda}(\cdot)$ satisfies (4.21); (4.27) yields that $\hat{\lambda}(t_0)$ satisfies (4.22), and $\hat{\lambda}(t_f)$ satisfies (4.23). Finally, (4.28) shows that $(\hat{\lambda}^0, \hat{\lambda}(\cdot))$ satisfies (4.25). By construction $\hat{\nu}_0$ and $\hat{\nu}_f$ satisfy $\hat{\nu}_0 \geqq 0$, $\hat{\nu}_f \geqq 0$ and (4.24), which completes our proof. □

4.29. Remark: The relations (4.21) - (4.25) are known as the Pontryagin-Maximum-Principle-in-Differential-Form. □

For the problem (4.1), algorithm (3.2) assumes the following specific form.

4.30. Algorithm: (Solves (4.1); $\beta \in (0,1)$ is a step size parameter).

Step 0: Compute $(\xi_0, u_0) \in \mathcal{B}_1$ such that $g_0(\xi_0) \leqq 0$ and $g_f(x(t_f, \xi_0, u_0)) \leqq 0$, and set $i = 0$.

Comment: The algorithm (4.30) can be used to compute such an $(\xi_0, u_0(\cdot))$ by solving the problem

4.31
$$\{\min \int_{t_0}^{t_f} x^0(t)dt \mid \frac{d}{dt}\underline{x} = \underline{h}(x,u,t);$$
$$g_0^j(x(t_0)) - x^0(t_0) \leq 0,\ j = 1,2, \ldots, m';$$
$$g_f^j(x(t_f)) - x^0(t_f)) \leq 0,\ j = 1,2, \ldots, m''\},$$

where $\underline{x} = (x^0, x)$ and $\underline{h} = (0, h)$, and for which an initial point $(\tilde{\underline{\xi}}_0, \tilde{u}_0(\cdot))$, $\tilde{\underline{\xi}}_0 = (\tilde{x}_0^0, \tilde{\xi}_0)$, can be chosen as follows: let $\tilde{\xi}_0$, $\tilde{u}_0(\cdot)$ be arbitrary, and let $\tilde{x}_0^0 = \max\{g_0^j(\tilde{\xi}_0),\ j = 1,2, \ldots, m';\ g_f^j(x(t_f, \tilde{\xi}_0, u),\ j = 1,2, \ldots, m''\}$. Since the optimal value of (4.31) is strictly negative, a $(\xi_0, u_0(\cdot))$ for Step 0 above can be computed by means of a finite number of iterations.

Step 1: For $z_i = (\xi_i, u_i)$, compute $\nabla f^0(z_i)$, $\nabla f_1^j(z_i)$, $j = 1,2, \ldots, m'$, $\nabla f_2^j(z_i)$, $j = 1,2, \ldots, m''$, according to (4.14) - (4.18).

Step 2: Compute $\mu^0(z_i)$, $\mu_1^j(z_i)$, $j = 1,2, \ldots, m'$, $\mu_2^j(z_i)$, $j = 1,2, \ldots, m''$, as a solution of

4.32
$$\phi(z_i) = \max\{ \sum_{j=1}^{m'} \mu_1^j g_0^j(\xi_i) + \sum_{j=1}^{m''} \mu_2^j g_f^j(x(t_f,\xi_i,u_i)) + -\frac{1}{2} \| \mu^0 \nabla f^0(z_i) + \sum_{j=1}^{m'} \mu_1^j \nabla f_1^j(z_i) + \sum_{j=1}^{m''} \mu_2^j \nabla f_2^j(z_i) \|_2^2 \mid \mu^0 + \sum_{j=1}^{m'} \mu_1^j + \sum_{j=1}^{m''} \mu_2^j = 1,\ \mu^0 \geq 0,\ \mu_1^j \geq 0 \ \ j = 1,2, \ldots, m',\ \mu_2^j \geq 0,\ j = 1,2, \ldots, m''\},$$

where $\|z\|_2^2 = \langle z,z \rangle_2$ is defined as in (4.3).

Step 3: If $\phi(z_i) = 0$, set $\hat{\xi} = \xi_i$, $\hat{u}(\cdot) = u_i(\cdot)$ and stop; else, go to Step 4 (see (4.14) - (4.18))

Step 4: Set

4.33
$$\omega_i = \mu^0(z_i)p(t_0,\xi_i,u_i) - \sum_{j=1}^{m'} \mu_1^j(z_i) \frac{\partial g_0^j(\xi_i)^T}{\partial x} + + \sum_{j=1}^{m''} \mu_2^j(z_i) q_j(t_0,\xi_i,u_i),$$

4.34
$$\nu_i(\cdot) = \mu^0(z_i) \left[\frac{\partial h}{\partial u} (x(\cdot,\xi_i,u_i),u_i(\cdot),\cdot)^T \times p(\cdot,\xi_i,u_i) - \frac{\partial h^0}{\partial u} (x(\cdot,\xi_i,u_i),u(\cdot),\cdot)^T \right] + + \sum_{j=1}^{m''} \mu_2^j(z_i) \frac{\partial h}{\partial u} (x(\cdot,\xi_i,u_i),u_i(\cdot),\cdot)^T q_j(\cdot,\xi_i,u_i),$$

and go to Step 5.

Step 5: Compute the smallest integer k, such that

$$4.35 \quad \max\{\int_{t_0}^{t_f} [h^0(x(t,\xi_i + \beta^k\omega_i, u_i + \beta^k\nu_i), u_i + \beta^k\nu_i, t)dt - h^0(x(t,\xi_i,u_i), u_i, t)]dt,$$

$$g_0^j(\xi_i + \beta^k\omega_i),\ j = 1,2,\ \ldots,\ m';\ g_f^j(x(t_f,\xi_i + \beta^k\omega_i, u_i + \beta^k\nu_i)),$$

$$j = 1,2,\ \ldots,\ m''\} - \frac{\beta^k}{2}\phi(z_i) \leq 0.$$

Step 6: Set $\xi_{i+1} = \xi_i + \beta^k\omega_i$, set $u_{i+1}(\cdot) = u_i(\cdot) + \beta^k\nu_i(\cdot)$, set $i = i+1$, and go to Step 1. □

The following result is obvious.

4.35 Proposition: Theorem (3.17) holds for algorithm (4.30), with the set Δ defined as the set of points $(\hat{\xi},\hat{u}) \in \mathcal{B}_1$ satisfying the Pontryagin-Maximum-Principle-in-Differential Form for problem (4.1), (see 4.29). □

5. <u>An Extension to Problems with Instantaneous Constraints on the Control</u>.

We shall now show that the so called "Valentine's trick" can be used to adapt algorithm (4.30) for the solution of the following optimal control problem:

5.1 $$\min\{\int_{t_0}^{t_f} h^0(x(t),u(t),t)dt \,|\, \frac{d}{dt}x(t) = h(x(t),u(t),t),$$

$$t \in [t_0,t_f];\ g_0(x(t_0)) \leqq 0,\ g_f(x(t_f)) \leqq 0;\ u \in L_\infty^s[t_0,t_f];$$

$$b^k \leq \langle a_k,u(t)\rangle \leq c^k,\ k = 1,2,\ldots,r, \text{ for all } t \in [t_0,t_f]\},$$

where h^0,h,g_0,g_f are as in (4.1); $a_k \in \mathbb{R}^s$ for $k = 1,2,\ldots,r$; $c_k \in \mathbb{R}^1$ for $k = 1,2,\ldots,r$, $b_k \in \mathbb{R}^1$ for $k = 1,2,\ldots,r'$, $r' \leqq r$, and $b_k = -\infty$ for $k = r' + 1,\ldots,r$.

5.2. <u>Assumptions</u>:

(i) We shall assume that (4.9) is satisfied.

(ii) The vectors a_k, $k = 1,2,\ldots,r$ are linearly independent.

(iii) There exists a control $\bar{u} \in L_\infty^s\,[t_0,t_f]$ and an initial state $\bar{\xi} \in \mathbb{R}^n$ such that $g_0(\bar{\xi}) \leqq 0$, $g_f(x(t_f,\bar{\xi},u)) \leqq 0$, and $b^k < \langle a_k,\bar{u}(t)\rangle < c^k$ for $k = 1,2,\ldots,r$ and all $t \in [t_0,t_f]$. □

To apply the Valentine trick, we must use certain substitutions for the inequalities on the control. Thus, consider the constraints

5.2 $$b^k \leq \langle a_k,u(t)\rangle \leqq c^k,\ k = 1,2,\ldots,r',\ t \in [t_0,t_f]$$

5.3 $$\langle a_k,u(t)\rangle \leq c^k,\ k = r' + 1,\ldots,r,\ t \in [t_0,t_f].$$

Suppose that $u \in L_\infty^s [t_0,t_f]$ satisfies (5.2), (5.3), then we can associate with this u functions v^k: $[t_0,t_f] \to \mathbb{R}^1$, $k = 1,2,\ldots,r'$ and w^k: $[t_0,t_f] \to \mathbb{R}^1$, $k = 1,2,\ldots,r - r'$, such that

$$5.4 \qquad \cos v^k(t) = \frac{2}{c^k-b^k} \langle a_k,u(t) \rangle - \frac{c^k+b^k}{c^k-b^k} , \; k = 1,2,\ldots r', \quad t \in [t_0,t_f],$$

$$5.5 \qquad (w^k(t))^2 = c^{k+r'} - \langle a_{k+r'},u(t) \rangle , \; k = 1,2,\ldots,r - r', \quad t \in [t_0,t_f].$$

We shall now use these functions to construct a problem which is equivalent to 5.1. Let A^T be the s×r matrix whose columns are $\frac{2}{c^k-b^k} a_k$, $k = 1,2,\ldots,r'$ and a_k, $k = r' + 1, r' + 2,\ldots,r$. Then its transpose, A, is an r×s matrix which can be partitioned as follows $A = [A' \,\vdots\, A'']$, (rearranging the components of $u(\cdot)$, if necessary) where A" is an r×r nonsingular matrix. We partition u similarly, i.e., we set $u = (u',u'')$, with $u'' \in \mathbb{R}^s$ and $u' \in \mathbb{R}^{s-r}$. Then, if u,v^k, w^k, satisfy (5.4) and (5.5), we obtain

$$5.6 \qquad u''(t) = A''^{-1}(\cos v'(t),\ldots, \cos v^{r'}(t), \omega^1(t)^2,\ldots, \omega^{r-r'}(t)^2)^T) - A''^{-1} A'u'(t) \triangleq f(u'(t),v(t),w(t)).$$

Now consider the problem

$$5.7 \qquad \min\{ \int_{t_0}^{t_f} h^0(x,(u',f(u',v,\omega)),t)dt \,|$$

$$\dot{x} = h(x,(u',f(u',v,\omega),t), \; t \in [t_0,t_f];$$

$$g_0(x(t_0)) \leqq 0; \; g_f(x(t_f)) \leqq 0; \; (u',v,\omega) \in L_\infty^s \,[t_0,t_f]\}$$

where all the quantities are as in (5.1) and (5.6). It is trivial to show that if $(\hat{\xi},\hat{u},\hat{v},\hat{\omega})$ is any optimal solution of (5.7), $(\hat{\xi},(\hat{u},\cos \hat{v},\hat{\omega}^2$ is also optimal for problem (5.1). However, not all the points $(\tilde{\xi}, \tilde{u}',\tilde{v},\tilde{\omega})$ which satisfy the Pontryagin principle for problem (5.7), result in a pair $(\tilde{\xi},\tilde{u}) \triangleq (\tilde{\xi},(\tilde{u}',\cos \tilde{v},\tilde{\omega}^2))$ which satisfy the maximum principle for problem (5.1). Hence, although algorithm (4.30) is directly applicable to problem (5.7) (with h^0, h, redefined, of course), it is desirable to modify it so as to prevent convergence to points which do not satisfy the optimality conditions for problem (5.1). This can be done as stated in the algorithm below, where, for the sake of simplifying our expressions, and without loss of generality, we assume that there is only one constraint of the form (5.2) and only one constraint of the form (5.3), i.e. we assume that (5.2), (5.3) have the following specific form:

5.8 $$-1 \leq u^{s-1}(t) \leq 1, \; u^s(t) \geq 0, \; t \in [t_0,t_f]$$

and that the remaining components of u(t) are unconstrained. In this case we have $u' = (u^1,u^2,\ldots,u^{s-2})$, $z = (\xi,u',v,\omega)$, $f^0(z) = \int_{t_0}^{t_f} h^0 \,(x(t,\xi,u))dt$, $f_1(z) = g_0(\xi)$, and $f_2(z) = g_f(x(t_f,\xi,u))$, where $u \equiv (u^1,\cos v,\omega^2)$.

5.9 Algorithm (Solves (5.1) for $r' = 1$, $r = 2$, $b^1 = -1$, $c^1 = 1$, $a_1 = (0,0,\dots,0,1,0)^T$, $a_2 = (0,0,\dots,0,-1)^T$, $c^2 = 0$; $\beta \in (0,1)$ is a step size parameter).

Step 0: Compute a $(\xi_0,u_0(\cdot))$ such that $g_0(\xi_0) \leq 0$; $g_f(x(t_f;\xi_0,u_0) \leq 0$; $|u_0^{s-1}(t)| < 1$, $u^s(t) > 0$, $t \in [t_0,t_f]$*; set $v_0(\cdot) = \sin^{-1} u_0^{s-1}(\cdot)$; set $w_0 = u_0^s(\cdot)^{1/2}$; set $i = 0$. (Note that we dropped the superscript 1 on v_0 and ω_0 since these are redundant in this case.)

Step 1: Compute $x_i(t)$, $t \in [t_0,t_f]$, by solving

5.10 $$\frac{d}{dt} x_i(t) = h(x_i(t),u_i(t),t),\ t \in [t_0,t_f],\ \text{with } x_i(t_0) = \xi_i.$$

Step 2: Compute $p_i(t) \in \mathbb{R}^n$, $q_{\ell,i}(t) \in \mathbb{R}^n$, $\ell = 1,2,\dots, m''$, $t \in [t_0,t_f]$, by solving

5.11 $$\frac{d}{dt} p_i(t) = -\frac{\partial h}{\partial x}(x_i(t),\ u_i(t),t)^T\ p_i(t) + \frac{\partial h^0}{\partial x}(x_i(t),u_i(t),t)^T,$$
$$t \in [t_0,t_f],\ p_i(t_f) = 0;$$

5.12 $$\frac{d}{dt} q_{\ell,i}(t) = -\frac{\partial h}{\partial x}(x_i(t),u_i(t),t)^T q_{\ell,i}(t),\ t \in [t_0,t_f],$$
$$q_{\ell,i}(t_f) = -\frac{\partial g_f^\ell(x_i(t_f))^T}{\partial x},\ \ell = 1,2,\dots,m''.$$

---

*To compute $(\xi_0,u_0(\cdot))$ choose $\varepsilon > 0$ small and use algorithm (4.30) on the following problem

$$\min\{\int_{t_0}^{t_f} h^0(x,(u',(1-\varepsilon)\cos v,\omega^2+\varepsilon),t)dt \,|\, \dot{x} = h(x,(u',(1-\varepsilon)\cos v,\omega^2+\varepsilon),t)$$
$$g_0(x(t_0)) \leq 0;\ g_f(x(t_f)) \leq 0\}.$$

Step 3: Compute $\nabla f^0(z_i)$, $\nabla f_1^k(z_i)$, $k = 1,2,\ldots,m'$, $\nabla f_2^\ell(z_i)$, $\ell = 1,2,\ldots,m''$, with $z_i = (u_i', v_i, w_i)$ and $u_i = (u_i', \cos v_i, w_i^2))$ according to

5.13 $$\nabla f^0(z_i)^T = (-\,p_i(t_0)^T,\ (-\,p_i(\cdot)^T \frac{\partial h}{\partial u'}(x_i(\cdot),u_i(\cdot),\cdot),$$
$$-\sin v_i(\cdot)p_i(\cdot)^T \frac{\partial h}{\partial u^{s-1}}(x_i(\cdot),u_i(\cdot),\cdot),$$
$$2w_i(\cdot)p_i(\cdot)^T \frac{\partial h}{\partial u^s}(x_i(\cdot),u_i(\cdot),\cdot)));\ ^*$$

5.14 $$\nabla f_1^k(z_i)^T = \left(\frac{\partial g_0^k(\xi_i)}{\partial x},\ 0\right),\ k = 1,2,\ldots,m'$$

5.15 $$\nabla f_2^\ell(z_i)^T = (-\,q_{\ell,i}(0)^T,\ (-\,q_{\ell,i}(\cdot)^T \frac{\partial h}{\partial u'}(x_i(\cdot),u_i(\cdot),\cdot),$$
$$-\sin v_i(\cdot)q_{\ell,i}(\cdot)^T \frac{\partial h}{\partial u^{s-1}}(u_i(\cdot),u_i(\cdot),\cdot),$$
$$2w_i(\cdot)q_{\ell,i}(\cdot)^T \frac{\partial h}{\partial u^s}(x_i(\cdot),u_i(\cdot),\cdot))),\ \ell = 1,2,\ldots,m''.$$

Step 4: Compute $\mu^0(z_i)$, $\mu_1^\ell(z_i)$, $\ell = 1,2,\ldots,m'$, $\mu_2^k(z_i)$, $k = 1,2,\ldots,m''$, as a solution of

5.16 $$\phi(z_i) = \max\{\sum_{k=1}^{m'} \mu_1^k g_0^k(\xi_i) + \sum_{\ell=1}^{m''} \mu_2^\ell g_f^\ell(x(t_f,\xi_i,u_i))$$
$$-\frac{1}{2}\|\mu^0\nabla f^0(z_i) + \sum_{k=1}^{m'}\mu_1^k\nabla f_1^k(z_i) + \sum_{\ell=1}^{m''}\mu_2^\ell\nabla f_2^\ell(z_i)\|_2^2 \Big|$$

*by $(\xi,u(\cdot))^T$ we mean $(\xi^T,u(\cdot)^T)$.

$$\mu^0 + \sum_{k=1}^{m'} \mu_1^k + \sum_{\ell=1}^{m''} \mu_2^k = 1,\ \mu^0 \geq 0,\ \mu_1^k \geq 0,\ k = 1,2\ldots,\ m',$$

$$\mu_2^\ell \geq 0,\ \ell = 1,2,\ldots,m''\}.$$

<u>Step 5</u>: If $\phi(z_i) = 0$, set $\hat{x}(\cdot) = x_i(\cdot)$, $\hat{u}(\cdot) = (u_i'(\cdot), \cos v_i(\cdot), w_i^2(\cdot))$ and stop; else (c.f. (4.33), (4.34)), set

5.17 $$\omega_i = \mu^0(z_i)p_i(t_0) - \sum_{k=1}^{m'} \mu_1^k(z_i) \frac{\partial g_0}{\partial x} (\xi_i)^T + \sum_{\ell=1}^{m''} \mu_2^\ell(z_i) q_{\ell,i}(t_0)$$

5.18 $$\nu_i^T = (\nu_i'^T, \nu_i^{s-1}, \nu_i^s)$$

with

$$\nu_i'^T(\cdot) = \mu^0(z_i)\left[\frac{\partial h}{\partial u'} (x_i(\cdot),u_i(\cdot),\cdot)^T p_i(\cdot) - \frac{\partial h^0}{\partial u'} (x_i(\cdot),u_i(\cdot),\cdot)^T\right]$$

$$+ \sum_{\ell=1}^{m''} \mu_2^\ell(z_i) \frac{\partial h}{\partial u'} (x_i(\cdot),u_i(\cdot),\cdot)^T q_{\ell,i}(\cdot)$$

$$\nu_i^{s-1}(\cdot) = - \sin v_i(\cdot)\{\mu^0(z_i)\left[\frac{\partial h}{\partial u^{s-1}} (x_i(\cdot),u_i(\cdot),\cdot)^T p_i(\cdot)\right.$$

$$\left. - \frac{\partial h^0}{\partial u^{s-1}} (x_i(\cdot)u_i(\cdot),\cdot)^T\right] + \sum_{\ell=1}^{m''} \mu_2^\ell(z_i) \frac{\partial h}{\partial u^{s-1}} (x_i(\cdot),u_i(\cdot),\cdot)^T q_{\ell,i}(\cdot)\}$$

$$\nu_i^s(\cdot) = 2\omega_i(\cdot)\{\mu^0(z_i)\left[\frac{\partial h}{\partial u^s} (x_i(\cdot),u_i(\cdot),\cdot)^T p_i(\cdot)\right.$$

$$\left. - \frac{\partial h^0}{\partial u^s} (v_i(\cdot),\cdot)^T\right] + \sum_{\ell=1}^{m''} \mu_2^\ell(z_i) \frac{\partial h}{\partial u^s} (x_i(\cdot),u_i(\cdot),\cdot)^T q_{\ell,i}(\cdot)\}$$

and go to Step 6.

Step 6: Compute the smallest integer k_i satisfying

5.19
$$\max\{ \int_{t_0}^{t_f} [h^0(x(t,\xi_i+\beta^{k_i}\omega_i,(u_i'+\beta^{k_i}\nu_i',\ \cos(v_i+\beta^{k_i}\nu_i^{s-1}),\ (\omega_i+\beta^{k_i}\nu_i^{s})^2),$$
$$(u_i'(t)+\beta^{k_i}\nu_i'(t),\ \cos(v_i(t)+\beta^{k_i}\nu_i^{s-1}(t)),\ (\omega_i(t)+\beta^{k_i}\nu_i^{s}(t))^2,t)$$
$$- h^0(x_i(t),u_i(t),t))dt,\ g_0(\xi_i+\beta^{k_i}\omega_i),\ g_f(x(t_f,\xi_i+\beta^{k_i}\omega_i,$$
$$(u_i'+\beta^{k_i}\nu_i',\cos(v_i+\beta^{k}\nu_i^{s-1}),\ (w_i+\beta^{k_i}\nu_i^{s})^2))\} \leq 0;$$

5.20
$$\beta^{k_i} \leq \left(2 \times \operatorname*{ess\,sup}_{t \in [0,1]} \{\max\{\left|\frac{\nu_i^{s-1}(t)}{\sin v_i(t)}\right|,\ -\frac{\nu_i^{s}(t)}{w_i(t)}\ \}\}\right)^{-1}$$

Step 7: Set $\xi_{i+1} = \xi_i + \beta^{k_i}\omega_i$, $u_{i+1}'(\cdot) = u_i'(\cdot) + \beta^{k_i}\nu_i'(\cdot)$, $v_{i+1}(\cdot) = v_i(\cdot) + \beta^{k_i}\nu_i^{s-1}(\cdot)$, $w_{i+1}(\cdot) = w_i(\cdot) + \beta^{k_i}\nu^{s}(\cdot)$.

Step 8: Set $u_{i+1}(\cdot)^T = (u_{i+1}'(\cdot)^T,\ \cos v_{i+1}(\cdot),\ w_{i+1}(\cdot)^2)$.

Step 9: Set $i = i+1$ and go to Step 1. □

5.21 Lemma: Let $\Delta = \{z = (\xi,u',v,w) \mid \xi \in \mathbb{R}^n,\ u' \in L_\infty^{s-2}[t_0,t_f],$
$v \in L_\infty^1[t_0,t_f],\ \omega \in L_\infty^1[t_0,t_f],\ \phi(z) = 0,\ g_0(\xi) \leq 0,$
$g_f(x(t_f,\xi,(u',\cos v,w)) \leq 0\}$ where $\phi(\cdot)$ is defined as in (5.16). Then the conclusions of theorem (3.17) remain true for algorithm (5.9).

Proof: Algorithm (5.9) differs from algorithm (4.30) only in the additional bound (5.20) on β^k. Now, since this bound, in turn, has a denominator which can be bounded from above by a continuous function of (ξ_i, u_i', v_i, w_i), i.e., since

5.21 $$\max\left\{\left|\frac{\nu_i^{s-1}(t)}{\sin v_i(t)}\right|, -\frac{\nu_u^s(t)}{w_i(t)}\right\} \leq$$

$$\leq \sum_{k=s-1}^{s} \left\{\left|(p_i(t)^T \frac{\partial h}{\partial u^k}(x_i(t),u_i(t),t) - \frac{\partial h^0}{\partial u^k}(x_0(t),u_i(t),t))\right|\right.$$

$$\left. + \sum_{\ell=1}^{m''}\left|q_{i,\ell}(t)^T \frac{\partial h}{\partial u^k}(x_i(t),u_i(t),t)\right|\right\}$$

it follows from arguments essentially duplicating the proof of theorem (3.17) that the conditions of theorem (2.16) are satisfied. □

5.22 Lemma: Let $\{(\xi_i,u_i',v_i,\omega_i)\}_{i=0}^{\infty}$ be a sequence generated by algorithm (5.9) in the process of solving problem (5.8). Suppose that K is an infinite subset of the positive integers such that $\lim_{i\in K} \|(\xi_i,u_i',v_i,\omega_i) - (\hat{\xi},\hat{u}',\hat{v},\hat{\omega})\|_2 = 0$ and $\sup_{i\in K} \|(\xi_i,u_i',v_i,\omega_i)\|_1 < \infty$. Furthermore, let K' be an infinite subset of K, such that $\lim_{i\in K'} \{(\mu^0(z_i),\mu_1(z_i),\mu_2(z_i)\} = \{\hat{\mu}^0,\hat{\mu}_1,\hat{\mu}_2\}$, where $z_i = (\xi_i,(u_i',v_i,\omega_i))$ and $\mu^0(z_i),\mu_1(z)$ and $\mu_2(z_i)$ are defined as in Step 4 of (5.9). Then

5.23 $$\left\{\frac{\partial h^0}{\partial u^{s-1}}(\hat{x}(t),\hat{u}(t),t) - \frac{\partial h}{\partial u^{s-1}}(\hat{x}(t),\hat{u}(t),t)^T \times\right.$$

$$\left.(\hat{\mu}^0\hat{p}(t) + \sum_{\ell=1}^{m''}\hat{\mu}_2^{\ell}\hat{q}_\ell(t))\right\}(-1)^k \leq 0$$

for almost all $t \in \{t|\hat{v}(t) = k\pi\}$, $k = 0,1$,

and

5.24 $$-\frac{\partial h^0}{\partial u^s}(\hat{x}(t),\hat{u}(t),t) - \frac{\partial h}{\partial u^s}(\hat{x}(t),\hat{u}(t),t)^T(\hat{\mu}_0\hat{p}(t) + \sum_{\ell=1}^{m''}\hat{\mu}_2^\ell\hat{q}_\ell(t)) \geq 0$$

for almost all $t \in \{t|\hat{w}(t) = 0\}$, $\hat{u} = (\hat{u}',\cos\hat{v},\hat{\omega}^2)$, $\hat{x}(t) \equiv x(t,\hat{\xi},\hat{u})$,

and $\hat{p}$, $\hat{q}_\ell$ defined by (5.11) and (5.12) for $u_i(t) \equiv \hat{u}(t)$, $x_i(t) \equiv \hat{x}(t)$.

<u>Proof</u>: Let H: $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{m'+m''} \times \mathbb{R}^1 \to \mathbb{R}^1$ be defined by

5.25 $H(x,u,\psi,t) = -h^0(x,u,t) + \langle\psi,h(x,u,t)\rangle$. Then, from (5.18) and the instructions in Step 7 of (5.9), we find that for $t \in [t_0,t_f]$,

5.26 $$v_{i+1}(t) = v_i(t) - \beta^{k_i}\sin v_i(t)\frac{\partial H}{\partial u^{s-1}}(x_i(t),u_i(t),\psi_i(t),t),$$

5.27 $$w_{i+1}(t) = w_i(t)\ 2\beta^{k_i}w_i(t)\frac{\partial H}{\partial u^s}(x_i(t),u_i(t),\psi_i(t),t),$$ with,

for $t \in [t_0,t_f]$,

5.28 $$\psi_i(t) = \mu^0(z_i)p_i(t) + \sum_{\ell=1}^{m''}\mu_2^\ell(z_i)q_{\ell,i}(t)\ .$$

Now, the purpose of the bound (5.20) on β^{k_i} was to ensure that for almost all $t \in [t_0,t_f]$,

5.29 $$0 < \frac{1}{2}v_i(t) \leq v_{i+1}(t) \leq \frac{1}{2}(\pi+v_i(t)) < \pi,$$

5.30 $$\frac{1}{2}w_i(t) \leq w_{i+1}(t).$$

Since we must have $\lim_{i \in K'} v_i(t) = \hat{v}(t)$ for almost all $t \in [t_0, t_f]$, suppose that $t' \in [t_0, t_f]$ is such that $\lim_{i \in K'} v_i(t') = \hat{v}(t') = k\pi$, with $k = 0$ or 1. It now follows from (5.26), since $\beta^{k_i} < 1$, and since $\frac{\partial H}{\partial u^{s-1}}(x_i(t'), u_i(t'), \psi_i(t'), t')$ is bounded for $i \in K'$, that we must also have

5.31 $$\lim_{i \in K'} v_{i+1}(t') = k\pi,$$

for almost all t' such that $\lim_{i \in K'} v_i(t') = k\pi$.*

By construction, $v_0(t) \in (0,\pi)$ for all $t \in [t_0, t_f]$, and hence $\sin v_0(t') > 0$. Let K'' be an infinite subsequence of the positive integers defined by $K'' = K' \cup \{i \mid (i-1) \in K'\}$. It now follows from (5.29) and (5.31) that $\{(-1)^k (k\pi - v_i(t))\}_{i \in K''}$ is a strictly positive sequence which converges to zero, and hence there must exist an infinite subsequence $K''' \subset K'$ such that

5.32 $$(-1)^{k+1}(k\pi - v_{i+1}(t') < (-1)^{k+1}(k\pi - v_i(t')) \text{ for all } i \in K'''.$$

Combining (5.32) with (5.26), and recalling that $\sin v_i(t') > 0$ for all i, we conclude that

5.33. $$(-1)^k \frac{\partial H}{\partial u^{s-1}}(x_i(t'), u_i(t'), \psi_i(t'), t) \geq 0 \text{ for all } i \in K'''.$$

It now follows from the continuity of $\frac{\partial H}{\partial u^{s-1}}$ that

*Note that $\{t \mid \lim_{i \in K} v_{i+1}(t) \neq \lim_{i \in K} v_i(t)\}$ is a null set

5.34. $(-1)^k \frac{\partial H}{\partial u^{s-1}} (\hat{x}(t'),\hat{u}(t'),\hat{\psi}(t'),t') \geq 0,$

where $\hat{\psi}(t') = \hat{\mu}^0\hat{p}(t') + \sum_{\ell=1}^{m''} \hat{\mu}_2^{\ell}\hat{q}_{\ell,i}(t')$. This establishes (5.23); (5.24) can be established in a similar way. □

The following result is now obvious.

5.35. Corollary: Let $\{(\xi_i,u_i)\}$ be a sequence generated by algorithm (5.9) in solving problem (5.1). Then, either $\{(\xi_i,u_i)\}$ is finite and its last element satisfies the Pontryagin maximum principle in differential form*, or $\{(\xi_i,u_i)\}$ is infinite and every pair of points $(\hat{\xi},\hat{u})$ which satisfies, for some $K \subset \{0,1,2,..\}$ either (i) $\lim_{i\in K} \|(\xi_i,u_i) - (\hat{\xi},\hat{u})\|_1 = 0$, or (ii) $\lim_{i\in K} \|(\xi_i,u_i) - (\hat{\xi},\hat{u})\|_2 = 0$ and $\sup_{i\, K} \|(\xi_i,u_i)\|_1 < \infty$, also satisfies the Pontryagin maximum principle in differential form. □

*In the maximum principle in differential form, the condition of maximum on the hamiltonian is replaced by the condition $\frac{\partial H(\hat{x},\hat{u},\hat{\psi},t)}{\partial u} \delta u \leq 0$ for all admissible δu and for almost all $t \in [t_0,t_f]$.

Conclusion.

In the form stated, algorithm (4.30) is readily implementable provided that the integrations of the various differential equations involved are done with reasonable care. The results in [3] indicate that the accuracy of integration can be relaxed in the method of steepest descent and the algorithm speeded up considerably, provided that an adaptive mechanism of integration is introduced into the algorithm. It appears that a similar device should also be possible for algorithms of the type described in this paper, though the exact manner in which this can be done is still to be worked out.

In constructing algorithm (5.9), a substitution formula (Valentine's trick) was used to extend algorithm (4.30) to problems with affine instantaneous inequality constraints and, in addition, a perturbation method was added to ensure that convergence was possible only to points satisfying both a first and a second order optimality condition for the desired problem (5.7). This eliminated points which satisfy a first order condition for (5.7) but not for (5.1). Industrial experience with the substitution formula, used in conjuction with simpler algorithms, indicates that it performs quite well, and certainly better than a penalty function. Incidentally, penalty functions could also have been used to cope with constraints on the controls, by following the pattern of algorithm (4.3.91) in [7].

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Appendix A.

The functions and norms used below are as in Sections 2 and 3.

A1. **Theorem:** Let $M > 0$ be such that $\Omega_M = \{z \in \Omega \mid \|z\|_1 < M\} \neq \phi$, and let $S_M : \Omega_M \to 2^{\Omega_M}$ be the restriction of S to Ω_M, where S was defined in (3.18). If assumptions (2.3) hold, then $S_M(\cdot)$ is upper semi-continuous on Ω_M with respect to the norm $\|\cdot\|_2$.

Proof: We recall that by assumption (2.3), the functions f^j and ∇f^j, $j = 0, 1, 2, ..$, are all continuous on Ω_M with respect to $\|\cdot\|_2$. Now, let $\Gamma_M : \Omega_M \to 2^{\mathbb{R}^{m+1}}$ be defined by

$$\text{A.2} \qquad \Gamma_M(z) = \{\mu = (\mu^0, \mu^1, \ldots, \mu^m) \mid \sum_{j=0}^{m} \mu^j \nabla f^j(z) \in S_M(z)\}$$

We note that $\Gamma_M(\cdot)$ is a closed, compact valued map, and therefore it is upper semi-continuous (see [1] corollary to theorem 7, p. 112). Next, let $X_M : \Omega_M \times \mathbb{R}^{m+1} \to \mathbb{R}^n$ be defined by

$$\text{A.3} \qquad X_M(z,\mu) = \sum_{j=0}^{m} \mu^j \nabla f^j(z),$$

with $\mu = (\mu^0, \mu^1, \ldots, \mu^m)$. Then we see that $S_M(z) = X_M(z, \Gamma(z))$ and hence S_M is upper semi-continuous, according to theorem 1' p. 113 of [1], since $z \mapsto (z, \Gamma(z))$ is upper semi-continuous by theorem 4', p. 114 of [1] and X_M is continuous, which completes our proof. □