GLOBAL INVERSE FUNCTION THEOREM

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Felix F. Wu and Charles A. Desoer

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Electronics Research Laboratory

College of Engineering
University of California, Berkeley
94720

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Felix F. Wu and Charles A. Desoer

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory,
University of California, Berkeley, California 94720

ABSTRACT

A simple proof of global inverse function theorem in $\mathbb{R}^n$ is given. A global homeomorphic version of the theorem is proved first. A global diffeomorphic version follows by an application of the classical local inverse function theorem.

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The problem of determining that a given function from $\mathbb{R}^n$ into $\mathbb{R}^n$ has an inverse is very useful in applications. In 1959, Palais established the necessary and sufficient condition for a function to be a diffeomorphism of R^n onto itself, which appeared as an episode in a paper[1] dealing with the determination of spaces of intertwining operators on differential forms. This global version of the classical inverse function theorem has been applied widely in nonlinear network theory[2-12] and is generally referred to as Palais Theorem by circuit theorists. Palais originally stated it without proof as a corollary in [1]. We believe that a simple proof of this useful theorem will be helpful to the readers of this Journal.

This note presents a proof that is intuitively appealing and easily understood with a modest background in mathematical analysis[13,14]. We first prove the necessary and sufficient condition for a global homeomorphism[2]; the case of a global diffeomorphism follows easily by an application of the classical inverse function theorem.

THEOREM

Let f be a map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then f is a homeomorphism[1] of $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if f is

(1) a local homeomorphism[2] and

(2) a proper map[3].

Proof: ⇒ By assumption f is a (global) homeomorphism, hence it is a local homeomorphism. Because f^{-1} is continuous, it maps any compact set into a compact set. [13, p. 78; 14, Theorem 4.1, p. 207].

⇐ We prove this in three steps: (1) f is surjective (onto), (2) f is injective (one-to-one), (3) f^{-1} is continuous. To facilitate the presentation, we denote the domain of f by X and the range by Y; of course, $X = Y = \mathbb{R}^n$.

(1) Surjective: Let Y_1 be the image of f, i.e., $Y_1 = f(X)$, or more specifically, $Y_1 = \{y \in Y | f^{-1}(y)$ is a nonempty subset of $X\}$. We know that Y is connected. If Y_1 is both open and closed, knowing also the fact that Y_1 is not empty, we can conclude that $Y_1 = Y$ [13, p. 59], i.e., f is surjective.

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1. A homeomorphism of X onto Y is, by definition, a continuous bijective map f: $X \to Y$ such that f^{-1} is also continuous.

2. A map f: $X \to Y$ is said to be a local homeomorphism if whenever $x \in X$ and $y \in Y$ are such that $f(x) = y$ then there exist open neighborhoods U of x and V of y such that f restricted to U is a homeomorphism of U onto V.

3. A continuous map is said to be proper if the inverse image of any compact set is compact.

(i) Y_1 open: Let $y_1 \in Y_1$, then there exists an $x_1 \in X$ such that $f(x_1) = y_1$. Now f is a local homeomorphism means that there exist open neighborhoods U of x_1 and V of y_1 such that f is a homeomorphism from U onto V. So $V \subset Y_1$ and Y_1 is thus open.

(ii) Y_1 closed: Let y be an accumulation point of Y_1, then there exists a sequence $\{y_i\}_1^\infty$ with $y_i \in Y_1$ $\forall i$, and $y_i \to y$. Consider $K = \{y_i\}_1^\infty \cup \{y\}$, which is clearly closed and bounded in Y, hence compact. [13, p. 58; 14 Theorem 4.5, p. 208]. By assumption, $f^{-1}(K)$ is compact in X. Now pick $x_i \in f^{-1}(y_i)$. $\{x_i\}_1^\infty$ is a sequence in a compact set $f^{-1}(K)$ in a metric space $\mathbb{R}^n$, therefore $\{x_i\}_1^\infty$ has a convergent subsequence, say $\{x_{i_j}\}_{j=1}^\infty \to x$. [13, p. 56; 14, Th. 4-4, p. 208]. But $\{f(x_{i_j})\}_{j=1}^\infty$ is a subsequence of $\{y_i\}_1^\infty$, therefore converges to the same limit y. f is a continuous map because it is a local homeomorphism, hence $f(x) = f(\lim_{j\to\infty} x_{i_j}) = \lim_{j\to\infty} f(x_{i_j}) = y$, therefore $y \in Y_1$. Hence Y_1 is closed. [13, p. 47; 14, p. 203].

(2) Injective: Suppose that f is not injective, hence there exist two distinct points x_1, x_2 such that $f(x_1) = f(x_2)$. Without loss of generality, we can assume $f(x_1) = f(x_2) = 0$. Let α: $[0,1] \to X$ be defined by $\alpha(t) = (1-t)x_1 + tx_2$ and $\beta = f \circ \alpha$. Geometrically, α is the line segment joining x_1 to x_2 and β, its image in Y under f, is a closed curve through 0. (Fig. 1). Let B: $[0,1] \times [0,1] \to Y$ be defined by $B(t,\tau) = (1-\tau)\beta(t)$. Thus for each τ, $B(\cdot,\tau)$ is obtained by shrinking the closed curve β toward the origin. The rough idea of the proof is to shrink the curve β toward the origin, the corresponding curve α will be continuously deformed into some curve joining x_1 to x_2; the contradiction will be reached in the limit when β degenerates into a single point.

(i) Construction of the inverse image of B (Fig. 2): Let us define for each t a map $A(t,\cdot)$: $[0,1] \to X$ by the following process of piecing together the local inverses of B. First let $A(t,0) = \alpha(t)$. Since f is a local homeomorphism, there exist homeomorphic neighborhoods of $\alpha(t)$ and $\beta(t)$, U_1 and V_1 respectively. Define $A(t,\cdot)$: $[0,\tau_1] \to U_1$ to be the local inverse image of $B(t,\tau)$ for $\tau \in [0,\tau_1]$ where τ_1 is so chosen that $B(t,\tau) \in V_1$, for all $\tau \in [0,\tau_1]$. Thus we have $f(A(t,\tau_1)) = B(t,\tau_1)$ and we can define $A(t,\cdot)$ on $[\tau_1,\tau_2]$ with $\tau_2 > \tau_1$ as the local inverse of $B(t,\cdot)$ around $B(t,\tau_1)$. Repeat the same procedure; at each step, we extend τ from τ_k to τ_{k+1} with $\tau_{k+1} > \tau_k$. We are going to show by contradiction that the domain of $A(t,\cdot)$ can always be extended to include 1. Suppose that the above process fails to do so. Then the increasing sequence $\{\tau_k\}$ is bounded by 1 and has a least upper bound T, so $\{\tau_k\} \to T \leq 1$. But $B(t,\cdot)$ is continuous, $\lim_{k\to\infty} B(t,\tau_k) = B(t,T)$; and since $\{A(t,\tau_k)\}_{k=1}^{\infty}$ is a sequence in a compact set $f^{-1}(B(t,\tau) : \tau \in [0,1])$, it has a subsequence converging to a limit $A(t,T)$. Now because f is continuous, $f(A(t,T)) = B(t,T)$, hence the domain of $A(t,\cdot)$ is extended to include T; moreover, in the case when $T < 1$, it can even be extended beyond T by local homeomorphism. Thus, we can define a map A: $[0,1] \times [0,1] \to X$ with the property that $f \circ A = B$, and also $A(0,\tau) = x_1$, $A(1,\tau) = x_2$, $\forall\tau$.

(ii) Continuity of $A(\cdot,\tau)$: We will show that for each τ, $A(\cdot,\tau)$: $[0,1] \to X$ is continuous by open-set arguments [13, p. 70; 14, pp. 201-202]. Let $\mathcal{O}$ be any open set in X and let its inverse image under $A(\cdot,\tau)$ be denoted by $\mathcal{T}$; equivalently, $\mathcal{T}$ is the inverse image under $A(\cdot,\tau)$ of the intersection of $\mathcal{O}$ with the image of $A(\cdot,\tau)$. Now for each t, let the homeomorphic neighborhoods of $A(t,\tau)$ and $B(t,\tau)$ be U_t and V_t, respectively. Note that

$\bigcup_{t\in[0,1]} (U_t \cap \mathcal{O})$ has $\mathcal{T}$ as its inverse image under $A(\cdot,\tau)$ and $\bigcup_{t\in[0,1]} f(U_t \cap \mathcal{O})$ also has $\mathcal{T}$ as its inverse image under $B(\cdot,\tau)$ since $B = f\circ A$. But $f(U_t \cap \mathcal{O})$ are open in Y, so does $\bigcup_{t\in[0,1]} f(U_t \cap \mathcal{O})$. Because $B(\cdot,\tau)$ is continuous, $\mathcal{T}$ is open in [0,1]. This completes our proof that $A(\cdot,\tau)$ is continuous.

Now for $\tau = 1$, $B(t,1) = 0$, $\forall t$. Geometrically, this is done by shrinking β to the origin. The corresponding $A(t,1)$ is still a continuous curve joining two distinct points x_1 and x_2. But the inverse image of a single point under a local homeomorphism f can not be a continuous curve. To demonstrate this, suppose it were true, every neighborhood of x_1 would contain points of $f^{-1}(0)$ other than x_1 itself, then it would be impossible for homeomorphic neighborhoods of x_1 and 0 to exist. Thus we have proved that f is injective.

Remark: Here we have in fact tacitly constructed a covering homotopy A of B (15, Th. 3, p. 59).

(3) <u>Continuity of f^{-1}</u>: Recall that continuity is a local property. [13, p. 68; 14, pp. 201-202]. The fact that f^{-1} exists globally (by (1) and (2)) together with the local homeomorphism assumption asserts that f^{-1} is continuous. Q.E.D.

Lemma 1

Let f be a continuous map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then f is a proper map if and only if $\lim_{\|x\|\to\infty} \|f(x)\| = \infty$.

Proof: $\Rightarrow$ By contradiction. Suppose that there is a sequence $\{x_k\}$ with $\|x_k\| \to \infty$, yet $\|f(x_k)\| \leq M < \infty$. Consider the closed and bounded ball

$B_M = \{y \mid \|y\| \leq M\}$, because f is proper, $f^{-1}(B_M)$ is compact. However, $\{x_k\}$ is contained in $f^{-1}(B_M)$, but $\|x_k\| \to \infty$ contradicts the compactness of $f^{-1}(B_M)$.

$\Leftarrow$ f is continuous implies that for each closed set K, $f^{-1}(K)$ is closed. Suppose K is bounded yet $f^{-1}(K)$ is not, then there exists a sequence $\{x_k\}$ in $f^{-1}(K)$ with $\|x_k\| \to \infty$. Clearly $\{f(x_k)\} \subset K$. But by assumption $\|f(x_k)\| \to \infty$, which contradicts boundedness of K. Q.E.D.

Lemma 2

Let f be a C^k map $(k \geq 1)$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, then f is a local C^k-diffeomorphism[4] if and only if $\det(\frac{\partial f}{\partial x}) \neq 0$.

Proof: This is the well-known classical local inverse function theorem. [13, p. 211 and Ex. 17, p. 217; 14, p. 167].

Corollary:

Let f be a C^k map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then f is a C^k-diffeomorphism if and only if

(1) $\det(\frac{\partial f}{\partial x}) \neq 0 \quad \forall x$

(2) $\lim_{\|x\| \to \infty} \|f(x)\| = \infty$

Proof: It follows from the Theorem, Lemma 1 and 2, as well as the fact that differentiability is a local property. [13, p. 198; 14, p. 142].

Q.E.D.

4. A C^k-map is, by definition, a map with continuous derivatives up to order k. A C^k-diffeomorphism is, by definition, a bijective C^k map such that the inverse is also C^k.

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FOOTNOTES

1. A homeomorphism of X onto Y is, by definition, a continuous bijective map f: $X \to Y$ such that f^{-1} is also continuous.

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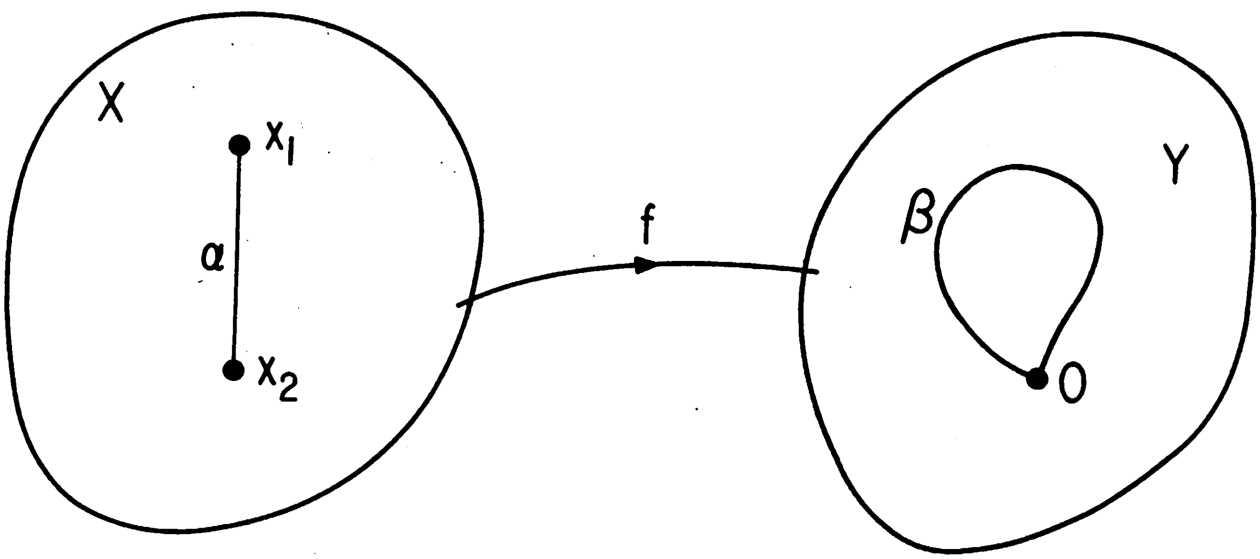
FIGURE CAPTIONS

Fig. 1. For the proof by contradiction, it is assumed that $x_1 \neq x_2$ and that $f(x_1) = f(x_2) = 0$. The line segment α which joins x_1 to x_2 is mapped by f onto the closed curve β.

Fig. 2. As τ goes from 0 to 1, $B(t,\tau) = (1-\tau)\ \beta(t)$ travels in a straight line from $\beta(t)$ to 0. For the same fixed t, the corresponding curve $A(t,\tau)$ is constructed from $B(t,\tau)$ by successive local homeomorphisms.


X
x_1
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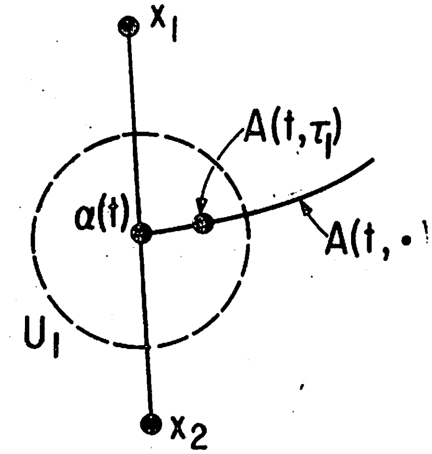
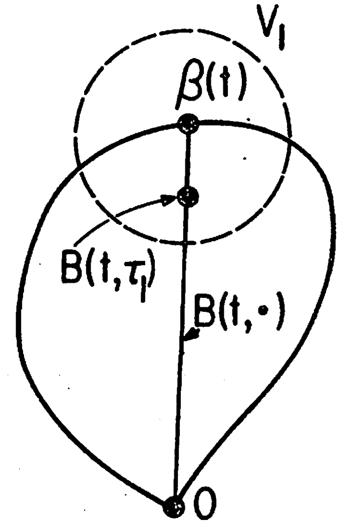


x1
A(t, τ1)
α(t)
A(t, •)
U1
x2



V1
β(t)
B(t, τ1)
B(t, •)
O