DESIGN OF LINEAR MULTIVARIABLE SYSTEMS

by

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CHAPTER 0

INTRODUCTION

0. Introduction

In the past few years, there has been considerable interest in the design of linear time-invariant multivariable systems via state or output feedback. This thesis presents solutions to several important design problems for linear time-invariant multivariable systems and, in each case, provides an algorithm which generates the appropriate state or output feedback laws for the desired purpose. All these algorithms consist of a finite number of steps and can be readily implemented on digital computers. In the following, we give a short description of the various problems considered in this thesis.

1. Statement of the problems

Consider a linear time-invariant multivariable system specified by the following equations,

1.1a $$\dot{x}(t) = Ax(t) + Bu(t)$$

1.1b $$y(t) = Cx(t)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times m}$, $u(t) \in \mathbb{R}^{m}$ is the input, $x(t) \in \mathbb{R}^{n}$ is the state and $y(t) \in \mathbb{R}^{q}$ is the output. We will use (A,B,C) to denote the equations in (1.1a,b), since they are completely determined by the matrices A, B and C. The zero-state input-output properties of this system are completely specified by its transfer function matrix H(s) and

$$H(s) \triangleq C(sI-A)^{-1}B. \tag{1.2}$$

It is easy to check that $H(s)$ is a $q\times m$ matrix whose elements are strictly proper rational functions in s.

In chapter I, we solve the problem of minimal realization of linear time-invariant multivariable systems. This problem can be stated as follows: Given any transfer function matrix $H(s)$, whose elements are strictly proper rational functions in s, find a triple (A,B,C) as in (1.1a,b) such that $H(s) = C(sI-A)^{-1}B$ and A is of least possible size.

In chapter II, we consider the problem of exact model matching via state or output feedback. For a given system specified by (1.1a,b) and for any given $q\times m$ rational matrix $H_2(s)$, the problem is to find a state feedback law

$$u(t) = Gv(t) + Fx(t),\ G \in \mathbb{R}^{m\times m},\ F \in \mathbb{R}^{m\times n} \tag{1.3}$$

or an output feedback law

$$u(t) = Gv(t) + Ky(t),\ G \in \mathbb{R}^{m\times m},\ K \in \mathbb{R}^{m\times q} \tag{1.4}$$

such that the over-all system transfer function matrix $C(sI-A-BF)^{-1}BG$ for the state feedback case, and $C(sI-A-BKC)^{-1}BG$ for the output feedback case, is exactly equal to the given rational matrix $H_2(s)$. This is a basic question in the design of multivariable feedback system. In the design of a state or output feedback law, we often want to know the class of overall system transfer function matrices which can be obtained by applying appropriate state or output feedback laws to a given system. For transfer function matrix in the above class, we want to know the class of state or output feedback laws which accomplish the matching.

In chapter III, we first consider the problem of diagonal decoupling via output feedback and pole assignability. This problem consists in finding an appropriate output feedback law, if it exists, for a given system in order to bring the over-all system transfer function matrix in diagonal form and to assign some of the closed-loop poles of the decoupled system. Then we consider the problem of triangular decoupling via state feedback. This is a problem of finding a state feedback law to bring the over-all system transfer function in an upper triangular form. This problem has applications in process control.

In chapter IV, we are dealing with a more general formulation of decoupling problems. Instead of making the over-all system transfer function matrix in diagonal (triangular) form, we only require it to be in the quasi-diagonal (quasi-triangular) form. We consider the output feedback case with and without dynamic compensation.

2. Contributions of this thesis

At the end of each chapter, we give references to previous work and we discuss the relation of our contributions with previous work.

In chapter I, II and III (except section 2 of chapter III), all the derivations are based on a canonical form for transfer function matrix (see (I.3.42)). This unified approach not only provides solutions to various design problems, but also gives deeper insight into the structure of linear multivariable systems. The contributions in each chapter of this thesis can be summarized as follows.

(i) In chapter I, we first derive a canonical form for transfer function matrix (see (I.3.42)), which is similar to the "structure theorem" due to Wolovich and Falb [Wo.1] but our derivation is more straight forward. Based on this canonical form and some factorization results due to Popov [Po.1], we derive a new algorithm for the minimal realization of linear multivariable systems. We also give a new proof to the stability theorem (I.4.1) due to Kalman, Hsu and Chen [Ka.2,Hs.1,Ch.1].

(ii). In chapter II, we give complete solutions to the exact model matching both via state and output feedback. In both cases, we have algorithms which consist of a finite number of steps and generate the whole class of state or output feedback laws for matching purposes.

(iii). In chapter III, we give an alternate conditions for diagonal decoupling via output feedback. The first necessary and sufficient conditions for the solvability of this problem is due to Falb and Wolovich [Fa.1]. Our approach has the advantage of relating the output feedback laws to the closed-loop poles. The triangular decoupling problem via state feedback is first formulated and solved by Morse and Wonham [Mo.2] in a geometric approach. We solve the same problem using Silverman's inversion algorithm [Si.1] and we show that the conditions for triangular decoupling via state feedback is equivalent to the conditions for invertibility of linear multivariable systems.

(iv) In chapter IV, we solve the diagonal and triangular decoupling problems via <u>output</u> feedback with or without dynamic compensation. We follow closely the geometric approach developed by Wonham and Morse

[Wo.5,Mo.2,Mo.3], where they considered only the decoupling problems via state feedback. In the present work, a constructive procedure for finding these decoupling matrices (and new dynamic elements) is given. The problem of minimizing the order of dynamic compensation (i.e., the number of new integrators associated with the feedback law) is still unsolved. In solving the above problems, we use the concept of controllability subspace of Wonham and Morse but have to extend it to the output feedback case.

In the sequel, if k is a positive integer, $\overline{k}$ is the set of integers $\{1,2,\ldots,k\}$. $\{f(i,j)\}$, $(i \in \overline{q})$, $(j \in \overline{m})$ denotes a $q \times m$ matrix, whose (i,j) element is $f(i,j)$.

CHAPTER I

REALIZATION OF LINEAR MULTIVARIABLE SYSTEMS

0 Introduction

This chapter considers the problem of minimal realization of linear time-invariant finite-dimensional systems from their given transfer function matrices. We use some basic results on polynomials and polynomial matrices as the tool to solve this problem. Our method is essentially based on some factorization results due to Popov [Po.1] and a canonical form of rational matrices, see (3.42) below. This canonical form is similar to the "structure theorem" due to Wolovich and Falb [Wo.1] but our derivation is more straight forward. The literature is discussed at the end of this chapter.

We use $\mathbb{R}[s]$ to denote the commutative ring of polynomials in a single variable s with coefficients in the field of real numbers $\mathbb{R}$. A matrix whose elements are in $\mathbb{R}[s]$ is called a polynomial matrix. $\mathbb{R}(s)$ denotes the field of rational functions in s over $\mathbb{R}$; every element of $\mathbb{R}(s)$ can be expressed (in many ways) as the quotient $f(s)/g(s)$ of two polynomials in $\mathbb{R}[s]$, with $g(s) \neq 0$. An element of $\mathbb{R}(s)$ is said to be a strictly proper rational function if the degree of its numerator is less than the degee of its denominator. A matrix whose elements are in $\mathbb{R}(s)$ is called a rational matrix. For a more detailed discussion on polynomials and rational functions, see Mostow [Mo.1] or MacLane [Ma.1].

1 Polynomial matrices

In this section, we introduce some results on matrices with elements in a ring of polynomials $\mathbb{R}[s]$. Most of these results can be found in MacDuffee [Ma.2].

1.1 Definition If three matrices with elements in $\mathbb{R}[s]$ satisfy the identity $A(s) = C(s)D(s)$, then $D(s)$ is called a right divisor of $A(s)$, and $A(s)$ is called a left multiple of $D(s)$. A greatest common right divisor (g.c.r.d.) $D(s)$ of two matrices $A(s)$ and $B(s)$ is a common right divisor which is a left multiple of every common right divisor of $A(s)$ and $B(s)$. If $D(s)$ is a unimodular matrix, (i.e., $\det D(s) = \text{constant} \neq 0$), then the pair of matrices $A(s)$ and $B(s)$ are said to be right coprime.

1.2 Remark In contrast to the multiplication of polynomials, the multiplication of polynomial matrices is not commutative, in general. That is the reason that we have to specify right divisor, left multiple, greatest common right divisor, and right coprime in Definition (1.1). It is clear that we can also define left divisor, right multiple, greatest common left divisor and left coprime in a natural way.

1.3 Theorem (MacDuffee [Ma.2])

Every pair of matrices $D(s)$, $m\times m$, and $N(s)$, $q\times m$, with elements in $\mathbb{R}[s]$ have a g.c.r.d. $R(s)$, $m\times m$, expressible in the form

$$R(s) = P(s)D(s) + Q(s)N(s), \tag{1.4}$$

where $P(s)$ and $Q(s)$ are $m\times m$, $m\times q$ polynomial matrices respectively.

<u>Proof</u> Consider the (m+q)xm polynomial matrix

$$F(s) = \begin{array}{c} \left[\begin{array}{c} D(s) \\ \hdashline N(s) \end{array}\right] \begin{array}{l} \}\, m \\ \}\, q \end{array} \\ \underbrace{\qquad}_{m} \end{array}$$

Let f_{ij} be the (i,j) element of F(s), ($i \in \overline{m+q}$), ($j \in \overline{m}$). In the following, we will use a sequence of elementary row operations to bring F(s) to the upper triangular form as shown in (1.6) below. Elementary row operations consist of three basic operations (i) multiplication of any row of F(s) by a <u>nonzero constant</u>, (ii) interchange of any two rows of F(s), (iii) addition to any row of F(s), say the i-th row, of any other row of F(s), say the j-th row, multiplied by any arbitrary polynomial p(s). The procedure is described as follows,

1.5 <u>Procedure</u>

<u>Step 1</u> M(s) = F(s)

<u>Step 2</u> If all elements in the first column of M(s) [except the one in the (1.1) position] are identically zero, go to step 6, otherwise go to step 3.

<u>Step 3</u> Among all elements in the first column of M(s), pick the one which has the least degree and is not identically zero. By a permutation of two appropriate rows, we bring this polynomial to the (1.1) position. Call the resulting matrix $\hat{M}(s)$ with $\hat{m}_{i1}$ in the (i,1) position.

Step 4 Divide each polynomial $\hat{m}_{i1}$ by $\hat{m}_{11}$, $(i=2,\ldots,j)$, where j is the number of rows in $\hat{M}(s)$,

$$\hat{m}_{i1} = g_{i1}\hat{m}_{11} + r_{i1} \quad (i=2,\ldots,j)$$

where degree of r_{i1} < degree of $\hat{m}_{11}$. Now we subtract from the i-th row of $\hat{M}(s)$ the first row of $\hat{M}(s)$ multiplied by $g_{i1}(i=2,\ldots,j)$. As a result of these elementary row operations, we get a new matrix, called $\tilde{M}(s)$, whose first column is $(\hat{m}_{11},r_{21},\ldots,r_{j2})$.

Step 5 $M(s) = \tilde{M}(s)$, go to step 2.

Comment Each sequence of operations of step 2 to step 5 reduces the degree of the polynomial in the (1,1) position of M(s) by at least 1. Therefore, after a finite number of iterations of step 2 to step 5, we will go to step 6.

Step 6 If M(s) has only one row or one column, go to step 9, otherwise go to step 7.

Step 7 Deleting the first row and the first column of M(s), where M(s) is of size jxk, we get a (j-1)x(k-1) matrix $\overline{M}(s)$.

Step 8 $M(s) = \overline{M}(s)$, go to step 2.

Step 9 Stop.

The sequence of elementary row operations described in Procedure (1.5) brings F(s) to the upper triangular form as shown below.

$$
1.6 \qquad
\begin{array}{c} m\left\{\right. \\ \\ q\left\{\right. \end{array}
\underbrace{\left[\begin{array}{c|c} U_{11}(s) & U_{12}(s) \\ \hline U_{21}(s) & U_{22}(s) \end{array}\right]}_{m \quad\quad q}
\cdot
\left[\begin{array}{c} D(s) \\ \hline N(s) \end{array}\right]
=
\underbrace{\left[\begin{array}{cccc} d_1 x & \cdot\cdot & & x \\ & d_2 & & \cdot \\ & & \cdot & x \\ 0 & & & d_m \\ & & & \end{array}\right]}_{m}
\begin{array}{l} \left.\right\}m \\ \\ \left.\right\}q \end{array}
\triangleq
\underbrace{\left[\begin{array}{c} R(s) \\ \hline 0 \end{array}\right]}_{m}
\begin{array}{l} \left.\right\} m \\ \\ \left.\right\} q \end{array}
$$

where the first factor U(s) is unimodular. U(s) represents the sequence of elementary row operations performed on F(s). Hence there exists a polynomial matrix $V(s) = U^{-1}(s)$ such that

$$
1.7 \qquad
\left[\begin{array}{c} D(s) \\ \hline N(s) \end{array}\right] = \left[\begin{array}{c|c} V_{11}(s) & V_{12}(s) \\ \hline V_{21}(s) & V_{22}(s) \end{array}\right] \left[\begin{array}{c} R(s) \\ \hline 0 \end{array}\right].
$$

whence

$$D(s) = V_{11}(s)R(s), \; N(s) = V_{21}(s)R(s),$$

i.e., R(s) is a common right divisor of D(s) and N(s). From (1.6).

$$U_{11}(s)D(s) + U_{12}N(s) = R(s),$$

we see that every common right divisor of D(s) and N(s) is a right divisor of R(s). Hence R(s) is a g.c.r.d. of D(s) and N(s).

Q.E.D.

1.8 <u>Corollary</u> In Theorem (1.3), if det $D(s) \not\equiv 0$. then for any two g.c.r.d. of D(s) and N(s), say $R_1(s)$ and $R_2(s)$, there exists a unimodular matrix U(s), such that

$$U(s)R_1(s) = R_2(s)$$

Proof Since $R_1(s)$ is a right divisor of $D(s)$, i.e., $D(s) = D_1(s)R_1(s)$ for some polynomial matrix $D_1(s)$, and $\det D(s) = \operatorname{Det} D_1(s)\cdot\det R_1(s) \neq 0$, we have $\det R_1(s) \neq 0$. From the definition of g.c.r.d., we have

1.9 $$R_1(s) = U_2(s)R_2(s)$$

1.10 $$R_2(s) = U_1(s)R_1(s),$$

for some polynomial matrices $U_1(s)$ and $U_2(s)$. By substituting (1.10) into (1.9),

1.11 $$R_1(s) = U_2(s)U_1(s)R_1(s)$$

and multiplying both sides of (1.11) on the left by $R_1^{-1}(s)$, we have

$$U_2(s)U_1(s) = I.$$

Hence, both $U_1(s)$ and $U_2(s)$ are unimodular matrices.

Q.E.D.

1.12 Corollary Let $N(s)$ and $D(s)$ be two matrices with elements in $\mathbb{R}[s]$. Then $N(s)$ and $D(s)$ are right coprime if, and only if, there exists two matrices $P(s)$ and $Q(s)$ with elements in $\mathbb{R}[s]$, such that

$$P(s)N(s) + Q(s)D(s) = I$$

Proof This corollary follows directly from Theorem (1.3) and the proof is omitted.

1.13 Definition (Wolovich[Wo.2]) Let $D(s)$ be an $m\times m$ matrix with elements in $R[s]$, and $\det D(s) \neq 0$, then $D(s)$ is said to be column proper if

$$\text{degree } (\det D(s)) = \sum_{j=1}^{m} p_j ,$$

where $p_j (j \in \overline{m})$ is the highest degree of the polynomials in the j-th column of D(s).

1.14 Assertion If D(s) is an mxm matrix with elements in R[s], det $D(s) \not\equiv 0$, and D(s) is not column proper, then there exists a unimodular matrix U(s), such that $\hat{D}(s) \triangleq D(s)U(s)$ is column proper.

1.15 Remark As in Definition (1.13), we can define an mxm matrix D(s) to be row proper. As in Assertion (1.14) we can show that if D(s) is not row proper and det $D(s) \not\equiv 0$, then there exists a unimodular matrix V(s), such that $\hat{D}(s) \triangleq V(s)D(s)$ is row proper.

Proof of Assertion (1.14) Let $p_j (j \in \overline{m})$ be the highest degree of the polynomials in the j-th column of D(s), and let $\sum_{k=1}^{p_j+1} d(i,j,k)s^{k-1}$ be the (i,j) element of D(s). Let D_0 be an mxm constant matrix with $d(i,j,p_j+1)$ in the (i,j) position. Since D(s) is not column proper, or equivalently, det $D_0 = 0$, there exist a set of real numbers α_j, $(j \in \overline{m})$, not all zero, such that

1.16 $$\sum_{j=1}^{m} \alpha_j d_j = \underline{0}$$

where $d_j \triangleq [d(1,j,p_{j+1}),\cdots, d(m,j,p_{j+1})]^T$ is the j-th column of D_0. Note that

$$\begin{aligned}&\det D(s) \not\equiv 0 \text{ and definition of the } p_j\text{'s}\\ \Rightarrow\ & d_j \neq \underline{0} \text{ for all } j \in \overline{m}\\ \Rightarrow\ & J \triangleq \{j \mid \alpha_j \neq 0\} \text{ contains at least two elements.}\end{aligned}$$

Let us pick the largest p_j for all $j \in J$, say p_{j_0}, and multiply the j_0-th column of D(s) with α_{j_0}, then add to it with $\alpha_j s^{p_{j_0}-p_j}$ times j-th column of D(s) for all $j \in J\setminus\{j_0\}$. The above elementary column operation on D(s) leads to a new polynomial matrix $\tilde{D}(s)$ which has the same elements as D(s) except in the j_0-th column. Let $\tilde{p}_j (j \in \overline{m})$ be the highest degree of the polynomials in the j-th column of $\tilde{D}(s)$. From (1.16) and the set of appropriately chosen multiples $\alpha_j s^{p_{j_0}-p_j}$, it is easy to see that $\tilde{p}_{j_0} < p_{j_0}$. Hence

$$\sum_{j=1}^{m} \tilde{p}_j < \sum_{j=1}^{m} p_j \, ,$$

i.e., the above elementary column operations reduce the sum of $p_j, (j \in \overline{m})$. Note that $\sum_{i=1}^{m} \tilde{p}_j \geq n \triangleq \text{degree}(\det \tilde{D}(s)) = \text{degree}(\det D(s))$ and $\tilde{p}_j \geq 0$ for all $j \in \overline{m}$. After a finite number of elementary column operations, we get a matrix $\hat{D}(s) \triangleq D(s)U(s)$ which is column proper, where the unimodular matrix U(s) represents the sequence of elementary column operations performed on D(s). Q.E.D.

1.17 Remark In Assertion (1.14), it is easy to see that we can choose U(s) appropriately such that $\hat{D}(s) \triangleq D(s)U(s)$ satisfies the following two conditions,

(i) $\hat{D}(s)$ is column proper

(ii) $p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$ and $p_{r+1} = \cdots = p_m = 0$ for some $r \leq m$

where $p_j (j \in \overline{m})$ is the highest degree of the polynomials in the j-th column of $\hat{D}(s)$.

2 Factorization of Rational Matrices

Using the above results on polynomial matrices, we show below that we are able to factor any rational matrix as a product of two-matrices, $H(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$, where $\tilde{N}(s)$ and $\tilde{D}(s)$ are polynomial matrices and right coprime. Similarly, we can factor H(s) as $\overline{D}(s)\overline{N}^{-1}(s)$, where $\overline{N}(s)$ and $\overline{D}(s)$ are polynomial matrices and left coprime.

Let H(s) be a qxm matrix, with elements in $\mathbb{R}(s)$. H(s) can be written as follows,

2.1 $$H(s) = \begin{bmatrix} \frac{n_{11}(s)}{d_1(s)} & \cdot & \cdot & \frac{n_{1m}(s)}{d_m(s)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{n_{q1}(s)}{d_1(s)} & \cdot & \cdot & \frac{n_{qm}(s)}{d_m(s)} \end{bmatrix}$$

$$= \begin{bmatrix} n_{11}(s) & \cdots & n_{1m}(s) \\ \vdots & & \vdots \\ n_{q1}(s) & \cdots & n_{qm}(s) \end{bmatrix} \begin{bmatrix} d_1(s) & & \bigcirc \\ & \ddots & \\ \bigcirc & & d_m(s) \end{bmatrix}^{-1}$$

$$\triangleq \quad N(s)D(s)^{-1},$$

where $d_j(s)$ $(j \in \overline{m})$ is the least common multiple of the denominators of the elements in the j-th column of H(s). As in the proof of Theorem (1.3), we can find polynomial matrices $\tilde{N}(s)$, $\tilde{D}(s)$ and R(s), which are of size qxm, mxm and mxm, respectively, such that

2.2 $$N(s) = \tilde{N}(s)R(s)$$

2.3 $$D(s) = \tilde{D}(s)R(s),$$

where R(s) is a greatest common right divisor of N(s) and D(s). Note that $\det D(s) \not\equiv 0 \Rightarrow \det R(s) \not\equiv 0$. From (2.1)-(2.3), we have

2.4 $$H(s) = \tilde{N}(s)\tilde{D}^{-1}(s).$$

2.5 <u>Theorem</u>. (Popov[1]).

Let H(s) be a qxm matrix with elements in $\mathbb{R}(s)$. As shown in (2.4) H(s) can be written as a product of $\tilde{N}(s)$ and $\tilde{D}^{-1}(s)$, where $\tilde{N}(s)$ and $\tilde{D}(s)$ are matrices with elements in $\mathbb{R}[s]$ and every greatest common right divisor (g.c.r.d.) of $\tilde{N}(s)$ and $\tilde{D}(s)$ is a unimodular matrix, (i.e., the two polynomial matrices $\tilde{N}(s)$ and $\tilde{D}(s)$ are right coprime), then

(a) For any other factorization of $H(s)$ of the form

$$H(s) = N(s)D^{-1}(s),$$

where $N(s)$ and $D(s)$ are matrices with elements in $\mathbb{R}[s]$ and $\det D(s) \neq 0$, there exists an mxm polynomial matrix $R(s)$, such that

$$N(s) = \tilde{N}(s)R(s) \text{ and } D(s) = \tilde{D}(s)R(s),$$

(b) if the two polynomial matrices $N(s)$ and $D(s)$ are also right coprime, then $R(s)$ is a unimodular matrix.

Proof. (a). Let $T(s)(\tilde{T}(s))$ be a g.c.r.d. of $N(s)$ and $D(s)(\tilde{N}(s)$ and $\tilde{D}(s))$, where $\tilde{T}(s)$ is a unimodular matrix, but $T(s)$ is not, in general. Let $\operatorname{adj} D(s)(\operatorname{adj} \tilde{D}(s))$ be the adjoint matrix of $D(s)(\tilde{D}(s))$. It is easy to see that

$\tilde{T}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s)$ is a g.c.r.d. of $\tilde{N}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s)$ and $\tilde{D}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s)$,

and

$T(s)\cdot[\operatorname{adj} D(s)]\cdot\det \tilde{D}(s)$ is a g.c.r.d. of $N(s)\cdot[\operatorname{adj} D(s)]\cdot \det\tilde{D}(s)$ and $D(s)\cdot[\operatorname{adj} D(s)]\cdot\det \tilde{D}(s)$.

Since

$$\tilde{N}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s) = N(s)\cdot[\operatorname{adj} D(s)]\cdot\det \tilde{D}(s)$$

$$\tilde{D}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s) = D(s)\cdot[\operatorname{adj} D(s)]\cdot\det \tilde{D}(s),$$

and $\det\{\tilde{D}(s)\cdot[\operatorname{adj} \tilde{D}(s)]\cdot\det D(s)\} = [\det \tilde{D}(s)\cdot\det D(s)]^m \neq 0$, from Corollory (1.8) we conclude that their g.c.r.d. can differ at most by a unimodular matrix $U(s)$, i.e.,

$$U(s)\tilde{T}(s)\cdot[\text{adj } \tilde{D}(s)]\cdot\det D(s) = T(s)\cdot[\text{adj } D(s)]\cdot\det \tilde{D}(s).$$

Hence

$$\frac{[\text{adj } \tilde{D}(s)]}{\det \tilde{D}(s)} = \tilde{T}^{-1}(s)\cdot U^{-1}(s)\cdot T(s) \times \frac{[\text{adj } D(s)]}{\det D(s)},$$

consequently

$$\tilde{D}^{-1}(s) = \tilde{T}^{-1}(s)\cdot U^{-1}(s)\cdot T(s)\cdot D^{-1}(s),$$

or

2.6 $$D(s) = \tilde{D}(s)R(s).$$

where $R(s) = \tilde{T}^{-1}(s)U^{-1}(s)T(s)$ is a polynomial matrix, this follows from the assumption that $\tilde{T}(s)$ and $U(s)$ are unimodular matrices. From (2.6) and $\tilde{N}(s)\tilde{D}^{-1}(s) = N(s)D^{-1}(s)$, we have

$$N(s) = \tilde{N}(s)R(s).$$

This proves (a).

(b). Follows from the definition. Q.E.D.

3 Realization of linear multivariable systems

Consider a factorization of $H(s)$ in the form (2.4) where $\tilde{N}(s)$ and $\tilde{D}(s)$ are right coprime. From Assertion (1.14) we can find a unimodular matrix $U(s)$, such that the product $\hat{D}(s) \triangleq \tilde{D}(s)U(s)$ is column proper. Hence the qxm matrix $H(s)$ in (2.4) can be written as

$$H(s) = \tilde{N}(s)U(s)\{\tilde{D}(s)U(s)\}^{-1}$$

3.1 $$= \hat{N}(s)\hat{D}^{-1}(s).$$

where $\hat{D}(s) \triangleq \tilde{D}(s)U(s)$ is column proper and again $\hat{N}(s) \triangleq \tilde{N}(s)U(s)$ and $\hat{D}(s)$ are right coprime.

3.2 <u>Assertion</u> In (3.1) if each element in $H(s)$ is a strictly proper rational function in s, then the qxm polynomial matrix $\hat{N}(s)$ has the following property: Each element in the j-th column of $\hat{N}(s)$ has a degree no greater than (p_j-1) when $p_j \geq 1$, and the j-th column of $\hat{N}(s)$ is identically zero when $p_j = 0$, where p_j, $(j \in \overline{m})$, is the highest degree of the polynomials in the j-th column of $\hat{D}(s)$.

<u>Proof</u> We write $\hat{D}^{-1}(s) = \{\mathrm{adj}\ \hat{D}(s)\}/\det \hat{D}(s)$. Since $\hat{D}(s)$ is column proper, adj $\hat{D}(s)$ is an mxm polynomial matrix, which is row proper, and the highest degree of the polynomials in its j-th row is $n-p_j$, where $n = \sum_{j=1}^{m} p_j$ = degree (det $\hat{D}(s)$). First consider the case $p_j \geq 1$, assume that there is an element $n_{ij}(s)$ in the i-th row and j-th column of $\hat{N}(s)$ having degree greater than p_j-1. Then in the i-th row of the product $\hat{N}(s)\{\mathrm{adj}\ \hat{D}(s)\}$, there is at least one element with degree greater than or equal to n. Since the degree of det $\hat{D}(s)$ is only n, this contradicts the assumption that $H(s) = \hat{N}(s)\{\mathrm{adj}\ \hat{D}(s)\}/\det \hat{D}(s)$ is strictly proper. If there are more than one element in the same row of $\hat{N}(s)$ having degree $\geq p_j-1$, in view of the fact that adj $\hat{D}(s)$ is a row proper matrix, we get the same contradiction, because any set of rows of coefficients of the power $(n-p_j)$ in adj $\hat{D}(s)$ are linearly independent. The case for some $p_j = 0$ is similar. Q.E.D.

From the above reasonings, the polynomial matrices $\hat{N}(s)$ and $\hat{D}(s)$ in (3.1) can be written as

3.3 $$\hat{N}(s) = \{\sum_{k=1}^{p_j} \hat{n}(i,j,k)s^{k-1}\},\ (i \in \overline{q}),\ (j \in \overline{m}).$$

and

3.4 $$\hat{D}(s) = \{\sum_{k=1}^{p_j+1} \hat{d}(i,j,k)s^{k-1}\},\ (i \in \overline{m}),\ (j \in \overline{m}).$$

3.5 <u>Realization algorithm of linear multivariable systems</u>

<u>Step 1</u> Given the qxm matrix H(s), whose elements are strictly proper rational functions in $\mathbb{R}(s)$, put it in the form

3.1' $$H(s) = \hat{N}(s)\hat{D}^{-1}(s)$$

where $\hat{N}(s)$ and $\hat{D}(s)$ are qxm, mxm polynomial matrices respectively, which are right coprime, and $\hat{D}(s)$ is column proper. (see (3.1)). In detail, we write

$$\hat{N}(s) = \{\sum_{k=1}^{p_j} \hat{n}(i,j,k)s^{k-1}\},\ (i \in \overline{q}),\ (j \in \overline{m}).$$

$$\hat{D}(s) = \{\sum_{k=1}^{p_j+1} \hat{d}(i,j,k)s^{k-1}\},\ (i \in \overline{m}),\ (j \in \overline{m}).$$

Furthermore we assume that the columns of $\hat{N}(s) \triangleq \tilde{N}(s)U(s)$ and

$\hat{D}(s) \triangleq \tilde{D}(s)U(s)$ (see (3.1)) are permuted by choosing appropriate $U(s)$, such that $p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$ and $p_{r+1} = \cdots = p_m = 0$, where $r \leq m$. (see Remark (1.17)).

Step 2 Let $n = p_1 + p_2 + \cdots + p_r = \text{degree}\{\det \hat{D}(s)\}$. Define a $q \times n$ constant matrix C with $\hat{n}(i,j,k)$, $(i \in \bar{g})$, $(j \in \bar{r})$, $(k \in \bar{p}_j)$ in the i-th row and $(\sum_{v=1}^{j-1} p_v)+k$-th column. Define the real $m \times m$ matrix $\hat{G}$ with $\hat{d}(i,j,p_j+1)$ in the i-th row and j-th column. Note that $\hat{D}(s)$ is column proper implies $\det \tilde{G} \neq 0$. Define a $m \times n$ matrix $\tilde{F} = -\tilde{G}^{-1}\{\hat{d}(i,j,k)\}$, where $\{\hat{d}(i,j,k)\}$ denotes an $m \times n$ matrix with $\hat{d}(i,j,k)$ in the i-th row and $(\sum_{v=1}^{j-1} p_v)+k$-th column, $(i \in \bar{m})$, $(j \in \bar{r})$, $(k \in \bar{p}_j)$.

$$C = \underbrace{\left[C_1 \,\vdots\, C_2 \,\vdots\, \cdots \,\vdots\, C_r \right]}_{n} \Big\} q$$

$$C_j = \underbrace{\begin{bmatrix} \hat{n}(1,j,1) & \hat{n}(1,j,2) & \cdots & \hat{n}(1,j,p_j) \\ \hat{n}(2,j,1) & \hat{n}(2,j,2) & \cdots & \hat{n}(2,j,p_j) \\ \cdots & \cdots & \cdots & \cdots \\ \hat{n}(q,j,1) & \hat{n}(q,j,2) & \cdots & \hat{n}(q,j,p_j) \end{bmatrix}}_{p_j} \Bigg\} q \qquad (j \in \bar{r})$$

$$\tilde{G} = \underbrace{\begin{bmatrix} \hat{d}(1,1,p_1+1) & \hat{d}(1,2,p_2+1) & \cdots & \hat{d}(1,m,p_m+1) \\ \hat{d}(2,1,p_1+1) & \hat{d}(2,2,p_2+1) & \cdots & \hat{d}(2,m,p_m+1) \\ \cdots & \cdots & \cdots & \cdots \\ \hat{d}(m,1,p_1+1) & \hat{d}(m,2,p_2+1) & \cdots & \hat{d}(m,m,p_m+1) \end{bmatrix}}_{m} \Bigg\} m$$

$$\tilde{F} \triangleq -\tilde{G}^{-1} \cdot \underbrace{\left[\hat{D}_1 \vdots \hat{D}_2 \vdots \cdots \vdots \hat{D}_r\right]}_{n} \Big\} m$$

$$\hat{D}_j = \underbrace{\begin{bmatrix} \hat{d}(1,j,1) & \hat{d}(1,j,2) & \cdots & \hat{d}(1,j,p_j) \\ \hat{d}(2,j,1) & \hat{d}(2,j,2) & \cdots & \hat{d}(2,j,p_j) \\ \cdot & \cdot & \cdots & \cdot \\ \hat{d}(m,j,1) & \hat{d}(m,j,2) & \cdots & \hat{d}(m,j,p_j) \end{bmatrix}}_{p_j} \Bigg\} m \qquad (j \in \bar{r})$$

Step 3 From the $p_j (j \in \bar{m})$ and r defined in step 1, define two matrices $\tilde{A}$ and $\tilde{B}$ of size nxn, nxm, respectively,

3.6 $$\tilde{A} = \text{block diag}[\tilde{A}_1, \ldots, \tilde{A}_r]$$

3.7 $$\tilde{B} = [\tilde{B}_r \,\vdots\, 0], \; \tilde{B}_r = \text{block diag}[\tilde{b}_1, \ldots, \tilde{b}_r]$$

3.8 $$\tilde{A}_j = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \quad \tilde{b}_j = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

where $\tilde{A}_j \in \mathbb{R}^{p_j x p_j}$, $\tilde{B}_r \in \mathbb{R}^{nxr}$, $\tilde{b}_j \in \mathbb{R}^{p_j}$, $(j \in \overline{r})$.

Step 4 Using $\tilde{G}$, $\tilde{F}$, $\tilde{A}$ and $\tilde{B}$ from step 2 and step 3, we calculate

$$A = \tilde{A} + \tilde{B}\,\tilde{F}$$

$$B = \tilde{B}\,\tilde{G}^{-1}$$

END OF ALGORITHM

In Theorem (3.9) below, we will prove that the matrices A, B and C given by the above algorithm is actually a minimal realization of the given transfer function H(s).

We have seen that every qxm matrix H(s), whose elements are strictly proper rational functions in $\mathbb{R}(s)$, can be put in the form

$$3.1'' \qquad H(s) = \hat{N}(s)\hat{D}^{-1}(s)$$

where $\hat{N}(s)$ and $\hat{D}(s)$ are matrices with elements in $\mathbb{R}[s]$, $\hat{N}(s)$ and $\hat{D}(s)$ are right coprime, and $\hat{D}(s)$ is column proper. (see step 1 in Algorithm (3.5)). The following theorem is a modified version of Proposition 2 due to Popov [Po.1].

3.9 Theorem Let H(s) be a qxm matrix, whose elements are strictly proper rational functions in $\mathbb{R}(s)$. Then

(a) the matrices A, B and C given by Algorithm (3.5) is a minimal realization of the given transfer function H(s), and

(b) $\det(sI-A) = k\cdot\det\hat{D}(s)$, where A is given by Algorithm (3.5), $\hat{D}(s)$ is shown in (3.1"), and k is a nonzero real number such that the polynomial $k\cdot\det\hat{D}(s)$ has leading coefficient 1. (i.e., the characteristic polynomial of the minimal realization of H(s) can differ with det $\hat{D}(s)$ by at most a nonzero constant factor.)

The order of the factors in the factorization of H(s), (see (3.1"), is arbitrary: similar results hold for the opposite order. More precisely, we have the

3.10 <u>Corollary</u>

(a). Every qxm matrix H(s), whose elements are strictly proper rational functions in $\mathbb{R}(s)$, can be put in the following form

$$H(s) = \bar{D}^{-1}(s)\bar{N}(s)$$

where $\bar{N}(s)$ and $\bar{D}(s)$ are matrices with elements in $\mathbb{R}[s]$, $\bar{N}(s)$ and $\bar{D}(s)$ are left coprime, and $\bar{D}(s)$ is row proper.

(b). The characteristic polynomial of the minimal realization of H(s) is equal to $k \cdot \det \bar{D}(s)$, where k is a nonzero constant such that the polynomial $k \cdot \det \bar{D}(s)$ has leading coefficient 1.

In order to prove Theorem (3.9), we need the following two lemmas.

3.11 <u>Lemma</u> (Luenberger's second canonical form [Lu.1])

Consider a linear time-invariant system specified by the following equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.12}$$

where $A \in \mathbb{R}^{nxn}$ and $B \in \mathbb{R}^{nxm}$ are constant matrices. We assume that the pair (A,B) is completely controllable and rank (B) = $r \leq m$.

Then there exists an nxn constant nonsingular matrix Q, such that the substitution of $z(t) = Qx(t)$ into (3.12) gives rise to the following equation

$$\dot{z}(t) = \hat{A}z(t) + \hat{B}u(t), \tag{3.13}$$

where $\hat{A} = QAQ^{-1}$, $\hat{B} = QB$ and

$$\hat{A} = \left[\begin{array}{c|c|ccc|c} \hat{A}_{11} & \hat{A}_{12} & . & . & . & \hat{A}_{1r} \\ \hline \hat{A}_{21} & \hat{A}_{22} & . & . & . & \hat{A}_{2r} \\ \hline \cdots & \cdots & . & . & . & \cdots \\ \cdots & \cdots & . & . & . & \cdots \\ \hline \hat{A}_{r1} & \hat{A}_{r2} & . & . & . & \hat{A}_{rr} \end{array}\right], \quad \hat{B} = \left[\begin{array}{c} \hat{B}_1 \\ \hline \hat{B}_2 \\ \hline . \\ . \\ \hline \hat{B}_r \end{array}\right] \tag{3.14}$$

$$\hat{A}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ 0 & . & . & . & \cdots & 0 & 0 \\ \alpha(i,j,1) & . & . & . & \cdots & . & \alpha(i,j,p_j) \end{bmatrix} \quad i \neq j,\ (i \in \overline{r}),(j \in \overline{r}) \tag{3.15}$$

$$\hat{A}_{jj} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ 0 & . & . & . & \cdots & 0 & 1 \\ \alpha(j,j,1) & . & . & . & \cdots & . & \alpha(j,j,p_j) \end{bmatrix} \quad (j \in \overline{r}) \tag{3.16}$$

$$\hat{B}_j = \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & \vdots\vdots\vdots\vdots\vdots\vdots & \vdots \\ 0 & \cdots\cdots\cdots & 0 \\ \beta(j,1) & \cdots\cdots\cdots & \beta(j,m) \end{bmatrix} (j \in \overline{r}) \tag{3.17}$$

where $\hat{A}_{ij} \in \mathbb{R}^{p_i x p_j}$, $\hat{A}_{jj} \in \mathbb{R}^{p_j x p_j}$ and $\hat{B}_j \in \mathbb{R}^{p_j xm}$. The set of bottom rows in $\hat{B}_j$, $[\beta(j,1),\cdots,\beta(j,m)]$, $(j \in \overline{r})$ are linearly independent. The positive integers $p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$ together with $p_{r+1} = p_{r+2} = \cdots = p_m = 0$ are called <u>controllability indices</u> of the controllable pair (A,B). Note that $\sum_{j=1}^{r} p_j = n$.

<u>Proof</u> It is assumed that the system specified by (3.12) is completely controllable, i.e., the nx(n·m) controllability matrix

$$S \triangleq [B \,|\, AB \,|\, A^2B \,|\cdots|\, A^{r-1}B] \tag{3.18}$$

has rank n. With the following procedure, we can select a unique set of n linearly independent vectors from the n·m columns of the matrix S in (3.18).

3.19 <u>Procedure</u>

The vectors are examined in the order:

$$b_1, b_2, \ldots, b_m, Ab_1, Ab_2, \ldots, Ab_m, \ldots, \ldots, A^{n-1}b_m \tag{3.20}$$

<u>Step 1</u> Select b_1 if it is a nonzero vector, otherwise omit it from

the selection.

Step 2 Consider the next vector. If the next vector is linearly independent of all previously selected vectors, retain it, otherwise omit it from the selection.

Step 3 Repeat step 2 until we have selected n linearly independent vectors.

According to our procedure, this set of vectors is clearly unique, and is called a Lexicographic basis of the controllable pair (A,B). A set of Lexicographic basis can be arranged in the following order

3.21 $$b_{i_1}, Ab_{i_1}, \ldots, A^{p_1-1} b_{i_1}, b_{i_2}, Ab_{i_2}, \ldots, A^{p_2-1} b_{i_2}, \ldots, A^{p_r-1} b_{i_r}$$

such that $p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$, and the set of distinctive integers $i_1, i_2, \ldots, i_r$ is a subset of $\{1,2,\ldots,m\}$. The positive integers $\{p_i\}_{i=1}^{r}$ together with $p_{r+1} = p_{r+2} = \cdots = p_m$ are called controllability indices of the controllable pair of the controllable pair (A,B). For detail, see [Wo.4,Br.1].

Define an nxn matrix P, whose columns are composed of the Lexicographic basis of (A,B) , i.e.,

$$P = \left[b_{i_1} \,\vdots\, Ab_{i_1} \,\vdots\, \cdots \,\vdots\, A^{p_1-1} b_{i_1} \,\vdots\, b_{i_2} \,\vdots\, Ab_{i_2} \,\vdots\, \cdots \,\vdots\, A^{p_2-1} b_{i_2} \,\vdots\, \cdots \,\vdots\, A^{p_r-1} b_{i_r} \right].$$

Write P^{-1} in terms of its row vectors

$$P^{-1} = \begin{bmatrix} e_{11} \\ \hdashline e_{12} \\ \hdashline \vdots \\ \hdashline e_{1p_1} \\ \hdashline e_{21} \\ \hdashline e_{22} \\ \hdashline \vdots \\ \hdashline e_{2p_2} \\ \hdashline \vdots \\ \hdashline e_{rp_r} \end{bmatrix} .$$

For simplicity, the row vectors e_{ip_i}, $(i \in \overline{r})$, are labeled

$$e_i = e_{ip_i} . \quad (i \in \overline{r})$$

The vectors $e_1, e_2, \ldots, e_r$ are used to construct the transformation matrix

3.22 $$Q = \begin{bmatrix} e_1 \\ \hdashline e_1 A \\ \hdashline \vdots \\ \hdashline e_1 A^{p_1 - 1} \\ \hdashline e_2 \\ \hdashline e_2 A \\ \hdashline \vdots \\ \hdashline e_r A^{p_r - 1} \end{bmatrix}$$

In order to use Q as a transformation matrix, we have to show that Q is nonsingular. This can be done by the following reasoning: Suppose there are constants a_{ij} such that

3.23 $$\sum_{i=1}^{r}\sum_{j=1}^{p_i} a_{ij}e_iA^{j-1} = 0.$$

Taking inner product of both sides of (3.23) with b_{i_k} $(k \in \overline{r})$ produces

3.24 $$a_{kp_k} = 0. \quad (k \in \overline{r}).$$

since by definition of the e_i's each term in the inner product is zero except the one involving $e_kA^{p_k-1}b_{i_k}$ which is unity. From (3.24), (3.23) can be written as

3.25 $$\sum_{i=1}^{r}\sum_{j=1}^{p_i-1} a_{ij}e_iA^{j-1} = 0$$

Taking the inner product of both sides of (3.25) with Ab_{i_k} produces

$$a_{k,p_k-1} = 0. \quad (k \in \overline{r})$$

Continuing this manner, it is proved that each $a_{ij} = 0$ $(i \in \overline{r})$ $(j \in \overline{p_i})$, i.e., Q is a nonsingular matrix.

With the transformation matrix Q defined in (3.22), it is a simple matter to verify that the matrices $\hat{A} \triangleq QAQ^{-1}$ and $\hat{B} = QB$

have the special form shown in (3.14)-(3.17). Note that since rank $B = r$ and Q is nonsingular, so the set of bottom rows in $\hat{B}_j$ $(j \in \overline{r})$ are linearly independent.

3.26 <u>Remark</u> From the set of controllability indices $p_1,\ldots,p_r$ of (A,B), we define two constant matrices $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{B} \in \mathbb{R}^{n\times m}$ as shown in (3.6)-(3.8). From the coefficients of $\hat{A}_{ij}(i \in \overline{r})$ $(j \in \overline{r})$ and $\hat{B}_j(j \in \overline{r})$, we define a constant matrice $\tilde{F} \in \mathbb{R}^{m\times n}$ and a constant nonsingular matrix $\tilde{G} \in \mathbb{R}^{m\times m}$ as follows

3.27 $$\tilde{F} = [\tilde{F}_1 \,\vdots\, \tilde{F}_2 \,\vdots\, \cdots \,\vdots\, \tilde{F}_r]$$

3.28 $$\tilde{F}_j = \underbrace{\begin{bmatrix} \alpha(1,j,1) & \alpha(1,j,2) & \cdots & \alpha(1,j,p_j) \\ \alpha(2,j,1) & \alpha(2,j,2) & \cdots & \alpha(2,j,p_j) \\ \cdots & \cdots & \cdots & \cdots \\ \alpha(r,j,1) & \alpha(r,j,2) & \cdots & \alpha(r,j,p_j) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{p_j} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} r \;\; (j \in \overline{r}) \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} (m-r) \end{matrix}$$

3.29 $$\tilde{G} = \underbrace{\begin{bmatrix} \beta(1,1) & \beta(1,2) & \cdots & \beta(1,m) \\ \beta(2,1) & \beta(2,2) & \cdots & \beta(2,m) \\ \cdot & \cdot\;\cdot & \cdots & \cdot\;\cdot \\ \beta(r,1) & \beta(r,2) & \cdots & \beta(r,m) \\ \hdashline & & J & \end{bmatrix}}_{m} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} r \\ \left.\vphantom{\begin{matrix}a\end{matrix}}\right\} m-r \end{matrix}$$

Note that the first r rows of $\tilde{G}$ consist of the set of bottom rows in $\hat{B}_j (j \in \bar{r})$. Since the set of row vectors $[\beta(j,1),\ldots,\beta(j,m)]$ are linearly independent, we can find a (m-r)xm constant matrix J such that the matrix $\tilde{\tilde{G}}$ defined in (3.29) is nonsingular. With $\tilde{A},\tilde{B},\tilde{F}$ and $\tilde{\tilde{G}}$ defined above, we have the following equalities,

3.30 $$QAQ^{-1} \triangleq \hat{A} = \tilde{A} + \tilde{B}\tilde{F}$$

3.31 $$QB \triangleq \hat{B} = \tilde{B}\tilde{G}$$

This observation is useful in the proof of the following lemma.

3.32 <u>Lemma</u> Consider a linear dynamical system specified by the completely controllable representation in (3.12) and (3.33) below

3.33 $$y(t) = Cx(t)$$

where $C \in \mathbb{R}^{qxn}$ and $y(t) \in \mathbb{R}^q$ is the output. Then the transfer function $H(s) \triangleq C(sI-A)^{-1}B$ of the above system can be put in the following form

3.34 $$H(s) = N(s)D^{-1}(s)$$

where N(s) and D(s) are qxm, mxm matrices with elements in $\mathbb{R}[s]$ and degree (det D(s)) = n, where n is the dimension of the state space of the system in (3.12) and (3.33).

<u>Proof</u> $H(s) \triangleq C(sI-A)^{-1}B$

$$= CQ^{-1}(sI-QAQ^{-1})^{-1}QB$$

$$= \tilde{C}(sI-\hat{A})^{-1}\hat{B}$$

3.35 $$= \tilde{C}(sI-\tilde{A}-\tilde{B}\tilde{F})^{-1}\tilde{B}\tilde{G}$$

where $\hat{A} \triangleq QAQ^{-1}$, $\hat{B} \triangleq QB$ as defined in (3.13), $\tilde{C} \triangleq CQ^{-1}$, and the last step follows from (3.30) and (3.31) with $\tilde{A},\tilde{B},\tilde{F}$ and $\tilde{G}$ defined in Remark (3.26) and (3.27)-(3.29).

Now we write (3.35) as follows

$$H(s) = \tilde{C}(sI-\tilde{A}-\tilde{B}\tilde{F})^{-1}\tilde{B}\tilde{G}$$

3.36 $$= \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{I-\tilde{F}(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}\tilde{G}$$

3.37 $$= \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{\tilde{G}^{-1}-\tilde{G}^{-1}\tilde{F}(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}.$$

In (3.36) we make use of the identity,

3.38 $$(sI-\tilde{A}-\tilde{B}\tilde{F})^{-1}\tilde{B} = \{I-(sI-\tilde{A})^{-1}\tilde{B}\tilde{F}\}^{-1}(sI-\tilde{A})^{-1}\tilde{B}$$

$$= (sI-\tilde{A})^{-1}\tilde{B}\{I-\tilde{F}(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}.$$

which can be easily visualized from the following picture,

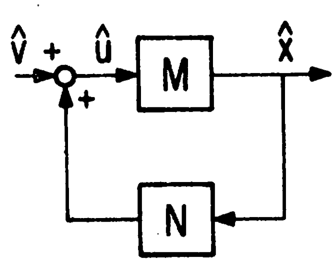

Figure I.3.1

where M denotes the transfer function of the forward loop, and N denotes the transfer function of the feedback loop. We have

$$\hat{u} = N\hat{x} + \hat{v}$$

$$\hat{x} = M\hat{u} = MN\hat{x} + M\hat{v}.$$

Hence

$$\hat{x} = (I-MN)^{-1}M\hat{v}$$

3.39 $$= M(I-NM)^{-1}\hat{v}$$

i.e. the transfer function of the closed loop system is given by $(I-MN)^{-1}M$ or $M(I-NM)^{-1}$. If we let $M = (sI-\tilde{A})^{-1}\tilde{B}$ and $N = \tilde{F}$, then (3.38) follows immediately.

Let us go back to (3.37). The first term in (3.37) can be written as

3.40 $$\tilde{C}(sI-\tilde{A})^{-1}\tilde{B} = \left\{\left(\sum_{k=1}^{p_j} n(i,j,k)s^{k-1}\right)\Big/ s^{p_j}\right\}, \quad (i \in \overline{q}), \ (j \in \overline{m}),$$

where $n(i,j,k)$, $(i \in \overline{q})$, $(j \in \overline{r})$, $(k \in \overline{p_j})$ denotes the element of the qxn matrix $\tilde{C}$ in the i-th row and $\left(\sum_{v=1}^{j-1} p_v\right)$ + k-th column, and $n(i,j,k) = 0$ for all $i \in \overline{q}$ and $j = r+1,\ldots,m$. Similarly, the second term in (3.37) can be written as

3.41 $$\left\{\tilde{G}^{-1}-\tilde{G}^{-1}\tilde{F}(sI-\tilde{A})^{-1}\tilde{B}\right\} = \left\{\left(\sum_{k=1}^{p_j+1} d(i,j,k)s^{k-1}\right)\Big/ s^{p_j}\right\},$$
$$(i \in \overline{m}), \ (j \in \overline{m}),$$

where $d(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{r})$, $(k \in \overline{p_j})$ denotes the element of

$-\tilde{G}^{-1}\tilde{F}$ in the i-th row and $\left(\sum_{v=1}^{j-1} p_v\right)$ + k-th column, and $d(i,j,p_j+1)$,

$(i \in \bar{m})$, $(j \in \bar{m})$ denotes the (i,j) element of $\tilde{G}^{-1}$.

From (3.37), (3.40) and (3.41), it follows that

3.42 $$H(s) = N(s)D^{-1}(s)$$

where $N(s) \triangleq \left\{\left(\sum_{k=1}^{p_j} n(i,j,k)s^{k-1}\right)\right\}$, $(i \in \bar{q})$, $(j \in \bar{m})$ is a qxm polynomial matrix, $D(s) \triangleq \left\{\left(\sum_{k=1}^{p_j+1} d(i,j,k)s^{k-1}\right)\right\}$, $(i \in \bar{m})$, $(j \in \bar{m})$ is an mxm polynomial matrix. Furthermore, since $\tilde{G}^{-1}$ is nonsingular, D(s) is column proper and whose determinant is of degree $n = \sum_{j=1}^{r} p_j$. Q.E.D.

3.43 Remark Realization algorithm (3.5) is based mainly on the developments in the proof of Lemma (3.32). We have shown that for any given system specified by the completely controllable representation $\dot{x} = Ax + Bu$ and $y = Cx$ in (3.12) and (3.33), its transfer function $H(s) \triangleq C(sI-A)^{-1}B$ can be written as a product of two matrices N(s) and $D^{-1}(s)$, where N(s) and D(s) are polynomial matrices whose elements are determined by the given matrices A, B and C. Furthermore, D(s) is column proper and the degree of det D(s) is equal to $n \triangleq$ dimension of the state-space of system (3.12), (3.33). On the other hand, for any given matrix H(s) whose elements are in

$\mathbb{R}(s)$, we can factor H(s) as a product of two matrices $\hat{N}(s)$ and $\hat{D}^{-1}(s)$, where $\hat{N}(s)$ and $\hat{D}(s)$ are polynomial matrices, and $\hat{D}(s)$ is column proper. (see step 1 of Realization algorithm (3.5)). From the derivations in the proof of Lemma (3.32), we can extract $A \triangleq \tilde{A} + \tilde{B}\tilde{F}$, $B \triangleq \tilde{B}\tilde{F}$ and C from $\tilde{N}(s)$ and $\tilde{D}(s)$ in order to get a realization of $H(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$. (see step 2 - step 4 of Realization algorithm (3.5)). The reason that we require $\hat{N}(s)$ and $\hat{D}(s)$ to be right coprime (see step 1 of Realization algorithm (3.5)) is to make the above realization to be of minimal dimension. (For detail, see the proof of Theorem (3.9) below)

3.44 Remark If we assume that the system in (3.12) and (3.33) is completely observable, then we can put its transfer function in the form,

$$H(s) = C(sI-A)^{-1}B$$
$$= \bar{D}^{-1}(s)\bar{N}(s)$$

where $\bar{N}(s)$ and $\bar{D}(s)$ are qxm, mxm matrices with elements in $\mathbb{R}[s]$ and degree $(\det \bar{D}(s)) = n$, where n is the dimension of the state space of the system (3.12) and (3.33). A simple way to do it is the following: Apply Lemma (3.32) to the system (A^T,C^T,B^T), where A^T, C^T and B^T are the transpose of A,C and B respectively, then

$$H^T(s) = B^T(sI-A^T)^{-1}C^T$$
$$= \tilde{N}(s)\tilde{D}^{-1}(s).$$

Hence $H(s) = C(sI-A)^{-1}B = \bar{D}^{-1}(s)\bar{N}(s)$, where $\bar{D}(s) = \tilde{D}^T(s)$ and $\bar{N}(s) = \tilde{N}^T(s)$.

Proof of Theorem (3.9)

(a) The matrices A, B and C given by Algorithm (3.5) can be shown by direct calculation to have the property that $C(sI-A)^{-1}B = H(s)$, i.e., (A,B,C) is a realization of the given transfer function H(s). (For the motivations of this algorithm, see Remark (3.43)). It remains to show that this realization (A,B,C), where A is an nxn matrix and $n \triangleq$ degree $(\det \hat{D}(s))$, is of minimal dimension. Suppose that there is another realization (A',B',C') of H(s) of smaller dimension, (i.e., A' is an n'xn' matrix and n' < n), then from Lemma (3.32), H(s) can be written as

$$H(s) = N(s)D^{-1}(s) \tag{3.34'}$$

where degree $(\det D(s)) = n'$. (Note that in order to apply Lemma (3.32), we assume that the pair (A',B') is completely controllable. If this is not the case, we first apply Theorem 4 in [De.1], pp. 172-173, to extract a completely controllable subsystem (A_0,B_0,C_0) from (A',B',C'), where A_0 is of smaller size than A', then we apply Lemma (3.32)). From (3.1") and (3.34') we have

$$H(s) = \hat{N}(s)\hat{D}^{-1}(s) = N(s)D^{-1}(s)$$

where $\hat{N}(s)$ and $\hat{D}(s)$ are right coprime. From Theorem (2.5), there exists a polynomial matrix R(s), such that

$$N(s) = \hat{N}(s)R(s) \text{ and } D(s) = \hat{D}(s)R(s)$$

where $\det R(s) \not\equiv 0$. Hence

$$n = \text{degree } (\det \hat{D}(s)) \leq \text{degree } (\det D(s)) = n',$$

this contradicts the assumption that $n > n'$. Hence the realization (A,B,C) given by Algorithm (3.5), where A is of size $n \times n$ and $n =$ degree $(\det \hat{D}(s))$, is a realization of $H(s)$ with minimal state space dimension.

(b) This part can be proven by direct calculations of $\det(sI-A)$ and $\det \hat{D}(s)$.

3.45 Remark Consider the following set of differential equations

3.46 $$M(p)y(t) = N(p)u(t)$$

where $M(p)$ and $N(p)$ are matrices with elements in $\mathbb{R}[p]$ of dimension $q \times q$ and $q \times m$ respectively and $p \triangleq d/dt$ is the differentiation operator. Assume that $\det M(p) \not\equiv 0$, then (3.46) can be written as

3.47 $$y(t) = M^{-1}(p)N(p)u(t).$$

If each element in $M^{-1}(p)N(p)$ is a strictly proper rational function in $\mathbb{R}(p)$, then the set of differential equations (3.46) can be put in state form

3.48 $$\dot{x}(t) = Ax(t) + Bu(t)$$

3.49 $$y(t) = Cx(t)$$

such that the system represented by (3.48), (3.49) is equivalent to the system specified by (3.46). For a precise definition of system equivalence, see [Za.1] pp. 90-91. Also we may be interested to find a minimal state space representation of the transfer function $M^{-1}(p)N(p)$ in (3.47). With several simple modifications, the results in this chapter can be used to solve these two problems. In this chapter, a given transfer function matrix $H(s)$ is put in the form $N(s)D^{-1}(s)$. It is clear that we can also put it as $\hat{D}^{-1}(s)\hat{N}(s)$, and replace column proper, greatest common right divisor, etc. by row proper, greatest common left divisor, etc. Then Realization algorithm (3.5) with some suitable modifications can be used for the present purposes.

To illustrate Realization algorithm (3.5), we consider the following example.

3.50 Example Consider the transfer function matrix

3.51
$$H(s) = \begin{bmatrix} \frac{-3s^2-6s-2}{(s+1)^3} & \frac{s^3-3s-1}{(s-2)(s+1)^3} & \frac{1}{(s-2)(s+1)^2} \\ \frac{s}{(s+1)^3} & \frac{s}{(s-2)(s+1)^3} & \frac{s}{(s-2)(s+1)^2} \end{bmatrix}$$

3.52 Step 1a $H(s)$ can be put in the form, (see (2.1)),

$$H(s) = N(s)D^{-1}(s)$$

3.53
$$= \begin{bmatrix} -3s^2-6s-2 & s^3-3s-1 & 1 \\ s & s & s \end{bmatrix} \begin{bmatrix} (s+1)^3 & 0 & 0 \\ 0 & (s-2)(s+1)^3 & 0 \\ 0 & 0 & (s-2)(s+1)^2 \end{bmatrix}^{-1}$$

<u>Step 1b</u> Using the procedures given in the proof of Theorem (1.3), we calculate a g.c.r.d. of the two polynomial matrices N(s) and D(s) in (3.53). (see 1.6).

$$
3.54 \quad \left[\begin{array}{ccc|cc}
1 & 0 & 0 & 0 & -s^2-3s-3 \\
2 & 0 & 0 & 1 & -2s^2-3s \\
0 & 0 & 1 & 0 & 0 \\
\hline
-2s+4 & -1 & -2s+1 & -s+2 & 2s^3-s^2-6s \\
s & 0 & s & s & -s^3+3s+1
\end{array}\right]
\left[\begin{array}{ccc}
s^3+3s^2+3s+1 & 0 & 0 \\
0 & s^4+s^3-3s^2-5s-2 & 0 \\
0 & 0 & s^3-3s-2 \\
\hline
-3s^2-6s-2 & s^3-3s-1 & 1 \\
s & s & s
\end{array}\right]
$$

$$
= \left[\begin{array}{ccc}
1 & -s^3-3s^2-3s & -s^3-3s^2-3s \\
0 & -s^3-3s^2-3s-1 & -2s^3-3s^2+1 \\
0 & 0 & s^3-3s-2 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

On the left hand side of (3.54), the 5x5 unimodular matrix represents the sequence of elementary row operations performed on the second factor on the left hand side of (3.54). On the right hand side of (3.54), the 3x3 upper triangular matrix is a g.c.r.d. of the two polynomial matrices N(s) and D(s) in (3.53). Multiplying the inverse of the 5x5 unimodular matrix on both sides of (3.54), we have (see (1.7)),

$$3.55\quad \left[\begin{array}{ccc} s^3+3s^2+3s+1 & 0 & 0 \\ 0 & s^4+s^3-3s^2-5s-2 & 0 \\ 0 & 0 & s^3-3s-2 \\ \hline -3s^2-6s-2 & s^3-3s-1 & 1 \\ s & s & s \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc} s^3+3s^2+3s+1 & -s^3-3s^2-3s & -s^3-3s^2-3s & 0 & s^2+3s+3 \\ 0 & -s+2 & -2s+1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline -3s^2-6s-2 & 3s^2+6s+1 & 3s^2+6s & 0 & -3s-6 \\ s & -s & -s & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & -s^3-3s^2-3s & -s^3-3s^2-3s \\ 0 & -s^3-3s^2-3s-1 & -2s^3-3s^2+1 \\ 0 & 0 & s^3-3s-2 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

<u>Step 1c</u> The transfer function H(s) in (3.51) can be written as, (see (2.4)),

$$H(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$$

$$3.56\quad = \begin{bmatrix} -3s^2-6s-2 & 3s^2+6s+1 & 3s^2+6s \\ s & -s & -s \end{bmatrix} \begin{bmatrix} s^3+3s^2+3s+1 & -s^3-3s^2-3s & -s^3-3s^2-3s \\ 0 & -s+2 & -2s+1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

where $\tilde{N}(s)$ and $\tilde{D}(s)$ are right coprime.

Step 1d Since the polynomial matrix $\tilde{D}(s)$ in (3.56) is not column proper, we must postmultiply both $\tilde{N}(s)$ and $\tilde{D}(s)$ by a unimodular matrix U(s) such that the product $\hat{D}(s) \triangleq \tilde{D}(s)U(s)$ is column proper. Using the procedures given in the proof of Assertion (1.14), we can find

$$U(s) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence H(s) in (3.56) can be written as, (see (3.1)),

$$H(s) = \tilde{N}(s)U(s)\ \{\tilde{D}(s)U(s)\}^{-1}$$
$$= \hat{N}(s)\hat{D}^{-1}(s)$$

3.57 $$= \begin{bmatrix} 3s^2-6s-2 & -1 & 0 \\ s & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3+3s^2+3s+1 & 1 & -1 \\ 0 & -s+2 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

From the matrices $\hat{N}(s)$ and $\hat{D}(s)$ in (3.57), it is easy to see that

3.58 $$p_1 = 3,\ p_2 = 1,\ p_3 = 0 \text{ and } r = 2$$

This completes step 1 of the Realization algorithm (3.5).

Step 2 From (3.58), we calculate $n = p_1 + p_2 = 4$. From the coefficients of the polynomials of $\hat{N}(s)$ in (3.57), we find the 2x4 constant matrix

3.59
$$C = \begin{bmatrix} -2 & -6 & -3 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

From the coefficients of the polynomials of $\hat{D}(s)$ in (3.57), we find the 3x3 nonsingular matrix

3.60
$$\tilde{G} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

and the 3x4 constant matrix

3.61
$$\tilde{F} = -\tilde{G}^{-1} x \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Step 3</u> From the p_i $(i = 1,2,3)$ and r in (3.58), we find $\tilde{A}$ and $\tilde{B}$ as follows,

3.62
$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right]$$

<u>Step 4</u> Using $\tilde{G},\tilde{F},\tilde{A}$ and $\tilde{B}$ from step 2 and step 3, we calculate

3.63 $$A = \tilde{A}+\tilde{B}\tilde{F} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -3 & -3 & -1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right], \quad B = \tilde{B}\tilde{G} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 0 & -1 & -3 \end{array}\right]$$

The matrices A, B and C in (3.59) and (3.63) is a minimal realization of the given transfer function in (3.51).

4. A new proof of a stability theorem

The results in previous sections can be used to prove a stability theorem due to Kalman, Hsu and Chen [Ka.2, Hs.1, Ch.2]. We state this theorem as follows,

4.1 Theorem (Kalman, Hsu and Chen)

Consider the linear time-invariant multivariable feedback system shown in Fig. (4.1). The system S_1 is assumed to be completely characterized by its pxp strictly proper rational matrix $\hat{G}_1(s)$. Let $\hat{G}_f(s)$ be the transfer function matrix of the feedback system. Then the characteristic polynomial of any minimal realization of $\hat{G}_f(s)$, denoted by $\Delta_f(s)$, is given by

4.2 $$\Delta_f(s) = \Delta_1(s) \cdot \det [I + \hat{G}_1(s)],$$

where $\Delta_1(s)$ is the characteristic polynomial of any minimal realization of $\hat{G}_1(s)$.

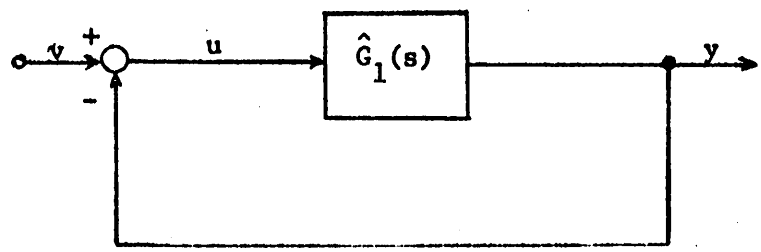

Figure 1.4.1

<u>Proof</u> From step 1 of Algorithm (3.5), $\hat{G}_1(s)$ can be factored as

4.3 $$\hat{G}_1(s) = \hat{N}(s)\ \hat{D}^{-1}(s)$$

where $\hat{N}(s)$ and $\hat{D}(s)$ are two pxp polynomial matrices, which are right coprime, and $\hat{D}(s)$ is column proper. From part (b) of Theorem (3.9), we have

4.4 $$\Delta_1(s) = k_1 \det \hat{D}(s)$$

where k_1 is a nonzero constant.

The transfer function matrix of the feedback system $\hat{G}_f$ is

4.5 $$\begin{aligned}\hat{G}_f(s) &= \hat{G}_1(s)\ [I + \hat{G}_1(s)]^{-1} \\ &= \tilde{N}(s)\tilde{D}^{-1}(s)\ [I + \tilde{N}(s)\tilde{D}^{-1}(s)]^{-1} \\ &= \tilde{N}(s)\ [\tilde{D}(s) + \tilde{N}(s)]^{-1}\end{aligned}$$

By construction, $\tilde{N}(s)$ and $\tilde{D}(s)$ are right coprime, and from Corollary (1.12), this is equivalent to the existence of P(s) and Q(s),such that

$$P(s)\ \tilde{N}(s) + Q(s)\ \tilde{D}(s) = I$$

$\Longleftrightarrow$ $$[P(s) + Q(s) - Q(s)]\ \tilde{N}(s) + Q(s)\ \tilde{D}(s) = I$$

$\Longleftrightarrow$ $$[P(s) - Q(s)]\ \tilde{N}(s) + Q(s)\ [\tilde{N}(s) + \tilde{D}(s)] = I$$

$\Longleftrightarrow$ $\tilde{N}(s)$ and $\tilde{N}(s) + \tilde{D}(s)$ are right coprime.

From the degree constraints on the elements of $\tilde{N}(s)$ and $\tilde{D}(s)$, and $\tilde{D}(s)$ is column proper, it is clear that $\tilde{N}(s) + \tilde{D}(s)$ is also column proper. Therefore, from Theorem (3.9), we have

$$
\begin{aligned}
\Delta_f(s) &= k_2 \cdot \det [\tilde{N}(s) + \tilde{D}(s)] \\
&= k_2 \cdot \det \tilde{D}(s) \cdot \det [I + \tilde{N}(s)\, \tilde{D}^{-1}(s)] \\
&= (k_1 \cdot k_2) \cdot \Delta_1(s) \cdot \det [I + \hat{G}_1(s)]
\end{aligned} \tag{4.6}
$$

where k_1 and k_2 are some nonzero constants. Since $\hat{G}(s)$ is a strictly proper rational matrix, we have $\lim_{s\to\infty} \hat{G}(s) = 0$. Therefore,

$$
\lim_{s\to\infty} \frac{\Delta_f(s)}{\Delta_1(s)} = (k_1 \cdot k_2) \cdot \lim_{s\to\infty} \det [I + \hat{G}_1(s)] = k_1 \cdot k_2 \ . \tag{4.7}
$$

Since $\Delta_f(s)$ and $\Delta_1(s)$ are assumed to be monic polynomials and $k_1 \cdot k_2$ is a nonzero constant, there follows $k_1 \cdot k_2 = 1$. Hence (4.6) gives the desired result. Q.E.D.

4.8 <u>Remark</u> (a) It is easy to see that $\hat{D}(s)$ need not be reduced to the column proper form; indeed if the factorization (4.2) is given with $\hat{N}(s)$ and $\hat{D}(s)$ right coprime, the above reasoning goes through. (The reduction to column proper form requires the multiplication of $\hat{D}(s)$ on the right by an appropriate unimodular matrix.)

(b) If $\hat{G}_1(s)$ were proper but not strictly proper, then the reasoning above would hold provided one assumes that $\det [1 + \hat{G}_1(\infty)] \neq 0$. (If this condition does not hold, the closed loop transfer function is not proper.)

5. <u>Discussion of the literature</u>

The realization of a rational transfer-function matrix into a minimal state-space form has been discussed by many authors. Gilbert [Gi.1] and Kalman [Ka.1] related controllability and observability to minimality. If the denominators of the elements in the transfer function matrix have no common factors, a minimal realization can be obtained by partial fractions [Gi.1, Ka.1, Za.1]. Kalman [Ka.1] had also observed that, in general, a minimal realization could be computed by starting with an arbitrary realization and reducing to a minimal realization. In [Ka.2] Kalman proposed a method for realization of a transfer function matrix by using the Smith-McMillan canonical form. The relationship between the McMillan degree [Mc.1] and Hankel Matrices has been pointed out by Youla and Tissi [Yo.1] and Ho and Kalman [Ho.1]. The module-theoretic viewpoint appears in [Ka.3]. In [Ka.4] there is a systematic presentation of the algebraic structure of linear system theory as well as the B. L. Ho algorithm for minimal realization of an impulse response matrix. Panda and Chen [Pa.1] and Kuo [Ku.1] have proposed methods for realization transfer function matrix into Jordan Form. Polak [Po.2] has an algorithm for obtaining state-space representations for systems whose dynamics are expressed by a matrix differential equation. His method uses a Gauss elimination method to "triangularize" the matrix differential operator. Popov has some results on the factorization of rational function matrix [Po.1] and has a realization algorithm for a special class of linear systems, prop. 3 [Po.1]. Rosenbrock proposed some realization methods based on the <u>system matrix</u>. His methods mainly consist of a sequence of elementary row and column operations in order to make the system matrix have least order. Rosenbrock has also developed several results on the factroizations of rational matrices, his results are close to those in Section 2 of this chapter. [Ro.1]. Recently Wolovich [Wo.3] proposed a metnod for obtaining state-space representation of linear time-invariant systems whose dynamics are expressed in a matrix differential equation. Wolovich's method is simpler than Rosenbrock's method. Wolovich used the idea of column proper. This facilitates the computations required to get the minimal realizations. Although the work in this chapter is independent of [Wo.3], they have many results in common. The main contributions in this chapter are the following, (a) We derive a canonical form of transfer function matrices, (see (3.42)), which is similar to the "structure theorem" due to Wolovich and Falb [Wo.1], but our derivation is more simple and straight forward. (b) Based on some factorization results due to Popov [Po.1], we derive a systematic realization algorithm with rigorous proofs.

CHAPTER II

THE EXACT MODEL MATCHING OF LINEAR MULTIVARIABLE SYSTEMS

0. Introduction

In the design of linear time-invariant multivariable systems via state or output feedback, several well known problems, as the pole assignment problem, the decoupling problem, etc., have been discussed by many authors [Wo.5,6,Fa.1,Gi.2,Da.1,2.] . A general problem is that of finding a state or output feedback law for a given system, in order to make the over-all system satisfy certain requirements. In this chapter, we make a first step in solving this general problem, we give a complete solution to the problem of "exact model matching" for finite dimensional linear time-invariant systems. It is a question of finding a state or output feedback law for a given system, in order to make the over-all system transfer function exactly equal to a given transfer function. This problem was proposed by Wolovich [Wo.2] in the state feedback case; however, as he pointed out, he has not yet fully solved the problem of finding, in general, the required state feedback law. Wolovich's algorithm can only be applied to invertible systems. A discussion of the literature is given at the end of this chapter.

In section 1, we solve the exact model matching problem via state feedback. In section 2, we give a modified algorithm for the exact model matching problem via state feedback, this modified algorithm requires less computations than that in section 1. In section 3, we solve the exact model matching problem via output feedback.

1. Exact model matching via state feedback

We solve this problem basically in two steps, (1) Apply a state feedback law (G_1,F_1) and a coordinate transformation Q to a given system (A,B,C), in order to transform it into a new system $(\tilde{A},\tilde{B},\tilde{C})$, where $\tilde{A}$ and $\tilde{B}$ are in a very simple canonical form, (2) Apply another state feedback law (G_2,F_2) to the new system $(\tilde{A},\tilde{B},\tilde{C})$; the simple structure of $\tilde{A}$ and $\tilde{B}$ yields a simple relationship between (G_2,F_2) and the transfer function of the resulting system $(\tilde{A}+\tilde{B}F_2,\tilde{B}G_2,\tilde{C})$; and this relationship can be further expressed as a real matrix equation. Given any transfer function matrix H(s), whose elements are strictly proper rational functions in s, we put their coefficients into that real matrix equation, and its solution is the state feedback law (G_2,F_2). When this state feedback law (G_2,F_2) is applied to $(\tilde{A},\tilde{B},\tilde{C})$, the resulting system has a transfer function exactly equal to H(s). This is Wolovich's "exact model matching" problem. The conditions for solvability of the aforesaid matrix equation give precisely the conditions under which H(s) can be "matched". The equation yields also the whole class of state feedback laws that match H(s).

PRELIMINARY ANALYSIS

Consider a linear dynamical system specified by the following differential equation,

1.1a $$\dot{x}(t) = Ax(t) + Bu(t)$$

1.1b $$y(t) = Cx(t)$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^q$ and A, B and C are real constant matrices of appropriate size. We assume that the pair (A,B) is completely controllable and rank $(B) = r \leq m$. From Lemma (I.3.11) and Remark (I.3.26), there exist a constant matrix $\tilde{F} \in \mathbb{R}^{m\times n}$ and two nonsingular matrices $\tilde{G} \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$, such that the following equalities hold (see (I.3.30) and (I.3.31)),

1.2 $$QAQ^{-1} = \tilde{A} + \tilde{B}\tilde{F}$$

1.3 $$QB = \tilde{B}\tilde{G}$$

where the matrices $\tilde{F}$, $\tilde{G}$ and Q are defined in (I.3.27)-(I.3.29) and (I.3.22). The matrices $\tilde{A}$ and $\tilde{B}$ are in the canonical forms shown in (I.3.6)-(I.3.8).

With $\tilde{F}$, $\tilde{G}$ and Q so determined, we apply a state feedback law $u(t) = F_1x(t) + G_1v(t) \triangleq (-\tilde{G}^{-1}\tilde{F}Q)x(t) + (\tilde{G}^{-1})v(t)$ to the system (1.1a,b) and with the substitution $z(t) = Qx(t)$, the resulting system is governed by

1.4a $$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}v(t)$$

1.4b $$y(t) = \tilde{C}z(t)$$

where $\tilde{C} \triangleq CQ^{-1}$, $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{B} \in \mathbb{R}^{n\times m}$ are in the canonical forms shown in (I.3.6)-(I.3.8). Note that the matrices $\tilde{A}$ and $\tilde{B}$ are specified by the set of <u>controllability indices</u> $p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$, $p_{r+1} = \cdots = p_m = 0$ of (A,B).

From now on, we shall work with the system (1.4a,b). We are going to investigate the relationship between the state feedback law

1.5 $$v(t) = G_2w(t) + F_2z(t)$$

where $G_2 \in R^{m\times m}$ is a nonsingular matrix and $F_2 \in R^{m\times n}$, which we apply to system (1.4a,b) and the resulting system transfer function

1.6 $$\hat{H}(s) \triangleq \tilde{C}(sI-\tilde{A}-\tilde{B}F_2)^{-1}\tilde{B}G_2.$$

Note that

$$\hat{H}(s) = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{I-F_2(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}G_2$$

$$= \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{G_2^{-1}-G_2^{-1}F_2(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}$$

The above identity has been justified in (I.3.36)-(I.3.39).

1.7 Notation

Let $\tilde{c}(i,j,k)$, $(i \in \overline{q})$, $(j \in \overline{r})$, $(k \in \overline{p_j})$ be the element of the q×n matrix $\tilde{C}$ in the i-th row and $(\sum_{v=1}^{j-1} p_v)$ + k-th column; thus $\tilde{c}(i,j,k)$ is in the i-th row and is the k-th element of the j-th block. Similarly, let $f(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{r})$, $(k \in \overline{p_j})$ be the element of $-G_2^{-1}F_2$ in the i-th row and $(\sum_{v=1}^{j-1} p_v)$ + k-th column; for later convenience, let $f(i,j,p_j+1)$, $(i\in \overline{m})$, $(j \in \overline{m})$ denote the element of G_2^{-1} in the i-th row and j-th column, and let $\tilde{c}(i,j,k) = 0$, $(i \in \overline{q})$, $(j = r+1,\dots,m)$.

1.8 Comment

From any given set of $f(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{m})$, $(k \in \overline{p_j+1})$, such that $M \equiv \{f(i,j,p_j+1)\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ is an m×m nonsingular matrix, then we can extract $G_2 = M^{-1}$, and $F_2 = -M^{-1}\{\tilde{f}(i,j,k)\}$, where $\{f(i,j,k)\}$, $(i \in \overline{m})$, $(j \in \overline{r})$, $(k \in \overline{p_j})$ is the m×n matrix with $f(i,j,k)$ in the i-th row and $(\sum_{v=1}^{j-1} p_v)+k$-th column. In other words, such a set of $f(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{m})$, $(k \in \overline{p_j+1})$ determines a unique feedback law (G_2,F_2).

With the above notations, the q×m matrix $\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}$ in (1.6) can be written as

$$\tilde{C}(sI-\tilde{A})^{-1}\tilde{B} = \{(\sum_{k=1}^{p_j} \tilde{c}(i,j,k)s^{k-1})/s^{p_j}\}, \ (i \in \overline{q}), \ (j \in \overline{m}),$$

and similarly for the second term in (1.6)

$$\{G_2^{-1} - G_2^{-1}F_2(sI-\tilde{A})^{-1}\tilde{B}\} = \{(\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1})/s^{p_j}\}, \ (i \in \overline{m}), \ (j \in \overline{m}).$$

Now the transfer function in (1.6) can be written as

1.9 $$\hat{H}(s) = \hat{C}(s)\hat{F}^{-1}(s)$$

where $\hat{C}(s) = \{\sum_{k=1}^{p_j} \tilde{c}(i,j,k)s^{k-1}\}$, $(i \in \overline{q})$, $(j \in \overline{m})$ is a q×m polynomial matrix and $\hat{F}(s) = \{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1}\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ is an m×m polynomial matrix whose determinant is not identically zero.

EXACT MODEL MATCHING

The problem of exact model matching via state feedback can be stated as follows: Given a q×m matrix H(s), each element of H(s) is a strictly proper rational function in s, does there exist a state feedback law (G_2,F_2) as in (1.5), such that the system (1.4a,b) can be transformed by the state feedback law (G_2,F_2) into a new system whose transfer function is equal to H(s)?

Let $\psi(s) = s^\gamma + \alpha_\gamma s^{\gamma-1} + \alpha_{\gamma-1}s^{\gamma-2} + \cdots + \alpha_1$, $(\gamma \leq n)$, be the least common multiple of the denominators of $h_{ij}(s)$, $(i \in \overline{q})$, $(j \in \overline{m})$, where $h_{ij}(s)$ is in the i-th row and j-th column of H(s), and the numerator and denominator of $h_{ij}(s)$ are coprime. We put H(s) in the following form

1.10 $$H(s) = \frac{1}{\psi(s)} N(s)$$

where $N(s) = \{\sum_{k=1}^{\gamma} n(i,j,k)s^{k-1}\}$, $(i \in \overline{q})$, $(j \in \overline{m})$ is a q×m polynomial matrix. From (1.9) and (1.10), we equate $\hat{H}(s)$ with H(s), and try to find an appropriate state feedback law (G_2,F_2), or equivalently, $\hat{F}(s)$ such that

1.11 $$N(s)\hat{F}(s) = \hat{C}(s)\psi(s)$$

Equating the coefficients of the corresponding polynomials on both sides of (1.11), we get the following matrix equation

1.12 $$N^*f^* = C^*\psi^*$$

where

$$N^* = \begin{bmatrix} N_1^* \\ \hdashline N_2^* \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline N_q^* \end{bmatrix}, \quad N_i^* = \text{block diag } [N_i^1, \cdots, N_i^m], \quad (i \in \overline{q})$$

$$N_i^j = [N(i,j,1) \quad N(i,j,2) \quad \cdots \quad N(i,j,m)] \quad (i \in \overline{q}), \ (j \in \overline{m})$$

$$N(i,j,k) = \overbrace{\left[\begin{array}{c:c:c:c} n(i,k,\gamma) & 0 & \cdot & 0 \\ n(i,k,\gamma-1) & n(i,k,\gamma) & \cdot & \cdot \\ \vdots & n(i,k,\gamma-1) & \cdot & 0 \\ n(i,k,1) & & \cdot & n(i,k,\gamma) \\ 0 & n(i,k,1) & \cdot & n(i,k,\gamma-1) \\ \vdots & 0 & \cdot & \vdots \\ & \vdots & \cdot & \\ 0 & 0 & \cdot & n(i,k,1) \end{array}\right]}^{p_j+1} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} (\gamma+p_j) \quad \begin{array}{l} (i \in \overline{q}), \ (j \in \overline{m}) \\ (k \in \overline{m}) \end{array}$$

1.13 $$f^* = \begin{bmatrix} f_1^* \\ \hdashline f_2^* \\ \hdashline \cdot \\ \cdot \\ \hdashline f_m^* \end{bmatrix}, \quad f_j^* = \begin{bmatrix} f_j^1 \\ \hdashline f_j^2 \\ \hdashline \cdot \\ \cdot \\ \hdashline f_j^m \end{bmatrix} \quad (j \in \overline{m})$$

1.14 $$f_j^i = \underbrace{\left.\begin{bmatrix} f(i,j,p_j+1) \\ f(i,j,p_j) \\ \cdot \\ \cdot \\ f(i,j,1) \end{bmatrix}\right\}}_{1} (p_j+1) \qquad (j \in \overline{m}),\ (i \in \overline{m})$$

$$C^* = \left[\begin{array}{c} C_1^* \\ \hdashline C_2^* \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline C_q^* \end{array}\right], \qquad C_i^* = \left[\begin{array}{c} C_i^1 \\ \hdashline C_i^2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline C_i^m \end{array}\right] \qquad (i \in \overline{q})$$

$$C_i^j = \underbrace{\left.\left[\begin{array}{c:c:c:c} c(i,j,p_j) & 0 & & 0 \\ c(i,j,p_j-1) & c(i,j,p_j) & & \vdots \\ \cdot & c(i,j,p_j-1) & & 0 \\ \cdot & \cdot & \cdot & c(i,j,p_j) \\ c(i,j,1) & \cdot & \cdot & c(i,j,p_j-1) \\ 0 & c(i,j,1) & \cdot & \cdot \\ \vdots & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & c(i,j,1) \end{array}\right]\right\}}_{(\gamma+1)} (\gamma+p_j) \qquad (i \in \overline{q}),\ (j \in \overline{m})$$

and

$$\psi^* = \begin{array}{c} \left.\begin{bmatrix} 1 \\ \alpha_\gamma \\ \alpha_{\gamma-1} \\ \cdot \\ \cdot \\ \alpha_1 \end{bmatrix}\right\} \gamma+1 \\ \underbrace{\hphantom{\alpha_{\gamma-1}}}_{1} \end{array}$$

In (1.12), C^* is uniquely determined by the system (1.4a,b); ψ^* is uniquely determined by the given transfer function $H(s) = \frac{1}{\psi(s)} N(s)$; N^* is uniquely determined by the given transfer function $H(s) = \frac{1}{\psi(s)} N(s)$ and the integers $p_i (i \in \overline{m})$, and f^* is uniquely determined by the state feedback law (G_2,F_2) and the integers $p_i (i \in \overline{m})$, where $p_i (i \in \overline{m})$ specify the canonical structure of (1.4a,b).

1.15 <u>Definition</u>

Any solution f^* of (1.12), which is in the partitioned form shown in (1.13) and (1.14) and which satisfies the condition that $M = \{f(i,j,p_j+1)\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ is a nonsingular matrix, is called a <u>regular solution</u>.

1.16 <u>Theorem</u>

Consider the system (1.4a,b) and a given transfer function H(s) as in (1.10). There exists a state feedback law (G_2,F_2) with G_2 nonsingular (see(1.5)), which when applied to system (1.4a,b), yields a new system whose transfer function $\hat{H}(s)$ (given in (1.6)), is equal to $H(s)$, if, and only if, there exists a regular solution f^* of (1.12).

Proof

$\Rightarrow$ Assume there exists such a feedback law (G_2,F_2), then (1.11) is satisfied. In (1.11), $\hat{F}(s) = \{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1}\}$, $(i \in \overline{m})$, $(j \in \overline{m})$, is uniquely determined by (G_2,F_2) with $\{f(i,j,p_j+1)\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ nonsingular (see Notation (1.7)). Since (1.11) and (1.12) are equivalent, i.e., $\hat{F}(s)$ satisfies (1.11), if, and only if f^* satisfies (1.12), where f^* is determined by $f(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{m})$, $(k \in \overline{p_j+1})$ via equations (1.13), (1.14), this gives one regular solution f^* of (1.12).

$\Leftarrow$ Assume there exists a regular solution f^* of (1.12), whose components are $f(i,j,k)$, $(i \in \overline{m})$, $(j \in \overline{m})$, $(k \in \overline{p_j+1})$, as shown in (1.13), (1.14). Since f^* satisfies (1.12), then $\hat{F}(s) \triangleq \{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1}\}$, $(i \in \overline{m})$, $(j \in \overline{m})$, satisfies (1.11). (1.11) can be written as

$$\frac{1}{\psi(s)} N(s) = \hat{C}(s)\{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1}\}^{-1},$$

and after some simple manipulations,

$$H(s) = \frac{1}{\psi(s)} N(s) = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1})/s^{p_j}\}^{-1}$$

1.17 $$= \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{G_2^{-1} - G_2^{-1}F_2(sI-\tilde{A})^{-1}\tilde{B}\}^{-1},$$

where G_2^{-1} and $G_2^{-1}F_2$ are defined by

1.18 $$G_2^{-1} \triangleq \{f(i,j,p_j+1)\},\ (i \in \overline{m}),\ (j \in \overline{m}),$$

and $G_2^{-1}F_2 \triangleq -\{f(i,j,k)\},\ (i \in \overline{m}),\ (j \in \overline{r}),\ (k \in \overline{p_j})$ as the m×n matrix with $f(i,j,k)$ in the i-th row and $(\sum_{v=1}^{j-1} p_v) + k$-th column. Since f^* is regular, the matrix $\{f(i,j,p_j+1)\},\ (i \in \overline{m}),\ (j \in \overline{m})$, is nonsingular, so G_2 is well defined by (1.18). With (G_2,F_2) so defined, (1.17) shows that (G_2,F_2) is a feedback law which matches the resulting transfer function (given in (1.6)) with the given $H(s)$. Q.E.D.

A PRELIMINARY CHECK

Assume that the condition in Theorem (1.16) is satisfied, i.e.,

1.6' $$H(s) = \hat{H}(s) = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}\{G_2^{-1}-G_2^{-1}F_2(sI-\tilde{A})^{-1}\tilde{B}\}^{-1}.$$

Let $h_i(s)$ and $\tilde{h}_i(s)$ be the i-th row of $H(s)$ and $\tilde{H}(s) \triangleq \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}$, respectively. Consider $H(s)$. Imagine all its elements expressed in a Taylor series in $1/s$. For each $i \in \overline{q}$, consider the list of the leading terms of each element in the i-th row. If $h_i(s) \equiv 0$, we set $\alpha_i = -1$ otherwise let $\alpha_i + 1$ be the smallest degree which appears in that list, i.e.,

$$\alpha_i = \begin{cases} -1 & \text{if } h_i(s) \equiv 0 \\ \min\{k \mid \lim\limits_{s\to\infty} s^{k+1} h_i(s) \neq 0, \ k \text{ is a finite positive integer}\} \end{cases}$$

Let B^* and $\tilde{B}^*$ be q×m constant matrices with $\lim\limits_{s\to\infty} s^{\alpha_i+1} h(s)$ and $\lim\limits_{s\to\infty} s^{\alpha_i+1} \tilde{h}(s)$ in the i-th row, $(i \in \bar{q})$, respectively. Note that B^* and $\tilde{B}^*$ are uniquely determined from the transfer functions H(s) and $\tilde{H}(s)$, respectively.

From (1.6')

$$s^{\alpha_i+1} h_i(s) = s^{\alpha_i+1} \tilde{h}_i(s)\{G_2^{-1} - G_2^{-1} F_2 (sI-\tilde{A})^{-1}\tilde{B}\}^{-1}$$

$$\Rightarrow \quad \lim_{s\to\infty} s^{\alpha_i+1} h_i(s) = \lim_{s\to\infty} s^{\alpha_i+1} \tilde{h}_i(s) G_2$$

$$\Rightarrow \quad B^* = \tilde{B}^* G_2 .$$

Thus it is clear that range (B^*) = range $(\tilde{B}^*)$ is a necessary condition for the existence of feedback law (G_2, F_2) which will match $\tilde{H}(s)$ to H(s). Since B^* and $\tilde{B}^*$ can be easily computed from H(s) and $\tilde{H}(s)$, this is a useful preliminary check.

1.19 ALGORITHM FOR FINDING REGULAR SOLUTIONS

Finding a feedback law (G_2, F_2) is equivalent to finding a regular solution f^* of (1.12), since there is an explicit one-to-one correspondence between them. The following algorithm generates the whole class of regular solutions of (1.12).

Step 1 Find any solution f_p^* of (1.12), which may or may not be regular. If there exists no solution of (1.12), then obviously there exists no regular solution of (1.12).

Step 2 Find a basis $G = \{g_1^*, g_2^*, \cdots, g_t^*\}$ of $\mathcal{N}(N^*)$, where $\mathcal{N}(N^*)$ is the null space of the matrix N^* defined in (1.12). The set G will be empty, if, and only if, the columns of N^* are linearly independent.

Step 3 Let

1.20 $$f^* \triangleq f_p^* + \sum_{v=1}^{t} \beta_v g_v^*,$$

where $\beta_v (v \in \bar{t})$ are some real numbers to be determined later. Let f^*, f_p^* and g_v^*, $(v \in \bar{t})$, be partitioned in the form shown in (1.13), (1.14). Form an m×m matrix

$$M = \{f(i,j,p_j+1)\} = \{f_p(i,j,p_j+1) + \sum_{v=1}^{t} \beta_v g_v(i,j,p_j+1)\}$$

$$(i \in \bar{m}), \ (j \in \bar{m}),$$

where $f(i,j,p_j+1)$, $f_p(i,j,p_j+1)$ and $g_v(i,j,p_j+1)$ are the components of f^*, f_p^* and $g_v^*(v \in \bar{t})$, respectively, at the appropriate positions as shown in (1.14).

Step 4 Calculate the determinant of M defined in Step 3. The determinant of M, det $M(\beta_1, \cdots, \beta_t)$, is a polynomial in $\beta_1, \cdots, \beta_t$ with real coefficients, and its degree is less than or equal to m.

<u>Step 5</u> If $\det M(\beta_1,\cdots,\beta_t)$ is not identically zero, then the class of $\{\hat{\beta}_1,\cdots,\hat{\beta}_t\}$ for which $\det M(\hat{\beta}_1,\cdots,\hat{\beta}_t) \neq 0$, together with (1.20), generates the whole class of <u>regular solutions</u> of (1.12). If $\det M(\beta_1,\cdots,\beta_t) \equiv 0$, there exists no <u>regular solution</u> of (1.12).

1.21 <u>Example</u> Consider a linear time-invariant system specified by (1.4a,b) with

1.22
$$\tilde{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right].$$

We will find the whole class of state feedback laws (G_2,F_2) such that the over-all system transfer function $\tilde{C}(sI-\tilde{A}-\tilde{B}F_2)^{-1}\tilde{B}G_2$ is equal to

1.23
$$H(s) = \frac{1}{s^3-s}\begin{bmatrix} s^2 & s \\ s & 1 \end{bmatrix}.$$

From (1.22) and (1.23), it is easy to see that the matrices N^*, f^*, C^* and ψ^* in (1.12) are given by

1.24 $$N^* = \left[\begin{array}{ccc|ccc|cc|cc}
0 & 0 & 0 & 2 & 0 & 0 & & & & \\
1 & 0 & 0 & 0 & 2 & 0 & & \bigcirc & & \\
0 & 1 & 0 & 0 & 0 & 2 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
 & & & & & & 0 & 0 & 2 & 0 \\
 & \bigcirc & & & & & 1 & 0 & 0 & 2 \\
 & & & & & & 0 & 1 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 2 & 0 & 0 & & & & \\
1 & 0 & 0 & 0 & 2 & 0 & & \bigcirc & & \\
0 & 1 & 0 & 0 & 0 & 2 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & & & & \\
\hline
 & & & & & & 0 & 0 & 0 & 0 \\
 & \bigcirc & & & & & 0 & 0 & 2 & 0 \\
 & & & & & & 1 & 0 & 0 & 2 \\
 & & & & & & 0 & 1 & 0 & 0
\end{array}\right], \quad
C^* = \left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
 & & & \\
 & \bigcirc & & \\
 & & & \\
 & & & \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\hline
 & & & \\
 & \bigcirc & & \\
 & & & \\
 & & &
\end{array}\right], \quad
f^* = \left[\begin{array}{c}
f(1,1,3) \\
f(1,1,2) \\
f(1,1,1) \\
\hline
f(2,1,3) \\
f(2,1,2) \\
f(2,1,1) \\
\hline
f(1,2,2) \\
f(1,2,1) \\
\hline
f(2,2,2) \\
f(2,2,1)
\end{array}\right]$$

$$\psi^* = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Using Algorithm (1.19), we will find the whole class of regular solutions of (1.12). In step 1, we find a solution f_p^* of (1.12),

1.25 $$f_p^* = \left[\begin{array}{ccc|ccc|cc|cc} 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{array}\right]^T.$$

In step 2, we find a basis $G = \{g_1^*, g_2^*, g_3^*\}$ of $\mathcal{N}(N^*)$,

1.26

$$g_1^* = \left[-2 \;\; 0 \;\; 0 \;\vdots\; 0 \;\; 1 \;\; 0 \;\vdots\; 0 \;\; 0 \;\vdots\; 0 \;\; 0\right]^T$$

$$g_2^* = \left[0 \;\; -2 \;\; 0 \;\vdots\; 0 \;\; 0 \;\; 1 \;\vdots\; 0 \;\; 0 \;\vdots\; 0 \;\; 0\right]^T$$

$$g_3^* = \left[0 \;\; 0 \;\; 0 \;\vdots\; 0 \;\; 0 \;\; 0 \;\vdots\; -2 \;\; 0 \;\vdots\; 0 \;\; 1\right]^T$$

In step 3, we define $f^* \triangleq f_p^* + \sum_{v=1}^{3} \beta_v g_v^*$

1.27

$$= \left[-2\beta_1 \;\; -2\beta_2 \;\; 0 \;\vdots\; \tfrac{1}{2} \;\; \beta_1 \;\; \beta_2-\tfrac{1}{2} \;\vdots\; -2\beta_3 \;\; 0 \;\vdots\; 0 \;\; \beta_3\right]^T$$

and form a 2×2 matrix

1.28

$$M = \begin{bmatrix} -2\beta_1 & -2\beta_3 \\ \frac{1}{2} & 0 \end{bmatrix}$$

In step 4, we calculate the determinant of M,

1.29

$$\det M = \beta_3.$$

In step 5, since det $M = \beta_3$ is not identically zero, the class of $\{\hat\beta_1, \hat\beta_2, \hat\beta_3\}$ with $\hat\beta_3 \neq 0$, together with (1.27), generates the whole class of regular solutions of (1.12). In particular, we choose $\hat\beta_1 = 0$, $\hat\beta_2 = \frac{1}{2}$ and $\hat\beta_3 = -\frac{1}{2}$, then f^* in (1.27) is given by

1.30

$$f^* = \left[0 \;\; -1 \;\; 0 \;\vdots\; \tfrac{1}{2} \;\; 0 \;\; 0 \;\vdots\; 1 \;\; 0 \;\vdots\; 0 \;\; -\tfrac{1}{2}\right]^T$$

From (1.30) and Notation (1.7), we calculate the desired state feedback law (G_2, F_2) as follows

$$G_2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

and

$$F_2 = - G_2 \times \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

1.31 <u>Remark</u> In the above example, since the given system defined in (1.4a,b) and (1.22) is not invertible, Wolovich's matching algorithm [Wo.2] is not applicable in this case.

1.32 <u>Remark</u> In solving the exact model matching problem, we first apply the state feedback law $u(t) = F_1\, x(t) + G_1 v(t)$ and the coordinate transformation $z(t) = Qx(t)$ to the given system (A, B, C) in (1.1a,b), in order to transform it into a new system $(\tilde{A}, \tilde{B}, \tilde{C})$ in (1.4a,b) where $\tilde{A}$ and $\tilde{B}$ are in a very simple canonical form. In fact, we can solve the exact model matching problem without going through this step. This can be shown as follows.

Consider the given system in (1.1a,b), We want to find a state feedback law $u(t) = F\, x(t) + G\, v(t)$ with $F \in \mathbb{R}^{mxn}$, $G \in \mathbb{R}^{mxm}$ and G being nonsingular, such that the over-all system transfer function

1.33 $$C(sI - A - BF)^{-1} BG$$

is exactly equal to a given rational matrix $H(s) = \frac{1}{\psi(s)} N(s)$, see (1.10).

Note that the over-all system transfer function in (1.33) can be written as

1.34 $$C(sI - A - BF)^{-1}BG = C(sI - A)^{-1}B\{G^{-1} - G^{-1} F(sI - A)^{-1} B\}^{-1}$$
$$= CW(s) B \{m(s) G^{-1} - G^{-1} F W(s) B\}^{-1}$$

where $m(s)$ is the minimal polynomial of A and $W(s) \triangleq (sI - A)^{-1} m(s)$. If we can find a pair (G, F) such that

1.35 $$N(s)\{m(s) G^{-1} - G^{-1}F W(s) B\} = C W(s) B \psi(s)$$

then the exact model matching problem is solved. This can be done by equating the coefficients of the corresponding polynomials on both sides of (1.35), we get the following real matrix equation

1.36 $$Sx = T\psi$$

where S is a real rectangular matrix whose coefficients are determined by $N(s)$, $m(s)$, $W(s)$ and B, x is a column vector whose components correspond to the elements of G^{-1} and $G^{-1}F$, T is a real rectangular matrix whose elements are determined by $CW(s)B$, and ψ is a real column vector whose components are determined by the polynomial $\psi(s)$. Using Algorithm (1.19), we can find the whole class of regular solutions of (1.36), or equivalently, the whole class of state feedback laws (G,F) for the matching purpose.

It is easy to see that the matrices of (1.12) are of smaller size than those of (1.36). Therefore the state feedback law (G_1,F_1) and the coordinate transformation Q should be used to obtain (1.12), which is of small size and hence is easy to solve.

2. <u>A modified algorithm for exact model matching via state feedback</u>

We assume that the reader is familiar with the results in Chapter 1 and the first section of this chapter, so that a modified algorithm for exact model matching can be easily derived. Although we give a complete solution to the problem of exact model matching in Section 1, the computations involved in solving the real matrix equation in (1.12) are complicated. The present modified algorithm requires less computations.

In Section 1, the exact model matching problem has been formulated as follows (see (1.11)): Given any q×m matrix H(s), with each element of H(s) being a strictly proper rational function in $\mathbb{R}(s)$, the problem is to find an m×m matrix $\hat{F}(s)$ whose elements are in $\mathbb{R}[s]$, such that the following three conditions are satisfied,

2.1 (i) the m×m polynomial matrix $\hat{F}(s)$ can be written as $\hat{F}(s) = \{\sum_{k=1}^{p_j+1} f(i,j,k)s^{k-1}\}$, $(i \in \bar{m})$, $(j \in \bar{m})$. i.e., the (i,j) element of $\hat{F}(s)$ is a polynomial with degree less than or equal to p_j, where p_j $(j \in \bar{m})$ is a set of integers specified by the given system in (1.1a,b)

2.2 (ii) the m×m constant matrix $\{f(i,j,p_j+1)\}$, $(i \in \bar{m})$, $(j \in \bar{m})$ is nonsingular, i.e., the polynomial matrix $\hat{F}(s)$ is column proper.

2.3 (iii) $H(s) = \hat{C}(s)\hat{F}^{-1}(s)$, where $\hat{C}(s) = \{\sum_{k=1}^{p_j} \tilde{c}(i,j,k)s^{k-1}\}$, $(i \in \bar{q})$, $(j \in \bar{m})$ is a q×m polynomial matrix whose elements are specified by the given system in (1.1a,b).

The following algorithm generates the whole class of solutions $\hat{F}(s)$ which satisfies (2.1)-(2.3).

2.4 A modified algorithm for exact model matching

Step 1 From step 1 of the Realization algorithm (I.3.5), H(s) can be factored in the form

2.5 $$H(s) = \hat{N}(s)\hat{D}^{-1}(s)$$

where $\hat{N}(s)$ and $\hat{D}(s)$ are right coprime, $\hat{D}(s)$ is column proper. In detail, we write

2.6 $$\hat{N}(s) = \left\{ \sum_{k=1}^{\tilde{p}_j} \hat{n}(i,j,k)s^{k-1} \right\}, \ (i \in \bar{q}), \ (j \in \bar{m}).$$

2.7 $$\hat{D}(s) = \left\{ \sum_{k=1}^{\tilde{p}_j+1} \hat{d}(i,j,k)s^{k-1} \right\}, \ (i \in \bar{m}), \ (j \in \bar{m})$$

where $\tilde{p}_1 \geq \tilde{p}_2 \geq \cdots \geq \tilde{p}_m \geq 0$.

Step 2 If $p_j \geq \tilde{p}_j \geq 0$ $(j \in \bar{m})$, go to step 3, otherwise go to step 7.

Step 3 Let $\psi_i(\phi_i)$, $(i \in \bar{q})$, be the greatest common divisor of the polynomials in the i-th row of $\hat{N}(s)(\hat{C}(s))$. If $\psi_i | \phi_i$ for all $i \in \bar{q}$, go to step 4, otherwise go to step 7.

Step 4 Let $\Psi(s) = \text{diag}\ (\psi_i)$ be a q×q diagonal matrix with ψ_i in the (i,i) position. Calculate

2.8 $$N_0(s) \triangleq \Psi^{-1}(s)\hat{N}(s) \text{ and } C_0(s) \triangleq \Psi^{-1}(s)\hat{C}(s).$$

Step 5 Find the whole class of polynomial matrices such that any member, say R(s), has the following three properties

2.9 (i). $$R(s) = \left\{ \sum_{k=1}^{p_j-\tilde{p}_i+1} r(i,j,k)s^{k-1} \right\}, \ (i \in \overline{m}), \ (j \in \overline{m})$$

2.10 (ii). $\{r(i,j,p_j-\tilde{p}_i+1)\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ is a real nonsingular matrix.

2.11 (iii). $N_0(s)R(s) = C_0(s)$.

If there is no solution in step 5, go to step 7.

Comment Equating the coefficients of the corresponding polynomials on both sides of (2.11), we get the following real matrix equation

2.12 $$N_0^* r^* = c^*$$

where the elements of N_0^*, r^* and c^* are uniquely determined by the coefficients of $N_0(s)$, $R(s)$ and $C_0(s)$ respectively. Using Algorithm (1.19) for finding regular solutions, we get the whole class of solutions r^* of (2.12) with the nonsingularity constraint in (2.10). From the class of r^*, we get the whole class of solutions R(s) satisfying (2.9)-(2.11).

Step 6 For the class of solutions R(s) generated in step 5, calculate

2.13 $$\hat{F}(s) = \hat{D}(s)R(s).$$

and go to step 8.

Step 7 Print (THERE IS NO SOLUTION $\hat{F}(s)$ SATISFYING (2.1)-(2.3)).

Step 8 Stop.

END OF THE ALGORITHM.

2.14 Theorem The class of $\hat{F}(s)$ generated in step 6 of Algorithm (2.4) is the whole class of solutions satisfying (2.1)-(2.3).

Proof We first show that the class of matrices $\hat{F}(s)$ generated in step 6 are solutions to (2.1)-(2.3). From (2.9), (2.10), (2.13) and the fact that $\hat{D}(s)$ is column proper, it is easy to verify that $\hat{F}(s)$ in (2.13) satisfies the conditions in (2.1) and (2.2). It remains to show that $\hat{F}(s)$ satisfies (2.3). Consider the following,

$$\begin{aligned}
\hat{C}(s)\hat{F}^{-1}(s) &= \hat{C}(s)\ (\hat{D}(s)R(s))^{-1} && \text{from (2.13)} \\
&= (\Psi(s)C_0(s))(\hat{D}(s)R(s))^{-1} && \text{from (2.8)} \\
&= (\Psi(s)N_0(s)R(s))(\hat{D}(s)R(s))^{-1} && \text{from (2.11)} \\
&= (\Psi(s)N_0(s))\hat{D}^{-1}(s) && \\
&= \hat{N}(s)\hat{D}^{-1}(s) && \text{from (2.8)} \\
&= H(s) && \text{from (2.5)}
\end{aligned}$$

i.e., the class of matrices $\hat{F}(s)$ generated in step 6 satisfies (2.3).

Next we show that any solution $\hat{F}_1(s)$ of (2.1)-(2.3) is contained in the class of $\hat{F}(s)$ generated in step 6. Suppose that $\hat{F}_1(s)$ satisfies (2.1)-(2.3), then from (2.3) and (2.5), we have

2.15
$$\begin{aligned} H(s) &= \hat{C}(s)\hat{F}_1^{-1}(s) \\ &= \hat{N}(s)\hat{D}^{-1}(s). \end{aligned}$$

From Corollary (I.1.8) and (2.15), there exists a polynomial matrix $R_1(s)$ such that

2.16
$$\hat{C}(s) = \hat{N}(s)R_1(s)$$

and

2.17
$$\hat{F}_1(s) = \hat{D}(s)R_1(s).$$

Note that since det $\hat{D}(s) \neq 0$, such a polynomial matrix $R_1(s)$ is unique. From the conditions on $\hat{F}_1(s)$ and $\hat{D}(s)$, (see (2.1), (2.2), (2.6) and (2.7)), and from (2.17), we can easily show that $R_1(s)$ satisfies (2.9) and (2.10). From (2.8) and (2.16), we can see that $R_1(s)$ satisfies (2.11). Hence $R_1(s)$ is contained in the class of solutions generated by step 5. Therefore, $\hat{F}_1(s) \triangleq \hat{D}(s)R_1(s)$ is contained in the class of $\hat{F}(s)$ generated in step 6. It remains to show that the existence of $\hat{F}_1(s)$ implies that the two necessary conditions in step 2 and step 3 are satisfied. Suppose that the condition in step 2 is not satisfied, i.e., there is a $j_0 \in \overline{m}$, such that $p_{j_0} < \tilde{p}_{j_0}$, then from the degree constraints on R(s) (see (2.9)), R(s) has the following form,

$$
m\left\{ \underbrace{\left[\begin{array}{c|c} \begin{matrix} x & \cdot & x \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} & \overbrace{\begin{matrix} \quad & \bigcirc & \quad \end{matrix}}^{m-j_0+1} \\ \hline \begin{matrix} x & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x & \cdot & \cdot \end{matrix} & \begin{matrix} \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \vdots & \cdot & x \end{matrix} \end{array}\right]}_{m} \right. \begin{array}{l} \left.\vphantom{\begin{matrix} x \\ x \\ x \end{matrix}}\right\} j_0 \\ \vphantom{\begin{matrix} x \\ x \\ x \end{matrix}} \end{array}
$$

i.e., the $j_0 x(m-j_0+1)$ submatrix on the right upper corner of R(s) is zero, this implies that $\{r(i,j,p_j-\tilde{p}_i+1)\}$ $(i \in \bar{m})$, $(j \in \bar{m})$ is singular. This contradicts (2.10). From (2.16), it is easy to see that the necessary condition in step 3 is satisfied. Q.E.D.

2.18 Remark In the above proof, we have shown that the class of solutions $\hat{F}(s)$ of (2.1)-(2.3) is the same class of matrices $\hat{F}(s)$ generated in step 6 of Algorithm (2.4). Since the degrees of the polynomials in (2.11) is less than the degrees of the polynomials in (2.3), this modified algorithm requires less computations than that in Section 1.

2.19 Remark In the modified matching algorithm (2.4), we make use of the factorization results developed in Chapter I. This approach has also been used by Wolovich [Wo.2] in solving the exact model matching problem.

3. Exact model matching via output feedback

In the previous two sections of this chapter, we have solved the problem of exact model matching via state feedback. In this section, we solve this problem using output feedback. The exact model matching of linear time-invariant system via output feedback can be stated as follows. Given any linear time-invariant system S_1 whose transfer function is a q×m matrix $H_1(s)$, and each element of $H_1(s)$ is a strictly proper rational function in $\mathbb{R}(s)$. The problem is to find an output feedback law (G,K) with G nonsingular, in order to make the over-all system transfer function exactly equal to a given transfer function $H_2(s)$. (see Figure II.3.1)

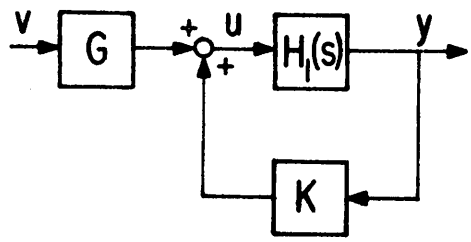

Figure II.3.1

The following algorithm generates the whole class of output feedback laws (G,K) with G nonsingular for exact model matching.

3.1 Algorithm for exact model matching via output feedback.

Step 1 From step 1 of the Realization algorithm (I.3.5), $H_u(s)$, (u=1,2), can be factored in the form

3.2 $$H_u(s) = N_u(s)D_u^{-1}(s) \quad (u=1,2).$$

the pair of polynomial matrices $N_u(s)$ and $D_u(s)$ are right coprime, $D_u(s)$ is column proper, (u=1,2). In detail, we write

$$N_u(s) = \left\{ \sum_{k=1}^{p_j(u)} n_u(i,j,k)s^{k-1} \right\}, \ (i \in \bar{q}), \ (j \in \bar{m}), \ (u=1,2)$$

$$D_u(s) = \left\{ \sum_{k=1}^{p_j(u)+1} d_u(i,j,k)s^{k-1} \right\}, \ (i \in \bar{m}), \ (j \in \bar{m}), \ (u=1,2)$$

where $p_1(u) \geq p_2(u) \geq \cdots \geq p_m(u) \geq 0$ for u=1,2.

Step 2 If $p_j(1) = p_j(2)$ for all $j \in \bar{m}$, go to step 3, otherwise go to step 5.

Step 3 Let $p_j = p_j(1) = p_j(2)$ for all $j \in \bar{m}$. Find the whole class of matrices such that any member, say $(R(s),\hat{G},\hat{K})$, consists of an m×m polynomial matrix R(s) and two constant real matrices $\hat{G}$ and $\hat{K}$ of size m×m and m×q respectively, and has the following properties,

3.3 (i) $$R(s) = \left\{ \sum_{k=1}^{p_j-p_i+1} r(i,j,k)s^{k-1} \right\}, \ (i \in \bar{m}), \ (j \in \bar{m}),$$

i.e., the (i,j) element of R(s) has degree $\leq p_j-p_i$.

3.4 (ii) $\hat{G}$ is a real nonsingular matrix, and

3.5 (iii) $$\underbrace{\begin{array}{l} q\{ \\ m\{ \end{array}\!\!\begin{bmatrix} 0 \\ \hline \hat{G} \end{bmatrix}}_{m} \underbrace{\begin{bmatrix} D_1(s) \end{bmatrix}}_{m} + \underbrace{\begin{array}{l} q\{ \\ m\{ \end{array}\!\!\begin{bmatrix} I \\ \hline \hat{K} \end{bmatrix}}_{q} \underbrace{\begin{bmatrix} N_1(s) \end{bmatrix}}_{m} = \underbrace{\begin{array}{l} q\{ \\ m\{ \end{array}\!\!\begin{bmatrix} N_2(s) \\ \hline D_2(s) \end{bmatrix}}_{m} \underbrace{\begin{bmatrix} R(s) \end{bmatrix}}_{m}$$

If there is no solution which satisfies (3.3)-(3.5), go to step 5.

3.6 Comment Equating the coefficients of the corresponding polynomials on both sides of (3.5), we have the following real matrix equation, 3.7 $$Sx = t$$

where S is a real rectangular matrix whose elements are determined by the coefficients of the polynomials in $D_1(s)$, $D_2(s)$, $N_1(s)$ and $N_2(s)$, x is a real column vector whose components correspond to the coefficients of $\hat{G}$, $\hat{K}$ and R(s), and t is a real column vector whose components are determined by the coefficients of the polynomials in $N_1(s)$. Using Algorithm (1.19) for finding regular solutions, we get the whole class of solutions x of (3.7) with nonsingularity constraint (3.4). From the class of x, we get the whole class of solutions $(\hat{G},\hat{K},R(s))$ satisfying (3.3)-(3.5).

3.8 Comment From (3.5), there follows

3.9 $$\hat{G}D_1(s) + \hat{K}N_1(s) = D_2(s)R(s).$$

Comparing the coefficients of the polynomials with degree p_j in j-th column on both sides of (3.9), we have

3.10 $$\hat{G}\cdot\{d_1(i,j,p_j+1)\} = \{d_2(i,j,p_j+1)\}\cdot\{r(i,j,p_j-p_i+1)\}.$$

Since $D_1(s)$ and $D_2(s)$ are column proper, i.e., $\{d_1(i,j,p_j+1)\}$ and $\{d_2(i,j,p_j+1)\}$ are m×m real nonsingular matrices, and $\hat{G}$ is a real nonsingular matrix, there follows that $\{r(i,j,p_j-p_i+1)\}$, $(i \in \overline{m})$, $(j \in \overline{m})$ is a real nonsingular matrix.

Step 4 From the whole class of matrices $(\hat{G},\hat{K},R(s))$ generated in step 3, we calculate

3.11 $$G = \hat{G}^{-1} \quad \text{and} \quad K = \hat{G}^{-1}\hat{K},$$

and go to step 6.

Step 5 Print (THERE IS NO OUTPUT FEEDBACK LAW (G,K) FOR EXACT MODEL MATCHING).

Step 6 Stop.

END OF THE ALGORITHM

3.12 Theorem The class of matrices (G,K) generated in step 4 of Algorithm (3.1) is the whole class of output feedback laws for the exact model matching.

Proof We first show that any pair of matrices (G,K) generated in step 4 of Algorithm (3.1), which when applied to system S_1, yields a new system whose transfer function is equal to $H_2(s)$. Let $H_0(s)$ denote the overall system transfer function, then

$$
\begin{aligned}
H_0(s) &\triangleq H_1(s)[I+KH_1]^{-1}G \\
&= N_1(s)D_1^{-1}(s)[I+KN_1(s)D_1^{-1}(s)]^{-1}G && \text{from (3.2)} \\
&= N_1(s)D_1^{-1}(s)[G^{-1} + G^{-1}KN_1(s)D_1^{-1}(s)]^{-1} \\
&= N_1(s)[G^{-1}D_1(s) + G^{-1}KN_1(s)]^{-1} \\
&= N_1(s)[\hat{G}D_1(s) + \hat{K}N_1(s)]^{-1} && \text{from (3.11)} \\
&= N_2(s)R(s)[D_2(s)R(s)]^{-1} && \text{from (3.5)}
\end{aligned}
$$

$$= N_2(s)D_2^{-1}(s)$$

$$= H_2(s). \qquad\qquad \text{from (3.2)}$$

It remains to show that any output feedback law (G,K) with G nonsingular for the exact model matching is contained in the class of matrices generated in step 3 of Algorithm (3.1). Let (G_1,K_1) be an output feedback law for the exact model matching, i.e.,

3.13 $$H_2(s) = H_1(s)[I + K_1H_1(s)]^{-1}G_1$$

$$= N_1(s)D_1^{-1}(s)[I + K_1N_1(s)D_1^{-1}(s)]^{-1}G_1 \qquad \text{from (3.2)}$$

$$= N_1(s)[D_1(s) + K_1N_1(s)]^{-1}G_1$$

$$= N_1(s)[G_1^{-1}D_1(s) + G_1^{-1}K_1N_1(s)]^{-1}$$

3.14 $$= N_1(s)[\hat{G}_1D_1(s) + \hat{K}_1N_1(s)]^{-1},$$

where $\hat{G}_1 \triangleq \hat{G}_1^{-1}$ and $\hat{K}_1 \triangleq G_1^{-1}K_1$. In (3.2), $H_2(s) = N_2(s)D_2^{-1}(s)$, where $N_2(s)$ and $D_2(s)$ are right coprime. Thus from (3.2), (3.14) and Theorem (I.2.5), there exists a polynomial matrix $R_1(s)$ such that

3.15 $$N_1(s) = N_2(s)R_1(s)$$

3.16 $$\hat{G}_1D_1(s) + \hat{K}_1N_1(s) = D_2(s)R_1(s).$$

Since $N_1(s)$ and $D_1(s)$ are right coprime, from Corollary (I.1.12) there exists two polynomial matrices P(s) and Q(s) such that

3.17 $$P(s)N_1(s) + Q(s)D_1(s) = I.$$

Thus

$$[P(s)+Q(s)\hat{G}_1^{-1}\hat{K}_1-Q(s)\hat{G}_1^{-1}\hat{K}_1]N_1(s) + [Q(s)\hat{G}_1^{-1}][\hat{G}_1D_1(s)] = I$$

$$\Rightarrow \quad [P(s)-Q(s)\hat{G}_1^{-1}\hat{K}_1]N_1(s) + [Q(s)\hat{G}_1^{-1}][\hat{K}_1N_1(s)+\hat{G}_1D(s)] = I$$

$$\Rightarrow \quad \{[P(s)-Q(s)\hat{G}_1^{-1}\hat{K}_1]N_2(s) + [Q(s)\hat{G}_1^{-1}]D_2(s)\}R_1(s) = I$$

from (3.15), (3.16)

3.18 $\Rightarrow$ $R_1(s)$ is a unimodular matrix.

From the degree constraint on $D_1(s)$ and $N_1(s)$, and from the fact that $D_1(s)$ is column proper, (see (3.2)), it is easy to see that the (i,j) element of $\hat{D}(s) \triangleq \hat{G}_1D_1(s) + K_1N_1(s)$ in (3.16) has degree $\leq p_j(1)$ and $\hat{D}(s)$ is column proper. By direct computation we can easily verify that in (3.16) the (i,j) element of the polynomial matrix

3.19 $$R_1(s) = D_2^{-1}(s)[\hat{G}_1D_1(s) + \hat{K}N_1(s)]$$

has degree $\leq p_j(1)-p_i(2)$. Since $R_1(s)$ is unimodular and $D_2(s)R_1(s) = D_1(s)$, we have $\sum_{j=1}^{m} p_j(1) = \sum_{j=1}^{m} p_j(2)$. We are going to show that $p_j(1) =$

$p_j(2)$ for all $j \in \overline{m}$. Suppose not, then there is at least one $j_0 \in \overline{m}$ such that $p_{j_0}(1) < p_{j_0}(2)$. From the degree constraints on the elements of $R(s)$, it is easy to see that $R_1(s)$ has the following form

$$R_1(s) = m\left\{ \underbrace{\left[\begin{array}{ccc|cccc} x & \cdot & x & & \overbrace{\qquad\qquad}^{m-j_0+1} & & \\ \cdot & \cdot & \cdot & & O & & \\ \cdot & \cdot & \cdot & & & & \\ \hline x & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & \cdot & \cdot & \cdot & \cdot & \cdot & x \end{array}\right]}_{m} \right.\begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} j_0 \\ \\ \\ \\ \end{array}$$

i.e., the $j_0 x(m-j_0+1)$ submatrix on the right upper corner of $R_1(s)$ is zero, this implies that $\det R_1(s) \equiv 0$ and contradicts (3.18). In summary, we have shown that the existence of output feedback law (G_1,K_1) for exact model matching implies that the conditions in step 2 and step 3 are satisfied. From (3.15) and (3.16), $(\hat{G}_1,\hat{K}_1,R_1(s))$ is seen to be a solution of (3.3)-(3.5). Therefore, $G_1 \triangleq \hat{G}_1^{-1}$ and $K_1 \triangleq \hat{G}_1^{-1}\hat{K}_1$ is contained in the class of matrices generated in step 4 of Algorithm (3.1).

Q.E.D.

4. Discussion of the literature

The problem of exact model matching via state feedback was proposed by Wolovich [Wo.2], but his algorithm for solving this problem can only be applied to invertible systems. In the present chapter, we have a complete solution to the problem of exactly model matching both via state and output feedback. Our matching algorithm can be applied to any linear time-invariant multivariable systems and generates the whole class of state or output feedback laws for the matching purpose.

In the first part of our work, we derive a canonical form of transfer function matrix (see (1.9)), which is similar to the "structure theorem" due to Wolovich and Falb [Wo.1]. The approach in this part of our work is close to that used by Wolovich in solving the exact model matching problem. Then we transform the problem of finding state or output feedback laws into the problem of finding "regular solutions" of a real matrix equation. An algorithm for finding the whole class of regular solutions of any real matrix equation is also given (see (1.19)).

A possible extension of this result is the inclusion of dynamics in the state or output feedback law when the static feedback law is insufficient for exact matching.

A more important design problem is that of finding some feedback law (with or without dynamics) so that the over all system transfer function satisfy some prescribed requirements, rather than matching exactly a given transfer function. This amounts to placing restrictions on the coefficients of N^* and ψ^* in (1.12), rather than specifying all of them, the problem is then to find some regular solutions f^* of (1.12).

CHAPTER III

DECOUPLING OF LINEAR MULTIVARIABLE SYSTEMS

0. Introduction

In the design of linear multivariable systems, we are often trying to have inputs control outputs independently, i.e., a single input affects only a single output. This is the diagonal decoupling problem. The problem of diagonal decoupling a linear time-invariant system using state or output feedback has been examined by several authors. Falb and Wolovich [Fa.1] gave necessary and sufficient conditions for the existence of state or output feedback laws for the diagonal decoupling problem. In section 1 of this chapter, we give an alternate condition for the existence of output feedback law for the diagonal decoupling problem. We give a complete characterization of the decoupled system transfer function and relate the output feedback law to the poles of the decoupled system.

Then we consider the problem of triangular decoupling. This is a problem of finding a state feedback law to bring the over-all system transfer function in a quasi-triangular form. This problem was first formulated and solved by Morse and Wonham [Mo.2] by using a geometric approach. In section 2 of this chapter, we are dealing with a more restrictive case, we require the over-all system transfer function in an upper triangular form. We solve this problem using Silverman's inversion algorithm [Si.1] and we show that the conditions for the existence of state feedback laws for triangular decoupling is equivalent to the conditions for invertibility of linear multivariable systems.

1. Diagonal decoupling via output feedback and pole assignability

Consider a linear time-invariant multivariable systems specified by the following equations.

1.1a $$\dot{x}(t) = Ax(t) + Bu(t)$$

1.1b $$y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^m$ is the output, A, B and C are real constant matrices of appropriate size. The problem of diagonal decoupling via output feedback can be stated as follows, find an output feedback law

1.2 $$v(t) = Gu(t) + Ky(t)$$

with $K, G \in \mathbb{R}^{m\times m}$ and G being nonsingular for the given system (1.1a,b) such that the over-all system transfer function $C(sI-A-BKC)^{-1}BG$ is diagonal and all its diagonal elements are not identically zero.

We first state the following lemma due to Falb and Wolovich [Fa.1].

1.3 Lemma (Falb and Wolovich)

Consider the system in (1.1a,b). Let $d_1, d_2, \ldots, d_m$ be defined as

1.4 $$d_i = \begin{cases} \min\{k: \ c_iA^kB \neq 0, \ k = 0,1,\ldots,n-1\} \\ n-1 \quad \text{if } c_iA^kB = 0 \quad \text{for all } k \end{cases}$$

where c_i denotes the i-th row of C. Let B^* be the m×m real constant matrix given by

1.5 $$B^* = \begin{bmatrix} c_1 A^{d_1} B \\ c_2 A^{d_2} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} .$$

If there is an output feedback law (G,K) in (1.2) which decouples (1.1a,b), then there is a diagonal matrix Λ such that $G = (B^*)^{-1}\Lambda$.

Proof Since the pair (G,K) decouples (1.1a,b), the over-all system transfer function can be written as

1.6 $$C(sI-A-BKC)^{-1}BG = \begin{bmatrix} \lambda_1 \frac{v_1}{w_1} & & & \bigcirc \\ & \lambda_2 \frac{v_2}{w_2} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_m \frac{v_m}{w_m} \end{bmatrix}$$

where λ_i is some nonzero real number, v_i and w_i are monic polynomials in s and they are coprime for all $i \in \bar{m}$. Let us consider the i-th row in (1.6),

$$c_i\left(\sum_{k=0}^{\infty} \frac{(A+BKC)^k}{s^{k+1}}\right) BG = [0,\ldots,0,\lambda_i \frac{v_i}{w_i} ,0,\ldots,0]$$

Multiplying both sides of (1.7) by s^{d_i+1} and taking the limit as $s \to \infty$, we have

1.7 $$c_i A^{d_i} BG = [0,\ldots,\lambda_i,0,\ldots,0].$$

From (1.7) and the definition of B^* in (1.5), we have

1.8 $$G = (B^*)^{-1}\Lambda,$$

where Λ is an $m\times m$ diagonal matrix with λ_i in the (i,i) position. Q.E.D.

Now we can state the following algorithm

1.9 <u>Algorithm for decoupling of linear multivariable systems via output feedback</u>.

<u>Step 1</u> Calculate $d_1,d_2,\ldots,d_m$ and the $m\times m$ real constant matrix B^* as defined in (1.4) and (1.5). If $\det B^* \neq 0$, go to step 2, otherwise go to step 7.

<u>Step 2</u> As in step 1 of Algorithm (I.3.5), the transfer function matrix $H(s) \triangleq C(sI-A)^{-1}B$ of (1.1a,b) can be factored as

1.10 $$H(s) = N(s)D^{-1}(s)$$

where $N(s)$ and $D(s)$ are $m\times m$ matrices with elements in $\mathbb{R}[s]$. $N(s)$ and $D(s)$ are right coprime.

<u>Step 3</u> Calculate $\psi_i \triangleq$ g.c.d. of $n_{i1}, n_{i2},\ldots,n_{im}$, where n_{ij} is the (i,j) element of $N(s)$. Let $\Psi(s)$ be an $m\times m$ diagonal matrix with $\psi_i(s)$ in the (i,i) position. Calculate $\Psi^{-1}(s)N(s)$. If $\Psi^{-1}(s)N(s)$ is a unimodular matrix, go to step 4, otherwise go to step 7.

Step 4 Let $M(s) \triangleq B^* D(s) N^{-1}(s) \Psi(s)$, where B^*, $D(s)$, $N(s)$ and $\Psi(s)$ are given by step 1, 2 and 3. Let m_{ij} be the (i,j) element of $M(s)$. If $\frac{m_{ij}}{\psi_j}$ is a constant, denoted by $-\tilde{k}_{ij}$, for all $i \in \overline{m}$, $j \in \overline{m}$, go to step 5, otherwise go to step 7.

Step 5 Choose appropriate real constant $\tilde{k}_{ii}$ $(i \in \overline{m})$ such that the zeros of the polynomials $w_i(s) \triangleq m_{ii} + \tilde{k}_{ii}\psi_i$ $(i \in \overline{m})$ are suitable to be the poles of the decoupled system (see (1.6)).

Step 6 Calculate $K \triangleq (B^*)^{-1} \times \tilde{K}$, where $\tilde{K}$ is an $m \times m$ real constant matrix with $\tilde{k}_{ij}$ in the (i,j) position, $(i \in \overline{m})$, $(j \in \overline{m})$. Choose a set of appropriate nonzero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ as in (1.6). Then calculate $G = (B^*)^{-1}\Lambda$, where $\Lambda = \mathrm{diag}\{\lambda_i\}$ and go to step 8.

Step 7 PRINT (THERE EXISTS NO OUTPUT FEEDBACK LAW FOR DECOUPLING PROBLEM).

Step 8 Stop.

1.11 Theorem If the system in (1.1a,b) can be decoupled by an output feedback law (G,K) in (1.2), i.e., the over-all system transfer function has the form in (1.6), then

(a) $v_i = \psi_i$, $(i \in \overline{m})$, where ψ_i is the g.c.d. of $n_{i1}, n_{i2}, \ldots, n_{im}$.

(b) $w_i = m_{ii} + \tilde{k}_{ii}\psi_i$, $(i \in \overline{m})$, where m_{ii} is defined in step 4 and $\tilde{k}_{ii}$ is a real number.

(c) step 6 of Algorithm (1.9) generates an output feedback law (G,K), which decouples system (1.1a,b).

<u>Proof</u> Applying an output feedback law (G,K) in (1.2) to the system (1.1a,b), the over-all system transfer function $H_c(s) \triangleq C(sI-A-BKC)^{-1}BG$ can be written as

1.12 $$H_c(s) = H(s)[I+KH(s)]^{-1}G$$

where $H(s) \triangleq C(sI-A)^{-1}B$. From step 2 of Algorithm (1.9), we have $H(s) = N(s)D^{-1}(s)$. Hence $H_c(s)$ can be written as

1.13 $$H_c(s) = N(s)[G^{-1}D(s)+G^{-1}KN(s)]^{-1}.$$

Since N(s) and D(s) are right coprime, and from Corollary (I.1.12), we can easily show that N(s) and $G^{-1}D(s)+G^{-1}KN(s)$ are also right coprime. Assume that the output feedback law (G,K) decouples (1.1a,b), i.e., $H_c(s)$ has the form in (1.6) which can be rewritten as

1.14 $$H_c(s) = \Lambda V(s)W^{-1}(s)$$

where $\Lambda = \text{diag}\{\lambda_i\}$, $V(s) = \text{diag}\{v_i\}$ and $W(s) = \text{diag}\{w_i\}$. We first show that $\Lambda V(s)$ and W(s) are right coprime. Let R(s) be a right common divisor of $\Lambda V(s)$ and W(s), i.e., $\Lambda V(s) = \tilde{V}(s)R(s)$ and $W(s) = \tilde{W}(s)R(s)$ for some polynomial matrix $\tilde{V}(s)$ and $\tilde{W}(s)$. Since v_i and w_i are coprime for all $i \in \overline{m}$, we have

$$\text{rank}\begin{bmatrix}\tilde{V}(s)\\ \hline \tilde{W}(s)\end{bmatrix} x R(s) = \text{rank}\begin{bmatrix}\Lambda V(s)\\ \hline W(s)\end{bmatrix} = m \qquad \forall s \in \mathbb{C}$$

$$\Rightarrow \text{rank } R(s) = m \qquad \forall s \in \mathbb{C}$$

$\Rightarrow$ det $R(s)$ = nonzero constant

$\Rightarrow$ $R(s)$ is a unimodular matrix

1.15 $\Rightarrow$ $\Lambda V(s)$ and $W(s)$ are right coprime.

In (1.13) and (1.14), we factor $H_c(s)$ as products of polynomial matrices which are right coprime, then from Theorem (I.2.5), there exists a unimodular matrix $U(s)$ such that

1.16 $$N(s) = \Lambda V(s)U(s)$$

1.17 $$G^{-1}D(s) + G^{-1}KN(s) = W(s)U(s)$$

From (1.16), $V^{-1}(s)N(s) = \Lambda U(s)$, this shows that $V^{-1}(s)N(s)$ is a polynomial matrix, therefore v_i divides ψ_i the greatest common divisor of $n_{i1}, n_{i2}, \ldots, n_{im}$. We can rewrite $V^{-1}(s)N(s)$ as follows.

1.18 $$V^{-1}(s)N(s) = \text{diag}\{\psi_i/v_i\} \times \psi^{-1}(s) \times N(s)$$

where $\Psi(s) \triangleq \text{diag}\{\psi_i\}$. It is easy to see that both $\text{diag}\{\psi_i/v_i\}$ and $\Psi^{-1}(s)N(s)$ are polynomial matrices. Since $V^{-1}(s)N(s)$ is unimodular, we conclude that both $\text{diag}\{\psi_i/v_i\}$ and $\Psi^{-1}(s)N(s)$ are unimodular matrices. Therefore ψ_i/v_i = constant $\neq 0$. We assume that both ψ_i and v_i are monic polynomials, so that $v_i = \psi_i$ $(i \in \overline{m})$. This proves part (a).

From (1.16) and (1.17), we have

1.19 $$G^{-1}D(s)N^{-1}(s)V(s)\Lambda + G^{-1}KN(s)N^{-1}(s)V(s)\Lambda = W(s)$$

Substituting (1.8) into (1.19), there follows

1.20 $$B^{*}D(s)N^{-1}(s)V(s) + B^{*}KV(s) = W(s).$$

From part (a) of this theorem, $V(s) = \Psi(s)$, therefore (1.20) can be written as

1.21 $$B^{*}D(s)N^{-1}(s)\Psi(s) + B^{*}K\Psi(s) = W(s).$$

Let $M(s) \triangleq B^{*}D(s)N^{-1}(s)\Psi(s)$ and $\tilde{K} \triangleq B^{*}K$, then we have

1.22 $$M(s) + \tilde{K}\Psi(s) = W(s).$$

From (1.22), it is easy to see that

1.23 $$m_{ij} + \tilde{k}_{ij}\psi_j = 0 \quad i \neq j$$

1.24 $$m_{ii} + \tilde{k}_{ii}\psi_i = w_i$$

for all $i, j \in \overline{m}$. This proves part (b).

From the above arguments, it is clear that the existence of an output feedback law (G,K), which decouples system (1.1a,b), implies the conditions in step 1, 3 and 4 are satisfied. Substituting the pair of matrices $G = (B^{*})^{-1}\Lambda$ and $K = (B^{*})^{-1}\tilde{K}$ into (1.13), and from (1.16), (1.17) and (1.22), we can easily show that the over-all system transfer function has the form in (1.6). This proves part (c). Q.E.D.

1.25 <u>Remark</u> We have shown that the numerators v_i's of the diagonal elements in the decoupled system transfer function, (see (1.6)), is equal to $\psi_i \triangleq$ g.c.d. of $n_{i1}, n_{i2}, \ldots, n_{im}$, (see step 3 of Algorithm (1.9)). This set of polynomials ψ_i, $(i \in \overline{m})$, is completely determined by the given system (1.1a,b) and is independent of the output feedback law (G,K). The denominators w_i's in the decoupled system transfer function (1.6) are of the form $m_{ii} + \tilde{k}_{ii}\psi_i$, where m_{ii} and ψ_i are completely determined by the given system (1.1a,b), see step 4 of Algorithm (1.9), and $\tilde{k}_{ii}$, $(i \in \overline{m})$, can be any real numbers by choosing appropriate feedback matrix K. The set of constant multiples λ_i, $(i \in \overline{m})$, in (1.6) can be any set of nonzero constants by choosing appropriate $G \triangleq (B^*)^{-1} x \operatorname{diag}\{\lambda_i\}$.

2. Relationship between Triangular Decoupling and Invertibility of Linear Multivariable Systems

For linear multivariable systems, which have square transfer function matrices, the necessary and sufficient conditions for the existence of state feedback laws for triangular decoupling are shown to be equivalent to the conditions of invertibility. A procedure of finding a state feedback law for this purpose is also given.

Consider the system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1a}$$

$$y(t) = Cx(t) + Du(t) \tag{2.1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A,B,C,D are real constant matrices of appropriate size. The triangular decoupling problem via state feedback can be stated as follows: Find matrices $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{m \times m}$, so that the system in (2.1a,b) together with the state feedback law

$$u(t) = Fx(t) + Gv(t), \tag{2.2}$$

results in a closed-loop transfer function matrix (relating the new input v and the output y) which has a nonsingular upper triangular form. This problem was first formulated by Morse and Wonham [Mo.2] in a slightly more general form, and was solved by using a geometric approach.

In the present, we modify the inversion algorithm by Silverman [Si.1] and apply it to the triangular decoupling problem.

Two sequences of matrices $[C_k \mid D_k]$ and $[\overline{C}_k \mid \overline{D}_k]$ $(k=1,2,\cdots)$ can be obtained from (A,B,C,D) as follows: Let $\overline{D}_0 \triangleq D$ and define D_k in terms of $\overline{D}_{k-1}$ by

2.3 $$d_k^j = \underline{0} \text{ if } \overline{d}_{k-1}^j \in \text{Span}\{\overline{d}_{k-1}^\ell, j < \ell \leq m\},\ j \in \{1,\ldots,m-1\}$$

2.4 $$d_k^j = \overline{d}_{k-1}^j \text{ if } \overline{d}_{k-1}^j \notin \text{Span}\{\overline{d}_{k-1}^\ell, j < \ell \leq m\},\ j \in \{1,\ldots,m-1\}$$

2.5 $$d_k^m = \overline{d}_{k-1}^m$$

$$(k=1,2,\cdots)$$

where d_k^j and $\overline{d}_k^j$, $(j \in \overline{m})$, are the j-th row of D_k and $\overline{D}_k$, respectively. It is clear that we can always find a nonsingular upper triangular matrix S_{k-1} such that $D_k = S_{k-1}\overline{D}_{k-1}$. Set $\overline{C}_0 \triangleq C$ and define C_k in terms of $\overline{C}_{k-1}$ by

2.6 $$[C_k \,\vdots\, D_k] \triangleq S_{k-1}[\overline{C}_{k-1} \,\vdots\, \overline{D}_{k-1}],\ (k=1,2,\cdots).$$

Note that in the above definition, C_k is not uniquely determined in terms of (A,B,C,D), because it also depends on the choice of S_{k-1}.

Next we define $[\overline{C}_k \,\vdots\, \overline{D}_k]$ in terms of $[C_k \,\vdots\, D_k]$, $(k=1,2,\cdots)$, let $J \triangleq \{1,\ldots,m\}$, and $J_k \triangleq \{j \mid d_k^j \neq \underline{0},\ j \in J\}$, then

2.7 $$\overline{c}_k^j = c_k^j \quad \forall j \in J_k$$

2.8 $$\overline{c}_k^j = c_k^j A \quad \forall j \in (J \setminus J_k)$$

2.9 $$\overline{d}_k^j = d_k^j \quad \forall j \in J_k$$

2.10 $$\bar{d}_k^j = c_k^j B \quad \forall j \in (J \backslash J_k)$$

where $\bar{c}_k^j$ and c_k^j are the j-th row of $\bar{C}_k$ and C_k, respectively.

Then we define an m×m diagonal differentiation operation $M_k(P) \triangleq \text{diag}(m_k^j)$ $(k=1,2,\cdots)$ with

2.11 $$m_k^j = 1 \quad \forall j \in J_k$$

2.12 $$m_k^j = P \triangleq (d/dt) \quad \forall j \in (J \backslash J_k)$$

Consider the output equation in (2.1b)

2.1b' $$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= \bar{C}_0 x(t) + \bar{D}_0 u(t) \text{(from the definitions of } \bar{C}_0 \text{ and } \bar{D}_0) \end{aligned}$$

multiply both sides of (2.1b') by S_{i-1} and $M_i(P)$, $(i \in \bar{k})$, together with the definitions of $\bar{C}_k$ and $\bar{D}_k$, we can show that

2.13 $$N_k(P)y(t) \triangleq \left(\prod_{\ell=0}^{k-1} M_{k-\ell}(P) S_{k-\ell-1} \right) y(t) = \bar{C}_k x(t) + \bar{D}_k u(t)$$

$$(k = 1,2,\cdots)$$

In (2.13), if there exists an integer $\alpha \geq 0$ such that $\bar{D}_\alpha$ is non-singular, then (2.13) can be written as

$$u(t) = (\bar{D}_\alpha)^{-1} [N_\alpha(P)y(t) - \bar{C}_\alpha x(t)]$$

If we define the state feedback law in (2.2) with

2.14 $$F = -(\bar{D}_\alpha)^{-1}\bar{C}_\alpha, \; G = (\bar{D}_\alpha)^{-1}$$

and from (2.13), set $k=\alpha$,

2.15 $$N_\alpha(P)y(t) = \bar{C}_\alpha x(t) + \bar{D}_\alpha[-(\bar{D}_\alpha)^{-1}\bar{C}_\alpha x(t) + (\bar{D}_\alpha)^{-1}v(t)] = v(t)$$

i.e. the closed-loop system transfer function matrix $\tilde{H}(s)$ relating the new input v and the output y is equal to $(N_\alpha(s))^{-1}$. From the constructions of $N_\alpha(P) \triangleq \left(\prod_{\ell=0}^{\alpha-1} M_{\alpha-\ell}(P)S_{\alpha-\ell-1}\right)$, each S_i, $(i=0,1,\cdots)$, is nonsingular and has an upper triangular form, and each $M_i(P)$, $(i=1,2,\cdots)$, is a diagonal differentiation operator with diagonal elements being 1 or P, it is easy to see that the product $N_\alpha(P)$ has a nonsingular upper triangular form, there follows that $\tilde{H}(s) \triangleq (N_\alpha(s))^{-1}$ is also in a nonsingular upper triangular form, this satisfies the requirements for triangular decoupling.

It remains to answer the question that under what conditions on (A,B, C,D), there always exists an integeger $\alpha \geq 0$ such that $\bar{D}_\alpha$ is nonsingular.

Let $W_0 = D$ and

2.16 $$W_k = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ CA^{k-1}B & CA^{k-2}B & CA^{k-3}B & \cdots & D \end{bmatrix}$$

$(k \in \bar{n})$

2.17 <u>Theorem</u> (Sain and Massey [Sa.1]

The system in (2.1a,b) is invertible (i.e. $\det[D+C(sI-A)^{-1}B] \neq 0$ a.e.) if and only if, for some integer α, $0 \leq \alpha \leq n$,

$$\text{rank } W_\alpha - \text{rank } W_{\alpha-1} = m.$$

The following Lemma and its proof are from a paper by Singh [Si.3] with a little modification.

2.18 Lemma Let W_k $(k \in \overline{n})$ be defined in (2.16) and D_ℓ $(\ell \in \overline{n+1})$ be defined in (2.3)-(2.10). Then

$$\text{rank } W_k = \sum_{\ell=1}^{k+1} \text{rank } D_\ell$$

Proof of Lemma 2.18) Premultiplying W_k by a $(k+1)m \times (k+1)m$ square matrix $U_0 \triangleq \text{diag}[S_0,\cdots,S_0]$ and performing elementary row operations on U_0W_k: shift the $[(k-\ell)m+j]$-th row of U_0W_k to the position of the $[(k-\ell-1)m+j]$-th row, for all $j \in (J\setminus J_1)$ and $\ell \in \{0,\cdots,k-1\}$, also shift the j-th row of U_0W_k to the position of the $(km+j)$-th row, for all $j \in (J\setminus J_1)$. After this process, (or equivalently, premultiplying U_0W_k by an elementary matrix R_1) we have

2.19
$$R_1U_0W_k = \begin{bmatrix} \overline{D}_1 & 0 & 0 & & 0 & 0 \\ \overline{C}_1B & \overline{D}_1 & 0 & & 0 & 0 \\ \overline{C}_1AB & \overline{C}_1B & \overline{D}_1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \overline{C}_1A^{k-2}B & \overline{C}_1A^{k-3}B & \overline{C}_1A^{k-4}B & \cdots & \overline{D}_1 & 0 \\ X & X & X & \cdots & x & D_1 \end{bmatrix}$$

where X denotes some m×m matrix, note that among the last m rows in the above matrix, there are exactly $\#(J_1)$ rows which are nonzero (where $\#(J_1)$ denotes the number of elements in the set J_1), moreover, these $\#(J_1)$ rows are linearly independent, this follows from the constructions of D_1. Continuing in this manner, it can be shown that

$$\left(\prod_{\ell=0}^{k-1} U_{k-\ell}R_{k-\ell}\right)\cdot U_0W_k \triangleq U_k\cdot R_k\cdots U_1\cdot R_1\cdot U_0\cdot W_k$$

$$\triangleq \mathrm{diag}[S_k,I,\cdots,I]\cdot R_k\cdot \mathrm{diag}[S_{k-1},S_{k-1},I,\cdots,I]\cdot R_{k-1}\cdots$$

$$\mathrm{diag}[S_1,\cdots,S_1,I]\cdot R_1\cdot \mathrm{diag}[S_0,\cdots,S_0]\cdot W_k$$

2.20
$$= \begin{bmatrix} D_{k+1} & 0 & 0 & \cdots & 0 & 0 \\ X & D_k & 0 & \cdots & 0 & 0 \\ X & X & D_{k-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ X & X & X & \cdots & D_2 & 0 \\ X & X & X & \cdots & X & D_1 \end{bmatrix}$$

where R_i, $(i \in \overline{k})$, corresponds to the shifting of the $[(k-\ell)m+j]$-th row to the position of $[(k-\ell-1)m+j]$-th row, and the shifting of the j-th row to the position of $[(k+1-i)m+j]$-th row, for all $j \in (J\backslash J_i)$ and $\ell \in \{i-1,\cdots, k-1\}$. In (2.20) we use X to denote some m×m matrix, note that there are exactly $\#(J_\ell)$ nonzero rows among these m rows containing D_ℓ, $(\ell \in \overline{k+1})$, moreover, all these nonzero rows are linearly independent, there follows

$$\operatorname{rank} W_k = \operatorname{rank}\left(\prod_{\ell=0}^{k-1} U_{k-\ell}R_{k-\ell}\right)\cdot U_0 W_k$$

$$= \sum_{\ell=1}^{k+1} \#(J_\ell) = \sum_{\ell=1}^{k+1} \operatorname{rank} D_\ell \qquad \text{Q.E.D.}$$

2.21 Theorem The system described by (2.1a,b) is invertible if and only if there exists a control law $u(t) = Fx(t) + Gv(t)$, such that the closed-loop system transfer function matrix relating the new input v and the output y has a nonsingular upper triangular form. (i.e. the system in (2.1a,b) can be triangularly decoupled).

Proof $\Rightarrow$ From Theorem (2.17), there exists α, $0 \leq \alpha \leq n$, such that

$$m = \operatorname{rank} W_\alpha - \operatorname{rank} W_{\alpha-1}$$

$$= \operatorname{rank} D_{\alpha+1} \qquad \text{(from Lemma (2.18))}$$

$$= \operatorname{rank} \overline{D}_\alpha \qquad (\because D_{\alpha+1} = S_\alpha \overline{D}_\alpha, \text{ and } S_\alpha \text{ is a nonsingular matrix})$$

this guarantees the existence of $(\overline{D}_\alpha)^{-1}$ for some positive integer $\alpha \leq n$, then apply the feedback law specified in (2.14), the closed loop system is triangularly decoupled.

$\Leftarrow$ Assume the system in (2.1a,b) is not invertible, (i.e. $H(s) \triangleq D + C(sI-A)^{-1}B$ is singular), and since the singularity of a transfer function matrix is invariant under state feedback, so there exists no state feedback law as in (2.2), such that the closed loop system transfer function matrix has a nonsingular upper triangular form. Q.E.D.

2.22 Remark Comparing the result in Theorem 2.21 with that in Theorem 1 by Morse and Wonham [Mo.2], it is clear that when $D = \underline{0}$, $\det[C(sI-A)^{-1}B] \neq 0$ a.e. is equivalent to the following condition: Let $\mathcal{N}_i$ be the null space of the i-th row of C, $(i \in \overline{m})$, then

$$\bar{\mathcal{R}}_i + \mathcal{N}_i = \mathcal{E}, \quad (i \in \overline{m})$$

where $\bar{\mathcal{R}}_i$, $(i \in \overline{m-1})$, denotes the maximal controllability subspace of (A,B) satisfying $\bar{\mathcal{R}}_i \subset \bigcap_{j \in J_i} \mathcal{N}_j$, $(i \in \overline{m-1})$, with $J_i \triangleq \{i+1, \cdots, k\}$, and $\bar{\mathcal{R}}_m \triangleq \{A|\mathcal{B}\}$. The reason is that both conditions are necessary and sufficient for the existence of a solution to the triangular decoupling problem. It is worthwhile to find a direct way of proving this equivalence other than the above arguments.

2.23 Remark We are using a modified Silverman's inversion algorithm to solve the triangular decoupling problem. As indicated by Silverman [Si.1] his algorithm can be extended to time-varying case by imposing some regular conditions on the system. A fruitful subject of research might be the extension of these methods to the problem of the triangular decoupling of time-varying systems.

3. Discussion of the literature

The problem of diagonal decoupling a linear time-invariant system using state feedback was introduced by Morgan [Mo.4]. Morgan and Rekasius [Mo.4,Re.1] have given some sufficient conditions for the decoupling of linear time-invariant system by state feedback. Falb and Wolovich [Fa.1] gave a necessary and sufficient condition for diagonal decoupling via both state and output feedback. Gilbert [Gi.2] and subsequently Wolovich and Falb [Wo.1] examined the assignability of closed loop poles while simultaneously decoupling a system via state feedback. In section 1, we give an alternate condition for the existence of output feedback law for the diagonal decoupling problem. We give a complete characterization of the decoupled system transfer function and relate the output feedback law to the poles of the decoupled system. Our approach in solving this problem is similar to that used by Wolovich and Falb [Wo.1].

Howze and Pearson [Ho.2] and Silverman and Payne [Si.3] have examined the problem of diagonal decoupling via output feedback with dynamic compensation. Sato and Lopresti [Sa.2] examined the partial decoupling problem. Wonham and Morse [Wo.5,Mo.3] have a general formulation of the diagonal decoupling problem via state feedback and have solved this problem by using a geometric approach. Their results have been extended to the output feedback case in Chapter IV of this thesis. The triangular decoupling problem via state feedback was also formulated and solved by Morse and Wonham [Mo.2] using geometric approach. In section 2, we solve this problem using Silverman's inversion algorithm [Si.1] and we show that the conditions for the existence of state feedback law for triangular decoupling is equivalent to the conditions of invertibility.

CHAPTER IV

GEOMETRIC THEORY FOR DECOUPLING VIA OUTPUT FEEDBACK

0. INTRODUCTION

In the last chapter, we consider the triangular and the diagonal decoupling problems via state and output feedback, respectively. In this chapter, we consider some more general formulations of the decoupling problems via output feedback. Instead of bringing the overall system transfer function matrix in the diagonal (triangular) form, we only require it to be in the quasi-diagonal (quasi-triangular) form. The present work was motivated by the results of Wonham and Morse [Wo.5, Mo.2, Mo.3], where they have formulated two kinds of decoupling problems (a) quasi-diagonal and (b) quasi-triangular decoupling problems, both via state feedback and have solved these problems by a geometric approach. They introduced the concept of controllability subspace and its relation to the pole assignment problem.

In this chapter, we also use a geometric approach and extend the results of [Wo.5, Mo.2, Mo.3] to the output feedback case, namely, we solve the following problems:

(a) diagonal decoupling via output feedback

(b) triangular decoupling via output feedback

(c) diagonal decoupling via output feedback with dynamic compensation, and

(d) triangular decoupling via output feedback with dynamic compensation.

In the above four cases, the necessary and sufficient conditions for the existence of decoupling matrices (and new dynamic elements) are found. A constructive procedure for finding these decoupling matrices (and new dynamic elements) is given. In solving the above problems, we also extend the concept of controllability subspace to output feedback case, which is seen to be of importance in linear multivariable system theory.

In this chapter, we follow closely the work of Wonham and Morse [Wo.5, Mo.2, Mo.3], where they consider the decoupling problems via state feedback.

1. Controllability subspace

In this section, we introduce the notions of invariant subspace, controllability subspace, etc., via output feedback. These are useful tools in solving decoupling problems in the following sections.

Consider a linear time-invariant multivariable systems specified by the following equations

1.1a $$\dot{x}(t) = Ax(t) + Bu(t)$$

1.1b $$y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$ and A, B and C are real constant matrices of appropriate size.

If we apply a feedback control law $u(t) = Ky(t) + v(t)$ to the above system, where $K \in \mathbb{R}^{mxq}$ and $v(\cdot)$ is the external input, then the overall system is governed by

1.2a $$\dot{x}(t) = (A+BKC)\, x(t) + Bv(t)$$

1.2b $$y(t) = Cx(t)$$

1.3 Lemma

Given any (A,B,C) with $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $C \in \mathbb{R}^{qxn}$ and given any subspace $\mathcal{V} \subset \mathbb{R}^n$, there exists a real constant matrix $K \in \mathbb{R}^{mxq}$

such that

1.4 $$(A+BKC)\mathcal{V} \subset \mathcal{V}$$

if and only if

1.5 $$A\mathcal{V} \subset \mathcal{B} + \mathcal{V}$$

1.6 $$A[\mathcal{V} \cap \mathcal{N}(C)] \subset \mathcal{V}$$

where $\mathcal{B}$ denotes the range of B, $\mathcal{N}(C)$ denotes the null space of C, $A: \mathbb{R}^n \to \mathbb{R}^n$ is the linear map determined by the matrix A relative to the canonical basis of $\mathbb{R}^n$, (i.e., we use the same symbols for matrices as for linear maps), and $\mathcal{B} + \mathcal{V} \triangleq \{b+v \mid b \in \mathcal{B},\ v \in \mathcal{V}\}$.

<u>Proof</u> We show first that (1.4) $\Rightarrow$ (1.5) and (1.6). Pick a vector $v_1 \in \mathcal{V}$. From (1.4) there exists a vector $w_1 \in \mathcal{V}$ such that

$$(A+BKC)v_1 = w_1$$

or

$$Av_1 = (-BKCv_1 + w_1) \in \mathcal{B} + \mathcal{V}$$

Since v_1 is arbitrary, (1.5) is established. Then we pick a vector $v_2 \in \mathcal{V} \cap \mathcal{N}(C)$. Again from (1.4) we know that $Av_2 = w_2$ for some $w_2 \in \mathcal{V}$, so (1.6) is established.

Now we are going to show that (1.5) (1.6) $\Rightarrow$ (1.4). Write $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1 \triangleq \mathcal{V} \cap \mathcal{N}(C)$. Let $v_1, \ldots, v_p$ be a basis of $\mathcal{V}_1$, and let $v_{p+1}, \ldots, v_k$ be a basis of $\mathcal{V}_2$. From (1.6)

1.7 $$Av_i = w_i \qquad (i=1, \ldots, p)$$

where $w_i \in \mathcal{V}$. From (1.5) we have

1.8 $$Av_i = Bu_i + w_i \qquad (i=p+1, \ldots, k)$$

where $u_i \in \mathbb{R}^m$, $w_i \in \mathcal{V}$. If we can construct an mxq matrix K, such that

1.9 $$K[Cv_{p+1} \vdots Cv_{p+2} \vdots \cdots \vdots Cv_k] = -[u_{p+1} \vdots \cdots \vdots u_k],$$

then from (1.7), (1.8) and (1.9, we have

1.10 $$(A+BKC)v_i = w_i \qquad (i=1, \ldots, k)$$

where $w_i \in \mathcal{V}$ and $v_1, \ldots, v_k$ is a basis of $\mathcal{V}$. Therefore, (1.4) is established.

In order to guarantee the existence of K in (1.9), it is sufficient to show that the set of column vectors $Cv_{p+1}, \ldots, Cv_k$ are linearly independent. Since then the row vectors in $[Cv_{p+1} \vdots Cv_{p+2} \vdots \cdots \vdots Cv_k]$ span $\mathbb{R}^{k-p}$, and multiplying it by K serves as an appropriate linear combination of the row vectors in $[Cv_{p+1} \vdots C_{p+2} \vdots \cdots \vdots Cv_k]$ to make (1.9) valid.

It is easy to show that $Cv_{p+1}, \ldots, Cv_k$ are linearly independent. Since $v_{p+1}, \ldots, v_k$ is a basis of $\mathcal{V}_2$, where $\mathcal{V}_2 \cap \mathcal{N}(C) = \{0\}$, thus $\sum_{i=p+1}^{k} \alpha_i Cv_i = C \sum_{i=p+1}^{k} \alpha_i v_i = \underline{0} \Rightarrow \sum_{i=p+1}^{k} \alpha_i v_i = \underline{0} \Rightarrow \alpha_i = 0, \ (i=p+1, \ldots, k)$.

Q.E.D.

1.11 <u>Definition</u>

$\mathcal{V} \subset \mathbb{R}^n$ is said to be <u>(A,B,C)-invariant</u> iff $\mathcal{V}$ satisfies (1.5) and (1.6).

$\mathcal{V} \subset \mathbb{R}^n$ is said to be <u>(A,B,I)-invariant</u> iff $\mathcal{V}$ satisfies (1.5).

$\mathcal{V} \subset \mathbb{R}^n$ is said to be <u>(A,I,C)-invariant</u> iff $\mathcal{V}$ satisfies (1.6).

1.12 <u>Theorem</u>

Given any (A,B,C) with $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $C \in \mathbb{R}^{qxn}$ and given any subspace $\mathcal{J} \subset \mathbb{R}^n$, there always exists a unique <u>maximal</u> (A,B,C)-

invariant subspace $\mathcal{V}$ contained in $\mathcal{J}$. More precisely, any subspace $\mathcal{V}$ which satisfies (1.13), (1.14) and (1.15)

1.13 $$\mathcal{V} \subset \mathcal{J}$$

1.14 $$A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$$

1.15 $$A[\mathcal{V} \cap \mathcal{N}(C)] \subset \mathcal{V}$$

is contained in the maximal solution $\overline{\mathcal{V}}$.

<u>Proof</u> Using the following algorithm, we can compute the maximal (A,B,C)-invariant subspace in $\mathcal{J}$.

<u>Step 1</u> $k = 1$

<u>Step 2</u> $\mathcal{V}_1^{(0)} = \mathcal{J}$

<u>Step 3</u> $\mathcal{P} = \mathcal{B}$

<u>Step 4</u> $i = 0$

<u>Step 5</u> $\mathcal{V}_k^{(i+1)} = \mathcal{V}_k^{(i)} \cap A^{-1}(\mathcal{P} + \mathcal{V}_k^{(i)})$

<u>Step 6</u> If $\mathcal{V}_k^{(i+1)} = \mathcal{V}_k^{(i)}$, go to Step 8, otherwise go to Step 7

<u>Step 7</u> $i = i+1$, go to Step 5.

<u>Step 8</u> If k is an odd number, go to Step 9, otherwise go to Step 15.

<u>Step 9</u> $\mathcal{V}_k = \mathcal{V}_k^{(i)}$

<u>Step 10</u> $\hat{\mathcal{V}}_k = \mathcal{V}_k \cap \mathcal{N}(C)$

<u>Step 11</u> Pick $\tilde{\mathcal{V}}_k$ such that $\mathcal{V}_k = \hat{\mathcal{V}}_k \oplus \tilde{\mathcal{V}}_k$

<u>Step 12</u> $\mathcal{P} = \tilde{\mathcal{V}}_k$

<u>Step 13</u> $k = k+1$

<u>Step 14</u> $\mathcal{V}_k^{(0)} = \hat{\mathcal{V}}_{k-1}$ go to Step 4

<u>Step 15</u> $\mathcal{V}_k = \mathcal{V}_k^{(i)} + \tilde{\mathcal{V}}_{k-1}$

Step 16 If $\mathcal{V}_k = \mathcal{V}_{k-1}$, go to Step 19, otherwise go to Step 17

Step 17 $k = k+1$

Step 18 $\mathcal{V}_k^{(0)} = \mathcal{V}_{k-1}$, go to Step 3

Step 19 $\bar{\mathcal{V}} = \mathcal{V}_k$ and stop.

From Step 1 to Step 7 of the above algorithm, we generate a sequence of subspaces $\mathcal{V}_1^{(0)}, \mathcal{V}_1^{(1)}, \ldots,$ in $\mathcal{J}$. This sequence is obviously monotone-decreasing, (i.e., $\mathcal{V}_1^{(i)} \supset \mathcal{V}_1^{(i+1)}\ \forall\ i \geq 0$). And since $\mathcal{J}$ is of finite dimension, there exists an integer $j \leq \dim \mathcal{J}$ such that $\mathcal{V}_1^{(i)} = \mathcal{V}_1^{(j)}$ for all $i \geq j$. We write $A^{-1}\mathcal{V}$ for the subspace $\{z : z \in \mathbb{R}^n,\ Az \in \mathcal{V}\} \subset \mathbb{R}^n$. Then $\mathcal{V}_1^{(j)} = \mathcal{V}_1^{(j)} \cap A^{-1}(\mathcal{B} + \mathcal{V}_1^{(j)}) \Rightarrow A\,\mathcal{V}_1^{(j)} \subset \mathcal{B} + \mathcal{V}_1^{(j)}$, i.e., $\mathcal{V}_1^{(j)}$ is an (A,B,I)-invariant subspace in $\mathcal{J}$. Let $\mathcal{V}$ be any (A,B,I)-invariant subspace in $\mathcal{J}$, then $\mathcal{V} = \mathcal{V} \cap A^{-1}(\mathcal{B} + \mathcal{V})$. Thus if $\mathcal{V}_1^{(i)} \supset \mathcal{V}$, then $\mathcal{V}_1^{(i+1)} \triangleq \mathcal{V}_1^{(i)} \cap A^{-1}(\mathcal{B} + \mathcal{V}_1^{(i)}) \supset \mathcal{V}$, and since $\mathcal{V}_1^{(0)} \triangleq \mathcal{J} \supset \mathcal{V}$, so $\mathcal{V}_1^{(i)} \supset \mathcal{V}$ for all i. From the above arguments, we know that $\mathcal{V}_1^{(j)}$ is the maximal (A,B,I)-invariant subspace in $\mathcal{J}$, and we call it $\mathcal{V}_1$. It is obvious that $\mathcal{V}_1$ contains any (A,B,C)-invariant subspace in $\mathcal{J}$.

Now we are going to find the <u>maximal</u> (A,I,C)-invariant subspace in $\mathcal{V}_1$. In Step 10 to Step 15 of the above algorithm, we first compute the <u>maximal</u> subspace $\mathcal{V}$ in $\hat{\mathcal{V}}_1 \triangleq \mathcal{V}_1 \cap \mathcal{N}(C)$ such that $A\mathcal{V} \subset \mathcal{V} + \tilde{\mathcal{V}}_1$, where $\tilde{\mathcal{V}}_1$ is given by Step 11. Using the iterate formula in Step 4 to Step 6, we find that the maximal solution $\mathcal{V}$ is $\mathcal{V}_2^{(i)}$. Then in Step 15, we define $\mathcal{V}_2 \triangleq \mathcal{V}_2^{(i)} + \tilde{\mathcal{V}}_1$. It is easy to show that $\mathcal{V}_2$ is the maximal (A,I,C)-invariant subspace in $\mathcal{V}_1$. Since any (A,I,C)-invariant subspace $\mathcal{V} \subset \mathcal{V}_1$ can be written as $\mathcal{V} = \hat{\mathcal{V}} \oplus \tilde{\mathcal{V}}$, where $\hat{\mathcal{V}} \triangleq \mathcal{V} \cap \mathcal{N}(C) \subset \hat{\mathcal{V}}_1$ and

$\tilde{\mathcal{V}} \subset \tilde{\mathcal{V}}_1$. And from the (A,I,C)-invariant property of $\mathcal{V}$, we have $A\hat{\mathcal{V}} \subset \hat{\mathcal{V}} + \tilde{\mathcal{V}} \subset \hat{\mathcal{V}} + \tilde{\mathcal{V}}_1$. The iterate formula in Step 4 to Step 6 gives the maximal subspace $\mathcal{V}_2^{(i)}$ in $\hat{\mathcal{V}}_1$ such that $A\,\mathcal{V}_2^{(i)} \subset \mathcal{V}_2^{(i)} + \tilde{\mathcal{V}}_1$, so $\mathcal{V}_2^{(i)} \supset \hat{\mathcal{V}}$. Therefore $\mathcal{V}_2 \triangleq \mathcal{V}_2^{(i)} + \tilde{\mathcal{V}}_1 \supset \hat{\mathcal{V}} + \tilde{\mathcal{V}} \triangleq \mathcal{V}$, i.e., $\mathcal{V}_2$ contains any (A,I,C)-invariant subspace in $\mathcal{V}_1$, so $\mathcal{V}_2$ is the maximal one. Note that $\mathcal{V}_2$ contains any (A,B,C)-invariant subspace in $\mathcal{J}$.

In the following,we find the maximal (A,B,I)-invariant subspace in $\mathcal{V}_2$, call it $\mathcal{V}_3$. Then we find the maximal (A,I,C)-invariant subspace in $\mathcal{V}_3$, call it $\mathcal{V}_4$, and so on. Continuing this process, we construct a monotone-decreasing sequence of subspace $\mathcal{V}_i$, $i = 1,2,3, \ldots$, where each $\mathcal{V}_i$ contains any (A,B,C)-invariant subspace in $\mathcal{J}$. And since $\mathcal{J}$ is of finite dimension,there exists $\rho \leq \dim \mathcal{J}$ such that $\mathcal{V}_\rho = \mathcal{V}_i$ for all $i \geq \rho$. This $\mathcal{V}_\rho$ is (A,B,C)-invariant, moreover, it is the maximal (A,B,C)-invariant subspace in $\mathcal{J}$, which is denoted by $\bar{\mathcal{V}}$.

Q.E.D.

If we apply a feedback control law $u(t) = Ky(t) + Gv(t)$ to the system specified by (1.1ab), where $v(\cdot)$ is the external input, then the overall system is governed by

1.16a $$\dot{x}(t) = (A+BKC)\,x(t) + BGv(t)$$

1.16b $$y(t) = Cx(t)$$

where $K \in \mathbb{R}^{m\times q}$ and $G \in \mathbb{R}^{m\times m'}$.

The controllable subspace from the input $v(\cdot)$ is

$$\mathcal{R} = \{A+BKC|\ \{BG\}\}$$

where $\{BG\}$ denotes the range of BG, and $\{A+BKC|\ \{BG\}\} \triangleq \sum_{j=1}^{n} (A+BKC)^{j-1} \{BG\}$. From Lemma 2.1 in [Wo.5], $\mathcal{R} = \{A+BKC|\ \{BG\}\} = \{A+BKC|\mathcal{B} \cap \mathcal{R}\}$. Then we can define an <u>(A,B,C) controllability subspace</u> as follows:

1.17 <u>Definition</u>

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ be real constant matrices and let $\mathcal{R}$ be a subspace in $\mathbb{R}^n$, then $\mathcal{R}$ is said to be an <u>(A,B,C) controllability subspace</u> or (A,B,C) c.s., if there exists $K \in \mathbb{R}^{m\times q}$ such that

1.18 $$\{A+BKC \mid \mathcal{B} \cap \mathcal{R}\} = \mathcal{R}$$

An (A,B,C) controllability subspace can be characterized as follows

1.19 <u>Theorem</u>

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ and $\mathcal{R}$ be a subspace in $\mathbb{R}^n$. Then $\mathcal{R}$ is an (A,B,C) controllability subspace if and only if

1.20 $$A\mathcal{R} \subset \mathcal{B} + \mathcal{R}$$

1.21 $$A[\mathcal{R} \cap \mathcal{N}(C)] \subset \mathcal{R}$$

and

1.22 $$\mathcal{R} = \mathcal{R}^{(\rho)}, \text{ where } \rho = \dim \mathcal{R} \text{ and}$$

1.24 $$\mathcal{R}^{(0)} = \{0\},\ \mathcal{R}^{(i)} = \mathcal{R} \cap (A\mathcal{R}^{(i-1)} + \mathcal{B}). \quad (i \in \bar{n})$$

Write $\underline{K}(\mathcal{R})$ for the class of matrices K such that $(A+BKC)\mathcal{R} \subset \mathcal{R}$. To prove the theorem we need two preliminary results.

1.25 <u>Lemma</u>

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ be real constant matrices and let $\mathcal{R}$ be an (A,B,C)-invariant subspace. If $\tilde{\mathcal{R}}$ is a subspace contained in $\mathcal{R}$, then

$$\mathcal{D}(K) \triangleq \mathcal{R} \cap \mathcal{B} + (A+BKC)\tilde{\mathcal{R}}$$
$$= \mathcal{R} \cap (A\tilde{\mathcal{R}} + \mathcal{B})$$

Proof It is easy to see that $\mathcal{D}(K) \subset \mathcal{R} \cap (A\tilde{\mathcal{R}}+\mathcal{B})$. To prove the reverse inclusion, we show first that $\mathcal{D}(K)$ is the same for all $K \in \underline{K}(\mathcal{R})$. If $K_1, K_2 \in \underline{K}(\mathcal{R})$ and $x \in \tilde{\mathcal{R}}$, then

$$B(K_2-K_1)Cx = (A+BK_2C)x-(A+BK_1C)x \in \mathcal{R},$$

so that $B(K_2-K_1)C\tilde{\mathcal{R}} \subset \mathcal{R} \cap \mathcal{B}$. Therefore

$$\begin{aligned}\mathcal{D}(K_2) &= \mathcal{R} \cap \mathcal{B} + [A + BK_1C + B(K_2-K_1)C]\tilde{\mathcal{R}} \\ &\subset \mathcal{R} \cap \mathcal{B} + (A + BK_1C)\tilde{\mathcal{R}} + B(K_2-K_1)C\tilde{\mathcal{R}} \\ &= \mathcal{R} \cap \mathcal{B} + (A + BK_1C)\tilde{\mathcal{R}} \\ &= \mathcal{D}(K_1)\end{aligned}$$

and similarly $\mathcal{D}(K_1) \subset \mathcal{D}(K_2)$. Now let $x \in \mathcal{R} \cap (A\tilde{\mathcal{R}}+\mathcal{B})$, i.e., $x \in \mathcal{R}$ and $x = A\tilde{r} + b$ for some $\tilde{r} \in \tilde{\mathcal{R}}$ and $b \in \mathcal{B}$. By Lemma (1.3), $A[\mathcal{R} \cap \mathcal{N}(C)] \subset \mathcal{R}$. So if $\tilde{r} \in \mathcal{R} \cap \mathcal{N}(C)$, then $A\tilde{r} \in \mathcal{R}$. Therefore $b \in \mathcal{R} \cap \mathcal{B}$ and $A\tilde{r} = (A + BKC)\tilde{r}$ for any $K \in \mathbb{R}^{mxq}$ i.e., $x = (A\tilde{r}+b) \in \mathcal{R} \cap \mathcal{B} + (A+BKC)\tilde{\mathcal{R}}$. Assume $\tilde{r} \notin \mathcal{R} \cap \mathcal{N}(C)$, and write

$$\mathcal{R} = [\mathcal{R} \cap \mathcal{N}(C)] \oplus \mathcal{R}_r$$

with $\tilde{r} \in \mathcal{R}_r$. From Lemma (1.3), $A\mathcal{R} \subset \mathcal{B} + \mathcal{R}$. Let $\tilde{r} = r_1, r_2, \ldots, r_\rho$ be a basis of $\mathcal{R}_r$; then $Ar_i = Bu_i + t_i$ $(i\epsilon\bar{\rho})$ for some $u_i \in \mathbb{R}^m$ and $t_i \in \mathcal{R}$, with $Bu_1 = -b$ and $t_1 = x$. As in the proof of Lemma (1.3), K_x can be chosen so that $BK_xCr_i + Bu_i = \underline{0}$ $(i\epsilon\bar{\rho})$. Then $(A+BK_xC)r_i = t_i$ $(i\epsilon\bar{\rho})$, there follows that $K_x \in \underline{K}(\mathcal{R})$, and $x \triangleq t_1 = (A+BK_xC)r_1 = (A+BK_xC)\tilde{r} \in (A+BK_xC)\tilde{\mathcal{R}} + \mathcal{R} \cap \mathcal{B} \triangleq \mathcal{D}(K_x)$. And since $\mathcal{D}(K)$ is the same for all $K \in \underline{K}(\mathcal{R})$, therefore $\mathcal{R} \cap (A\tilde{\mathcal{R}}+\mathcal{B}) \subset \mathcal{D}(K)$ for all $K \in \underline{K}(\mathcal{R})$.

Q.E.D.

1.26 Lemma Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ be real constant matrices and let $\mathcal{R}$ be an (A,B,C)-invariant subspace. If $K \in \underline{K}(\mathcal{R})$, then

1.27 $$\sum_{j=1}^{i} (A+BKC)^{j-1} (\mathcal{B} \cap \mathcal{R}) = \mathcal{R}^{(i)} \qquad i\varepsilon\bar{n}$$

where the sequence $\mathcal{R}^{(i)}$, $(i\varepsilon\bar{n})$, is defined by (1.24).

Proof Equation (1.27) is true for i=1. If it is true for i = k-1, then by Lemma (1.25)

$$\begin{aligned}\sum_{j-1}^{k} (A+BKC)^{j-1} (\mathcal{B} \cap \mathcal{R}) &= \mathcal{B} \cap \mathcal{R} + (A+BKC)\,\mathcal{R}^{(k-1)} \\ &= \mathcal{R} \cap (A\,\mathcal{R}^{(k-1)} + \mathcal{B}) \\ &= \mathcal{R}^{(k)} .\end{aligned}$$

Q.E.D.

Proof of Theorem (1.19)

$\Rightarrow$ Let $\mathcal{R}$ be an (A,B,C) controllability subspace. From the definition of (A,B,C) controllability subspace, there exists K $\mathbb{R}^{m\times q}$ such that

1.28 $$\mathcal{R} = \{A+BKC \mid \mathcal{B} \cap \mathcal{R}\}$$

From (1.28), $\mathcal{R}$ is clearly an (A,B,C)-invariant subspace, i.e., $\mathcal{R}$ satisfies (1.20) and (1.21). By Lemma (1.26),

$$\mathcal{R} \triangleq \sum_{j=1}^{n} (A+BKC)^{j-1} (\mathcal{B} \cap \mathcal{R}) = \mathcal{R}^{(n)} = \mathcal{R}^{(\rho)}$$

where $\rho = \dim \mathcal{R}$.

$\Leftarrow$ From (1.20), (1.21) and Lemma (1.3), there exists $K_1 \in \mathbb{R}^{m\times q}$ such that

$$(A+BK_1C)\,\mathcal{R} \subset \mathcal{R},$$

i.e., $K_1 \in \underline{K}(\mathcal{R})$. From (1.22), (1.24) and Lemma (1.26),

$$\mathcal{R} = \mathcal{R}^{(\rho)} = \mathcal{R}^{(n)} = \sum_{j=1}^{n} (A+BK_1C)^{j-1} (\mathcal{B} \cap \mathcal{R})$$

$$\triangleq \{A+BK_1C \mid \mathcal{B} \cap \mathcal{R}\} ,$$

i.e., $\mathcal{R}$ is an (A,B,C) controllability subspace.

Q.E.D.

Let $\bar{\mathcal{V}}$ be the maximal subspace of $\mathcal{J}$ which is (A+BKC)-invariant for some K, and let $\underline{K}(\bar{\mathcal{V}})$ be the class of K for which $(A+BKC)\bar{\mathcal{V}} \subset \bar{\mathcal{V}}$, ($\bar{\mathcal{V}}$ is also said to be the maximal (A,B,C)-invariant subspace in $\mathcal{J}$).

1.29 Theorem Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ be real constant matrices and let $\bar{\mathcal{V}}$ and $\underline{K}(\bar{\mathcal{V}})$ be defined as above. If $K \in \underline{K}(\bar{\mathcal{V}})$, then the subspace

1.30 $$\bar{\mathcal{R}} \triangleq \{A+BKC \mid \mathcal{B} \cap \bar{\mathcal{V}}\}$$

is the maximal (A,B,C) c.s. in $\mathcal{J}$.

Proof As in the proof of Lemma (1.25), we can show that the right hand side of (1.30) is the same for any $K \in \underline{K}(\bar{\mathcal{V}})$, so that $\bar{\mathcal{R}}$ is uniquely defined.

Suppose that

$$\hat{\mathcal{R}} = \{A+B\hat{K}C \mid \mathcal{B} \cap \mathcal{R}\}$$

is an (A,B,C) controllability subspace in $\mathcal{J}$. Since $\hat{\mathcal{R}}$ is $(A+B\hat{K}C)$-invariant and $\bar{\mathcal{V}}$ is maximal, there follows $\bar{\mathcal{V}} \supset \hat{\mathcal{R}}$. Write $\bar{\mathcal{V}}$ and $\hat{\mathcal{R}}$ as direct sums

$$\bar{\mathcal{V}} = [\bar{\mathcal{V}} \cap \mathcal{N}(C)] \oplus \mathcal{V}_1$$
$$\hat{\mathcal{R}} = [\hat{\mathcal{R}} \cap \mathcal{N}(C)] \oplus \mathcal{R}_1$$

with $\mathcal{V}_1 \supset \mathcal{R}_1$. Then we pick the following sets of vectors such that

$$\begin{array}{ll} v_1, \ldots, v_{p'} & \text{is a basis of } \hat{\mathcal{R}} \cap \mathcal{N}(C) \\ v_1, \ldots, v_{p'}, \ldots, v_p & \text{is a basis of } \bar{\mathcal{V}} \cap \mathcal{N}(C), \text{ where } p \geq p' \\ v_{p+1}, \ldots, v_{m'} & \text{is a basis of } \mathcal{R}_1 \\ v_{p+1}, \ldots, v_{m'}, \ldots, v_m & \text{is a basis of } \mathcal{V}_1, \text{ where } m \geq m' \end{array}$$

Now we are going to find $K \in \underline{K}(\bar{\mathcal{V}})$ for which

$$KCx = \hat{K}Cx \qquad \forall\, x \in \hat{\mathcal{R}}.$$

Since $A\bar{\mathcal{V}} \subset \mathcal{B} + \bar{\mathcal{V}}$, there exists $u_i \in \mathbb{R}^m$, $t_i \in \bar{\mathcal{V}}$ such that

1.31 $$Av_i = Bu_i + t_i \qquad (i = m'+1, \ldots, m)$$

By the same reasoning as we used in the proof of Lemma (1.3), the existence of K in the following equation is guaranteed by the linear independence of $Cv_{p+1}, \ldots, Cv_{m'}, Cv_{m'+1}, \ldots, Cv_m$.

1.32 $$KC[v_{p+1} \vdots \cdots \vdots v_{m'} \vdots v_{m'+1} \vdots \cdots \vdots] = [\hat{K}Cv_{p+1} \vdots \cdots \vdots \hat{K}Cv_{m'} \vdots -u_{m'+1} \vdots \cdots \vdots -u_m]$$

For such K,

$$(A+BKC)x = (A+B\hat{K}C)x \quad \text{for all} \quad x \in \hat{\mathcal{R}}.$$

thus

$$(A+BKC)\hat{\mathcal{R}} = (A+B\hat{K}C)\hat{\mathcal{R}} \subset \hat{\mathcal{R}} \subset \bar{\mathcal{V}}$$

i.e.,

$$(A+BKC)v_i \in \bar{\mathcal{V}} \qquad (i=p+1, \ldots, m')$$

From (1.31) and (1.32)

$$\begin{aligned}(A+BKC)v_i &= Av_i - Bu_i \\ &= t_i \in \bar{\mathcal{V}} \qquad (i=m'+1, \ldots, m)\end{aligned}$$

so that for $i \in \bar{k}$ each new input v_i (an r_i-vector) can control y_i completely and does not affect any of the y_j's for $j \neq i$.

More precisely, the diagonal decoupling problem via output feedback can be formulated as follows,

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ real constant matrices as in (1.1a,b), where C is partitioned into k submatrices $C_1, \ldots, C_k$, as in (2.1), find a matrix K and controllability subspace $\mathcal{R}_1, \ldots, \mathcal{R}_k$ of (A,B,C), such that

2.3 $$\mathcal{R}_i = \{A+BKC \mid \mathcal{B} \cap \mathcal{R}_i\} \qquad (i\in\bar{k})$$

2.4 $$\mathcal{R}_i + \mathcal{N}_i = \mathbb{R}^n \qquad (i\in\bar{k})$$

2.5 $$\mathcal{R}_i \subset \bigcap_{j\neq i} \mathcal{N}_j \qquad (i\in\bar{k})$$

where $\mathcal{N}_i \triangleq \mathcal{N}(C_i)$, and with the following assumptions

(i) $\mathcal{N}_i \neq \mathbb{R}^n$ $\quad (i\in\bar{k})$

(ii) The subspaces $\mathcal{N}_i^{\perp}$ are mutually independent,

(i.e., $\mathcal{N}_i^{\perp} \cap \sum_{j\neq i} \mathcal{N}_j^{\perp} = \{0\}$, $i\in\bar{k}$),

or equivalently, the row-space of the k matrices C_k are mutually independent.

(iii) $\{A \mid \mathcal{B}\} = \mathbb{R}^n$

In the following theorem, $\bar{\mathcal{R}}_i$ denotes the maximal (A,B,C) c.s. such that

$$\bar{\mathcal{R}}_i \subset \bigcap_{j\neq i} \mathcal{N}_j \qquad (i\in\bar{k})$$

The $\bar{\mathcal{R}}_i$ are constructed according to Theorem (1.29)

Therefore $(A+BKC)\ \bar{\mathcal{V}} \subset \bar{\mathcal{V}}$ and $KCx = \hat{K}Cx$ for all $x \in \hat{\mathcal{R}}$, there follows

$$\begin{aligned}\hat{\mathcal{R}} &\triangleq \{A+B\hat{K}C \mid \mathcal{B} \cap \hat{\mathcal{R}}\}\\ &= \{A+BKC \mid \mathcal{B} \cap \hat{\mathcal{R}}\}\\ &\subset \{A+BKC \mid \mathcal{B} \cap \bar{\mathcal{V}}\}\\ &\triangleq \bar{\mathcal{R}}\end{aligned}$$

i.e., $\bar{\mathcal{R}}$ is the maximal (A,B,C) controllability subspace contained in $\mathcal{J}$.

Q.E.D.

2. Diagonal decoupling via output feedback

In this section, we solve the problem of diagonal decoupling via output feedback. This problem can be stated as follows: Consider the output equation in (1.1b) with C partitioned into k submatrices

2.1 $$C = \begin{bmatrix} C_1 \\ \hline \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hline C_k \end{bmatrix}$$

where C_i is of dimension $q_i \times n$ ($i=1, \ldots, k$; $k \geq 2$; $q_1 + \ldots + q_k = q$). Then equation (1.1a,b) can be written as

2.2a $$\dot{x}(t) = Ax(t) + Bu(t)$$

2.2b $$y_i(t) = C_i x(t) \qquad (i \in \bar{k})$$

The problem is to find an <u>output feedback law</u>

$$\begin{aligned}u(t) &= Ky(t) + [G_1 \vdots \cdots \vdots G_k] \begin{bmatrix} v_1(t) \\ \hline \vdots \\ \hline v_k(t) \end{bmatrix}\\ &= Ky(t) + \sum_{i=1}^{k} G_i\, v_i(t)\end{aligned}$$

2.6 Theorem

If $\dim(\mathcal{B}) = k$, then the problem (2.3)-(2.5) has a solution if and only if

2.7 $$\bar{\mathcal{R}}_i + \mathcal{N}_i = \mathbb{R}^n \qquad (i\varepsilon\bar{k})$$

and

2.8 $$\mathcal{B} = \sum_{i=1}^{k} \mathcal{B} \cap \bar{\mathcal{R}}_i$$

Furthermore, if K, $\mathcal{R}_1, \ldots, \mathcal{R}_k$ is any solution of (2.3)-(2.5), then

2.9 $$\mathcal{R}_i = \bar{\mathcal{R}}_i \qquad (i\varepsilon\bar{k})$$

Proof $\Rightarrow$ This part of proof is the same as the proof of Theorem (5.1) in [Wo.5].

$\Leftarrow$ We will show that (2.7) and (2.8) are sufficient conditions for the existence of a solution to (2.3)-(2.5). Let $\bar{\mathcal{V}}_i$ be the maximal subspace such that

2.10 $$A\,\bar{\mathcal{V}}_i \subset \mathcal{B} + \bar{\mathcal{V}}_i \qquad (i\varepsilon\bar{k})$$

2.11 $$A[\bar{\mathcal{V}}_i \cap \mathcal{N}(C)] \subset \bar{\mathcal{V}}_i \qquad (i\varepsilon\bar{k})$$

2.12 $$\bar{\mathcal{V}}_i \subset \bigcap_{j\neq i} \mathcal{N}_j \qquad (i\varepsilon\bar{k})$$

If we can show that the $\bar{\mathcal{V}}_i$ are compatible, in the sense that there exists a K such that

$$(A{+}BKC)\,\bar{\mathcal{V}}_i \subset \bar{\mathcal{V}}_i \qquad (i\varepsilon\bar{k})$$

then for this K together with $\bar{\mathcal{R}}_i \triangleq \{A{+}BKC \mid \mathcal{B} \cap \bar{\mathcal{V}}_i\}$, $(i\varepsilon\bar{k})$ is a solution of (2.3)-(2.5)

For this we show first that $\bar{\mathcal{V}}_i^* \triangleq \sum_{j\neq i} \bar{\mathcal{V}}_j$ $(i\epsilon\bar{k})$ are compatible.

From (2.8)

$$\mathcal{B} = \sum_{i=1}^{k} \mathcal{B} \cap \bar{\mathcal{R}}_i$$

$$= \sum_{i=1}^{k} \mathcal{B} \cap \bar{\mathcal{V}}_i \qquad (\because\ \bar{\mathcal{V}}_i \supset \bar{\mathcal{R}}_i)$$

$$= \mathcal{B} \cap \bar{\mathcal{V}}_i + \sum_{j\neq i} \mathcal{B} \cap \bar{\mathcal{V}}_j$$

$$\subset \mathcal{B}_i + \bar{\mathcal{V}}_i^*$$

where $\mathcal{B}_i \triangleq \mathcal{B} \cap \bar{\mathcal{V}}_i$. Then from (2.10)

2.13 $$A\,\bar{\mathcal{V}}_i^* \subset \mathcal{B}_i + \bar{\mathcal{V}}_i^* .$$

From (2.11), we have

2.14 $$A\Big[\sum_{j\neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}(C)\Big] \subset \bar{\mathcal{V}}_i^* \qquad (i\epsilon\bar{k})$$

Now we will prove the following result,

2.15 $$\sum_{j\neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}(C)) \supset [\bar{\mathcal{V}}_i^* \cap \mathcal{N}(C)]$$

Note that the left hand side of (2.15) can be written as

2.16 $$\sum_{j\neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}(C)) = \sum_{j\neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}_1 \cap \mathcal{N}_2 \ldots \cap \mathcal{N}_k)$$

$$= \sum_{j\neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}_j) \quad \text{from (2.12).}$$

Let $x \in [\bar{\mathcal{V}}_i^* \cap \mathcal{N}(C)]$, then x can be written as follows

2.17 $$x = \sum_{j \neq i} x_j \in \bigcap_j \mathcal{N}_j$$

with

2.18 $$x_j \in \bar{\mathcal{V}}_j \subset \bigcap_{p \neq j} \mathcal{N}_p \qquad j \in \{1,2, \ldots, k\} \setminus \{i\}$$

Consequently, $\mathcal{N}_p$ contains x_j for $j \in \{1,2, \ldots, k\} \setminus \{p,i\}$. And from (2.17), $\mathcal{N}_p$ also contains x, there follows

2.19 $$x_p \in \mathcal{N}_p \qquad p \in \{1,2, \ldots, k\} \setminus \{i\}$$

Then from (2.18) and (2.19) it is clear that $x \in \sum_{j \neq i} (\bar{\mathcal{V}}_j \cap \mathcal{N}_j)$,

i.e., (2.15) is established. From (2.14), (2.15),

2.20 $$A[\bar{\mathcal{V}}_i^* \cap \mathcal{N}(C)] \subset \bar{\mathcal{V}}_i^* \qquad (i \in \bar{k})$$

By Lemma (1.3) and (2.13), (2.20), there exist B_i with $\{B_i\} = \mathcal{B}_i$, and K_i such that

2.21 $$(A + B_i K_i C)\, \bar{\mathcal{V}}_i^* \subset \bar{\mathcal{V}}_i^* \qquad (i \in \bar{k})$$

Find $\hat{\mathcal{V}}$ and $\hat{\mathcal{V}}_i^*$ for which

$$\bar{\mathcal{V}} \triangleq \bar{\mathcal{V}}_1 + \ldots + \bar{\mathcal{V}}_k = [\bar{\mathcal{V}} \cap \mathcal{N}(C)] \oplus \hat{\mathcal{V}}$$

$$\bar{\mathcal{V}}_i^* \triangleq \sum_{j \neq i} \bar{\mathcal{V}}_j = [\bar{\mathcal{V}}_i^* \cap \mathcal{N}(C)] \oplus \hat{\mathcal{V}}_i^*$$

with $$\hat{\mathcal{V}} \supset \hat{\mathcal{V}}_i^* \qquad (i \in \bar{k})$$

Then pick a basis $\{v_1, \ldots, v_\mu\}$ of $\hat{\mathcal{V}}$, find a K such that

2.22 $$BKCv_\nu = \left(\sum_{i=1}^{k} B_i K_i C\right) v_\nu \triangleq Bu_\nu \qquad (\nu \in \bar{\mu}),$$

in other words, find a solution K to the following equation

$$K[Cv_1 \vdots \cdots \vdots Cv_\mu] = [u_1 \vdots \cdots \vdots u_\mu] .$$

Such a K always exists, since the set of vectors $Cv_1, \ldots, Cv_\mu$ are linearly independent, (also see the proof of Lemma (1.26)). From (2.22),

2.23
$$\begin{aligned}(A+BKC)\,\bar{\mathcal{V}}_i^* &= (A+B_iK_iC + \sum_{j\neq i} B_jK_jC)\,\bar{\mathcal{V}}_i^* \\ &\subset (A+B_iK_iC)\,\bar{\mathcal{V}}_i^* + \sum_{j\neq i} \mathcal{B}_j \\ &\subset \bar{\mathcal{V}}_i^* + \sum_{j\neq i} \bar{\mathcal{V}}_j \qquad (\because \mathcal{B}_i \triangleq \mathcal{B} \cap \bar{\mathcal{V}}_i) \\ &= \bar{\mathcal{V}}_i^* \qquad (i\varepsilon\bar{k})\end{aligned}$$

This proves the compatability of the $\bar{\mathcal{V}}_i^*$'s. Now define

$$\mathcal{V}_i \triangleq \bigcap_{j\neq i} \bar{\mathcal{V}}_j^* \qquad (i\varepsilon\bar{k})$$

Since each $\bar{\mathcal{V}}_j^*$ $(j=1, \ldots, i-1, i+1, \ldots, k)$ contains $\bar{\mathcal{V}}_i$, therefore $\mathcal{V}_i \supset \bar{\mathcal{V}}_i (i\varepsilon\bar{k})$. From (2.23),

2.24
$$(A+BKC)\,\mathcal{V}_i \subset \mathcal{V}_i \qquad (i\varepsilon\bar{k}) .$$

From (2.12), we have

2.25
$$\begin{aligned}\mathcal{V}_i &\triangleq \bigcap_{j\neq i} \bar{\mathcal{V}}_j^* \\ &= \bigcap_{j\neq i} (\sum_{\alpha\neq j} \bar{\mathcal{V}}_\alpha) \\ &\subset \bigcap_{j\neq i} (\sum_{\alpha\neq j} (\bigcap_{m\neq\alpha} \mathcal{N}_m)) \\ &= \bigcap_{j\neq i} \mathcal{N}_j\end{aligned}$$

the last equality was set up by the modular distributive rule for subspaces [Wo.5].

By (2.24) and (2.25), $\mathcal{V}_i$ satisfies the conditions imposed on $\bar{\mathcal{V}}_i$. Since $\bar{\mathcal{V}}_i$ are maximal, there follows $\bar{\mathcal{V}}_i \supset \mathcal{V}_i$. Therefore $\mathcal{V}_i = \bar{\mathcal{V}}_i$ $(i\epsilon\bar{k})$.

Q.E.D.

3. Triangular decoupling via output feedback

The problem of triangular decoupling via state feedback was first formulated and solved by Morse and Wonham [Mo.2], an alternate treatment can be found in Section 1 of Chapter III. In this section, we are dealing with the triangular decoupling problem via output feedback. Let us consider the system specified in (1.1a,b) and the partitioned outputs in (2.2b). We try to find an output feedback law

$$u(t) = Ky(t) + [G_1 \vdots G_2 \vdots \cdots \vdots G_k] \begin{bmatrix} v_1(t) \\ \hline \vdots \\ \hline v_k(t) \end{bmatrix}$$

$$= Ky(t) + \sum_{i=1}^{k} G_i \, v_i(t)$$

so that each new input v_i (a r_i-vector) can control y_i completely and does not affect y_j for $j > i$. Namely, the problem is to find matrices K and G_i such that the transfer function matrix relating to the new input v and output y is upper triangular. More precisely, the problem can be stated as follows,

Given $A \epsilon \mathbb{R}^{nxn}$, $B \epsilon \mathbb{R}^{nxm}$, $C \epsilon \mathbb{R}^{qxn}$ as in (1.1a,b), where C is partitioned into k submatrices $C_1, \ldots, C_k$ as in (2.1), find a matrix $K \epsilon \mathbb{R}^{mxq}$ and (A,B,C) controllability subspaces $\mathcal{R}_i$, $(i\epsilon\bar{k})$, such that

3.1 $$\mathcal{R}_i = \{A+BKC \mid \mathcal{B} \cap \mathcal{R}_i\} \qquad (i\varepsilon\bar{k})$$

3.2 $$\mathcal{R}_i + \mathcal{N}_i = \mathbb{R}^n \qquad (i\varepsilon\bar{k})$$

3.3 $$\mathcal{R}_i \subset \bigcap_{j=i+1,\dots,k} \mathcal{N}_j \qquad (i\varepsilon\overline{k-1})$$

where $\mathcal{N}_i \triangleq \mathcal{N}(C_i)$ with C_i defined in (2.1), $(i\varepsilon\bar{k})$.

Let $\bar{\mathcal{R}}_i$ $(i\varepsilon\overline{k-1})$ be the maximal (A,B,C) controllability subspace satisfying (3.3). A constructive procedure for calculating $\bar{\mathcal{R}}_i$ can be found in the proof of Theorem (1.12) and Theorem (1.29). Let $\bar{\mathcal{R}}_k = \{A \mid \mathcal{B}\}$ be the controllable subspace of system (1.1a,b).

3.4 <u>Theorem</u>

There exist $K \in \mathbb{R}^{m\times q}$ and (A,B,C) controllability subspace $\mathcal{R}_i$, $(i\varepsilon\bar{k})$, satisfying (3.1)-(3.3) if and only if

3.5 $$\bar{\mathcal{R}}_i + \mathcal{N}_i = \mathbb{R}^n \qquad (i\varepsilon\bar{k})$$

Furthermore, if (3.5) holds, one may choose

$$\mathcal{R}_i = \bar{\mathcal{R}}_i, \qquad (i\varepsilon\bar{k})$$

<u>Proof</u> $\Rightarrow$ This part of proof follows directly from the maximality of the $\bar{\mathcal{R}}_i$.

$\Leftarrow$ We will show that there exists a K such that (3.1) is satisfied with $\mathcal{R}_i = \bar{\mathcal{R}}_i$, $(i\varepsilon\bar{k})$, i.e., we will show that $\bar{\mathcal{R}}_i$ $(i\varepsilon\bar{k})$ are compatible.

From (3.3) it is clear that

3.6 $$\bar{\mathcal{R}}_1 \subset \bar{\mathcal{R}}_2 \subset \dots \subset \bar{\mathcal{R}}_k$$

Write $\bar{\mathcal{R}}_0 = \{0\}$, and let $\mathcal{E}_i$ $(i\varepsilon\bar{k})$ be any subspace such that

3.7 $$\bar{\mathcal{R}}_i = \mathcal{E}_i \oplus \bar{\mathcal{R}}_{i-1} \qquad (i\epsilon\bar{k})$$

Since $\bar{\mathcal{R}}_i$ $(i\epsilon\bar{k})$ are (A,B,C)-invariant, there exist $K_i \in \mathbb{R}^{mxq}$ such that

$$(A+BK_iC)\ \bar{\mathcal{R}}_i \subset \bar{\mathcal{R}}_i \qquad (i\epsilon\bar{k})$$

thus

3.8 $$(A+BK_iC)\ \mathcal{E}_i \subset \bar{\mathcal{R}}_i \qquad (i\epsilon\bar{k})$$

Let $\mathcal{F}_i$ $(i\epsilon\bar{k})$ be any subspace such that

3.9 $$\mathcal{E}_i = \mathcal{F}_i \oplus (\mathcal{E}_i \cap \mathcal{N}(C)) \qquad (i\epsilon\bar{k})$$

Then pick a basis $\{f_{i1}, \ldots, f_{i\rho_i}\}$ of $\mathcal{F}_i$, where $\rho_i \triangleq \dim \mathcal{F}_i$. Let $I \triangleq \{i|\rho_i \geq 1$, and $i=1, \ldots, k\}$. From (3.7), it is clear that the $\mathcal{E}_i$ $(i\epsilon\bar{k})$ are mutually independent, thus the set of vectors $\{f_{ij}|i\epsilon I, j=1, \ldots, \rho_i\}$ are linearly independent. Furthermore, the set of vectors $\{C\ f_{ij}|i\epsilon I, j=1, \ldots, \rho_i\}$ are linearly independent, this follows from the fact that f_{ij} $(j=1, \ldots, \rho_i)$ are basis for $\mathcal{F}_i$, where $\mathcal{F}_i \cap \mathcal{N}(C) = \{0\}$. The existence of K in the following equation

3.10 $$KCf_{ij} = K_iCf_{ij} \qquad (i\epsilon I \text{ and } j\epsilon\bar{\rho}_i)$$

is guaranteed by the linear independence of the set of vectors $\{Cf_{ij}|i\epsilon I$ and $j\epsilon\bar{\rho}_i\}$. (For a detailed argument, see the proof of Lemma (1.26)).

Now from (3.8) and (3.10), there follows

$$(A+BKC)\ \mathcal{E}_i \subset \bar{\mathcal{R}}_i \qquad (i\epsilon\bar{k})$$

Thus $(A+BKC)\ \bar{\mathcal{R}}_1 \subset \bar{\mathcal{R}}_1$ and by induction

$$\begin{aligned}(A+BKC)\ \bar{\mathcal{R}}_i &= (A+BKC)\ (\mathcal{E}_i + \bar{\mathcal{R}}_{i-1}) \\ &\subset \bar{\mathcal{R}}_i + \bar{\mathcal{R}}_{i-1} \\ &= \bar{\mathcal{R}}_i \qquad (i\epsilon\bar{k})\end{aligned}$$

This K together with the $\bar{\mathcal{R}}_i$ $(i\varepsilon\bar{k})$ is a solution of (3.1) — (3.3).

Q.E.D.

4. Diagonal decoupling via output feedback with dynamic compensation

In section 2, we solve the diagonal decoupling problem with static output feedback. In case that no such output feedback law exists, one may try to include some integrators in the feedback loop to solve the diagonal decoupling problem. The system in (1.1a,b) can be augmented by adjoining to it some new dynamic elements. The augmented system is as follows:

4.1a $$\begin{bmatrix} \dot{x} \\ \hline \dot{x}' \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline x' \end{bmatrix} + \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right] \begin{bmatrix} u \\ \hline u' \end{bmatrix}$$

4.1b $$\begin{bmatrix} y \\ \hline x' \end{bmatrix} = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & I \end{array}\right] \begin{bmatrix} x \\ \hline x' \end{bmatrix}$$

where $x(t) \varepsilon \mathbb{R}^n$, $x'(t) \varepsilon \mathbb{R}^{n'}$, $u(t) \varepsilon \mathbb{R}^m$, $u'(t) \varepsilon \mathbb{R}^{n'}$, $y(t) \varepsilon \mathbb{R}^q$, I is an $n'xn'$ identity matrix, and A,B and C are real constant matrices of appropriate size, as defined in (1.1a,b). Denote

4.2 $$A^e \triangleq \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right], \quad B^e \triangleq \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right], \quad C^e \triangleq \left[\begin{array}{c|c} C & 0 \\ \hline 0 & I \end{array}\right]$$

as the $(n+n') \times (n+n')$, $(n+n') \times (m+n')$ and $(q+n) \times (n+n')$ real constant matrices, respectively, given in (4.1a,b). Let $\mathcal{R}_i^e \subset \mathbb{R}^{n+n'}$ be an (A^e,B^e,C^e) controllability subspace contained in $\bigcap_{j\neq i} (\mathcal{N}(C_j) \oplus \mathbb{R}^{n'})$, where C_j, $(j\varepsilon\bar{k})$, is a $q_j xn$ matrix defined in (2.1), i.e., there exists an $(m+n') \times (q+n')$ real constant matrix K^e such that

$$\mathcal{R}_i^e = \{A^e + B^eK^eC^e | \mathcal{B}^e \cap \mathcal{R}_i^e\} \subset \bigcap_{j\neq i} \mathcal{N}(C_j) \oplus \mathbb{R}^{n'},$$

where $\mathcal{B}^e \triangleq$ range space of B^e. Similarly, let $\mathcal{S}_i \subset \mathbb{R}^n$ be an (A,B,I) controllability subspace contained in $\bigcap_{j\neq i} \mathcal{N}(C_j)$, i.e., there exists an mxn real constant matrix, K, such that

$$\mathcal{S}_i = \{A+BK \mid \mathcal{B}\cap \mathcal{S}_i\} \subset \bigcap_{j\neq i} \mathcal{N}(C_j).$$

The problem of diagonal decoupling via output feedback with dynamic compensation can be stated as follows:

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ real constant matrices and C is partitioned $C_1, \ldots, C_k$ as in (2.1), find a positive integer n'. an $(m\times n') \times (q+n')$ matrix K^e and (A^e,B^e,C^e) controllability subspaces $\mathcal{R}_1^e, \ldots, \mathcal{R}_k^e$, where A^e,B^e and C^e are defined in (4.2), such that

4.3 $$\mathcal{R}_i^e = \{A^e+B^eK^eC^e \mid \mathcal{B}^e \cap \mathcal{R}_i^e\} \qquad (i\in\bar{k})$$

4.4 $$\mathcal{R}_i^e + (\mathcal{N}(C_i) \oplus \mathbb{R}^{n'}) = \mathbb{R}^n \oplus \mathbb{R}^{n'} \qquad (i\in\bar{k})$$

4.5 $$\mathcal{R}_i^e \subset \bigcap_{j\neq i} (\mathcal{N}(C_j) \oplus \mathbb{R}^{n'}) \qquad (i\in\bar{k})$$

4.6 <u>Theorem</u>

Let $\bar{\mathcal{S}}_i$ $(i\in\bar{k})$ be the maximal (A,B,I)-controllability subspace contained in $\bigcap_{j\neq i} \mathcal{N}(C_j)$. Then (4.3)-(4.5) is solvable if and only if

4.7 $$\bar{\mathcal{S}}_i + \mathcal{N}(C_i) = \mathbb{R}^n \qquad (i\in\bar{k})$$

<u>Proof</u> $\Rightarrow$ We show first that if $\mathcal{R}^e$ is an (A^e,B^e,C^e) c.s. contained in $\bigcap_{j\neq i} (\mathcal{N}(C_j) \oplus \mathbb{R}^{n'})$, then $\mathcal{S} \triangleq P\mathcal{R}^e$ is an (A,B,I) c.s. contained in $\bigcap_{j\neq i} \mathcal{N}(C_j)$, where P is the projection map $\mathbb{R}^n \oplus \mathbb{R}^{n'} \to \mathbb{R}^n$ with the following matrix representation,

4.8
$$P = \begin{array}{c} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}n \\ \}n' \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{n'} \end{array} \end{array}$$

$\mathcal{R}^e$ is an (A^e,B^e,C^e) c.s. $\Rightarrow A^e\mathcal{R}^e \subset \mathcal{R}^e + \mathcal{B}^e$

$\Rightarrow PA^e\mathcal{R}^e \subset P\mathcal{R}^e + P\mathcal{B}^e$

$\Rightarrow AP\mathcal{R}^e \subset P\mathcal{R}^e + P\mathcal{B}^e$

$\Rightarrow A\mathcal{S} \subset \mathcal{S} + \mathcal{B}$,

i.e., $\mathcal{S} \triangleq P\mathcal{R}^e$ is (A,B,I)-invariant (see Definition(1.11)).

By Theorem (1.19), $\mathcal{R}^e = \lim (\mathcal{R}^e)^\mu$ $(\mu=0,1,2,\ldots)$, where $(\mathcal{R}^e)^0 = \{0\}$, $(\mathcal{R}^e)^{\mu+1} = \mathcal{R}^e \cap [A^e(\mathcal{R}^e)^\mu + \mathcal{B}^e]$. Let $\mathcal{S}^\mu \triangleq P(\mathcal{R}^e)^\mu$, $(\mu=0,1,2,\ldots)$

$$\begin{aligned} \mathcal{S}^{\mu+1} &\triangleq P(\mathcal{R}^e)^{\mu+1} \\ &= P\mathcal{R}^e \cap [PA^e(\mathcal{R}^e)^\mu + P\mathcal{B}^e] \\ &= \mathcal{S} \cap [A\mathcal{S}^\mu + \mathcal{B}] \end{aligned}$$

Furthermore,
$$\begin{aligned} \lim \mathcal{S}^{\mu+1} &= \lim P(\mathcal{R}^e)^\mu \\ &= P\,[\lim (\mathcal{R}^e)^\mu] \\ &= P\mathcal{R}^e \\ &\triangleq \mathcal{S}. \end{aligned}$$

Again by Theorem (1.19), $\mathcal{S} \triangleq P\mathcal{R}^e$ is an (A,B,I) c.s.
Thus $\mathcal{R}_i^e$ is an (A^e,B^e,C^e) c.s. implies $\mathcal{S}_i \triangleq P\mathcal{R}_i^e$ is an (A,B,I) c.s. for $i\in\bar{k}$.

From (4.4), (4.5)

4.9
$$P[\mathcal{R}_i^e + (\mathcal{N}(C_j) \oplus \mathbb{R}^{n'})] = \mathcal{S}_i + \mathcal{N}(C_j) = \mathbb{R}^n$$

4.10
$$P\mathcal{R}_i^e = \mathcal{S}_i \subset \bigcap_{j\neq i} \mathcal{N}(C_j).$$

i.e., $\mathcal{S}_i$ is an (A,B,I) c.s. contained in $\bigcap_{j\neq i} \mathcal{N}(C_j)$. From the maximality of the $\bar{\mathcal{S}}_i$ and (4.9), there follows

$$\bar{\mathcal{S}}_i + \mathcal{N}(C_i) = \mathbb{R}^n .$$

$\Leftarrow$ Define $n' = \sum_{i=1}^{k} d(\bar{\mathcal{S}}_i)$, where $d(\bar{\mathcal{S}}_i) \overset{\Delta}{=}$ dimension of $\bar{\mathcal{S}}_i$.

Let S_i be an $(n+n') \times d(\bar{\mathcal{S}}_i)$ real constant matrix such that $\{S_i\} = \bar{\mathcal{S}}_i$, where we consider $\bar{\mathcal{S}}_i$ as a subspace in $\mathbb{R}^n \oplus \mathbb{R}^{n'}$, and let M_i be an $(n+n') \times (n+n')$ real constant matrix as follows

4.11

$$M_i = \begin{array}{c} \\ \ell_i\left\{\begin{array}{c}\\ \end{array}\right. \\ \\ \\ \end{array}\underbrace{\left[\begin{array}{c} 0 \\ \hline 0 \\ \hline (S_i)^t \\ \hline 0 \end{array}\right]}_{n+n'}\begin{array}{l} \}\, n \\ \\ \}\, n' \\ \\ \end{array} \qquad (i=1, \ldots, k)$$

where $\ell_i \overset{\Delta}{=} \sum_{j=1,\ldots,i-1} d(\bar{\mathcal{S}}_j)$, and $(S_i)^t$ is the transpose of S_i.

With M_i so defined, it is clear that $\bar{\mathcal{S}}_i \cap \mathcal{N}(M_i) = \{0\}$, $\{M_i\} = M_i \bar{\mathcal{S}}_i$ and the ranges $\{M_i\}$, $(i\in\bar{k})$ are independent.

Define $\mathcal{R}_i^e \overset{\Delta}{=} (P + M_i)\bar{\mathcal{S}}_i$, then

4.12 $$A^e \mathcal{R}_i^e = A^e \bar{\mathcal{S}}_i = A\bar{\mathcal{S}}_i \subset \bar{\mathcal{S}}_i + \mathcal{B} \subset \mathcal{R}_i^e + \mathcal{B}^e$$

and

4.13 $$A^e[\mathcal{R}_i^e \cap \mathcal{N}(C^e)] \subset \mathcal{R}_i^e$$

where (4.13) follows from the fact that $\mathcal{R}_i^e \cap \mathcal{N}(C^e) = \{0\}$.

Since the $\mathcal{R}_i^e$ $(i\epsilon\bar{k})$ are clearly independent, then from (4.12), (4.13) there exists an $(m+n') \times (q+n')$ real constant matrix K^e such that

$$(A^e+B^eK^eC^e)\,\mathcal{R}_i^e \subset \mathcal{R}_i^e\,, \qquad (i\epsilon\bar{k})$$

Now we are going to show that $\mathcal{R}_i^e \triangleq (P+M_i)\,\bar{\mathcal{S}}_i$ is an (A^e,B^e,C^e) c.s. . Dropping the subscript i, suppose $\mathcal{S}$ is a c.s. of (A,B,I). From Theorem (1.19), $\mathcal{S} = \lim \mathcal{S}^\mu$, $\mathcal{S}^{\mu+1} = \mathcal{S} \cap (A\mathcal{S}^\mu + \mathcal{B})$, $(\mu=0,1,2, \ldots,)$, $\mathcal{S}^0 = \{0\}$. Let $\{M\} \subset \mathbb{R}^{n'}$ (see (4.11)), $\mathcal{R}^e \triangleq (P+M)\,\mathcal{S}$, and

$(\mathcal{R}^e)^{\mu+1} \triangleq \mathcal{R}^e \cap [A^e(\mathcal{R}^e)^\mu + \mathcal{B}^e]$ $\qquad (\mu=1,2, \ldots,)$

with $(\mathcal{R}^e)^0 \triangleq \{0\}$. Then $(\mathcal{R}^e) \supset (P+M)\mathcal{S}^0$; and if $(\mathcal{R}^e)^\mu \supset (P+M)\,\mathcal{S}^\mu$,

$$\begin{aligned}(\mathcal{R}^e)^{\mu+1} &\supset [(P+M)\,\mathcal{S}\,] \cap [A^e(P+M)\mathcal{S}^\mu + \mathcal{B}^e] \\ &= [(P+M)\,\mathcal{S}\,] \cap [A\,\mathcal{S}^\mu + \mathcal{B}^e] \\ &\supset (P+M)[\mathcal{S} \cap (A\mathcal{S}^\mu + \mathcal{B}^e)] \\ &= (P+M)[\mathcal{S} \cap (A\mathcal{S}^\mu + \mathcal{B})] \\ &= (P+M)\,\mathcal{S}^{\mu+1}\end{aligned}$$

By induction, $\mathcal{R}^e \supset (\mathcal{R}^e)^\mu \supset (P+M)\,\mathcal{S}^\mu \uparrow (P+M)\,\mathcal{S}$, i.e., $(\mathcal{R}^e)^\mu \uparrow \mathcal{R}^e$, so $\mathcal{R}^e$ is an (A^e,B^e,C^e) c.s. . Application of this argument to the $\bar{\mathcal{S}}_i$ and $\mathcal{R}_i^e$ yields the desired result.

The relation $P\,\mathcal{R}_i^e = \bar{\mathcal{S}}_i$ implies

$$\mathcal{R}_i^e \subset \bar{\mathcal{S}}_i \oplus \mathbb{R}^{n'} \subset \Big(\bigcap_{j\neq i} \mathcal{N}(C_j)\Big) \oplus \mathbb{R}^{n'} = \bigcap_{j\neq i}\big(\mathcal{N}(C_j) + \mathbb{R}^{n'}\big),$$

i.e., $\mathcal{R}_i^e$ satisfies (4.5). By (4.7)

$$(P+M_i)\,\bar{\mathcal{S}}_i + (P+M)\,\mathcal{N}(C_i) = (P+M_i)\,\mathbb{R}^n\,, \tag{4.14}$$

and addition of $\mathbb{R}^{n'}$ to both sides of (4.14) yields (4.4)

$$\mathcal{R}_i^e + (\mathcal{N}(C_i) \oplus \mathbb{R}^{n'}) = \mathbb{R}^n \oplus \mathbb{R}^{n'}\,.$$

Q.E.D.

4.14 Remark It is interesting to note that the condition in (4.7) is the same as the necessary and sufficient condition for the existence of decoupling matrices via state feedback with dynamic compensation in the diagonal decoupling problem, (see Theorem (1.1) in [Mo.3]. The new dynamic elements adjoined to the system in (1.1a,b) have two purposes, (a) performing as a precompensator for decoupling problem [Gi.2, Si.2, Wa.1], (b) performing as an observer [Lu.2, Wo.7].

It should be noted that the number of new integrators, n', adjoined to system (1.1a,b) in the proof of Theorem (4.6) is seen to be too large. Further research can be done on the problem of minimizing the number of new integrators for the diagonal decoupling problem via output feedback.

5. Triangular decoupling via output feedback with dynamic compensation

In section 3, we solve the problem of triangular decoupling via statis output feedback. In case that there exists no such output feedback law, one may try to include some integrators in the feedback loop to solve this problem. As we did in section 4, the system in (1.1a,b) is augmented by adjoining to it some new dynamic elements. More precisely, the problem of triangular decoupling via output feedback with dynamic compensation can be stated as follows:

Given $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $C \in \mathbb{R}^{qxn}$ real constant matrices, where C is partitioned into k submatrices $C_1, \ldots, C_k$ as in (2.1), find a positive integer n', an $(m+n') \times (q+n')$ real constant matrix K^e and (A^e,B^e,C^e) controllability subspace $\mathcal{R}_1^e, \mathcal{R}_2^e, \ldots, \mathcal{R}_k^e$, where A^e, B^e and C^e are defined in (4.2), such that

5.1 $$\mathcal{R}_i^e = \{A^e+B^eK^eC^e \mid \mathcal{B}^e \cap \mathcal{R}_i^e\} \qquad (i\epsilon\bar{k})$$

5.2 $$\mathcal{R}_i^e + (\mathcal{N}(C_i) \oplus \mathbb{R}^{n'}) = \mathbb{R}^n \oplus \mathbb{R}^{n'} \qquad (i\epsilon\bar{k})$$

5.3 $$\mathcal{R}_i^e \subset \bigcap_{j=i+1,\ldots,k} (\mathcal{N}(C_j) \oplus \mathbb{R}^{n'}) \qquad (i\epsilon\bar{k})$$

5.4 Theorem

Let $\bar{\mathcal{S}}_i$ $(i\epsilon\overline{k-1})$ be the maximal (A,B,I) controllability subspace contained in $\bigcap_{j=i+1,\ldots,k} \mathcal{N}(C_j)$, and let $\bar{\mathcal{S}}_k = \{A|\mathcal{B}\}$ be the controllable space of system (1.1a,b). Then (5.1)-(5.3) is solvable if and only if

5.5 $$\bar{\mathcal{S}}_i + \mathcal{N}(C_i) = \mathbb{R}^n \qquad (i\epsilon\bar{k})$$

Proof $\Rightarrow$ By the same reasoning as in the proof of Theorem (4.6), we can show that if $\mathcal{R}_i^e$ is an (A^e,B^e,C^e) c.s. satisfying (5.3), then $\mathcal{S}_i \triangleq P\mathcal{R}_i^e$ is an (A,B,I) c.s. contained in $\bigcap_{j=i+1,\ldots,k} \mathcal{N}(C_j)$, where P is a projection mapping defined in (4.8). From (5.2)

5.6 $$P[\mathcal{R}_i^e + (\mathcal{N}(C_i) \oplus \mathbb{R}^{n'})] = \mathcal{S}_i + \mathcal{N}(C_i) = \mathbb{R}^n .$$

From the maximality of the $\bar{\mathcal{S}}_i$ and (5.6), there follows

$$\bar{\mathcal{S}}_i + \mathcal{N}(C_i) = \mathbb{R}^n \qquad (i\epsilon\bar{k})$$

$\Leftarrow$ From the assumptions on $\bar{\mathcal{S}}_i$, it is clear that

$$\bar{\mathcal{S}}_1 \subset \bar{\mathcal{S}}_2 \subset \ldots \subset \bar{\mathcal{S}}_k .$$

Write $\bar{\mathcal{S}}_1 = \mathcal{F}_1 \oplus \mathcal{G}_1$, where $\mathcal{G}_1 \triangleq \bar{\mathcal{S}}_1 \cap \mathcal{N}(C)$. Similarly, write

5.7 $$\bar{\mathcal{S}}_j = \mathcal{F}_j \oplus \mathcal{F}_{j-1} \oplus \ldots \oplus \mathcal{F}_1 \oplus \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_{j-1} \oplus \mathcal{G}_j$$

where $\mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_j \triangleq \bar{\mathcal{S}}_j \cap \mathcal{N}(C)$, $\qquad (j\epsilon\bar{k})$.

Define $n' = \sum_{i=1}^{k} d(\mathcal{G}_i)$ where $d(\mathcal{G}_i) \triangleq$ dimension of $\mathcal{G}_i$. Let G_i

be an $(n+n') \times d(\mathcal{G}_i)$ real constant matrix such that $\{G_i\} = \mathcal{G}_i$, we consider here $\mathcal{G}_i$ as a subspace in $\mathbb{R}^n \oplus \mathbb{R}^{n'}$, and let M_i be an $(n+n') \times (n+n')$ real constant matrix as follows

5.8
$$M_i = \begin{array}{r} \\ \ell_i \{ \\ \\ \\ \end{array} \underbrace{\left[\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline (G_i)^t \\ \hdashline 0 \end{array}\right]}_{n+n'} \begin{array}{l} \} \, n \\ \\ \} \, n' \\ \\ \end{array} \qquad (i \in \bar{k})$$

where $\ell_i = \sum_{j=1,\ldots,i-1} d(\mathcal{G}_i)$, and $(G_i)^t$ is the transpose of G_i.

Define $\mathcal{E}_i \triangleq (P+M_i)(\mathcal{F}_i \oplus \mathcal{G}_i)$, $(i \in \bar{k})$, where P is a projection mapping in (4.8). It is an easy matter to show that

5.9
$$\mathcal{E}_i \cap \mathcal{N}(C^e) = \{0\} \qquad (i \in \bar{k})$$

and that the $\mathcal{E}_i$ $(i \in \bar{k})$ are mutually independent. Define $\mathcal{R}_i^e = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \ldots \oplus \mathcal{E}_i$, $(i \in \bar{k})$, then

5.10
$$A^e \mathcal{R}_i^e = A^e \bar{\mathcal{S}}_i = A \bar{\mathcal{S}}_i \subset \bar{\mathcal{S}}_i + \mathcal{B} \subset \mathcal{R}_i^e + \mathcal{B}^e \qquad (i \in \bar{k})$$

and from (5.9), (5.10),

5.11
$$A^e [\mathcal{E}_i \cap \mathcal{N}(C^e)] = \{0\} \subset \mathcal{E}_i \oplus \mathcal{R}_{i-1}^e \qquad (i \in \bar{k})$$

5.12
$$A^e \mathcal{E}_i \subset \mathcal{E}_i + \mathcal{R}_{i-1}^e + \mathcal{B}^e \qquad (i \in \bar{k})$$

with $\mathcal{R}_0^e = \{0\}$. Since the $\mathcal{E}_i$ $(i \in \bar{k})$ are independent, there exists K^e such that

$$(A^e + B^e K^e C^e)\, \mathcal{E}_i \subset \mathcal{E}_i \oplus \mathcal{R}_{i-1}^e \qquad (i \in \bar{k})$$

Thus $(A^e + B^eK^eC^e)\mathcal{R}_1^e \subset \mathcal{R}_1^e$ and by induction

$$\begin{aligned}(A^e+B^eK^eC^e)\,\mathcal{R}_i^e &= (A^e+B^eK^eC^e)(\mathcal{E}_i \oplus \mathcal{R}_{i-1}^e)\\ &\subset \mathcal{E}_i \oplus \mathcal{R}_{i-1}^e\\ &= \mathcal{R}_i^e \qquad (i\varepsilon\bar{k})\end{aligned}$$

i.e., we have shown that the $\mathcal{R}_i^e$ $(i\varepsilon\bar{k})$ is a set of compatible (A^e,B^e,C^e)-invariant subspaces.

Now we are going to show that $\mathcal{R}_i^e \triangleq \sum_{j=1}^{i} \mathcal{E}_j \triangleq \sum_{j=1}^{i} (P+M_j)(\mathcal{F}_j+\mathcal{G}_j)$

is an (A^e,B^e,C^e) c.s. . From (5.7), $\bar{\mathcal{S}}_i = \sum_{j=1}^{i} (\mathcal{F}_j \oplus \mathcal{G}_j)$ is an (A,B,I)

c.s., and from Theorem (1.19), $\bar{\mathcal{S}}_i = \lim\, (\bar{\mathcal{S}}_i)^\mu$, $(\bar{\mathcal{S}}_i)^{\mu+1} = \bar{\mathcal{S}}_i \cap [A(\bar{\mathcal{S}}_i)^\mu + \mathcal{B}]$ $(\mu=0,1,2,\ \ldots,)$, $(\bar{\mathcal{S}}_i)^0 = \{0\}$. We define $(\mathcal{F}_j \oplus \mathcal{G}_j)^\mu \triangleq (\mathcal{F}_j \oplus \mathcal{G}_j) \cap \{A(\bar{\mathcal{S}}_i)^{\mu-1} + \mathcal{B}\}$, $(\mu=1,2,\ \ldots,)$ and $(\mathcal{F}_j \oplus \mathcal{G}_j)^0 = \{0\}$, $(j\varepsilon\bar{i})$. Then $(\bar{\mathcal{S}}_i)^\mu = \sum_{j=1}^{i} (\mathcal{F}_j \oplus \mathcal{G}_j)^\mu$, $(\mu=0,1,\ \ldots,)$.

Let $(\mathcal{R}_i^e)^{\mu+1} \triangleq \mathcal{R}_i^e \cap [A^e(\mathcal{R}_i^e)^\mu + \mathcal{B}^e]$ $\qquad(\mu=1,2,\ \ldots,)$

with $(\mathcal{R}_i^e)^0 = \{0\}$. Then $(\mathcal{R}_i^e)^0 \supseteq \{0\} = \sum_{j=1}^{i} (P+M_j)(\mathcal{F}_j \oplus \mathcal{G}_j)^0$; and if

$(\mathcal{R}_i^e)^\mu \supset \sum_{j=1}^{i} (P+M_j)(\mathcal{F}_j \oplus \mathcal{G}_j)^\mu$, then

$$(\mathcal{R}_i^e)^{\mu+1} \supset \left[\sum_{j=1}^{i} (P+M_j)(\mathcal{F}_j \oplus \mathcal{G}_j)\right] \cap \left\{A^e\left[\sum_{j=1}^{i} (P+M_j)(\mathcal{F}_j \oplus \mathcal{G}_j)^\mu\right] + \mathcal{B}^e\right\}$$

$$= \left[\sum_{j=1}^{i} (P+M_j)(\mathcal{J}_j \oplus \mathcal{G}_j)\right] \cap \left\{A(\bar{\mathcal{S}}_i)^{\mu} + \mathcal{B}^e\right\}$$

$$\supset \sum_{j=1}^{i} (P+M_j) \left\{(\mathcal{J}_j \oplus \mathcal{G}_j) \cap [A(\bar{\mathcal{S}}_i)^{\mu} + \mathcal{B}^e]\right\}$$

$$= \sum_{j=1}^{i} (P+M_j) \left\{(\mathcal{J}_j \oplus \mathcal{G}_j) \cap [A(\bar{\mathcal{S}}_i)^{\mu} + \mathcal{B}]\right\}$$

$$= \sum_{j=1}^{i} (P+M_j)(\mathcal{J}_j \oplus \mathcal{G}_j)^{\mu+1}$$

By induction, $\mathcal{R}_i^e \supset (\mathcal{R}_i^e)^{\mu} \supset \sum_{j=1}^{i} (P+M_j)(\mathcal{J}_j \oplus \mathcal{G}_j)^{\mu} \uparrow \sum_{j=1}^{i} (P+M_j)(\mathcal{J}_j \oplus \mathcal{G}_j)$,

i.e., $(\mathcal{R}_i^e)^{\mu} \uparrow \mathcal{R}_i^e$. So $\mathcal{R}_i^e$ $(i\in\bar{k})$ is an (A^e,B^e,C^e) c.s. .

The relation $P\mathcal{R}_i^e = \bar{\mathcal{S}}_i$ implies

$$\mathcal{R}_i^e \subset \bar{\mathcal{S}}_i \oplus \mathbb{R}^{n'} \subset \left\{\bigcap_{j=i+1,\dots,k} \mathcal{N}(C_j)\right\} + \mathbb{R}^{n'} = \bigcap_{j=i+1,\dots,k} \left\{\mathcal{N}(C_j) \oplus \mathbb{R}^{n'}\right\}$$

i.e., $\mathcal{R}_i^e$ satisfies (5.3). From (5.5) and the relation $\bar{\mathcal{S}}_i = P\mathcal{R}_i^e$,

$$P\mathcal{R}_i^e + \mathcal{N}(C_i) = \mathbb{R}^n$$

or equivalently, for some $\mathcal{V}_i \subset \mathbb{R}^{n'}$,

5.13 $$\mathcal{R}_i^e + \mathcal{N}(C_i) = \mathbb{R}^n \oplus \mathcal{V}_i$$

Addition of $\mathbb{R}^{n'}$ to both sides of (5.13) yields (5.2),

$$\mathcal{R}_i^e + (\mathcal{N}(C_i) \oplus \mathbb{R}^{n'}) = \mathbb{R}^n \oplus \mathbb{R}^{n'} .$$

Q.E.D.

5.14 Remark The new dynamic elements adjoined to the system in (1.1a,b) in the proof of Theorem (5.4) are performing as an observer [Lu.2,Wo.7].

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