A MARTINGALE APPROACH TO POINT PROCESSES

by

Pierre Marie Bremaud

Memorandum No. UCB/ERL M345

4 August 1972

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Acknowledgements

In the first place I wish to thank Professor Eugene Wong whose strong interest in the applications of martingale theory to the problems of Electrical Engineering which are of stochastic nature was not only the starting point of this thesis but also a continuous encouragement to me. Without his patient and competent guidance this work would have been impossible.

It is also a pleasure to acknowledge the help given to me by Professor Millar and Professor Azencott for the enlightening discussions I had with them and for their invaluable teaching.

My studies at the University of California of Berkeley from October 1969 to August 1972 were fully sponsored by the Délégation Ministérielle à l'Armement and I am very grateful to this agency.

The Ingénieur général Louchart and the Ingénieur général Pellegrin provided me with a moral and material support for which I sincerely thank them.

Also, I wish to express my gratitude to Professor Barlow and Professor Blackwell for reading my thesis and discussing it with me.

ABSTRACT

Point processes are studied from the point of view of martingales using the fundamental results of Meyer and of Kunita and Watanabe. Such an approach not only illuminates certain basic questions concerning the existence of point processes with prescribed properties, but the underlying martingale calculus also permits the derivation of a number of important results in applications.

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Introduction

In Communication Theory, one often deals with receivers of the counting type where the arrival of photons, electrons or whatever "particles" is registered. Then problems of detection and of estimation arise. In Nuclear medicine, one gets information about the spreading into the organism of injected substances by radioactive tracing. In Operations Research, queuing and dispatching are problems where point processes arise in a natural way. In Neuro physiology the information is transmitted along the nerves by pulses, another manifestation of point processes. These examples suffice to show that point processes are widely used to model random phenomena which occur in practice.

Usually the following definition is proposed: Let X_t be the number of events in the interval $[0,t]$, let there exist a nonnegative process, λ_t such that:

$$\lim_{h\to 0} \frac{1}{h} \Pr\{X_{t+h} - X_t = 1 | X_0^t\} \overset{a.s.}{=} \lambda_t \tag{1}$$

and

$$\lim_{h\to 0} \frac{1}{h} \Pr\{X_{t+h} - X_t > 1 | X_0^t\} \overset{a.s.}{=} 0 \tag{2}$$

where X_0^t represents the past $\{X_s, s \in [0,t]\}$. The existence of such a process is often assumed on intuitive grounds. Strictly speaking, however, one needs to establish the existence of a probability space (Ω,F,P), a step process $X = \{X_t, t \in R^+\}$ with jumps +1 and a nonnegative process $\lambda = \{\lambda_t, t \in R^+\}$ such that (1) and (2) are verified. We shall establish the existence of such a process by construction with the aid of the theory of square integrable (or locally square integrable) martingales. We should emphasize that such a construction is of more than mere technical importance. In the process a number of basic issues are illuminated. For example, the whole question of whether a point process is self exciting or not is clarified by the result that every point process has a characterization as a self-exciting process.

The results of this thesis fall roughly into three categories: first, the calculus of stochastic integration (Courrege, Kunita-Watanabe) with respect to martingales is exploited to solve a number of problems (dispatching, pulse modulation, change of time, etc.). Secondly, the striking similarity between Poisson process and Wiener process is used in deriving a number of results which are Poisson counterparts of some well known results associated with Wiener processes. These include: Girsanov's theorem, innovation theorem, and likelihood ratio formula. While not all of these results are new, they attain their greatest generality under the martingale approach. Finally, formulas for <u>mutual information</u> between point processes are given and some extensions are considered.

CHAPTER I

LIKELIHOOD RATIOS AND MARTINGALE CHARACTERIZATION

0 In Section 1 of this chapter, we give some standard results concerning martingale theory and stochastic integration with respect to a martingale that will be used later in this work. In particular, in Sec. 1.3 the relation between stochastic integration and Stieltjes integration in the case of a martingale whose trajectories are a.s. of bounded variation is shown.

We have given two ways of defining stochastic integrals: the constructive way (1.1 and 1.2), which is due to Doob, Ito and Courrège, and the method of definition of 1.3, which is that of Kunita-Watanabe. The integrals obtained are the same by the uniqueness property mentioned in 1-3. However, we found it useful to give the simplest version (Doob) of Sec. 1.1 so that a reader only interested in Sec. 2 of Ch. I and Ch. III can avoid the more sophisticated aspects of martingale theory.

In Sec. 2, we show, by construction of a probability measure ρ the existence of point processes with random rate when this rate is bounded. The more general case is treated in Sec. 6. Two features should be noted: we deal in Secs. 2 and 6 with rates that may depend on more than the past of the counting process; also, we obtain a characterization of point processes equivalent to a Poisson process in terms of square integrable martingales. This characterization in terms of the counting process X and of the rate (possibly depending on something more than the past of X) will be central in the explanation of the relation between general point processes and self exciting point processes (the innovation theorem: Section 1, Chapter II). In Sec. 3, this characterization is used to give a result concerning the change of time in point processes (see Pajangelon [35], for related ideas). Sec. 4 is another application of the martingale characterization concerning the superposition of point processes. In Sec. 5 we give a kind of converse to the results of 2. We start with the basic measurable space of point process (Ω, f) defined as follows: is the set of right continuous step functions X with nonnegative integer values, jumps +1 and starting at 0 at time 0, F_t is the smallest σ-algebra that makes all the coordinate mappings $(X_s, s \leq t)$ measurable and $F = \bigvee_{t \in r} F_t$. On this space we can put a probability measure P_0 that makes X the counting process of a Poisson process with rate 1 . We ask the question: if P is absolutely continuous with respect to P_0, can we derive an expression for $\frac{dP}{dP_0}$? We are able to obtain an answer in the self-exciting case.

In Sec. 7 we sketch the proof of the same kind of theorem for process absolutely continuous with respect to a Markov chain.

1 Preliminaries: Martingales, Poisson Process and Stochastic Integrals

1.1 Stochastic integration: The Stochastic integral of Doob

1 Let $\{\Omega, F, P_0$ be a probability space and $X = \{X_t, t \in R^+\}$ be a stochastic process defined on it, and such that:

1) $X_0 \equiv 0$ and X has right continuous paths

2) X is a process with independent increments

3) for all s, $t \in R^+$ such that $s \leq t$, $X_t - X_s$ is a Poisson random variable with parameter t - s.

The couple (X, P_0) is called a Poisson process with rate one, or a standard Poisson process.

2 Let $\{\Omega, F, P\}$ be a probability space together with an increasing family of sub-σ-fields of F: $\{F_t, t \in R^+\}$. A process $M = \{M_t, t \in R^+\}$ is said to be a (P, F_t) martingale if

1) $E\,|M_t| < \infty, \qquad \forall t \in R^+$

2) $E\{M_t \mid F_s\} = M_s$ P a.s., $\forall s, t \in R^+$ such that $s \leq t$.

(Note that 2) implies that M is adapted to $\{F_t, t \in R^+\}$, i.e., M_t is F_t-measurable, $\forall t \in R^+$.)

If in 2) the symbol = is replaced by $\leq$, M is called a (P, F_t) supermartingale.

From now on the attention will be restricted to processes M with right continuous paths, unless they are explicitly defined otherwise. Also, all the processes will have all the good measurability properties.

It is an easy exercise to verify that the two processes $\{X_t - t, t \in R^+\}$ and $\{(X_t - t)^2 - t, t \in R^+\}$ are $(P_0, \sigma\{X_s, 0 \leq s \leq t\})$ martingales.

3 A $(P,F_t)L^2$ martingale M is a $(P,F_t$ martingale such that

$$E|M_t|^2 < \infty \qquad \forall t \in R^+$$

One also says that $M \in \mathcal{M}$, by definition of $\mathcal{M}$.

4 Let (Ω,F) be a measurable space and $\{F_t,\ t \in R+\}$ be an increasing family of sub σ-fields of F. A random variable T defined on (Ω,F) is called an F_t-stopping time if:

$$\{T \leq t\} \in F_t \qquad \forall t \in R^+$$

Given an F_t-stopping time T, the past at time T is, by definition, the following σ-field:

$$F_T = \{A \in F/A \cap \{T \leq t\} \in F_t, \qquad \forall t \in R^+\}$$

5 A process $M = \{M_t,\ t \in R^+\}$ adapted to an increasing family $\{F_t,\ t \in R^+\}$ is said to belong to $\mathcal{M}_{loc}$, or to be a local martingale, if one can exhibit a sequence of F_t-stopping times $\{T_n,\ n \in N\}$ such that:

1) $T_n \uparrow \infty \qquad$ P a.s.

2) $M^n = \{M_{t \wedge t_n},\ t \in R^+\} \in \mathcal{M}, \qquad \forall n \in N$

6 Let (Ω,F, P) be a probability space and $\{F_t,\ \in\ R^+\}$ be an increasing family of sub σ-fields of F. A process $\phi = \{\phi_t,\ t \in R^+\}$ is said to be a (P,F_t)-step process on $[a,b] \subset R^+$ if there exists a sequence $a = t_0 < t_1 < \ldots < t_n = b$ and random variables ϕ_i, $i = 0,1, \ldots, n-1$ such that:

1) ϕ_i is F_{t_i}-measurable

2) $\phi_t = \phi_i \quad$ for $\quad t_i \leq t < t_{i+1}, \qquad i = 0,1, \ldots, n-1$

3) $E\int_a^b \phi_t^2 \ dt < \infty$

We say that ϕ is a (P,F_t)-step process if it is a (P,F_t)-step process on any $[a,b] \subset R^+$. The following approximation lemma will be of central importance in the definition of stochastic integrals with respect to the Poisson process.

Lemma: Let $\phi = \{\phi_t, t \in R^+\}$ be a process adapted to $\{F_t, t \in R^+\}$ and such that $E\int_a^b \phi_t^2\, dt < \infty$. Then, there exists a sequence $\{\phi^n, n \in N\}$ of (P,F_t)-step processes on $[a,b]$ such that

$$E \int_a^b |\phi_t - \phi_t^n|^2\, dt \to 0 \quad \text{as} \quad n \to \infty$$

A proof of this result can be found in [51], pp. 142-143.

7 Let $\{X = (X_t, t \in R^+), P_0\}$ be a standard Poisson process and ϕ a (P_0,F_t)-step process, where $F_t = \sigma\{X_s, 0 \le s \le t\}$. The stochastic integral $\int_a^b \phi_t[d X_t - dt]$ is defined as:

$$\int_a^b \phi_t[d X_t - dt] = \sum_{i=0}^{n-1} \phi_{t_i}[X_{t_{i+1}} - X_{t_i} - (t_{i+1} - t_i)]$$

One can easily check the two following facts:

a) $\left\{\int_0^t \phi_s(dX_s - ds),\ t \in R^+\right\}$ is a (P,F_t) L^2 martingale

b) $E\left|\int_0^t \phi_s[dX_s - ds]\right|^2 = E\int_0^t \phi_s^2\, ds$

8 Now, if ϕ is simply a process adapted to $\{F_t, t \in R^+\}$ and such that

$$E\int_a^b \phi_s^2\, ds < \infty$$

then $\int_a^b \phi_s(dX_s - ds)$ is defined as the limit in quadratic mean of $\int_a^b \phi_s^n(dX_s - ds)$ where $\{\phi^n, n \in N\}$ is an approximating sequence of ϕ in the sense of Lemma 1. Indeed, for any $n,m \in N$, $\phi^n - \phi^m$ is also a (P,F_t) step process on $[a,b]$, and:

$$E\left|\int_0^t \phi_t^n(dX_t - dt) - \int_a^b \phi_t^m(dX_t - dt)\right|^2 =$$

$$= E\left|\int_a^b (\phi_t^n - \phi_t^m)(dX_t - dt)\right|^2 = E\int_a^b |\phi_t^n - \phi_t^m|^2\, dt \to 0$$

$$\text{as } n,m \to \infty$$

therefore $\left\{\int_a^b \phi_t^n\ (dX_t - dt),\ n \in N\right\}$ is Cauchy in $L^2(\Omega,F,P)$, and $\int_a^b \phi_t(dX_t - dt)$ is defined. That it is uniquely defined (i.e., does not depend on the approximating sequence $\{\phi_n, n \in N\}$) is a simple task left to the reader. Obviously, properties a) and b) are conserved in the passage to the limit (in quadratic mean).

1.2 More on Stochastic Integration: the Stochastic Integral of Ito-Courrège.

① Let (Ω,F,p) be a probability space, $\{F_t, t \in R^+\}$ an increasing family of sub σ fields of F, and $M = \{M_t, t \in R^+\}$ a $(P - F_t)$ martingale, with right continuous paths and square integrable.

Let $A = \{A_t, t \in R^+\}$ be the natural increasing process associated with it, i.e., A is a process such that:

1) $\{A_t, t \in R^+\}$ is P a.s. a right continuous increasing function

2) $\{M_t^2 - A_t, t \in R^+\}$ is a $(P\text{-}F_t)$-martingale

3) $E\int_0^t Y_s \, dA_s = E\int_0^t Y_{s-} dA_s$ for all $t \in R^+$, all bounded (P,F_t) martingales $Y = \{Y_t, t \in R^+\}$.

A process A satisfying 1) and 2) is known to exist, and 3) ensures its uniqueness (results of Meyer [32]. See also Courrège [5].)

2 If $f = \{f_s, s \in R^+\}$ is a process that is $\sum$-measurable, where $\sum$ is the σ field on $R^+ \times \Omega$ generated by the process adapted to $\{F_t, t \in R^+\}$ whose trajectories are left-continuous,* and if $E\int_0^t f_s^2 \, dA_s < \infty$ then the <u>stochastic integral</u> $f \cdot M = \left\{\int_0^t f_s dM_s, t \in R^+\right\}$ can be defined as follows: there exists a sequence $\{f^n\}$ of <u>stochastic step processes</u>, i.e., processes with the same properties as f and such that, moreover:

$$f_t^{(n)} = \sum_{k=0}^{m-1} f_{t_k}^{(n)} I_{]t_k,t_{k+1}]}(t)$$

where the t_k's form a g fixed sequence.

$0 = t_0 < t_1 < \ldots < t_k < \ldots < t_m < \infty$ and $f_{t_k}^{(n)}$ is bounded and F_{t_k}-measurable for all k; moreover:

$$E\int_0^t \left[f_s^{(n)} - f_s\right]^2 dA_s \to 0 \quad \text{as } n \to \infty$$

Then one defines $f^{(n)} \cdot M$ by:

$$f^{(n)} \cdot M(t) = \sum_{k=1}^{m-1} f_{t_k}^{(n)} (M_{t_{k+1}} - M_{t_k})$$

It is then easily proven that $\{f^{(n)} \cdot M(t), t \in R^+\}$ is a (P,F_t) martingale, with right continuous paths and such that

$$E[(f^{(n)} \cdot M)(t)]^2 = E\int_0^t [f_t^{(n)}]^2 \, dA_s$$

---

*Such a process is said to be <u>previsible</u> (or <u>predictable</u>).

If ν_A is the norm defined by

$$\nu_A(\phi) = E\int_0^t \phi_s^2 \, dA_s \, ,$$ then $\{f^{(n)} \cdot M\}$ is a Cauchy sequence and $f \cdot M$ is defined as the limit of the sequence $\{f^{(n)} \cdot M\}$ for this norm. See [6].

3 If M is quasi left continuous and A is a.c. with respect to the lebesque measure then if M can be defined for the class of adapted processes of (not necessary predictable) by a norm presenting extension. See [6].

Illustrations

a) $M = \{X_t - t, \ t \in R^+\}$ where $X = \{X_t, \ t \in R^+\}$ is a standard Poisson process (rate 1). M is quasi left continuous and its increasing process is $A = \{t, \ t \in R^+\}$.

b) $M = \{B_t, \ t \in R^+\}$, the standard brownian motion, is continuous (therefore quasi left continuous) and $A = \{t, \ t \in R^+\}$

1.3 The Relation Between Stochastic Integrals and Stieltjes Integrals.

1 Let $\mathcal{A}$ be the set of the processes with integrable variation on each $[0,t]$, $t \in R^+$; if $V \in \mathcal{A}$ let $L'(V)$ be the set of previsible processes f such that $E\int_0^t |f_s||dV_s| < \infty$ for all $t \in R^+$. If $V \in \mathcal{A}$ is a martingale and if $f \in L'(V)$, then $\left\{ \int_0^t f_s dV_s, t \in R^+ \right\}$ is a (P,F_t) martingale ([10], p. 89).

2 Also the following ([10], p. 90) relates Stieltjes integrals and stochastic integrals:

If $M \in \mathcal{A} \cap \mathcal{M}$ and if $f \in L^2(M) \cap L'(M)$ (where $f \in L'(M)$ means $\int_0^t |f_s||dM_s| < \infty$, $\forall t \in R^+$ and $f \in L^2(M)$ means $E\int_0^t |f_s|^2 dA_s < \infty$, $\forall t \in R^+$). Then $(f.M)_t = \int_0^t f_s dM_s$ where the integral on the right side of the equation is a Stieltjes integral.

3 It is an interesting exercise to try to express the (P_0, F_t) martingale $\{(X_t - t)^2 - t,\ t \in R^+\}$ as a stochastic integral. Using a formula of integration by parts for processes of bounded variation (see appendix] we have

$$(X_t - t)^2 = \int_0^t (X_t - t)(dX_t - dt) + \int_0^t (X_{t-} - t)(dX_t - dt)$$

$$= 2\int_0^t (X_t - t)(dX_t - dt) + \int_0^t (X_t - X_{t-})\, dX_t$$

therefore $(X_t - t)^2 - t = 2\int_0^t (X_t - t)(dX_t - dt) + (X_t - t).$

The reader familiar with the integration with respect to a Brownian motion $\{W_t,\ t \in R^+\}$ will note the similarity of this formula with:

$$W_t^2 - t = 2\int_0^t W_t\, dW_t$$

a classical example of how "ordinary" calculus does not apply. One sees that "ordinary" calculus does not apply in Stieltjes integration also. This was already noted by Wong and Zakai in [50].

2. Point Processes a.c. With Respect to the Poisson Process. With Bounded Random Rate

2.0 In engineering literature, one often deals with processes $X = \{X_t,\ t \in R^+\}$, called extensions of Poisson processes, or point processes with random rate $\lambda = \{\lambda_t,\ t \in R^+\}$ where λ is a measurable, non-negative process. These processes are defined on a probability space (Ω,F,P) and are supposed to satisfy the following properties:

1) $X_0 \equiv 0$, X is step, right continuous, $X_t - X_t = 0$ or 1

2) $\lim\limits_{h\to 0} \frac{1}{h} E\{1_{\{X_{t+h}-X_t=1\}} / X_s,\ 0 \le s \le t\} = \lambda_t$ P a.s.

3) $\lim\limits_{h\to 0} \frac{1}{h} E\{1_{\{X_{t+h}-X_t>1\}} / X_s,\ 0 \le s \le t\} = 0$ P a.s.

Property 1) says that X is the counting process of a point process.

Property 3) says that no more than one event should occur at the same point t.

Property 2) is a definition of the intensity process λ.

Two remarks immediately arise:

1) Is there such a probability measure P?

2) X is necessarily adapted to $\{\sigma(X_s, 0 \le s \le t),\ t \in R^+\}$; in other words, the process is self-exciting. Could we not allow λ_t to depend on something more than the past of X up to time t?

The following paragraph is devoted to an answer to these questions.

2.1 Construction of point processes with random rate

Let (Ω,F,P_0) be a probability space, (X,P_0) a standard Poisson process, and $\lambda = \{\lambda_t,\ t \in R^+\}$ a non-negative stochastic process with left-hand limits and adapted to the family $\{F_t, t \in R^+\}$ where $F_t \supset \sigma\{X_s, 0 \le s \le t\}$. Suppose also that $\{X_t - t, t \in R^+\}$ is a (P_0, F_t) L^2 martingale.

Let t_i, $i = 1,2, \ldots$ be the jump times of the process X. Consider the process $L = \{L_t, t \in R^+\}$ defined by:

$$L_t = \prod_{t_i \le t} \lambda(t_i) \quad \exp\left(-\int_0^t (\lambda_s - 1)\, ds\right)$$

L_t can be rewritten as:

$$L_t = 1 + \int_0^t L_{s-}(\lambda_{s-} - 1)(dX_s - ds)$$

where the integral in the second member is a Stieltjes integral.

If λ is assumed to be bounded uniformly by a constant A, then

$$|L_{s-}(\lambda_{s-} - 1)| \le A^{(X_t+1)} \quad \exp t \quad \text{on } s \in [0,t]$$

therefore:

$$E \int_0^t L_{s-}(\lambda_{s-} - 1)^2 \, dt \le t \exp 2t \; E\, A^{2(X_t+1)} < \infty$$

therefore the stochastic integral $\int_0^t L_{s-}(\lambda_{s-} - 1)(dX_s - ds)$, exists and is the same as the Stieltjes integral of (2). So

$$L = \{L_t, t \in R^+\} \text{ is a } (P_0, F_t)\ L^2 \text{ martingale,}$$

and it has the mean value $EL_t = EL_0 = 1$. This is summarized by:

Theorem 2-1-i

When $\lambda = \{\lambda_t, t \in R^+\}$ is a non-negative uniformly bounded process with left-hand limits, the process $L = \{L_t, t \in R^+\}$ given by (1) defines on (Ω, F) a probability measure P absolutely continuous with respect to P_0 by

$$E_0\left\{\frac{dP}{dP_0} / F_t\right\} = \prod_{t_i \leq T} \lambda_{t_{i-}} \exp\left(-\int_0^t (\lambda_s - 1)\, ds\right)$$

We also have:

Theorem 2-1-ii

Under P, the process $\left\{X_t - \int_0^t \lambda_s\, ds,\ t \in R^+\right\}$ is a (P, F_t) martingale

(This means that λ is the intensity of the jumps of X, under P, since:

$$\frac{1}{h} E\{X_{t+h} - X_t / F_t\} = \frac{1}{h} E\left\{\int_t^{t+h} \lambda_s\, ds / F_t\right\}$$

But as $h \to 0$, $\frac{1}{h}\int_t^{t+h} \lambda_s d_s \to \lambda_t$ which is F_t-measurable (by the dominated convergence theorem). Therefore:

$$\lim_{h\to 0} \frac{1}{h} E\{X_{t+h} - X_t / F_t\} = \lambda_t)$$

Proof

To prove that a process M is a (P, F_t) martingale it is enough to prove that $\left\{M_t\, E_0\left(\frac{dP}{dP_0} / F_t\right),\ t \in R^+\right\}$ is a (P, F_t) martingale, since:

$$\int_A M_t\, dP = \int_A M_t\, E_0\left(\frac{dP}{dP_0} / F_t\right) dP_0 \quad \text{for all } t \geq s,\ A \in F_s$$

We shall therefore proceed to show that N defined by

$$N_t = \prod_{t_i \leq t} \lambda_{t_{i-}} \exp\left\{-\int_0^t (\lambda_s - 1)ds\right\} \left[\sum_{t_i \leq t} 1 - \int_0^t \lambda_{s-} ds\right]$$

is a (P_0, F_t) martingale.

First at a jump t_i of X:

$$N_{t_i} - N_{t_{i-}} = N_{t_i}(\lambda_{t_{i-}} - 1) + \lambda_{t_{i-}} L_{t_{i-}} \tag{3}$$

and for $h \geq 0$, $t_i + h < t_{i+1}$:

$$\frac{dN_{t_i+h}}{dh} = -[\lambda(t_i+h) - 1]\, N_{t_i+h} - \lambda(t_i + h)\, L(t_i+h) \tag{4}$$

From (3) and (4):

$$N_t = 1 + \int_0^t N_{t-}(\lambda_t - 1)(dX_s - ds) + \int_0^t \lambda_{t-} L_{t-}(dX_s - ds) \tag{5}$$

where the second member is a Stieltjes integral. But the boundedness of λ also shows that this integral has a meaning as a stochastic integral. Therefore the equation is valid as a stochastic equation and N is a (P_0, F_t) martingale.

<u>Example: **renewal processes**</u>

Let $F(x) = \int_0^x f(u)\, du$ be the d.f of a certain r.v. and suppose $\frac{f(y)}{1 - F(y)}$ bounded for all $y \geq 0$. Define λ by:

$$\lambda_s = \frac{f(s - \sigma_s)}{1 - F(s - \sigma_s)}$$

where σ_s is the last jump of X before s. Let P be the measure defined on (Ω,F) by this λ. P is the measure that makes X a renewal process with renewal d.f. F. We will proceed to prove it.

Let $T_n = \inf\{t / X_t = n\}$: T_n is an F_t-stopping time, and so is $T_n + s$, for any $s \in R^+$.

$$P\{X(T_n + s) - X(T_n) = 0 / F_{T_n}\}$$

$$= E(1\{X(T_{n+s}) - X(T_n) = 0\} / F_{T_n})$$

$$= E_0\left(1\{X(T_n+s) - X(T_n) = 0\} \frac{L_{T_n+s}}{L_{T_n}} / F_{T_n} \right)$$

$$= E_0\left(\exp\left(- \int_{T_n}^{T_n+s} \frac{f(u - T_n)}{1 - F(u - T_n)} ds\right) / F_{T_n} \right) = 1 - F(s) \blacksquare$$

We shall use the characterization to solve a problem of modeling.

2.2 Markov point processes and general point processes

Given a family of functions

$$P_k: R^+ \to [0,1]], \quad k \in N^+ = \{0,1,2, \dots \}$$

such that

$$\sum_{k=0}^{\infty} P_k(t) = 1, \quad \forall t \in R^+ \tag{7}$$

Is there a point process such that:

$$P\{X(t) = k\} = P_k(t), \quad \forall k \in N^+, \forall t \in R^+ \ ? \tag{8}$$

and if so, what are all such processes? We shall, for the time being, try to find processes with uniformly bounded rate (but the proof would be the same in the more general case); because of this we have to impose the condition:

$$\sum_{j=0}^{k} \frac{dp_j(t)}{dt} P_k(t) \leq K \leq \infty$$

Suppose there exists a point process X with bounded rate λ satisfying (7) and (8). Let $g_k(n) = \delta_{k-n}$ $k, n \in N^+$. We have

$$g_k(X_t) = \sum_{\substack{s \leq t \\ X_s \neq X_{s-}}} \left[g_k(X_s) - g_k(X_{s-}) \right] \quad \text{if } k \neq 0,\ k \in N^+ \tag{10}$$

and:

$$g_0(X_t) - 1 = \sum_{\substack{s \leq t \\ X_s \neq X_{s-}}} \left[g_0(X_s) - g_0(X_{s-}) \right] \tag{10'}$$

Also $X_{s-} = X_s - 1$ when $X_s \neq X_{s-}$ and $E\{g_k(X_t)\} = P_k(t)$ and $E\{g_k(X_t - 1)\} = E\{g_{k+1}(X_t)\} = P_{k+1}(t)$. Integrating $g_k(X_t - 1) - g_k(X_t)$ with respect to $\left\{ X_t - \int_0^t \lambda_s ds,\ t \in R^+ \right\}$ we have, by the martingale property of the Stieltjes integral:

$$\begin{aligned} P_k(t) &= E\int_0^t [g_k(X_s^{+1}) - g_k(X_s)]\ \lambda(s,\omega)\ ds \\ &= E\int_0^t [g_k(X_s + 1) - g_k(X_s)]\ E\{\lambda_s/X_s\}\ ds \quad \text{for } k \in N^+ - \{0\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} P_0(t) - 1 &= E\int_0^t [g_0(X_s^{+1}) - g_0(X_s)]\ \lambda(s,\omega)\ ds \\ &= E\int_0^t g_0(X_s + 1) - g_0(X_s)]\ E\{\lambda_s/X_s\}\ ds \end{aligned} \tag{11'}$$

Therefore, if we let $\mu_k(s) = E(\lambda_s/X_s = k)$, the equations (11) and (11') become:

$$\begin{cases} P_k(t) = \int_0^t [\mu_{k-1}(s)\ P_{k-1}(s) - \mu_k(s)P_k(s)]\ ds & \text{for } k \in N^+-\{0\} \qquad (12) \\ P_0(t) - 1 = \int_0^t [-\mu_0(s)\ P_0(s)]\ ds & (12') \end{cases}$$

This gives

$$\mu_k = -\sum_{j=0'}^{k} \dot{P}_j/P_k \qquad k \in N^+$$

Of course, $-\sum_{j=0}^{k} \dot{P}_j$ has to be positive; in other words $\sum_{j=0}^{k} P_j(t)$ has to be decreasing in t, for all $k \in N^+$. This only says that $P\{X_t \geq k\}$ increases in t, an obviously necessary condition that the P_k's have to satisfy.

Therefore:

<u>Theorem 2-4-iii</u>

If the P_k's satisfy the necessary conditions

$\sum_{k=0}^{\infty} P_k(t) = 1$ for all $t \in R^+$, $k \in N^+$ (<u>compatibility</u>) and

$\sum_{j=k}^{\infty} P_j(t)\uparrow$ as $t\uparrow$ for all $k \in N^+$ (<u>growth</u>) and

$\sum_{j=0}^{k} \dot{P}_j/P_k \leq K$, for some $K > 0$ (boundedness)

then there exists a whole family of point processes with bounded rate absolutely continuous with reference to the Standard Poisson process, such that:

$$P(X_t = k) = P_k(t),\ \forall t \in R^+,\ \forall k \in N^+$$

Namely, if we let $\mu_k(s) = E\{\lambda_s/X_s = k\}$

$$\mu_k(t) \quad \frac{\sum_{j=k}^{\infty} \dot{P}_j(t)}{P_k(t)} \quad , k \in N^+, \; t \in R^+$$

3. Change of Time for Point Processes.

3.0 We shall now use the martingale characterization of Theorem (2-1-i) to relate the rate λ to a change of time (a process X being given together with a family $\{\tau_t, \; t \in R^+\}$ of F_t-stopping times a.s. increasing and right continuous, we say that the process Y is derived from X by the change of time $\{\tau_t, t \in R^+\}$ if $Y_t = X_{\tau(t)}$ P a.s.). We will need a characterization theorem due to Watanabe [49]. The proof that we shall give here is based on the same idea as in the proof of the characterization theorem for Brownian motion that Kunita and Watanabe give in [28].

3.1 A Characterization of standard Poisson process

Theorem 3-1

Let $X = \{X_t, \; t \in R^+\}$ be a right continuous step process defined on (Ω, F, P_0) and such that $X_0 = 0$ and X increases only by jumps of magnitude +1. Let $F_t = \sigma\{X_s, \; 0 \leq s \leq t\}$. If $\{X_t - t, \; T \in R^+\}$ is a local (P_0, F_t) martingale, then (X, P_0) is a standard Poisson process.

This theorem has already been proven by Watanabe in [28]. Kunita and Watanabe [28] have also given a theorem concerning the characterization of Brownian motion, the method of proof of which we shall use now:

$$de^{iuX_t} = [e^{iu} - 1]\, e^{iu X_{t-}}\, dX_t \quad \text{or} \tag{14}$$

$$e^{iu X_t} - e^{iu X_s} = \int_s^t (e^{iu} - 1)\, e^{iu X_{w-}}\, dX_w \tag{14'}$$

Also for any stopping time T_n

$$e^{iu\wedge X_{t\wedge T_n}} - e^{iu X_{s\wedge T_n}} = \int_s^t (e^{iu} - 1) e^{iu X_{w\wedge T_n}}\, dX_{w\wedge T_n} \tag{15}$$

Choose $\{T_n, n \in N\}$ such that $\{X_{t\wedge T_n} - t\wedge T_n, t \in R^+\}$ is a square integrable (P_0,F_t) martingale for each n.

Let $\{A_t \ \ t \in R^+\}$ be the increasing process associated with $\{X_t - t, t \in R^+\}$. Then $E\int_0^t |e^{iu X_t}|^2\, dA_{t\wedge T_n} \leq EA_{t\wedge T_n} < \infty$ for all $n \in N$,

Therefore $\int_0^t (e^{iu} - 1)e^{iu X_{w-\wedge T_n}}\{dX_{w\wedge T_n} - d(w\wedge T_n)\}$ is a (P_0,F_t) martingale and, for any $A \in F_s$

$$E\, I_A\{e^{iu X_{t\wedge T_n}} - e^{iu X_{s\wedge T_n}}\} = E\, 1_A \int_s^t (e^{iu} - 1)e^{iuX_w}\, d(w\wedge T_n) \tag{16}$$

Multiplying both sides by $e^{iu(X_{s\wedge T_n})}$ and letting $n \to \infty$ in (16):

$$E\left\{1_A\, e^{iu(X_t-X_s)}\right\} = P_0(A) \int_s^t (e^{iu} - 1)\, e^{iu(X_w-X_s)}\, dw \tag{17}$$

Therefore

$$E\, 1_A\, e^{iu(X_t-X_s)} = P_0(A)\, e^{(e^{iu}-1)(t-s)} \tag{18}$$

This being true for all $A \in F_s$, we see that X has independent increments and $X_t - X_s$ is a Poisson r.v. with parameter $(t - s)$. ■

3.2 Change of time

Theorem 3-2

Let (Ω,F,P) be a probability space and $X = \{X_t, t \in R^+\}$ a process defined on it, adapted to an increasing family $\{F_t, t \in R^+\}$ and such that

1) X is a step process, $X_0 \equiv 0$ and $X_t - X_{t-} = 0$ or $+1$

2) $\left\{X_t - \int_0^t \lambda_s \, ds,\ t \in R+\right\}$ is a $(P,F_t)L^2$ martingale

where $\lambda = \{\lambda_s, s \in R^+\}$ is a non-negative measurable process, adapted to $\{F_t, t \in R^+\}$ and such that $E \int_0^t \lambda_s ds < \infty$, $\forall t \in R^+$.

Let $\tau(t) = \inf\{s / \int_0^s \lambda_u \, du \geq t\}$ and $Y_t = X_{\tau(t)}$.

Then $\{Y_t - t,\ t \in R^+\}$ is a $(P,\mathcal{G}_t)L^2$ martingale where $\mathcal{G}_t = F_{\tau(t)}$

Moreover:

If $F_t = \sigma\{X_s,\ 0 \leq s \leq t\}$ then Y is a standard Poisson process.

This is a mere corollary of the Watanabe characterization.

Definition 3-2-1

A process X satisfying the conditions 1) and 2) of the above theorem is called a good point process with rate λ.

For instance, the process X defined by:

$$X_t = n \quad \text{for } t \in [n, n + 1[\tag{19}$$

is not a point process, since the existence of a measurable

intensity process λ such that

$$E\int_0^t \lambda_s \, ds < \infty \text{ and } \left\{X_t - \int_0 \lambda_s d_s, \; t \in R^+\right\} \text{is a } (P,F_t)$$

martingale would imply:

$$E(X_n - X_{n-h}) = 1 = E\left\{\int_{n-h}^{n} \lambda_s \, ds/F_{n-h}\right\} \tag{20}$$

therefore $E\left\{\int_{n-h}^{n} \lambda_s ds\right\} = 1, \; \forall h > 0$. Letting $h \to 0$ we would obtain a contradiction with the measurability of λ. ▲

<u>Remark</u>: Definition (3-2-i) is very restrictive, as we shall see by the martingale characterization of point processes that are equivalent to the Poisson process, since we can only say that, in general, $\left\{X_t - \int_0^t {}_s ds, \; t \in R^+\right\}$ is a (P,F_t) <u>local</u> martingale.

<u>Definition 3-2-ii</u>

a. Point processes that are equivalent to the Poisson process such that $\left\{X_t - \int_0^t {}_s ds, \; t \in R^+\right\}$ is a (P,F_t) local martingale but not a square integrable (P,F_t) martingale, are called <u>semi-good point processes</u>.

b. Point processes that are absolutely continuous with respect to a Poisson process but not equivalent are called <u>degenerate point processes of the first kind</u>.

c. Point processes that are not a.c. with respect to a Poisson process are called <u>degenerate point processes of the second kind</u>.

The next paragraph shows how the martingale characterization can be used theoretically. It is written in terms of good point process but it is easy to see that the same results apply in the case of semi-good point processes.

4 Superposition of Independent Point Processes, Self-Exciting

Let (Ω,F,P) be a probability space, $X = \{X_t, t \in R^+\}$ and $Y = \{Y_t, t \in R^+\}$ be point processes such that:

$$\left\{X_t - \int_0^t \lambda_s \, ds, \ t \in R^+\right\} \text{ is a } (P,X_0^t) \ L^2 \text{ martingale}$$

$$\left\{Y_t - \int_0^t \mu_s \, ds, \ t \in R^+\right\} \text{ is a } (P,Y_0^t) \ L^2 \text{ martingale}$$

where $\lambda = \{\lambda_t, t \in R^+\}$ is non-negative, measurable, adapted to $\{X_0^t, t \in R^+\}$ and $\mu = \{\mu_t, t \in R^+\}$ is non-negative, measurable, adapted to $\{Y_0^t, t \in R^+\}$. Suppose, moreover, that X and Y are independent.

Let

$$Z_t = X_t + Y_t \tag{21}$$

Then

$$E(Z_t - Z_s/Z_0^s) = E(X_t - X_s \ /Z_0^s) + E(Y_t - Y_s/Z_0^s) \tag{22}$$

But

$$E(X_t - X_s/Z_0^s) = E\{E(X_t - X_s/Z_0^s \vee X_0^s)/Z_0^s\} \tag{23}$$

Also

$$Z_0^s \vee X_0^s = Y_0^s \vee X_0^s \tag{24}$$

Therefore

$$E(X_t - X_s/Z_0^s \vee X_0^s) = E(X_t - X_s/Y_0^s \vee X_0^s) \tag{25}$$

As $X_t - X_s$ and X_0^s are independent of Y_0^s :

$$E(X_t - X_s/Y_0^s \vee Y_0^s) = E(X_t - X_s/X_0^s) = E\left\{\int_s^t \lambda_u \, du/F_s\right\} \tag{26}$$

Finally, combining (23) and (26):

$$E(X_t - X_s/Z_0^s) = E\left(\int_s^t \lambda_u \, d\mu/Z_s^s\right) = \int_0^t E(\lambda_u/Z_0^u) \, du \qquad (27)$$

Similarly for $E(Y_t - Y_s/Z_0^s)$. Therefore

$$Z_t - \int_0^t E(\lambda_s + \mu_s/Z_0^s) \, ds \quad \text{is a } (P,Z_0^t)L^2 \text{ martingale}$$

<u>Special case</u>: If X and Y are Poisson process with deterministic rate $\lambda(t)$ and $\mu(t)$, then Z is also a Poisson process with rate $\lambda(t) + \mu(t)$.

5 Absolute Continuity with respect to a Poisson Process: the Self-Exciting Case

5.0 In 2 we have been constructing (under restrictive conditions) a probability measure P on a measure space (Ω,F) sufficiently rich to support a process $X = \{X_t, t \in R^+\}$ such that X is a right continuous step process with $X_0 \equiv 0$ and $X_t - X_{t-} = 0$ or 1. This measure was absolutely continuous (by construction) with respect to P_0, the latter probability measure making X a Poisson process with rate 1.

Also, P made X a point process with parameter $\lambda = \{\lambda_s, s \in R^+\}$ not necessarily self-exciting (since F_t could be chosen such that $F_t \supset \sigma\{X_s, 0 \leq s \leq t\}$ provided (Ω,F) was sufficiently rich a measure space).

Now the question is:

Given a measure P on (Ω,F) such that

$$P << P_0$$

where P_0 is a measure making X a Poisson process with rate 1, what does P look like? More precisely, what is an expression for $\frac{dP}{dP_0}$?

We will solve this in the <u>self-exciting</u> case, i.e., the case where $F_t = \sigma(X_s, 0 \leq s \leq t)$. First, we shall recall some useful facts about absolute continuity.

5.1 Absolute Continuity

Let (Ω,F) be a measurable space and let $\{F_t, t \in R^+\}$ be a family of increasing sub σ-fields of F.

$$\int_A L_\tau \, dP_0 = \int_A L_{\tau\wedge a} \, dP_0 = \int_A L_a \, dP_0 = P(A) \leq 1 \qquad (30)$$

But on A, $L_\tau = \infty$; therefore one must have $P_0(A) = 0$. As a is arbitrary we get the following lemma:

<u>Lemma 1</u> $\quad P_0(\tau < \infty) = 0$ and $P(\tau < \infty) = 0$ (31)

Let $\mathcal{J} = \lim \uparrow \mathcal{J}_n$. As L is a non-negative martingale we have $L_{\mathcal{J}+s} = 0$ for all $s > 0$. (For a statement of this fact, see Blumenthal and Getoor, Reference [3], h. I.) Therefore, if we let $B = \{\mathcal{J} \wedge b \leq b\}$ where b is an arbitrary positive number, we have:

$$P(B) = \int_B L_{\mathcal{J}\wedge b} \, dP_0 = \int_B L_{\mathcal{J}} \, dP_0 = \int_B L_{\mathcal{J}+s} \, dP_0 = 0 \qquad (32)$$

Therefore $P(B) = 0$ and as b is arbitrary:

<u>Lemma 2</u> $\quad P(\mathcal{J} < \infty) = 0$ (33)

(Note: in this case we do not have $P_0(\mathcal{J} < \infty)$. However, this is true if $P_0 \ll P$, i.e., $P \sim P_0$, as one can easily check).

<u>5.2 The Likelihood Ratio Formula for Point Processes</u>

The following theorem should be understood as a kind of converse to Theorems 2-1-1 and 2-1-ii.

<u>Theorem 5-2-1</u>

Let (Ω, F) be the basic measurable space of the point processes $X = \{X_t, t \quad R^+\}$, the coordinate process, and $F_t = \sigma\{X_s, 0 \leq s \leq t\}$.

Two probability measures, P and P_0, being given on (Ω, F, P), one says that P is absolutely continuous w.r. to P_0 ($P \ll P_0$) iff for all $A \in F$, such that $P_0(A) = 0$ we have $P(A) = 0$.

P and P_0 are said to be equivalent iff $P \ll P_0$ and $P_0 \ll P$. Suppose $P \ll P_0$, then there exists a non-negative random variable denoted by $\frac{dP}{dP_0}$ (and called the Radon-Nikodym derivative of P w.r. to P_0) such that $E_0 \frac{dP}{dP_0} = 1$ and for all the P-integrable r.v.'s Y

$$E\,Y = E_0\left\{ Y \frac{dP}{dP_0} \right\} \tag{28}$$

The process $L = \{L_t,\ t \in R^+\}$ $\left(\text{where } L_t = E_0\left\{ \frac{dP}{dP_0} \Big/ F_t \right\}\right)$ is a (P_0, F_t) martingale. This martingale is right-continuous if the family $\{F_t,\ t \in R^+\}$ is right-continuous $\left(\bigcap_{h>0} F_{t+h} = F_t \right)$.

$$\text{Let:} \quad \begin{cases} \tau_n = \inf\{t / L_t \geq n\} \\ \mathcal{T}_n = \inf\left\{t / L_t \leq \frac{1}{n}\right\} \\ T_n = \tau_n \wedge \mathcal{T}_n \end{cases} \tag{29}$$

All these random variables are F_t-stopping times. Moreover, they all increase with n.

Let $\tau = \lim \uparrow \tau_n$ and a be a positive number. Let $A = \{\omega / L_t$ become infinite on $[0,a]\}$. A belongs to F_τ and F_a since $A = \{\tau \wedge a \leq a\}$. By the optional sampling theorem:

Let P_0 be the probability measure on (Ω,F) that makes X a Poisson process with rate one, and P another probability measure on (Ω,F) absolutely continuous with respect to P_0. Then there exists a non-negative measurable process $\lambda = \{\lambda_t, t \in R^+\}$ adapted to $\{F_t, t \in R^+\}$ and such that:

$$\int_0^t \lambda_s \, ds < \infty \quad \text{on } \Lambda_t = \left\{ E_0\left\{ \frac{dP}{dP_0} \middle/ F_t \right\} \neq 0 \right\} \tag{34}$$

and:

$$E_0\left\{ \frac{dP}{dP_0} \middle/ F_t \right\} = \prod_{t_i \leq t} \lambda_{t_i} \exp\left\{ -\int_0^t (\lambda_s - 1) \, ds \right\} \text{ on } \Lambda_t \tag{35}$$

where the t_i's are the times at which X has a jump.

We shall need three lemmas.

Lemma 3. Let (Ω,F) be the basic measurable space of point processes and P_0 the probability measure on it which makes the coordinate process X a Poisson process with rate 1. All the martingales of $\mathcal{M}$ have the form

$$M = \left\{ \int_0^t f_s (dX_s - ds), \; t \in R^+ \right\} \tag{36}$$

where $f = \{f_t, t \in R^+\}$ is a measurable process adapted to $\{F_t, t \in R^+\}$ and such that $E \int_0^t f_s^2 \, ds < \infty, \; \forall t \in R^+$.

Proof:

It follows from Appendix A-1.

Indeed:

$$u(X_t) - u(X_0) = \sum_{\substack{s \leq t \\ X_s \neq X_{s-}}} [u(X_s) - u(X_{s-})]$$

$$= \sum_{\substack{s \leq t \\ X_s \neq X_{s-}}} [u(X_s + 1) - u(X_{s-})] \tag{37}$$

Also:

$$\alpha u(X_t) - f(X_t) = u(X_s + 1) - u(X_s) \tag{38}$$

Therefore $X_t^{f,\alpha} = \int_0^t [u(X_{s-} + 1) - u(X_s)][dX_s - ds]$, i.e., (39)

$X^{f,\alpha} \in \mathcal{L}(\{X_t - t,\ t \in R^+\})$. Therefore, since

$\mathcal{M} = \mathcal{L}(\{X^{f,\alpha},\ f \text{ bounded}, \alpha > 0\})$ we have $\mathcal{M} = \mathcal{L}(\{X_t - t,\ t \in R^+\})$. ■

Lemma 4. Let T be an F_t-stopping time and let $\mathcal{M}(T)$ be the set of 0-mean, square integrable martingales with respect to $\{F_{t\wedge T},\ t \quad R^+\}$. $\mathcal{M}(T)$ consists of the martingales of $\mathcal{M}$ stopped at time T.

Proof:

Let $N \in \mathcal{M}(T)$ be orthogonal to all the $(P_0, F_{t\wedge T})$ martingales $M^T = \{M_{t\wedge T},\ t \quad R^+\}$ where $M \quad \mathcal{M}$, M uniformly integrable. This implies:

$$E\{N_t M_{t\wedge T} - N_s M_{s\wedge t}/F_{s\wedge T}\} = 0 \tag{40}$$

and letting $s = 0$:

$$E\{N_t\ M_{t\wedge T}/F_0\} = E\{N_t M_{t\wedge T}\} = 0 \tag{41}$$

But: $E\{N_t M_t\} = E\{E\{N_t M_t/F_{t\ T}\}\} = E\{N_t E\{M_t/F_{t\ T}\}\}$

$$= E\{N_t M_{t\ T}\}$$

Therefore:

$$E\{N_t M_t\} = 0, \quad \forall M \text{ uniformly integrable} \tag{42}$$

Letting $M_t = E\{I_A/F_t\}$ where $A \quad F$, we obtain:

$$N_t = 0 \quad P_0\text{-a.s.} \tag{43}$$

In other words, the set of $(P_0, F_{t\wedge T})$ martingales

$$\{M^T = \{M_{t\wedge T}, t \quad R^+\}, M \quad \mathcal{M}, M\ \mu.\text{i.}\}$$

is dense in $\mathcal{M}(T)$.

Therefore

$$N \quad \mathcal{M}(T) \Rightarrow Y_t = \lim_{n\to\infty} \text{q.m.} \int_0^t f_n(s)\ I(s<T)(dX_s - ds)$$

where

(44)

$$E_0 \int_0^t f_n^2(s)\ I(s < T)\ ds < \infty \tag{45}$$

Therefore

$$N_t = \int_0^t f_s\ I(s < T)(dX_s - ds) \text{ where } E_0 \int_0^t f_s^2\ ds < \infty \quad \blacksquare \tag{46}$$

<u>Lemma 5</u>. Let $\phi = \{\phi_t, t \in R^+\} \in \mathcal{U}_c^{loc}$ and let $g = \{g_t, \{t \in R^+\}$ be measurable. Then:

$M = \left\{ M_t = \exp\left\{ \sum_{\substack{s \le t \\ X_s \ne X_s}} g_s + \phi_t \right\}\right\}$ is a (P_0, F_t) local martingale iff

$$\int_0^t (e^{g_s} - s)\ ds < \infty \tag{47}$$

and

$$-\phi_t = \int_0^t (e^{g_s} - 1)\ ds \tag{48}$$

This is a particular case of Lemma 6.1, pp. 232-233 in Kunita-Watanabe [28].

Proof of the Theorem

Define $L = \left\{L_t,\ t \in R^+\right\}$ by $L_t = E_0\left\{\frac{dP}{dP_0}\Big/F_t\right\}$. L is a (P_0,F_t) martingale, right continuous.

By the optional sampling theorem $L^n = \left\{L_{t\wedge T_n},\ t \in R^+\right\}$ is a $(P_0,F_{t\wedge T_n})$ martingale, and by Jensen's inequality $Z^n = \left\{\log L_{t\wedge T_n},\ t \in R^+\right\}$ is a $(P_0,F_{t\wedge T_n})$ super MG. The last martingale has right continuous paths, is bounded uniformly (by construction of the T_n's) and is regular, since $\left\{F_t,\ t \in R^+\right\}$ has no times of discontinuities (Appendix A-1). Therefore, there exists one and only one Meyer's decomposition

$$Z^n = M^n - A^n \tag{49}$$

where $M^n = \left\{M_t^n,\ t \in R^+\right\}$ is a $(P_0,F_{t\wedge T_n})$ martingale and $A^n = \left\{A_t^n,\ t \in R^+\right\}$ is a natural increasing process with continuous sample paths.

From Lemmas 3 and 4

$$M_t^n = \int_0^t f_n(s)\ I(s < T_n)[dX_s - ds] \quad \text{where } f_n = \{f_n(s), s \in R^+\} \tag{50}$$

is measurable and adapted to $\left\{F_t,\ t \in R^+\right\}$ and:

$$E_0\int_0^t f_n^2(s)\ I(s < T_n)\ ds < \infty \tag{51}$$

By the uniqueness of Meyer's decomposition:

$$Z_{t\wedge T_n} = \int_0^t f_s\ I(s < T_n)[dX_s - ds] - A_{t\wedge T_n} \tag{52}$$

where $f = \{f_s,\ s \in R^+\}$ is a measurable process adapted to $\{F_t, t \in R^+\}$

such that

$$E_0 \int_0^t f_s^2 \, I(s < T_n) \, ds < \infty \quad \forall n \tag{53}$$

and A is a natural increasing process with continuous paths.

Lemma 5 gives the necessary conditions:

$$A_{t\wedge T_n} = \int_0^t [e^{f(s)I(s<T_n)} - 1] \, ds = \int_0^{t\wedge T_n} (e^{f(s)} - 1) < \infty \quad P \text{ a.s.} \tag{54}$$

Defining $\lambda(s) = e^{f(s)}$, and letting n go to ∞, one gets the announced result ($T_n \to \infty$ P-a.s., or more precisely, $T_n \wedge t \to t$ on $\wedge_t$). (See Lemmas 1 and 2.)

We shall now see that $\lambda = \{\lambda_s, s \in R^+\}$ is the <u>intensity process</u> or <u>rate</u> of the point process X. **More precisely:**

<u>Theorem 5-2-ii</u>

If $P \sim P_0$, then

$\{X_t - \int_0^t \lambda_s d_s, \; t \in R^+\}$ is a <u>local</u> (P,F_t) martingale and

$\left\{\int_0^t \lambda_s \, ds, \; t \in R^+\right\}$ is its associated increasing process.

<u>Proof</u>

The proof is the same as in the Markov case that follows (pp.).

<u>Remark</u>: We can say no more. For instance, we could expect that $\{X_t - \int_0^t \lambda_s \, ds, \; t \in R^+\}$ would be a $(P,F_t)L^2$ martingale. It may even happen that:

$$E\, X_t = \infty$$

(and consequently $E \int_0^t \lambda_s ds = \infty$ since for all n:

$$E\, X_{t\wedge T_n} = E\int_0^{t\wedge T_n} \lambda_s ds, \quad \text{and } T_n \uparrow \infty)$$

If $E\int_0^t \lambda_s ds < \infty$, then by the dominated convergence theorem, we obtain:

$$X_t - \int_0^t \lambda_s ds,\ t \in R^+ \quad \text{is a } (P,F_t) \text{ martingale}$$

But even so, we cannot say that it is a $(P,F_t)L^2$ martingale.

We now turn to a theorem which complements Theorem 5-2-i.

6 Girsanov theorem

So far, we have proven the existence of a parameter process such that:

$$\left(X_t - \int_0^t \lambda_s ds,\ t \in R^+ \right) \text{ is a local } (P,F_t)L^2 \text{ martingale.}$$

However, in modeling, one thinks of a parameter process and then says that there exists a point process. In paragraph 2 we have answered to this existence problem in the particular case where λ is uniformly bounded, (Theorems 2-1-i and 2-1-ii).

We shall extend Theorem 2-1-i.

Theorem 6-1

Let (Ω,F,P_0) be a probability space and $X = \{X_t,\ t \in R^+\}$ a right continuous step process, starting at 0, with jumps +s and such that $\{X_t - t,\ t \in R^+\}$ is a $(P_0,F_t)L^2$ martingale for some family of increasing sub-σ-fields of F, $\{F_t,\ t \in R^+\}$. Let $\lambda = \{\lambda_s,\ s \in R^+\}$ be a process adapted to $\{F_t,\ t \in R^+\}$, non-negative and such that $\int_0^t \lambda_s ds < \infty$ P_0 a.s. Then if

$$E_0\left\{\prod_{t_i \leq a} \lambda_{t_i} \exp\left(-\int_0^a (\lambda_s - 1)\, ds\right)\right\} = 1 \qquad (55)$$

for some $a > 0$, the process

$$L = \left\{ L_t = \prod_{t_i \leq t} \lambda_{t_i} \exp\left(-\int_0^t (\lambda_s - 1)\, ds\right),\ t \in [0,a]\right\} \qquad (56)$$

is a martingale.

<u>Proof:</u>

L_t can be rewritten as:

$$L_t = 1 + \sum_{t_i \leq t} \lambda_{t_i} L_{t_i-} + \int_0^t L_{s-}(\lambda_s - 1)\, ds \qquad (57)$$

and the rest is a consequence of Sec. 1.3.

7 Likelihood ratios for Markov chains

Let $X = \{X_t,\ t \in R^+\}$ be a conservative Markov chain defined on (Ω, F, P); that is to say, X is a Markov process taking its values in Z, the set of relative integers, and such that:

$$\sum_{y \in Z} q(x,y) = q(x) \qquad (58)$$

where

$$\begin{cases} q(x) = \lim\limits_{t \to 0} \dfrac{P_t(x,x) - 1}{t} & (59) \\[2ex] q(x,y) = \lim\limits_{t \to 0} \dfrac{P_t(x,y)}{t} & (60) \\[2ex] P_t(x,y) \triangleq P\{X_t = y / X_0 = x\} \triangleq P_x\{X_t = y\} & (61) \end{cases}$$

Using the notations of Appendix A1, we call

$$M = \{X_t, F_t; P_x, x \in z\}$$

a conservative Hunt chain (here $\zeta = \infty$, since the process is conservative).

Theorem

If $P << P_0$ there exists a family of non-negative measurable processes adapted to $\{F_t, t \in R^+\}$:

$$\{\lambda(y) = \{\lambda_s(\omega,y), s \in R^+\}; y \in z\}$$

such that:

1) $\int_0^t \lambda_s(\omega)\, ds < \infty$ P_0-a.s. on $\Lambda_t = \left(E_0\left\{\frac{dP}{dP_0} \Big/ F_t\right\} < \infty\right)$

(where $\lambda_s(\omega) = \sum_{y \in z} \lambda_s(\omega,y)$)

2) $$E_0\left\{\frac{dP}{dP_0} \Big/ F_t\right\} = \prod_{t_i \leq t} \frac{\lambda_{t_i}(\omega, X_{t_i})}{q(X_{t_{i-}}, X_{t_i})} \exp\left\{-\int_0^t [\lambda_s(\omega) - q(X_s)]ds\right\}$$

on Λ_t (62)

Sketch of the proof.

The proof follows the same lines as in the Poisson case:

$$\log L_{t\wedge T_n} = \log E_0\left\{\frac{dP}{dP_0} \quad F_{t\wedge T_n}\right\} = M_{t\wedge T_n} + A_{t\wedge T_n} \tag{63}$$

$$M_{t\wedge T_n} = \sum_{\substack{s \leq t \\ X_s \neq X_{s-}}} f_s^{(n)}(\omega, X_{s-}) - \int_0^t \sum_{y \in z} f_s^{(n)}(\omega,y)\, q(X_s,y)\, ds \tag{64}$$

(see Appendix A1.

By the uniqueness of Meyer's decomposition:

$$M_{t\wedge T_n} = \sum_{\substack{s \leq t\wedge T_n \\ X_s \neq X_{s-}}} f_s(\omega, X_{s-}) - \int_0^t \sum_{y \in z} f_s(\omega,y)\, q(X_s,y)\, ds \tag{65}$$

By the K-W rule and the uniqueness of Meyer's decomposition:

$$A_{t\wedge T_n} = \int_0^{t\wedge T_n} \sum_{y\in z} (e^{f_s(\omega,y)} - 1)\, q(X_s,y)\, ds \tag{66}$$

where $E_0 \int_0^{t\wedge T_n} \sum_{y\in z} (e^{f_s(\omega,y)} - 1)\; q(X_s,y)\, ds < \infty$

Therefore letting $e^{f_s(\omega,y)} q(X_s,y) = \lambda_s(\omega,y)$ we obtain the announced result (since $T_n \to t$ on Λ_t). **End of sketch** ∎

Now, let $X_1(s) = \sum_{\substack{s \le t \\ X_s \ne X_{s-}}} I\{X_s = y\} - \int_0^t \lambda(s,y)\, ds$ **(67)**

and

$$X_2(s) = E_0\left(\frac{dP}{dP_0} \Big/ F_s \right) \tag{68}$$

Then applying the differentiation rule of Doleans-Dade and Meyer **(see [10] or Appendix A1)**

$$\begin{aligned} X_1(s)X_2(s) = &\int_0^t X_2(s-)\, dX_1(s) + \int_0^t X_1(s-)\, dX_2(s) \\ &+ \sum_{X_s \ne X_{s-}} \{X_1(s)\, X_2(s) - X_1(s-)\, X_2(0-) \\ &- X_2(s-)[X_1(s) - X_1(s-)] \\ &- X_1(s-)[X_2(s) - X_2(s-)]\} \end{aligned} \tag{69}$$

but

$$\begin{aligned} \int_0^t X_2(s-)\, dX_1(s) &= \int_0^t X_2(s-)\, d[\sum_{X_s \ne X_{s-}} 1\, (X_s{=}y) - \int_0^t q(X_{s-},y)ds] \\ &+ \int_0^t X_2(s-)[q(X_{s-},y) - \lambda(s,y)]\, ds \\ &= \text{local}\, (P_0,F_t)\ \text{MG} + \int_0^t X_2(s-)[q(X_{s-},y) - \lambda(s,y)]ds \end{aligned}$$

Also since $\left\{E_0\left(\frac{dP}{dP_0}\Big/F_t\right), t \in R^+\right\}$ is a (P_0,F_t) local MG, we have

$$\int_0^t X_1(s-)\, dX_2(s) = (P_0,F_t) \text{ local MG.}$$

On the other hand, the term in $\sum$ in (69) can be rewritten as

$$\sum_{s \le t} [X_1(s-) - X_1(s-)]\,[X_2(s) - X_2(s-)]$$

But:

$$X_1(s) - X_1(s-) = I(X_s = y) \tag{70}$$

$$X_2(s) - X_2(s-) = \left[\frac{\lambda(s,X_s)}{q(X_s,X_s)} - 1\right] X_2(s-) \tag{71}$$

Therefore the term in $\sum$ in (69) has the form:

$$\sum_{s \le t} \dot{1}(X_s = y)\, X_2(s-)\left[\frac{\lambda(s,y)}{q(s,y)} - 1\right]$$

which, combined with the term

$$\int_0^t X_2(s-)[q(X_{s-},y) - \lambda(s,y)]\, ds$$

gives a local martingale.

Therefore:

$$\left\{\sum_{s< t} (X_s = y) - \int_0^t \lambda(s,y)\, ds,\ t \in R^+\right\} \text{ is a } (P,F_t) \text{ \underline{local} MG,}$$

and $\lambda(s,y)$ can be interpreted in the same way as $q(X_s,y)$--i.e., it is the probability that at time s there is a jump to the state y knowing the past up to t.

CHAPTER II

APPLICATION TO COMMUNICATION THEORY

0 The martingale characterization of "<u>semi-good point processes</u>" $\left(\text{i.e., } \left\{ X_t - \int_0^t \lambda_s \, ds,\ t \in R^+ \right\} \right.$ is a (P,F_t) <u>local</u> martingale, see Definition 3-2-ii, Ch. I is used to derive the innovation theorem (Thm. 1-1-i).$\left. \right)$ This theorem is trivial when martingale theory is used and it sheds light on problems of modeling related to self-exciting processes (see Sec. 3). Also it is used, together with Thm. 5, Ch. I and Thm. 6, Ch. I (the Girsanov theorem) to prove the detection formula (Thm. 2) which is analogous to the well-known detection formula for the case of a signal corrupted by white noise. In Sec. 4 the likelihood ratio formula is in turn used to give an expression for the mutual information for point processes on the real line. This result parallels closely the result of Duncan on the mutual information between processes described by white noise stochastic differential equations [13]. In Chapter IV, Sec. 1, we will comment on the close similarity between a signal modulating a point process and a signal corrupted by white noise. Sec. 5 treats the filtering problem for Poisson processes and Markov chains. The method used there parallels that of Zakai [53] (see also Wong [51]) and is different from that of Snyder [43] because it uses the pseudo-density, the advantage of which is seen in Example 1 of Sec. 5. Another advantage of this method is that it uses martingale theory, therefore unifying the theory of the filtering of point processes with the theory of the filtering of signals corrupted by white noise. We will not give the stochastic differential equation for filtering in the case of a Markov

message with density satisfying Fokker Plank equations (see Wong [51], p. 237 in the case of a signal corrupted by white noise). The results are formally the same as those found in [51], and the demonstration would be a mere replica of Zakai's paper [53]. We mention in connection with filtering, the works of Rubin (Markov chains) and Frost (processes with independent increments).

1 Innovation Theorem

Theorem 1.1

Let $\{X_t,\ t \in R^+\}$ be a stochastic process defined on a probability space (Ω,F,P).

Let $\{\lambda_t,\ t \in R^+\}$ be a measurable process on (Ω,F,P) adapted to a family $\{F_t,\ t \in R^+\}$ such that $F_t \supset \sigma(X_s, 0 \leq s \leq t)$; moreover, suppose that

$$\left\{X_t - \int_0^t \lambda_s\, ds,\ t \in R^+\right\} \text{ is a } (P,F_t)\ L^2 \text{ martingale;}$$

then, if $\quad E\left|\int_0^t \lambda_s\, ds\right| < \infty$

$$X_t - \int_0^t E(\lambda_s/\sigma\{X_u, 0 \leq u \leq s\})\, ds \quad \text{is a } (P, \sigma(X_s, 0 \leq s \leq t))$$

martingale.

Proof: Let us use the notation $X_0^t = \sigma\{X_s, 0 \leq s \leq t\}$. Then:

$$E\{X_t - X_s/X_0^s\}$$

$$= E\{E\{X_t - X_s/F_s\}/X_0^s\} \qquad (\text{Since } F_s \supset X_0^s)$$

$$= E\left\{\int_s^t \lambda_u\, du \,/\, X_0^s\right\}$$

$$= \int_s^t E\{\lambda_u/X_0^s\}\, du \qquad (\text{by Fubini since } E\int_0^t \lambda_s ds < \infty\ )$$

$$= \int_s^t E\{E\{\lambda_u/X_0^u\}/X_0^s\}\, du \quad (\text{Since } X_0^u \supset X_0^s \text{ for } u \geq s)$$

$$= E\{\int_s^t E\{\lambda_u/X_0^u\}\, du/X_0^s\} \qquad \blacksquare$$

Example 1: The innovation theorem of Kailath and Frost

In the modeling of a signal $S = \{S_t,\ t \in R^+\}$ corrupted by white noise, one encounters the observation process $X = \{X_t, t \in R^+\}$

satisfying $X_t = \int_0^t S_u\, du + B_t$, where $B = \{B_t, t \in R^+\}$ is a Brownian motion with respect to a family $\{\mathcal{B}_t, t \in R^+\}$ and $\{S_t, t \in R^+\}$ is adapted to a family $\{\mathcal{S}_t, t \in R^+\}$.

We can take the family $\{\mathcal{S}_t, t \in R^+\}$ and $\{\mathcal{B}_t, t \in R^+\}$ to be independent (no feedback case), or we can less restrictively impose that

$$\sigma\{B_u, u \geq t\} \text{ is independent of } F_t = \mathcal{S}_t \vee \mathcal{B}_t \tag{1}$$

In any case, what we have is:

$$X_t - \int_0^t S_u\, du = B_t \tag{2}$$

where $B = \{B_t, t \in R^+\}$ is a Brownian motion w.r. to $\{\mathcal{B}_t, t \in R^+\}$. In other words:

$$\left\{X_t - \int_0^t S_u\, du, t \in R^+\right\} \text{ is a } (P,F_t) \text{ martingale,} \tag{3}$$

square integrable, sample continuous, with associated increasing process $A = \{t, t \in R^+\}$ (From Kunita and Watanabe's characterization theorem [28] this suffices to ensure that, with respect to $\{F_t, t \in R^+\}$ $\left\{X_t - \int_0^t S_u\, du, t \in R^+\right\}$ is a Brownian motion.)

From the innovation theorem we have:

$$\left\{X_t - \int_0^t E(S_u/X_0^u)\, du, t \in R^+\right\} \text{ is a } (P,X_0^t) \text{ martingale.} \tag{4}$$

It is not difficult to show that it is square integrable. Also, by calculating the quadratic variation, we find that the associated increasing process is $A = \{t, t \in R^+\}$. Therefore:

$$\left\{ X_t - \int_0^t E(S_u/X_\sigma^u)\, du,\ T \in R^+ \right\} \text{ is a } (P, X_0^t) \text{ Brownian motion.} \quad (5)$$

See [8, 50] for more elaborated results.

Example 2: Point processes

Here we take $\lambda = \{\lambda_t, t \in R^+\}$ to be the intensity process of a point process $X = \{X_t, t \in R^+\}$ defined on (Ω, F, P). Therefore

$$\left\{ X_t - \int_0^t \lambda_s ds,\ t \in R^+ \right\} \text{ is a } (P, F_t) \text{ local martingale} \quad (6)$$

and by the innovation theorem

$$\left\{ X_t - \int_0^t E(\lambda_s/X_0^s)\, ds,\ t \in R^+ \right\} \text{ is a } (P, X_0^t) \text{ local martingale} \quad (7)$$

▲

2 The Detection Formula for Point Processes

Consider now the last example and suppose, moreover, that Ω contains the space Ω' of the right continuous step functions starting from 0 and with jumps +1; also suppose that X is the coordinate process of Ω' (we call Ω' the basic measurable space of point processes). Let P_0 be a measure on (Ω, F) that makes

$$\{X_t - t, t \in R^+\} \text{ a } (P_0, F_t) \text{ square integrable martingale}$$

(that is to say, under P_0, X is a Poisson process with rate 1); See Thm. 3-1 of Ch. I).

Suppose that P is a probability measure equivalent to P_0. Then $P^x \sim P_0^x$ where P^x and P_0^x are the restrictions of P and P_0 to $\{\Omega', \sigma\{X_s, s \in R^+\}\}$.

P_0^x still makes X a Poisson process with rate 1 since by the innovation theorem:

$$\{X_t - t,\ t \in R^+\} \text{ is a } (P_0^x,\ X_0^t)\ L^2 \text{ martingale.} \tag{9}$$

Also: $$\left\{X_t - \int_0^t E(\lambda_s/X_0^s)\, ds,\ t \in R^+\right\} \text{ is a } (P^x,\ X_0^t) \text{ local MG} \tag{10}$$

But: $$E_0\left(\frac{dP^x}{dP_0^x} \Big/ X_0^t\right) = E_0\left\{\frac{dP}{dP_0}\Big/X_0^t\right\} = \prod_{t_i \leq t} \mu_{t_i} \exp\left\{-\int_0^t (\mu_s - 1)\, ds\right\} \tag{11}$$

for some non-negative process $\mu = \{\mu_t,\ t \in R^+\}$ adapted to X_0^t.
Also from the Girsanov theorem:

$$\left\{X_t - \int_0^t \mu_s\, ds,\ t \in R^+\right\} \text{ is a } (P^x,\ X_0^t) \text{ local martingale.} \tag{12}$$

Therefore by subtracting (10) from (12), we get

$$\left\{\int_0^t [E(\lambda_s/X_0^s) - \mu_s]\ ds,\quad t \in R^+\right\} \text{ is a } (P^x, X_0^t) \text{ local MG.} \tag{13}$$

Now we may invoke the uniqueness of Meyer's decomposition to show that $E(\lambda_s/X_0^s) = \mu_s$.

Indeed, formally:

$$\int_0^t [E\lambda_s/X_0^s) - \mu_s]\, ds + 0 = 0 + \int_0^t [E(\lambda_s/X_0^s) - \mu_s]\, ds \tag{14}$$

that is to say

$$M_t^1 + A_t^1 = M_t^2 + A_t^2 \tag{15}$$

where M stands for "martingale" and A for "process of bounded variation." Therefore:

$$A^1 = 0 = A_t^2 = \int_0^t [E(\lambda_s/X_0^s) - \mu_s]\, ds \quad \text{for all } t. \quad \blacksquare \tag{16}$$

We summarize these results in the following.

<u>Theorem 2</u>

Let P be a measure on a probability space (Ω,F,P) containing Ω', the basic measurable space of point processes. Let X be the coordinate process of Ω'. Suppose that $P \ll P_0$ where P_0 makes X a Poisson process with rate 1 and suppose that under P, X admits $\lambda = \{\lambda_s, s \in R^+\}$ as its intensity process. Then

if $E_0\left\{\int_0^t \lambda_s\, ds\right\} < \infty$ and if we let $\hat{\lambda}_s = E(\lambda_s/X_0^s)$:

$$E_0\left\{\frac{dP^x}{dP_0^x} \Big/ X_0^t\right\} = \prod_{t_i \le t} \hat{\lambda}_{t_i} \exp\left(-\int_0^t (\hat{\lambda}_s - 1)\, ds\right) \text{where}$$

$$\hat{\lambda}_s = E\{\lambda_s/X_0^s\} .$$

<u>Remark</u>: It should be emphasized that this theorem is valid for a class of point processes that contains the doubly stochastic Poisson processes (see [44]).

3 Remark on Self-Exciting Point Processes

We shall give the definition of a self-exciting point process (semi good in the terminology of Def. 32 ii, Ch. I):

A point process is a family of r.v.'s $X = \{X_t,\ t \in R^+\}$. defined on a probability space (Ω, F, P) such that, for a given family $\{F_t,\ t \in R^+\}$ of increasing sub-σ-fields of F and a given measurable non-negative process $\lambda = \{\lambda_t,\ t \in R^+\}$ adapted to $\{F_t,\ t \in R^+\}$ and such that $\int_0^t \lambda_s\, ds < \infty$ p a.s., $\forall t \in R^+$, the following holds:

1) X is P a.s. a right continuous, step function with jumps +1 and such that $X_0 = 0$.

2) $Y = \left\{X_t - \int_0^t \lambda_s ds,\ t \in R^+\right\}$ is a $(P,\ F_t)$ local MG

If $F_t = \sigma\{X_s,\ 0 \leq s \leq t\}$, X is called a self-exciting point process.

In this description in terms of martingales, the family $\{F_t, t \in R^+\}$ is very important: it represents "what you observe" about the past of the process. The innovation theorem gives full meaning to the previous sentence. It says that any point process for which $F_t \supset \sigma(X_s,\ s \in [0,t])$ can be described as a self-exciting process. Physically, going from F_t to $\sigma(X_s,\ s \in [0,t])$ means that we forget about the fine structure of the point process and that we are an external observer just seeing the occurrence of the points and not knowing how they have been generated.

An example

Lewis in [29] has analyzed a model for computer failure patterns. There are primary failures that occur at the rate $\lambda_1(t)$ and give rise to the process $X_1(t)$ such that $\left\{X_1(t) - \int_0^t \lambda_1(s)ds,\ t \in R^+\right\}$ is a $\sigma\{X_1(s),\ 0 \leq s \leq t\}$ martingale. In turn, each of these primary failures generates secondary failures at the rate $g(t)$; that is to say, if at time t_i there is a primary failure, then the probability that there is a secondary failure due to this one between the times t and $t + dt$ where $t \geq t_i$, is $g(t - t_i)dt$. Let us call X_t the total number of failures between 0 and t. Let F_t be the σ-field that summarizes all the information about the failures; i.e., F_t gives the times of occurrences of all the failures and says which ones are primary failures. Then one sees that the rate of $X = \{X_t,\ t \in R^+\}$ is

$$\lambda(t) = \lambda_1(t) + \int_0^t g(t - u)\ dX_1(u) \tag{17}$$

When we say that $\lambda(t)$ is the rate of X_t, we mean with respect to $\{F_t,\ t \in R+\}$, i.e.:

$$\left\{X_t - \int_0^t \lambda(s)\ ds,\ t \in R^+\right\} \text{ is an } F_t\text{-martingale.} \tag{18}$$

If we try to describe the process as self-exciting, the new rate is:

$$\tilde{\lambda}(t) = \lambda_1(t) + \int_0^t g(t - u)\ P(t,u)\ dX(u) \tag{19}$$

where $P(t,u)$ is the probability that the failure that occurred at time u is a primary one, knowing the positions of all the failures between 0 and t. Of course the problem is to determine $P(t,u)$.

<u>Remark on anticipative self-exciting processes.</u>

In the definition of self-exciting point processes, by letting $F_t = \sigma(X_s,\ 0 \le s \le t)$ we allowed the <u>excitation</u> to depend only on the past. We could, however, in principle, think of <u>anticipative self-exciting point processes</u>, i.e., point processes where:

$\sigma\{X_s, s \in R^+\} \supset F_t \supset \sigma\{X_s,\ 0 \le s \le t\}.$

For instance, we may think of a rate $\lambda = \{\lambda_t,\ t \in R^+\}$ such that:

$$\lambda_t = \begin{cases} 1 & \text{if} \quad X_{t+a} - X_t > 0 \\ 2 & \text{if} \quad X_{t+a} - X_t = 0 \end{cases}$$

(Here: $F_t = \sigma\{X_s,\ 0 \le s \le t\} \vee \sigma\{X_{t+a} - X_t = 0\} \subset \sigma\{X_s, 0 < s < t+a\}$)

This rate is bounded, and one could be led to believe that the construction of Sec 2, Ch. I is still valid. However, we have imposed in this construction that

$\{X_t - t,\ t \in R^+\}$ be a (P_0, F_t) square integrable martingale

because we wanted to use the tools of stochastic integration, and $\{X_t - t,\ t \in R^+\}$ is not a $(P_0,\ \sigma\{X_s,\ 0 \le s \le t\} \vee \{\sigma\{X_{t+a} - X_t = 1\}\})$ martingale. **Question: Is there an anticipative self-exciting process which is absolutely continuous with respect to the Standard Poisson process?**

Conjecture: No.

4. Mutually exciting point processes

4.1 Let (Ω,F,P_0) be a probability space and let $Z = \{Z_t, t \in R^+\}$ a two-dimensional Markov process defined on it. Under P_0, we suppose that the 2 component processes $X = \{X_t, t \in R^+\}$ and $Y = \{Y_t, t \in R^+\}$ are independent Poisson processes with rate 1. Let $\{F_t, t \in R^+\}$ be a family of increasing sub-σ-fields of such that $F_t \supset \sigma\{Z_s, 0 \leq s \leq t\}$, $\forall t \in R^+$. Let $\lambda = \{\lambda_s, s \in R^+\}$ and $\mu = \{\mu_s, s \in R^+\}$ be two nonnegative measurable processes adapted to $\{F_t, t \in R^+\}$ with left hand limits and uniformly bounded (by K) Define $L = \{L_t, t \in R^+\}$ by:

$$L_t = \prod_{t_i \leq t} \lambda_{t_{i-}} \prod_{\tau_i \leq t} \mu_{t_{i-}} \exp\{-\int_0^t (\lambda_s + \mu_s - 1) ds\} \tag{21}$$

where the t_i's are the jump times of X and the τ_i's are the jump times of Y.

Theorem 4.1.i: L is a (P_0,F_t) L^2 martingale.

Proof:

$$L_{t_i} - L_{t_{i-}} = (\lambda_{t_{i-}} - 1) L_{t_{i-}} \tag{22}$$

$$L_{\tau_i} - L_{\tau_{i-}} = (\mu_{\tau_{i-}} - 1) L_{\tau_{i-}} \tag{23}$$

Also for $h > 0$ such that:

$$t_i + h < t_{i+1} \wedge \tau_{n_i}$$

where:

$n_i = \inf\{n/\tau_n > t_i\}$ we have:

$$L(t_i+h) = L(t_i) \exp - \int_{t_i}^{t_i+h} (\lambda_s+\mu_s-1)ds \qquad \text{i.e:}$$

$$\frac{dL(t_i+h)}{dh} = -(\lambda(t_i+h) + \mu(t_i+h)-1)\ L(t_i+h) \tag{24}$$

Similarly for $h > 0$ such that:

$$\tau_i + h < \tau_{i+1} \wedge t_{k_i}$$

where: $k_i = \inf\{k/t_h > \tau_i\}$ we have:

$$\frac{dL(\tau_i+h)}{dh} = -\ (\lambda(\tau_i+h) + \mu(\tau_i+h)-1)\ L(\tau_i+h) \tag{25}$$

Therefore:

$$L_t = 1 + \int_0^t L_{s-}(\lambda_{s-}-1)(dX_s-ds) \tag{26}$$

$$+ \int_0^t L_{s-}(\mu_{s-}-1)(dY_s-ds)$$

where the Stieltjes integrals can be understood as stochastic integrals (by the boundedness of λ and μ).

Define P probability measure on (Ω,F) by

$$E_0\left(\frac{dP}{dP_0}\ /F_t\right) = L_t, \quad \forall t \in R^+ \tag{27}$$

<u>Theorem 4.1.ii</u>: The processes:

$$\{X_t - \int_0^t \lambda_s ds, t \in R^+\} \text{ and } \{Y_t - \int_0^t \mu_s ds; t \in R^+\}$$

are (P,F_t) L^2 martingales.

Proof:

Same kind of proof as in 2 of Chapter I.

4.2. Mutual Information Between Two Point Processes

The mutual information in the pair $\{(X_t,Y_t),t \in R^+\}$ defined on a probability space (Ω,F,P_{xy}) is given by

$$I(X,Y) = E\left\{\log \frac{dP_{xy}}{dP_x dP_y}\right\} \tag{28}$$

where P_x abd P_y are the restrictions of P_{xy} to $\sigma(X_t,t \in R^+)$ and $\sigma(Y_t,t \in R^+)$ respectively. By our construction of P_{xy}, P_{xy} is absolutely continuous with respect to P^0_{xy} which makes (x,y) a process with independent Poisson coordinates. If we let P^0_x and P^0_y be the restrictions of P^0_{xy} to $\sigma(X_t,t \in R^+)$ and $\sigma(Y_t,t \in R^+)$ respectively, then $P^0_x << P_x$ and $P^0_y << P_y$ and $P^0_{xy} = P^0_x P^0_y$. Therefore the mutual information can be rewritten as:

$$I(X,Y) = E\left\{\log\left(\frac{dP_{xy}}{dP^0_{xy}} \cdot \frac{dP^0_x}{dP_x} \cdot \frac{dP^0_y}{dP_y}\right)\right\} \tag{29}$$

Actually we shall deal with restrictions of the probability measures considered to the past at time t, that is to say we shall find an expression for:

$$I(X,Y,t) = E\left\{\log E_0\left(\frac{dP_{xy}}{dP^0_{xy}} /F_t\right) - \log E_0\left(\frac{dP_x}{dP^0_x} /F_t\right) - \log E_0\left(\frac{dP_y}{dP^0_y} /F_t\right)\right\} \tag{30}$$

But we have by construction:

$$\log E_0\left(\frac{dP_{xy}}{dP^0_{xy}}/F_t\right) = \sum_{\substack{x_s \neq x_s \\ s \leq t}} \log \lambda_s + \sum_{\substack{y_s \neq y_{s-} \\ s \leq t}} \log \mu_s - \int_0^t (\lambda_s - 1)ds$$

$$- \int_0^t (\mu_s - 1)ds \tag{31}$$

Also by the innovation theorem and the Duncan formula

$$\log E_0\left(\frac{dP_x}{dP^0_x}/F_t\right) = \sum_{\substack{x_s \neq x_{s-} \\ s \leq t}} \log \hat{\lambda}_s - \int_0^t (\hat{\lambda}_s - 1)ds \tag{32}$$

where $\hat{\lambda}_s = E\{\lambda_s/\sigma(X_u, 0 \leq u \leq s)\}$ (33)

and similarly:

$$E_0\left(\frac{dP_y}{dP^0_y}/F_t\right) = \sum_{\substack{y_s \neq y_{s-} \\ s \leq t}} \log \hat{\mu}_s - \int_0^t (\hat{\mu}_s - 1)ds \tag{34}$$

where $\hat{\mu}_s = E\{\mu_s/\sigma(Y_u, 0 \leq u \leq s)\}$ (35)

Therefore:

$$I(X,Y,t) = E\left\{\sum_{\substack{x_s \neq x_{s-} \\ s \leq t}} \log \frac{\lambda_s}{\hat{\lambda}_s} + \sum_{\substack{y_s \neq y_{s-} \\ s \leq t}} \log \frac{\mu_s}{\hat{\mu}_s} - \int_0^t (\lambda_s - \hat{\lambda}_s)ds\right.$$

$$\left. - \int_0^t (\mu_s - \hat{\mu}_s)ds \right\} \tag{36}$$

But on the other hand $\{X_t - \int_0^t \lambda_s ds, t \in R^+\}$ and $\{Y_t - \int_0^t \lambda_s ds, t \in R^+\}$ are square integrable martingales, therefore:

$$E \sum_{\substack{x_s \neq x_{s-} \\ s \leq t}} \log \frac{\lambda_s}{\hat{\lambda}_s} = E \int_0^t \left(\log \frac{\lambda_s}{\hat{\lambda}_s}\right) \lambda_s ds \tag{37}$$

and similarly

$$E \sum_{\substack{y_s \neq y_{s-} \\ s \leq t}} \log \frac{\mu_s}{\hat{\mu}_s} = \int_0^t \left(\log \frac{\mu_s}{\hat{\mu}_s}\right) \mu_s ds \tag{38}$$

$$I(X,Y,t) = E\left\{\int_0^t (\log \frac{\lambda_s}{\hat{\lambda}_s} + \frac{\hat{\lambda}_s}{\lambda_s} - 1)\lambda_s ds + \int_0^t (\log \frac{\mu_s}{\hat{\mu}_s} + \frac{\hat{\mu}_s}{\mu_s} - 1)\mu_s ds\right\} \tag{39}$$

<u>Remark</u>:

We could have calculated in the same manner the information carried by a point process X about an other process y modulating X (i.e. the rate of X is $\lambda_t(Y_t)$). We would have obtained:

$$I(Y/X,t) = E\left\{\int_0^t (\log \frac{\lambda_s}{\hat{\lambda}_s} + \frac{\hat{\lambda}_s}{\lambda_s} - 1)\lambda_s ds\right\} \tag{40}$$

5. Filtering of Point processes

5.1 Doubly Stochastic Point Processes Definition

Let (Ω_2,F_2) be the basic measurable space of point processes and P_2^0 the measure on it which makes the coordinate process X a Poisson process with rate 1. We recall that if we define:

$$L_t = \prod_{\lambda_{ti} \leq t} \lambda_{ti} \exp\{-\int_0^t (\lambda_s-1)ds\} \tag{41}$$

where $\{\lambda_t,t \in R^+\}$ is a deterministic measurable function, it can be rewritten as:

$$L_t = 1 + \int_0^t L_{s-}(\lambda_s-1)(dX_s-ds) \tag{42}$$

From Theorem 6 of Chapter I:

$$E_0 \int_0^t L_{s-}(\lambda_s-1)(dX_s+ds) < \infty \quad \forall t \tag{43}$$

then $L = \{L_t,t \in R^+\}$ is a $(P_2^0,F_{2,t})$ martingale (Note that the condition above is satisfied if λ_t is bounded). Therefore, under condition (43) a measure P_2 on (Ω_2,F_2) can be defined by

$$P_2(A) = \int_A L_t \, dP_2^0 \text{ , } A \in F_{2,t} \tag{44}$$

Let (Ω_1,F_1,P_1) be a probability space and $Y = \{Y_t,t \in R^+\}$ a Markov process defined on it. To each trajectory Y we associate a function:

$$\lambda_t = \lambda(t,Y_t) \tag{45}$$

We suppose that for each of these functions, condition (43) is satisfied. Therefore at each trajectory, we may associate

$P_2(A,Y) = \int_A L_t(Y)\, dP_2^0$ a probability measure on (Ω_2, F_2, t) $P_2(A,Y)$ is for each $A \in F_{2,t}$ a measurable function from $(\Omega_1, F_{1,t})$ to $[0,1]$. Therefore we can define on $(\Omega_1 \times \Omega_2,\ F_{1,t} \otimes F_{2,t})$ a probability measure P defined by:

$$P(A_1 \times A_2) = \int_{A_1} P_2(A_2,\cdot)\, dP_1 \tag{46}$$

Also we can define on $(\Omega_1 \times \Omega_2,\ F_{1,t} \times F_{2,t})$

$$P^0(A_1 \times A_2) = P_1(A_1)\, P_2^0(A_2) \tag{46'}$$

Notations: $\Omega_1 \times \Omega_2 = \Omega,\ F_{1,t} \otimes F_{2,t} = F_t,\ X_0^t = \sigma\{X_s, 0 \leq s \leq t\}$.

Also all these probabilities can be inductively extended to (Ω, F) where $F : \bigvee_{t \in R^+} F_t$.

We call X, considered as a stochastic process on (Ω, F, P), a doubly stochastic point process with Markov rate.* (Note that we

---

*it is not true that $\{\lambda_t(Y_t),\ t \in R^+\}$ is Markovian in general, but we use the term Markov rate for the sake of brevity.

could have in the same way defined a doubly stochastic point process by defining $\lambda_t = \lambda(t,\omega_1)$. The construction is the same; only for the purpose of filtering we need a Markov process)

5.2 The filtering problem

Let X be a point process on (Ω,F,P). Let $g = \{g_t, t \in R^+\}$ be a measurable process adapted to $\{F_t, t \in R^+\}$ and $\{G_t, t \in R^+\}$ a family such that $G_t \subset F_t$, $\forall t \in R^+$.

Filtering g with respect to $\{G_t, t \in R^+\}$ is finding:

$$\hat{g}_t = E\{g_t | G_t\} \quad \forall t \in R^+. \tag{47}$$

In the case where X is a doubly stochastic Point process with Markov rate we say that we filter Y if for every borel function $f : R \to R$ we filter $\{f(Y_t), t \in R^+\}$ with respect to $\{X_0^t, t \in R^+\}$

5.3 The general equation of filtering

As $P_2(\cdot,Y) << P_2^0(\cdot)$ for each Y, $P << P^0$ and

$$L_t = E_0\{\frac{dP}{dP_0} \Big| F_t\} = \prod_{t_i \leq t} \lambda_{t_i}(Y_{t_i}) \exp\{-\int_0^t (\lambda_s(Y_s)-1)ds\}\ . \tag{48}$$

Define $U_t(Y_t, X_0^t) = E_0\{L_t | X_0^t V \sigma(Y_t)\}$, and (49)

$$\mu(dy,t) = P\{Y_t \in dy\} = P^0\{Y_t \in dy\} \tag{50}$$

The following is proven in [51] pp. 234 for instance:

$$E\{f(Y_t)/X_0^t\} = \frac{\int f(y)U_t(y,X_0^t)Pdy,t)}{\int U_t(y,X_0^t)P(dy,t)} \tag{51}$$

The quantity $U_t(y,X_0^t)$ is called the pseudo-density (of Y_t at time t knowing X_0^t).

5.3 The recursive equation for the pseudo density.

We have:

$$L_t = 1 + \int_0^t L_{s-}(\lambda(s,Y_s)-1)(dX_s-ds) \tag{52}$$

Therefore:

$$U_t(y,X_0^t) = 1 + \int_0^t E_0\, L_{s-}(\lambda(s,Y_s)-1)/Y_t = y,X_0^t\} \, (dX_s-ds) \tag{53}$$

Also: $E_0\{L_{s-}(\lambda_s(Y_s)-1)/Y_t = y,X_0^t\} = E_0\{L_{s-}(\lambda_s(Y_s)-1)/Y_t = y,X_0^s\}$ (54)
because under P^0, X is independent of Y (we use here the well known relation $E\{X/G_1 V G_2\} = E\{X/G_1\}$ true if X and G_1 are independent of G_2)

Now:

$$E_0\{L_{s-}(\lambda_s(Y_s)-1)/Y_t = y,X_0^s\} =$$

$$E_0\{E_0\{L_{s-}(\lambda_s(Y_s)-1)/\sigma(Y_t) V\ \sigma(Y_s)\ V\ X_0^s\}/Y_t = y,X_0^s\} =$$

$$E_0\{(\lambda_s(Y_s)-1)E_0\{L_{s-}/\sigma(Y_s)\ V\ X_0^s\}/Y_t = y,\ X_0^s\} \tag{55}$$

(because under P^0, L_s is independent of Y_t given Y_s).

So:

$$E_0\{L_{s-}(\lambda_s(Y_s)-1)/Y_t = y,X_0^s\} =$$

$$E_0\{E_0\{\lambda_s(Y_s)-1/\ (Y_s)\ V\ X_0^s\}/Y_t = y,X_0^s\} =$$

$$\int (\lambda_s(z)-1)U_t(z,X_0^s)P^0(Y_s \in dz/Y_t = y,X_0^s) \tag{56}$$

But $P^0(Y_s \in dz/Y_t = y,X_0^s) = P^0(Y_s \in dz/Y_{\in} = y)$ because under P^0, X and Y are independent. Also Y has the same distribution under P and under P^0 therefore, $P^0(Y_s \in dz/Y_t = y) = P(Y_s \in dz/Y_t = y) \triangleq P(dz,s/y,t)$ and finally:

$$U_t(y,X_0^t) = 1 + \int_0^t \int_R (\lambda_s(z)-1)U_s(z,X_0^s)\ P(dz,s/y,t)(dX_s-ds) \tag{57}$$

Example 1: exponential rate

The idea involved in the solution of this example is due to Wong [50] who solved a similar example for Wiener filtering (example 1, pp. 238-239, [50]). The present example has been studied by Snyder [43], who had to use approximations. It is not necessary, however, and this is an advantage of the use of the pseudo density instead of the density.

Here we want to obtain $\hat{U}_t = E(g(t)Z/X_0^t)$ in the case of a rate $\lambda_t(Z) = \exp(-Zt(t))$ where Z is a r.v. with distribution F. We have:

$$\hat{Z}_t = \frac{g(t)\displaystyle\int_{-\infty}^{+\infty} z\ U_t(z,X_0^t)F(dz)}{\displaystyle\int_{-\infty}^{+\infty} U_t(z,X_0^t)F(dz)} \tag{58}$$

where

$$U_t(z,X_0^t) = \exp\{-Z\int_0^t f(s)dX_s - \int_0^t (\exp(-zf(s))-1)ds\} \tag{59}$$

let

$$h(t,x) = -\frac{1}{v(t,x)} \frac{\partial v(t,x)}{\partial x} \tag{60}$$

where

$$v(t,x) = \int_{-\infty}^{+\infty} [\exp -zx - \int_0^t (\exp(-zf(s))-1)ds]F(dz) \tag{61}$$

Then:

$$\hat{U}_t = g(t)\ h\ (\tilde{X}_t,t) \tag{62}$$

where

$$\tilde{X}_t = \int_0^t f(s)dX_s. \tag{63}$$

Example 2: Approximate filtering

Suppose that the rate is of the form $zf(t)$ where Z is a r.v. with distribution $F(dz)$. We want to estimate $g(Z,t)$. For this we have the formula:

$$g(Z,t) = \frac{\int_{-\infty}^{+\infty} f(z,t)U_t(z,X_0^t)F(dz)}{\int_{-\infty}^{+\infty} U_t(z,X_0^t)F(dz)} \tag{64}$$

where

$$U_t(z,X_0^t) = \exp\{-\int_0^t \log f(s)dX_s - \log Z)\ X_t{}' - \int_0^t (2f(s)-1)ds\} \tag{65}$$

if we let

$$v(t,x) = \int_{-\infty}^{+\infty} \exp\{-\log Z\, x - \int_0^t (zf(s)-1)ds\}F(dz) \tag{67}$$

we have

$$\frac{\partial v^n(t,x)}{\partial x^n} = \int_{-\infty}^{+\infty} (-\log Z)^n \exp\{-(\log Z)x - \int_0^t (zf(s)-1)ds\}F(dz) \tag{68}$$

Therefore, if we can approximate uniformly in the range of Z the function g(Z,t) by a truncated series of powers of $(\log Z)^n$ say:

$$g(t,Z) = \sum a_n(t)\ (\log Z)^n \text{ we have } g(Z,t) = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{\partial v^n(t,X_t)}{\partial x^n}}{v(t,X_t)} \tag{69}$$

and the average error obtained by truncating (49) can be determined exactly.

6. Filtering for Doubly Stochastic Markov Chains

We start from a basic conservative chain with parameters q(x,z) instead of starting with a Poisson process with rate 1. Then we do exactly as in the case of Point process. For instance the rates have the form:

$$\lambda_t \equiv \lambda_t(y,Y_s) \tag{70}$$

and:

$$L_t = \pi \frac{\lambda_{ti}(X_{ti},Y_{ti})}{(X_{ti-},X_{ti})} \exp\{-\int_0^t [\lambda_s(X_{s-},Y_s) - q(X_s)]ds\} \tag{71}$$

where $\lambda_s(x,Y_s) = \sum_z \lambda_s(x,Y_s)$ and $q(x) = \sum_z q(x,z)$ (72)

(also we must have $\int_0 \lambda_s(X_{s-},Y_s)\, ds < \infty$ P_0 .as. for each Y)

Also:

$$L_t = 1 + \int_0^t L_{s-}\left[\frac{\lambda_s(X_s,Y_s)}{q(X_{s-},X_s)} - 1\right]\left[dN_t - q(X_s)\,)ds\right] \tag{73}$$

where N_t = number of jumps of X in [0,t] (Note that for each z,

$\sum_{\substack{x_s \neq x_{s-} \\ s \leq t}} I\{X_s = z\} - \int_0^t q(X_{s-},z)ds$ is a (P_0,F_z) L^2 martingale ; more

generally

$\sum_{\substack{Y_s \neq Y_{s-} \\ s \leq t}} I\{X_s \in A - \int_0^t q(X_{s-},z)ds$ is a (P_0,F_t) L^2 martingale. See

[28] or Appendix [A- 1]). We define $U_t(Y_t,X_0^t)$ as in the Poisson case by:

$U_t(Y_t,X_0^t) = E_0\{L_t/\sigma(Y_t) V X_0^t\}$ and the same calculations yield:

$$U_t(y,X_0^t) = 1 + \int_0^t \int_R U_s(z,X_0^s)\left(\frac{\lambda_s(X_s,z)}{q(X_{s-},X_s)} - 1\right)P(dz,s/y,t)[dN_s - q(X_s)ds] \tag{74}$$

CHAPTER III

TWO APPLICATIONS

0 This chapter is almost independent of the results of the rest of this thesis (only Secs. 1 and 2 of Ch. I need to be read).

We treat an example belonging to the field of Operations Research: the dispatching problem (See Ross [37], [38]); and another example belonging both to the operations research and the communication theory fields: pulse modulation or pulse filtering.

1 A Derivation of a Pulse Modulation Formula

① Let $X = \{X_t, t \in R^+\}$ be a Poisson process with rate λ defined on a probability space (Ω,F,P), and $h(t,s)$ be a given function. Let $Y = \{Y_t, t \in R^+\}$ be defined by

$$Y_t = \sum_{t_i \le t} h(t,t_i) \tag{1}$$

where the t_i's are the jumps of X. We wish to find an expression for

$$\phi_t(u) = E\{\exp(iuY_t)\} \tag{2}$$

Using the same arguments as in Sec. 2, Ch. I, we can show that

$$M_t = \exp\left\{iuY_t - \int_0^t (e^{iuh(t,s)} - 1)\lambda\, ds\right\} \tag{3}$$

is a square integrable martingale since it can be rewritten as

$$M_t = 1 + \int_0^t (e^{iuh(t,s)}-1)(dX_s - \lambda ds) \tag{4}$$

Therefore, as $EM_t = 1$

$$\phi_t(u) = \exp\left(\int_0^t (e^{iuh(t,s)} - 1)\, \lambda ds\right) \tag{5}$$

a very classical formula.

② We now proceed to the more general case:

Let (Ω_1,F_1) be the measurable space of sequences $(q_1,q_2,\ldots)$ i.e., $\Omega_1 \equiv R^\infty$, $F_1 = B^\infty$), together with a probability measure P_1 defined on it such that the coordinate r.v.'s $a_1, \ldots, a_n, \ldots$ are i.i.d with distribution function F.

Let (Ω_2,F_2) be the basic measurable space of point processes on the real line, and $X = \{X_t, t\in R^+\}$ the coordinate process on (Ω_2,F_2). Let P_2 be a probability measure on (Ω_2,F_2) that makes X a Poisson process with rate $\{\lambda(t),\ t \in R^+\}$; in other words, under P_2:

$$\left\{X_t - \int_0^t \lambda(s)\ ds,\ t \in R^+\right\}$$

is a square integrable martingale, and $A = \left\{\int_0^t \lambda(s)ds,\ t \in R^+\right\}$ is its associated increasing process.

Let $F_{2,t} = \sigma\{X_s, 0 \leq 0 \leq t\}$

Define (Ω,F,P) and F_t by:

$$\Omega = \Omega_1 \times \Omega_2$$

$$F = F_1 \times F_2$$

$$P = P_1 \times P_2$$

$$F_t = \Omega_1 \times F_{2,t}\ \text{(i.e., } A \in F_t \text{ iff } A = \Omega_1 \times A_2,\ A_2 \in F_{2,t})$$

Let $h(t,s,u)$ be a real function measurable in (t,s,u) and consider the process $Y = \{Y_t,\ t \in R^+\}$ defined on (Ω,F,P) by:

$$Y_t = \left(\sum_{t_i \leq t} h(t,t_i,a_{X_{t_i}})\right) \tag{6}$$

where the t_i's are the jump points of $X = \{X_t,\ t \in R^+\}$.

Let $\phi_t(u) = E\{\exp(i\,u\,Y_t)\}$.

We have:

$$\exp iuY(t,s) = \sum_{t_i \leq s} \left(e^{iuh(t_i,t,a_{X_{t_i}})} - 1\right) \exp iuY(t,t_{i-}) \tag{7}$$

For a fixed occurrence of the process X, one can integrate with respect to P_1, and noting the independence of the coordinate processes under P_1, one gets:

$$E_1 \exp iu\, Y(t,s) = \sum_{t_i \le s} \left[\int_0^\infty e^{iu\, h(t_i,t,a)} dF(a) - 1 \right] E_1 \exp iuY(t,t_{i-}) \tag{8}$$

which can be rewritten in the formalism

$$E_1 \exp iu\, Y(t,s) = \int_0^s \left[\int_0^\infty e^{iu\, h(s,t,a)} dF(a) - 1 \right] E_1 \exp iu\, Y(t,s)\, dX_s$$

Now, using the fact that under P_2, $\left\{ X_s - \int_0^t \lambda_s ds,\ t \in R^+ \right\}$ is a square integrable martingale:

$$E \exp iu\, Y(t,s) = E_2 E_1 \exp iu\, Y(t,s)$$
$$= \int_0^s \left[\int_0^\infty e^{iu\, h(s,t,a)} dF(a) - 1 \quad E \exp iu\, Y(t,s)\, \lambda(s) ds \right] \tag{9}$$

which is solved in

$$E \exp iu\, Y(t) = \exp \int_0^t \int_0^\infty \left(e^{iu\, h(s,t,a)} - 1 \right) dF(a)\, \lambda(s)\, ds \tag{10}$$

3 The technique used above can be used again in the case where the rate is random, but does not depend either on the past of time s, nor on the parameters $\{a_1, a_2, \ldots\}$. Then one gets

the general formula:

$$\phi_\tau(a) = \exp\left\{\int_0^t \left(e^{iu\ h(t,s,a)} - 1\right) dF(a)\ E_3\ \lambda(s)ds\right\} \tag{11}$$

where E_3 stands for the integration with respect to the space of the parameter process $\lambda = \{\lambda_s,\ s \in R^+\}$

2 The Dispatching Problem

2.1 The Problem.

Items arrive at a plant at a constant rate τ. At time T all the items are to be dispatched. However, at an intermediate time τ (a stopping time), to be chosen, all the items present may be dispatched. The time τ is to be chosen such that the total waiting time is minimized. In other words, if at time τ there are X_τ items present, the dispatching saves $X_\tau(T - \tau)$ units of time; therefore the problem is: find τ such that $E\, X_\tau(T - \tau)$ is minimized. A solution has been given by S. Ross in [38]; this author also studied the case where a constant lag b is allowed between the decision to dispatch and the dispatching itself.

The Generalizations.

If the cost of waiting from time s to time t ($s \leq t$) is ϕ_s^t, a decreasing function of s for t fixed, then the problem becomes

$$\text{minimize } E\, X_\tau \phi_\tau^T \tag{12}$$

One may wish to use a final time T which is a stopping time (for instance T = first time at which there are N objects in the plant); also the rate of arrivals is time-varying and random; also two or more intermediate times can be allowed.

2.2 The Case where T is Fixed

From the integration by parts formula of Appendix [A2]

$$X_t \phi_t^T = \sum_{\substack{s < T \\ X_s \neq X_{s-}}} \phi_{s-}^T + \int_0^t X_s \, d\phi_s^T \tag{13}$$

We shall assume for the sake of simplicity in the answers that ϕ_s^T is differentiable; then:

$$X_t\ \phi_t^T = \sum_{\substack{s \leq T \\ X_s \neq X_{s-}}} \phi_s^T + \int_0^t X_s\ \dot{\phi}_s^T\ ds \quad \text{where } \dot{\phi}_s^T = \frac{d\phi_s^T}{ds} \qquad (13')$$

$\{X_t, t \in R^+\}$ being a Poisson process with rate λ, $\{X_t - \lambda t, t \in R^+\}$ is a square integrable martingale and so is $\left\{\int_0^t \phi_s^T(dX_s - \lambda ds), t \in R^+\right\}$ since $s \to \phi_s^T$ is bounded on $[0,T]$.

The stopping time τ that is looked for is bounded ($\leq T$), therefore:

$$E\int_0^\tau \phi_s^T(dX_s - \lambda ds) = 0 \quad \text{i.e.,} \qquad (14)$$

$$E\int_0^\tau \phi_s^T\ \lambda ds = E \sum_{\substack{s \leq \tau \\ X_s \neq X_{s-}}} \phi_s^T \qquad (14')$$

So:

$$E\ X_\tau \phi_\tau^T = E\int_0^\tau [\phi_s^T \lambda + X_s\ \dot{\phi}_s^T]\ ds \qquad (15)$$

From this we see that the optimal stopping time is a time at which $s \to X_s$ crosses the curve $s \to -\phi_s/\dot{\phi}_s^T\lambda$. We have the more special result:

<u>Theorem</u>:

> If $s \to \log\phi_s^T$ is convex in $[0,T]$, the optimal dispatching time is given by
>
> $$\tau = \inf\left\{t / X_t \geq -\lambda \frac{\phi_s^T}{\dot{\phi}_s^T}\right\} \qquad (16)$$

Proof:

$s \to \log \phi_s^T$ being convex, $s \to \dfrac{\phi_s^T}{\dot{\phi}_s^T}$ is decreasing; it is also negative. Therefore $s \to X_s$ has to cross $s \to -\lambda \dfrac{\phi_s^T}{\dot{\phi}_s^T}$ once and only once at a point τ. Before τ, the integrand $\left(X_s + \lambda \dfrac{\phi_s^T}{\dot{\phi}_s^T} \right) \dot{\phi}_T^s$ is positive, after τ it is negative; hence the optimality of τ. ■

Remarks:

1) In the case studied by S. Ross, $\phi_s^T = (T - s)$; therefore:

$$\tau = \inf\{t/X_t \geq \lambda(T - t)\} \tag{17}$$

2) In the case where there is a time lag a between the dispatching decision and the actual dispatching, the solution is the same, once the following transformation is performed:

$$T \to T - a \;, \quad \phi_s^T \to \phi_{s+a}^T \tag{18}$$

3) Now let X_t be a generalized Poisson process, i.e., let the jump times of X_t occur at a random rate $\{\lambda_t, t \in R^+\} = \lambda$ where λ is a measurable random process adapted to $\{F_t, t \in R^+\}$ such that $\int_0^t \lambda_s \, ds < \infty$ a.s. and

$\left\{X_t - \int_0^t \lambda_s \, ds, \ t \in R^+\right\}$ is a square integrable martingale with increasing process $\left\{\int_0^t \lambda_s \, ds, \ t \in R^+\right\}$. The existence of such a process under general conditions for λ was demonstrated in Ch. I. For this case, the same arguments hold and one gets:

$$E\, X_\tau \phi_\tau^T = E\int_0^\tau [\phi_s^T \lambda_s + X_s \dot{\phi}_s^T]\, ds \tag{19}$$

and the same discussion as in the constant rate case follows. ▲

2.3 The case where T is a stopping time

We suppose that the rate is random for the time being. What is sought is the minimization of

$$E\, X_\tau \phi_\tau^T = E\left[\sum_{\substack{1 \leq \tau \\ X_s \neq X_{s-}}} \phi_s^T - \int_0^\tau X_s \dot{\phi}_s^T \, ds\right] \tag{20}$$

or equivalently:

$$E\, X_\tau \phi_\tau^T = E\left[\sum_{\substack{s \leq \tau \\ X_s \neq X_{s-}}} E\{\phi_s^T/F_s\} - \int_0^\tau X_s \, E(\dot{\phi}_s^T/F_s)\, ds\right] \tag{21}$$

Therefore

$$E\, X_\tau \phi_\tau^T = E\left\{\int_0^\tau [E(\phi_s^T/F_s)\, \lambda_s + X_s \, E(\dot{\phi}_s^T/F_s)]\, ds\right\} \tag{22}$$

Let us now specialize to the case $\lambda_s \equiv \lambda$, $\phi_s^T \equiv T - s$ and $T = \inf\{t/X_t = n\}$.

$$E\{T - s/F_s\} = E\{T - s/X_s\} = \frac{n = X_s}{\lambda}$$

$E\,X\,(T - \tau) = E\int_0 [n - 2X_s]\,ds$. Therefore the optimal stopping time is the first time at which there are more than $\frac{n}{2}$ items in the plant. ▲

2.4 The Case of Two Dispatching Times

We shall modify the problem as follows he arrival process has constant rate λ, the cost function is $\phi_s^t = t - s$, the final time T is fixed, but now we allow the choice of two dispatching times τ_1 and τ_2, Therefore we have to maximize

$$E\{X_{\tau_1}(T - \tau_1) + (X_{\tau_2} - X_{\tau_1})(T - \tau_2)\} \tag{23}$$

One sees that after the first dispatching time τ_1, τ_2 is chosen according to the same rule as in the one dispatching time problem; that is to say:

$$\tau_2 = \inf\{t \geq \tau_1 / X_t - X_{\tau_1} \geq (T - t)\} \tag{24}$$

If we call τ^* the optimal dispatching time in the one dispatching time problem and if we define:

$$f(T) = EX_{\tau^*}(T - \tau^*) \tag{25}$$

Then

$$\begin{aligned}
&E\{X_{\tau_1}(T - \tau_1) + (X_{\tau_2} - X_{\tau_1})(T - \tau_2)\} \\
&= E\{X_{\tau_1}(T - \tau_1)\} + E\{E\{(X_{\tau_2} - X_{\tau_1})(T - \tau_2)/\tau_1\}\} \\
&= E\{X_{\tau_1}(T - \tau_1) + f(T - \tau_1)\} \\
&= E\left\{\int_0^{\tau_1} [\lambda(T - s) - X_s - \dot{f}(T - s)]\,ds\right\} + f(T)
\end{aligned}$$

where $\dot{f}$ is the derivative of f. Therefore

$$\boxed{\tau_1 = \inf\{t / X_t \geq \lambda(T - s) - \dot{f}(T - s)\}}$$

▲

CHAPTER IV

CONCLUSION

0 The analogies between the point processes and signals corrupted by white noise mentioned in paragraph 0 of Chapter II are in paragraph 1. Paragraph 2 consists of historical remarks. Finally paragraph 3 shows that the theory of point processes on the real line may be just an appendix of the theory of martingales (as far as the theory is concerned).

1. <u>Analogies between the Poisson process and the Brownian motion</u>

In this paragraph when we use the notation (Ω,F,P_0) we mean two different things:

1) if we refer to a Poisson process, then (Ω,F) is the basic measurable space of point processes, i.e. Ω is the set of right continuous step functions $X = \{X_t, t \in R^+\}$ with jumps + 1 starting from 0, and $F = \bigvee_{t\in R^+} F_t$ where $F_t = \sigma\{X_s, 0 \leq s \leq t\}$. P_0 is the measure on (Ω,F) that makes X the counting process of a Poisson point process with rate 1. X is also called a Poisson process (with rate 1).

2) if we refer to a Brownian motion, then Ω is the space of continuous functions $X = \{X_t, t \in R^+\}$ starting from 0 and $F = \bigvee_{t\ R^+} F_t$ where $F_t = \sigma\{X_s, 0 \leq s \leq t\}$. P_0 is then the measure that makes X a standard Brownian motion.

We shall give a succession of theorems that show the formal analogy between Poisson processes (P.P.) and Brownian motion (B.M.). A theorem relative to the Brownian motion will be announced as <u>Theorem B.M.1</u> for instance. The corresponding theorem for Poisson process will be called <u>Theorem P.P.1.</u>

The proofs relative to the Theorems P.P. have been given in this work. The Theorems B.M. are standard results available in the literature. One reference is given for each of them which is not necessarily relative to the original article.

First we shall start with the <u>likelihood ratio theorems</u>

<u>Theorem B.M.1</u>

If $P << P_0$, then there exists a measurable process $\phi = \{\phi_t, t \in R^+$ adapted to $\{F_t, t \in R^+\}$ and such that, on the set $\Lambda_t = E_0\{\frac{dP}{dP_0}/F_t\} \neq 0\}$:

a) $\int_0^t \phi_s^2\, ds < \infty \quad P_0$ a.s.

b) $E_0\{\frac{dP}{dP_0}/F_t\} = \exp\{\int_0^t \phi_s d\, X_s - \frac{1}{2}\int_0^t \phi_s^2\, ds\}$

where the integral $\int_0^t \phi_s d\, X_s$ is a stochastic integral defined in probability

<u>Theorem PP1</u>

If $P << P_0$, then there exists a non-negative measurable previsible process $\lambda = \{\lambda_t,\ t \in R^+\}$ adapted to $\{F_t, t \in R^+\}$ and such that, on the set $\Lambda_t = \{E_0\{\frac{dP}{dP_0}/F_t\} \neq 0\}$

a) $\int_0^t \lambda_s ds < \infty \quad P_0$ a.s.

b) $E_0\{\frac{dP}{dP_0}/F_t\} = \prod_{t_i \leq t} \lambda_{ti} \exp\{-\int_0^t (\lambda_s - 1) ds\}$

Now the <u>Girsanov Theorems</u>:

Theorem B.M 2

Let $\phi = \phi\{_t, t \in R^+\}$ be a measurable process adapted to $\{F_t, t \in R^+\}$ and such that $\int_0^t \phi_s^2 ds < \infty$ P_0 a.s. Let $L_t = 1 + \exp\{\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t \phi_s^2 ds\}$ and suppose that $E_0 L_1 = 1$. Then, $\{L_t, t \in [0,1]\}$ is a (P_0, F_t) martingale and if we define P by $dP = L_1\, dP_0$, the process $\{X_t - \int_0^t \phi_s ds, t \in [0,1]\}$ is a Brownian motion with respect to $\{F_t, t \in R^+\}$ and P.

Theorem P.P.2

Let $\lambda = \{\lambda_t, t \in R^+\}$ be a measurable process, nonnegative, adapted $\{F_t, t \in R^+\}$ and such that $\int_0 f_s d_s < \infty$ P_0 a.s. Let $L_t =$

$$\prod_{t_i \leq t} \lambda_{ti} \exp - \int_0^t (\lambda_s - 1) ds$$ and suppose that $E_0 L_1 = 1$.

Then $\{L_t, t \in [0,1]\}$ is a (P_0, F_t) martingale and if we define P by $dP = L_1\, dP_0$, the process $\{X_t - \int_0^t \lambda_s d_s, t \in R^+\}$ is a (P_0, F_t) local martingale.

Remark: in these 2 last theorems, we could take instead of Ω a space Ω' that contains Ω, and instead of Fand of the F_t's, we could take G and G_t's such that $F \supset G$ and $F_t \subset G_t$. The theorems would be more general in the sense that the drift ϕ or the rate λ would not depend only on the past of X.

Now we shall quote the Detection Theorems

Theorem B.M.3

Let (Ω',G) and $\{G_t,t\in R^+\}$ be as in the above remark. Let $P << P_0$. Then, under P, $\{X_t - \int_0^t \phi_s ds, t \in R^+\}$ is a Brownian motion for some ϕ described in B.M.2 (Girsanov) and

$$E_0\{\frac{dP}{dP_0}/F_t\} = \exp\{\int_0^t \hat{\phi}_s dX_s - \frac{1}{2}\int_0^t \hat{\phi}_s^2 ds\}$$

where

$$\hat{\phi}_s = E(\phi_s/F_t)$$

Theorem P.P.3

Ω', G, G_t as in the remark. Let $P << P_0$. Then under P

$\{X_t - \int_0^t \lambda_s ds, t \in R^+\}$ is a (P,G_t) local martingale where λ is described in PP2. If $E\int_0^t \lambda_s ds < \infty$, $\forall t \in R^+$, then:

$$E_0\{\frac{dP}{dP_0}/F_t\} = \prod_{t_i\le t} \hat{\lambda}_{t_i} \exp\{-\int_0^t (\hat{\lambda}_s - 1)ds\}$$

where $\hat{\lambda}_s = E(\lambda_s/F_s)$

Now the Kunita-Watanabe Characterization Theorems

Theorem B.M.4

Let (Ω',G,P) be a probability space, $X = \{X_t,t \in R^+\}$ a measurable process adapted to a family $\{G_t,t \in R^+\}$, sample continuous and such $X_0 = 0$ and $\{X_t,t \in R^+\}$ and $\{X_t^2 - t,t \in R^+\}$ are (P,G_t) local martingales. Then X is a Brownian motion with respect to P and $\{G_t,t \in R^+\}$.

Theorem P.P.4

Let (Ω',G,P) be a probability space, $X = \{X_t, t \in R^+\}$ a measurable step process adapted to a family $\{G_t, t \in R^+\}$, right continuous, such that $X_0 = 0$, $X_t - X_{t-} = 0$ or 1 and $\{X_t - t, t \in R^+\}$ is a (P,F_t) local martingale. Then: X is the counting process of a process with rate 1.

The following representation theorems are due to Wentzel (BM5) and Kunita and Watanabe (PP5 is implicit in theorem of [ ], the proof is in lemma of Chapt. of this work).

Theorem B.M.5

All the (P_0,F_t) L^2 martingales have the form:

$$M_t = M_0 + \int_0^t \phi_s \, dX_s$$

where $\phi = \{\phi_t, t \in R^+\}$ is a measurable process adapted to $\{F_t, t \in R^+\}$ and such that $E_0 \int_0^t \phi_s^2 \, d_s < \infty$.

Theorem P.P.5

All the (P_0,F_t) L^2 martingales have the form:

$$M_t = M_0 + \int_0^t f_s (dX_s - ds)$$

where $f = \{f_t, t \in R^+\}$ is a measurable process adapted to $\{F_t, t \in R^+\}$ and such that $E_0 \int_0^t f_s^2 \, ds < \infty$.

We will only mention the analogy between the filtering for Markov signals corrupted by white noise and the filtering for Markov signals modulating a point process, this analogy is clear at the view of paragraph 5 of Chapt. II. The equations obtained are strikingly similar. This similarity was noted in [ ]; it was said that the rate appears in a nonlinear fashion in the observation process in the case of a point process as opposed to the case of the signal $\{S_t, t \in R^+\}$ corrupted by a white noise where the observation X_t is related to S_t by:

$$X_t = W_t + \int_0^t S_u \, du$$

($W = \{W_t, t \in R^+\}$ is a Brownian motion). Looking at the "equation":

$$X_t - \int_0^t \lambda_u \, du = \text{Martingale}$$

we see that in the case of a point process X modulated by λ, we have the same kind of linearity.

3. Historical remarks

This is, I believe, the first approach of Point process through Martingale Theory. However, the ideas here are in many cases not new: the paragraph "Formal analogies between the Poisson Process and the Brownian motion" should make this point clear.

In the first place, the work of Kunita and Watanabe [28], contains some ideas that are in this thesis: first, I have mentioned that the characterization of Poisson process (see the paragraph: "changing the clock") is due to Watanabe in [43], and the type of proof that I have given is formally due to Kunita and Watanabe in [28], where they were concerned with a characterization of the Brownian motion. Secondly, and most important, is their characterization of positive additive functionals of a Hunt process which are martingales. This may have been a source of inspiration for the literature concerned with likelihood ratios and is certainly the starting point for the likelihood ratio of self exciting point processes. With the remark that the martingales which are functionals of a Poisson process have the form $M_t = M_0 + \int_0^t f_s(dX_s - ds)$, I was able to mimic a proof found in the book by Wong [51] and due to Duncan [12].

Concerning likelihood ratios, I should mention the work of Skorokhod in [40] and [41]. In the first reference this author was concerned with the absolute continuity of two processes with independent increments; in the second reference, he dealt with Markov processes. However no tool of martingale theory was used. The first one to use such tools was Girsanov in 1960 in [18]. This is a fundamental paper for Stochastic control theorists, the importance of which has often been emphasized already.

The innovation theorem has been revisited by Kailath and Frost and these authors attribute the idea to Wold. Kailath and Frost first applied the idea to martingale theory (Brownian motion plus an integrated signal). I have in turn used this idea to prove the Detection Theorem analogous to the Duncan-Kailath detection theorem [14,24]. This theorem was given by Snyder [44] in the case of a doubly stochastic poisson process. However the proof of Snyder does not rest in an obvious manner on the innovation idea. It should be remarked that the innovation theorem could be applied in the same manner to obtain "Detection formulas" in the case of a Markov process absolutely continuous with a Markov chain, and more generally, of a process a.c. with a process with independent increments, (just replace the jump parameters and the drifts by their estimates). Also in reference to the mutual information between point processes one should mention Duncan who gives [13] the mutual information between processes satisfying stochastic differential equations. Concerning the filtering of Poisson processes, the first work is the work of Snyder [43]. Two remarks: first the filtering equation of the present work is obtained by mimicing a proof of the book by Wong [51] and due to Zakai [53]. The use of the pseudo-density (instead of the density as in [43]) allows to get rid of the term $\widehat{\lambda_t(Y_t)}$ in the second member of the equation. Secondly, some questions of existence (of point processes with random rates) have been assumed in [43]. For all other acknowledgement of priorities, concerning the filtering problem, we refer to [42]. The question concerning the problem of modeling in paragraph has been asked to me by Nelson Blachman. The dispatching problem and its solution in the case of a nonrandom rate and a linear cost function is due to S. Ross but the proof here is mine and solves more general cases. The two first pulse modulation formulas have already been proven in Karlin [25] and Takacs [47]. I have not seen a proof in the case of a time varying rate (independently random) although it may exist. What is new here is the trivial proof using martingale theory. The example of the "computer failures" process of paragraph I have heard from P. A. Lewis in a conference at the Dept of Statistics of the U. C. California. Finally some connections between Papangelon's work and the present work have to be mentioned. Ryll - Nardzewsky [54] and Papangelou [35] construct point processes on the real line by defining a probability on the measurable space (Ω,F) defined as follows: Ω is the set of countable sets of points of R (called ω) unbounded both on the right and on the left. F is the smallest σ field on Ω that makes the variables $N(B,\omega)$ measurable for all bounded borelian B of R, where $N(B,\omega)$ is the number of points of ω in B. This method yields point processes that are essentially <u>self-exciting</u> (i.e. the generation of points as time evolves depend on the past of the counting process). On the other hand, there are two directions where Papangelou's results achieve some generality: first he does not deal only with what is called a Palm probability by fixing a point at time 0 (the general study would require minor adaptations in the present thesis); secondly and most important, Papangelou's work is not restricted to processes absolutely continuous with respect to a Poisson process. Also Papangelou [35] is aware of the existence of a martingale relation between the counting process and the rate (remark of Rost, p. of [35]) and of deeper relation of his work with the general theory of martingales as developed by P. A. Meyer.

3. Point Processes and Martingale Theory: General Case

Let us start with a probability space (Ω,F,P), right continuous increasing family $\{F_t, t \in R^+\}$ and such that X is a counting type (i.e. X takes its values in Z^+, starts from 0, is right continuous and has jumps of magnitude +1). Let us note that in this setting $F_t \supset \sigma\{X_s, 0 \leq s \leq t\}$ but that there are no other restrictions on F_t: it could even anticipate on the future of X at time t. Let T_n be the F_t-stopping time defined by:

$$T_n = \inf\{t/X_t = n\} \text{ or } \infty.$$

$\{X_{t\wedge T_n}, t \in R^+\}$ is a right continuous bounded (P,F_t) martingale therefore there is one and only one integable natural increasing process $\{A_t^n, t \in R^+\}$ such that $M^n = A\{X_{t\wedge T_n} - A_t^n, t \in R^+\}$ is a square integrable (P,F_t) martingale. As $M_{t\wedge T_n}^{n+m} = M_t^n$, we can invoke the uniqueness of Meyer's decomposition to prove that on $\{T_n < t\}$,

$A_t^n = A_t^{n+m}$. Therefore, under the condition that X a.s. does not "explode" (i.e. P a.s., $T_n \to \infty$), there exists a natural increasing process $\{A_t, t \in R^+\}$, such that $\{X_t - A_t, t \in R^+\}$ is a (P, F_t) local martingale. Also such a process is unique and it is in that sense that we can say that it characterizes the point process (whose counting process is X) with respect to $\{F_t, t \in R^+\}$ through the relation $\{X_t - A_t, t \in R^+\}$ is a (P, F_t) local martingale. We have to insist on the role of the family $\{F_t, t \in R^+\}$: let us consider a family $\{G_t, t \in R^+\}$ such that $\sigma\{X_s, 0 \leq s \leq t\} \subset G_t \subset F_t$ for all $t \in R^+$. The process $\{E\{A_{t \wedge T_n}/G_t\}, t \in R^+\}$ is still a natural increasing process and $\{X_{t \wedge T_n} - E\{A_{t\, T_n}/G_t\}, t \in R^+\}$ is a (P, G_t) square integrable martingale. Therefore if A characterizes X with respect to $\{F_t, t \in R^+\}$, $B = \{E\{A_t/G_t\}, t \in R^+\}$ characterizes X with respect to $\{G_t, t \in R^+\}$. This result is the general innovation theorem for point processes (see 1 of Chapt. II for the motivation of such a terminology).

We shall call A the generalized integrated rate of X with respect to $\{F_t, t \in R^+\}$.

If moreover X is regular with respect to (P, F_t), i.e. if for any sequence of increasing F_t-stopping times T_n that converge to a F_t - stopping time T, we have $EX_{Tn} \to EX_T$, then (Meyer [ ]), A has almost surely continuous paths. As a trivial counter example to this situation we shall the deterministic process X that jumps of one unit at each integer valued time of R^+; choosing $T_n = k - \frac{1}{n}$ and $T = k$ for k an integer > 0 and using the right continuity of X we see that $EX_{Tn} \equiv k-1$ does not converge to $EX_T = k$. In that case we see that the only martingales are the constant processes and, therefore X is a natural increasing process and $X \equiv A$. At the other end of the spectrum there is the Poisson process with rate 1 which is quasi-left continuous (i.e. if $T_n \uparrow T$ a.s then $X_{T_n} \to X_T$ a.s.) and for which $A_t = t$; in this case A is not only sample continuous, but $A_t = \int_0^t \lambda_s ds$ for some process $\lambda = \{\lambda_s, s \in R^+\}$ adapted to $\{F_t, t \in R^+\}$ (here $\lambda_s \equiv 1$), and also $\{X_t - A_t, t \in R^+\}$ is a (P, F_t) square integrable martingale.

Let us go back to the general case where $\{X_t - A_t, t \in R^+\}$ is a (P, F_t) local martingale. Then from theorem 5, p. 87 of [10], the process $\{X_t - A_t)^2 = A_t, t \in R^+\}$ is a (P, F_t) local martingale if the processes X is regular (i.e. A_t is sample continuous) Such point processes will be called regular or processes with smooth integrated rate.

One question is: in what case is the integrated rate a.c. with respect with the lebesque measure? i.e. in what case can one write $A_t = \int_0^t \lambda_s ds$ for some nonnegative process $\lambda = \{\lambda_s, s \in R^+\}$ called the rate. We know that in the case of processes that are equivalent to R a Poisson process there exists a rate; we also know that the Poisson process is quasi left continuous (i.e. if $T_n \uparrow T$, $X_{T_n} \to X_T$) a consequence of the inaccessibility of the $\sigma\{X_s, 0 \le s \le t\}$ stopping times (i.e. $T_n \uparrow T \Rightarrow$) these exists some random N such that $T_n = T$ for $n \ge N$). Therefore all the point processes equivalent to the Poisson process are quasi left continuous (or have inaccessible stopping times). Does quasi left continuity (or inaccessibility) implies equivalence with the Poisson process?

The above suggests a classification of point processes

1) <u>deterministic</u> ($\Rightarrow X = A$). Does the converse hold, i.e. $X \equiv A \Rightarrow$ the process is deterministic ?

2) processes with smooth integrated rate ($\Longleftrightarrow$ regularity)

3) processes with a rate ($\Longleftrightarrow$ equivalent to Poisson?)

This is certainly an area of future research in Point processes. Also maybe many of the answers are already implicit in the work of P. A. Meyer et al. but exhibiting them would certainly be a contribution to the theory of Point processes on the Real line.

Appendix A.1

Hunt Processes and their Functionals which are Martingale

1. Hunt Processes

① Let E be a locally compact Hausdorff space and $\bar{E}=E\cup\{\partial\}$ its one point compactification. The topological σ-algebra on E (i.e., the σ-algebra generated by the open sets) is denoted by $\mathcal{E}$. $\bar{\mathcal{E}}$ is the topological σ-algebra on $\bar{E}$.

② Let (Ω,F) be a measurable space and let $X:R^+ \times \Omega \to \bar{E}$ be a measurable mapping such that:

1) $X(\cdot,\omega)$ is right continuous and has left-hand limits, for all $\omega \in \Omega$

2) $X(t,\omega) = \partial$ for all $t \geq \zeta(\omega)$ where $\zeta(\omega)$ is the killing time of x defined by $\zeta(\omega) = \inf\{t/X(t,\omega) = \partial\}$

③ Define the shift operator $\sigma_t: \Omega \to \Omega$ by

$$X(s, \sigma_t\omega) = X(s + t,\omega) \qquad \forall s,t \geq 0$$

This operator is well defined and F measurable

④ Let $\mathcal{B}_t$ be the σ-algebra generated by $\{X_s, s \leq t\}$, that is to say, the σ-algebra generated by the sets of the form $\{X(s,\omega) \in A\}$, $A \in \mathcal{E}$, $s \leq t$. Notation: $\mathcal{B}_t = \sigma(X_s, s \leq t)$.

⑤ Let $\{P_x, x \in E\}$ be a family of probability measures on (Ω,F) such that:

(a) $P_x(B)$, $B \in \mathcal{B}_t$ is $\mathcal{E}$-measurable

(b) $P_2(X(0,\omega) = x) = 1$, $\forall x \in \mathcal{E}$.

6 Let $F_\infty \triangleq \underset{\mu}{\cap} \mathcal{B}_\infty^\nu$ where $\mathcal{B}_\infty^\nu$ is the completion of $\mathcal{B}_\infty$ with respect to the measure $\nu = \int_E P_x \, d\mu_x$, μ being a Radon measure on $(E, \mathcal{E})$.

7 Let $F_t = \{B \in F_\infty / \forall_\mu, \exists \; B_\mu \in \mathcal{B}_t$ such that $\mu(B \, \Delta \, B_\mu) = 0\}$

8 A <u>stopping time</u> relative to the increasing family $(F_t, \, t \in \mathbb{R}^+)$ is a $\mathbb{R}^+$-valued random variable τ such that

$$\{\tau \leq t\} \in F_t, \; \forall t \in R^+$$

9 Define F_τ as the set of all the events A in F_∞ such that $A \cap \{\tau \leq t\} \quad F_t, \; \forall t \in R^+$

F_τ is a σ-algebra and τ is F_τ-measurable

10 We shall say that P_x has the <u>strong Markov property</u> iff:

$$E_x\{f \cdot \theta_\tau(\omega)/F_\omega\} = E_{X(\tau)} f(\omega)$$

where f is any bounded random variable and τ any stopping time with respect to $\{F_t, \, t \in R^+\}$.

11 We shall say that X is <u>quasi left continuous</u> with respect to P_x if whenever $\tau_n \uparrow \tau P_x$ a.s. where τ_n and τ are stopping time (relatively to $(F_t, \, t \in \mathbb{R}^+)$) then:

$$X_{\tau_n} \to X_\tau \quad \text{a.s. } P_n.$$

12 The quadruplet $M = (X_t, \zeta, \{F_t, \, t \in \mathbb{R}^+\}, \{P_x, \, x \in E\})$ is called a <u>Hunt process</u> if for each $x \in E$, P_x has the strong Markov property and X is quasi left continuous with respect to P_x.

2. Martingales

1 A functional (of X is a process $Y = \{Y_t(\omega),\ t \geq 0\}$ satisfying the following properties:

(a) Y is adapted to $\{F_t,\ t \in \mathbb{R}^+\}$, i.e., Y_t is F_t-measurable, $\forall t \in R^+$

(b) $Y(\omega)$ is right continuous a.s. P_x, $\forall x \in E$

2 A functional M is a martingale if it satisfies:

(a) $E_x|M_t| < \infty$, $\forall x \in E$, $\forall t \in \mathbb{R}^+$

(b) $E_x(M_t/F_s) = M_t$ a.e P_x, $\forall x \in E$

3 A martingale M is said to be in $\mathcal{M}$ if:

(a) $EM_t = 0 \quad \forall t \in R^+$

(b) $E_x|M_t|^2 < \infty \quad \forall x \in E$, $\forall t \in R^+$

4 A martingale M is said to belong to $\mathcal{M}_c$ (to be in $\mathcal{M}_c$) if it is in $\mathcal{M}$ and is a.s. P_x sample continuous, $\forall x \in E$.

5 $\mathcal{M}^{loc}$ is the set of local martingales, i.e., of martingales X such that one can exhibit a sequence of stopping times T_n satisfying

(a) $T_n \uparrow \infty$ a.s. P_x, $\forall x \in E$

(b) $M^n = \{M_{t\wedge T_n},\ t \in \mathbb{R}^+\} \in \mathcal{M}$, $\forall_n$

$\mathcal{M}_c^{loc} = \{M/M \in \mathcal{M}^{loc};$ M is P_x sample continuous, $\forall x \in E\}$

6 $\mathcal{U}^+$ is the set of natural increasing processes $A_t = \{A_t,\ t \in \mathbb{R}^+\}$ such that $E_x A_t < \infty$, $\forall t \in R^+$, $\forall x \in E$.

$\mathcal{U} = \mathcal{U}^+ - \mathcal{U}^-$

$\mathcal{U}^{+loc}$, $\mathcal{U}^{loc}$, $\mathcal{U}_c^{+loc}$, $\mathcal{U}_c^{loc}$ are defined in an obvious manner.

3. Orthogonality

1 We say that M and N, both in $\mathcal{M}$, are <u>orthogonal</u> if $\{M_t N_t,\ t \in \mathbb{R}^+\}$ is in $\mathcal{M}$. This is equivalent to saying that $\langle M,N \rangle = 0$ where $\langle M,N \rangle = \{\langle M,N\rangle_t,\ t \in R^+\}$ is the (unique) process in $\mathcal{U}$ satisfying:

$$E_x\{(M_t - M_s)(N_t - N_s)/F_s\} = E_x\{\langle M,N\rangle_t - \langle M,N\rangle_s/F_s\},$$
$$\forall x \in E,\ \ \forall_t \geq s.$$

2 $\mathcal{N} \subset \mathcal{M}$ is called a <u>subspace</u> of $\mathcal{M}$ if

(a) $M,N \in \mathcal{N} \Rightarrow M + N \in \mathcal{N}$

(b) $M \in \mathcal{N}$, ϕ satisfies $E_x \int_0^t |\phi_s|^2\, d\langle X\rangle_s < \infty,\ \forall x \in E$
$$\Rightarrow \left\{\int_0^t \phi_s\, dM_s,\ t \in \mathbb{R}^+\right\} \in \mathcal{N}$$

(c) $\mathcal{N}$ is closed for the topology defined by the semi- $\|f\|_{x,t} = E_x f_t^2$ ($\mathcal{M}$ together with this topology is a complete separable metric space if the spaces $L^2(\Omega,F,P_x)$ are separable).

3 Let H be a subset of $\mathcal{M}$. $\mathcal{L}$(H) is by definition the smallest subspace of $\mathcal{M}$ containing H. One can check that if $M \in \mathcal{M}$, $\mathcal{L}(M) =$

$$\left\{\left\{\int_0^t \phi_s dM,\ t \in \mathbb{R}^+\right\} / E_x\int_0^t \phi_s^2 d\langle X\rangle_s < \infty\ \ \forall x \in E,\ \forall t \in \mathbb{R}^+\right\}$$

4 Let $\mathcal{N}$ be a subset of M. $\mathcal{N}^\perp$ is the set of all elements of $\mathcal{M}$ which are orthogonal to each element of $\mathcal{N}$.
$\mathcal{N}^\perp$ is a subspace of $\mathcal{M}$.

By definition $\mathcal{M}_d = \mathcal{M}_c^\perp$. $\mathcal{M}_d$ will be called the <u>subspace of discontinuous martingales</u> of $\mathcal{M}$.

All these results are in Kunita-Watanabé [28].

4. Generation of Martingales.

Let $\{X_t, F_t, \theta_t, P_x\}$ be a Hunt process.

For any $\alpha > 0$ and any borel funcction f which is bounded, let

$$u(x) = G_\alpha(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t)\, dt \quad \text{and define}$$

$$X_t^{f,\alpha} = u(X_t) - u(X_0) - \int_0^t [\alpha u(X_s) - f(X_s)]\, ds$$

Then $X^{f,\alpha} = \{X_t^{f,}, \ t \in R^+\} \in \mathcal{M}$ and

Theorem

$\{X^{f,\alpha} / \text{ f bounded, } \alpha > 0\}$ generates $\mathcal{M}$

Proof in [28] , pp. 226-227.

5. The Levy system of a Hunt process

Let ρ be a metric on E and for each $x \in S$ let

$$U_\varepsilon(x,y) = \begin{cases} 1 \text{ if } \rho(x,y) > \varepsilon \\ 0 \quad \text{otherwise} \end{cases}$$

Define $N_\varepsilon(t,A) = \sum\limits_{\substack{s \leq t \\ X_s \neq X_{s-}}} U_\varepsilon(X_{s-},X_s)\, 1\{X_s \in A\}$

where $A \in \Gamma = \{A \in \xi;\ E_{X_s} N_s(t,A) < \infty \text{ for all } t > 0 \text{ and } x \in E\}$

Then there exists a non-negative continuous additive functional ϕ_t and a kernel $n(x,dy)$ such that

$M_\varepsilon(t,A) = N_\varepsilon(t,A) - \int_0^t \int_A U_\varepsilon(X_s,y)\, n(X_s dy)\, d\phi_s$ is a square integrable martingale for all $A \in \Gamma^\varepsilon$.

Also $\langle M_\varepsilon(t,A), M_{\varepsilon'}(t,A')\rangle = \int_0^t \int_{A\cap A'} U_{\varepsilon\vee\varepsilon'}(X_s, Y) \quad (X_s, dy)\, d\phi_s$

In the case of a conservative Markov chain we have

$$n(x,A) = \sum_{y\in A} q(x,y)$$

where $q(x,y) = \lim\limits_{t\to 0} \dfrac{P_t(X_t = y/X_0 = x)}{t}$

and $\phi_t = t$.

6. <u>The Doleans -Meyer differentiation formula</u>

A process $X = \{X_t,\ t \in R^+\}$ adapted to a family $\{F_t,\ t \in R^+\}$ is called a (P,F_t) <u>semimartingale</u> iff it can be decomposed as

$$X_t = X_0 + M_t + A_t$$

where X_0 is F_0-measurable, $M = \{M_t, t \in R^+\}$ is a (P,F_t) local martingale and $A = \{A_t,\ t \in R^+\}$ is a process of bounded variation. Then let $F\colon R^n \to C$ be twice continuously differentiable and X be a (P,F_t) n-vector semi-martingale (i.e., $X = (X^1, \ldots, X^n)$, and the X^i's are (P,F_t) semimartingales. We have the formula (Doleans-Dade and P. A. Meyer [10]).

$$F(X_t) = F(X_0) + \int_0^t \sum_{i=1}^{n} \frac{\partial}{\partial X_i} F(X_{s-})dX_s^i$$

$$+ \frac{1}{2}\int_0^t \sum_{\substack{i=1 \\ j=1}}^{n} \frac{\partial^2}{\partial x_i \partial x_j} F(X_{s-})\, d\, \langle X^{ic},X^{jc} \rangle_s$$

$$+ \sum_{s\leq t} \left[F(X_s) - F(X_{s-}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(X_{s-})(X_s^i - X_{s-}^i) \right]$$

where $\langle X^{ic},X^{jc} \rangle$ is the associated process of the couple (X^{ic},X^{jc}) where X^{ic} is the continuous local martingale part of the decomposition of X^i.

In connection with this see also the rule of Differentiation of Kunita and Watanabé in [28].

Appendix A.2

Let f(t) and g(t) be two functions of bounded variation, right continuous and with left-hand limit at each point. Then:

$$f(t)g(t) + f(0)g(0) = \int_0^t f(s-)dg(s) + \int_0^t g(s)df(s) \qquad (1)$$

Proof:

$$[f(t) - f(0)][g(t) - g(0)] = \iint_{[0,t]\times[0,t]} df(\mu)dg(v) =$$

$$= \iint_{D_t^-} df(u)dg(v) + \iint_{\overline{D}_t^+} df(\mu)dg(v) \qquad (2)$$

where: $\begin{cases} D_t^- = \{(\mu,v)/u < v \quad \text{and} \quad (u,v) \in [0,t] \times [0,t]\} \\ \overline{D}_t^+ = \text{complement of } D_t^- \text{ in } [0,t] \times [0,t] \end{cases}$

By Fubini $$\iint_{D_t^-} df(u)dg(v) = \int_{[0,t]} \left(\int_{[0,s[} df(u) \right) dg(v)$$

$$= \int_0^t [f(s\_) - f(0)]\, dg(s) \qquad (3)$$

Similarly $$\iint_{\overline{D}_t^+} df(a)dg(v) = \int_{[0,t]} \left(\int_{[0,s]} dg(u) \right) df(v) =$$

$$= \int_0^t [g(t) - g(0)]df(s) \qquad (4)$$

and (1) follows. ∎

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