MODELING THE STEP RESPONSE OF NONLINEAR QUASI-STATIC SYSTEMS

by

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MODELING THE STEP RESPONSE OF NONLINEAR QUASI-STATIC SYSTEMS

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ABSTRACT

A restriction of dynamic systems to what we call "quasi-static" systems is introduced. Two black-box models of the step response of nonlinear quasi-static systems are then proposed. Parameter identification is shown to be a simple matter with specific examples being presented. The first order model is characterized by a relatively simple form and can be made to mimic the system step response exactly.

The n^{th} order model has an orthogonal feature which separates the dynamics from the nonlinearity, thus making it suitable for circuit realization. Extension of the system domain to signals other than steps is also considered. An essential feature of the two models is the relatively large class of nonlinear dynamic systems encompassed by the quasi-static criterion.

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I. INTRODUCTION

The art of model making is as old as man's desire to understand the world around him. It is somewhat paradoxical that this quest is, in many ways, yet in its infancy. Volterra proposed the first and only general nonlinear modeling technique in the form of his functional series expansion [1]. Wiener later enhanced this method making the series orthogonal relative to Gaussian inputs [2]. Yet for most systems this technique is still impractical due to the enormous difficulties encountered in identifying the required number of kernels. In fact the largest number of terms attempted thus far has been three, and there is no guarantee that this will suffice to reasonably approximate the response [3 - 4].

We feel, therefore, that it is worthwhile to be less ambitious in the selection of a domain over which the system is defined. In particular we propose to model nonlinear systems relative to various classes of inputs, the simplest of which are the steps. Within this framework we can construct models which mimic the system response exactly for the particular class of inputs in question.

Prior to the introduction of these black box models, we discuss the assumptions under which we will be working. In particular we limit ourselves to step inputs, and we place certain requirements on the system stability and the system memory. These requirements constitute the "quasi-static" criterion which we introduce in Section II. In Sections III and IV we propose a first order model and an n^{th} order model respectively. Qualitative properties of the models are discussed and specific examples are presented. The problem of expanding the domain to signals other than steps is then discussed in Section V. Section VI is a summary of conclusions.

Throughout this paper we make use of the following notation: $\mathbb{R}^n$ = set of real n-tuples; Z = set of integers; U_+ = non-negative elements of U; $\in$ denotes "an element of"; U x V = set of ordered pair's (u,v), $u \in U$, $v \in V$; $\ni$ denotes "such that"; $\exists$ denotes "there exists"; $\forall$ denotes "for all"; $\Rightarrow$ denotes "implies"; $\Leftrightarrow$ denotes "if and only if"; $\triangleq$ denotes "is defined"; $\circ$ denotes "composed with"; and $\rightarrow$ denotes "tends to."

II. QUASI-STATIC SYSTEMS

A quasi-static system is a system whose dc input/output characteristic is single-valued. More precisely:

Definition 1

A system S with input u and output y is quasi-static $\Leftrightarrow \exists$ a single-valued function F: $\mathbb{R} \to \mathbb{R}$ called the dc input/output characteristic $\ni$

$$u(t) \triangleq \alpha, \qquad (t, \alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.1}$$

$$\Rightarrow \quad y(t) \to F(\alpha) \quad \text{as} \quad t \to \infty \tag{2.2}$$

We emphasize here that the past history of the input:

$$\mathcal{H} \triangleq \{u(t) \mid t \in (-\infty, 0)\} \tag{2.3}$$

is left arbitrary in the above definition.

There are many examples of quasi-static systems. Electronic, mechanical, and hydraulic devices whose dynamics can be characterized as "parasitic effects" are all quasi-static [5 - 9]. Delay lines constitute a less obvious albeit equally valid example of a quasi-static system. Roughly speaking any dynamic system exhibiting an input/output relationship in the form of a "loop" under sinusoidal excitation is quasi-static if that loop "collapses" to a single-valued curve as the tracing frequency tends to zero.[1] Thus a quasi-static system is a system which is "static at dc."

Systems which are not quasi-static include any system with more than one stable equilibrium state. Therefore, hysteretic systems [10] and multi-state logic circuits [11 - 13] are examples of systems which violate the quasi-static criterion. These systems all have the property that their dc input/output characteristics are multi-valued. In Fig. 1 we illustrate this with a quasi-static system (tunnel diode) and a non-quasi-static system (iron-core inductor).

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[1]Our observations of this very effect in various electronic devices such as junction, tunnel, and zener diodes is, in fact, the motivation behind the quasi-static criterion.

Implicit in the quasi-static criterion is the notion of system stability. We can see this more clearly if we restrict ourselves to the realm of differential equations. Consider, therefore, the differential-algebraic system below:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{\hat{f}}(\underset{\sim}{x}, u), \qquad \underset{\sim}{x}(0) = \underset{\sim}{x}_o \tag{2.4}$$

$$y = g(\underset{\sim}{x}, u) \tag{2.5}$$

Here "$\cdot$" denotes differentiation relative to time. We assume $\underset{\sim}{f}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz in $\underset{\sim}{x}$ and continuous in u ; and $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous in both $\underset{\sim}{x}$ and u.

Theorem 1

The system depicted in (2.4) and (2.5) is quasi-static if for every $\alpha \in \mathbb{R}$ the autonomous system:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, \alpha), \qquad \underset{\sim}{x}(0) = \underset{\sim}{x}_o \tag{2.6}$$

has an equilibrium point, $\underset{\sim}{h}(\alpha)$, which exhibits global asymptotic stability [14].

Proof

Let $\underset{\sim}{\phi}(t, \underset{\sim}{x}_o, \alpha)$ denote the solution of (2.6) for $\underset{\sim}{x}_o \in \mathbb{R}^n$ arbitrary. Since $\underset{\sim}{h}(\alpha)$ exhibits global asymptotic stability we have

$$\underset{\sim}{\phi}(t, \underset{\sim}{x}_o, \alpha) \to \underset{\sim}{h}(\alpha) \quad \text{as} \quad t \to \infty \tag{2.7}$$

Then from (2.4) through (2.7)

$$y(t) \to F(\alpha) \triangleq g\circ[\underset{\sim}{h}(\alpha), \alpha] \text{ as } t \to \infty \tag{2.8}$$

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The relationship between system stability and the quasi-static criterion assumes a particularly simple form in the linear case:

Theorem 2

A lumped linear system is quasi-static ⇔ it is strictly stable.[2]

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[2] A linear system is said to be "strictly stable" ⇔ all the poles of the system function lie in the open left half of the complex plane.

Proof

Let H(s) denote the system function. Then

$$Y(s) = H(s)\,U(s) \tag{2.9}$$

where

$$H(s) = \sum_{k=o}^{m} a_k s^k + \sum_{\ell=1}^{n} \frac{r_\ell}{s-p_\ell} \tag{2.10}$$

and

$$U(s) = \frac{\alpha}{s}, \qquad \alpha \in \mathbb{R} \tag{2.11}$$

Combining (2.9) through (2.11) and expanding via partial fractions about the poles of H(s) we get

$$Y(s) = \alpha\left\{\sum_{k=o}^{m} a_k s^{k-1} + \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\left[\frac{1}{s-p_\ell} - \frac{1}{s}\right]\right\} \tag{2.12}$$

Taking inverse Laplace transforms we then have

$$y(t) = \alpha\left\{a_o + \sum_{k=1}^{m} a_k \delta^{(k)}(t) + \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\left[e^{p_\ell t} - 1\right]\right\} \tag{2.13}$$

where $\delta^{(k)}(t)$ is a k^{th}-order impulse. Clearly

$$y(t) \to F(\alpha) \triangleq \alpha\left[a_o - \sum_{\ell=1}^{n} \frac{r_\ell}{p_\ell}\right] \qquad \text{as } t \to \infty \tag{2.14}$$

$\Leftrightarrow$

$$\mathrm{Re}\{p_\ell\} < o \qquad \ell = 1, 2, \ldots, n \tag{2.15}$$

⌑

Next let us investigate the behavior of a quasi-static system under sinusoidal excitation. In particular let

$$u(t) \triangleq \alpha\, \mathrm{Cos}(\omega t), \qquad (t, \alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.16}$$

and let y and y_s denote the complete and steady-state response respectively.

We define the steady-state response as:

$$y_s(t) \triangleq \lim_{\substack{n\to\infty \\ n\in Z_+}} \{y(t + \frac{2\pi n}{\omega})\}, \qquad t \in [0, \frac{2\pi}{\omega}) \tag{2.17}$$

Here we assume that for any periodic input, y_s is unique and periodic of the same fundamental period. Regarding the transient portion of the response we make the following uniform stability assumption: For every $\varepsilon > 0 \;\exists\; \omega_o > 0$ and $T > 0 \ni$

$$\omega \in (0, \omega_o) \Rightarrow |y(t) - y_s(t)| < \varepsilon, \qquad \forall\, t > T \tag{2.18}$$

Finally we assume the system itself is continuous in the following sense: For every $\varepsilon > 0$ and every $t_o > 0 \;\exists\; \delta > 0 \ni$

$$\sup_{t\in[0,t_o]} |u_1(t) - u_2(t)| < \delta \Rightarrow \sup_{t\in[0,t_o]} |y_1(t) - y_2(t)| < \varepsilon \tag{2.19}$$

Consider the closed curve $\Gamma(\omega) \subset \mathbb{R}^2$ defined parametrically as:

$$\Gamma(\omega) \triangleq \{(u(t), y_s(t)) \mid t \in [0, \frac{2\pi}{\omega})\} \tag{2.20}$$

where $u(\cdot)$ is as given in (2.16).

Theorem 3

Let S be a system satisfying the uniform stability and continuity assumptions of (2.18) and (2.19) respectively. Then S is quasi-static if the closed curve $\Gamma(\omega)$ "collapses" to a single-valued curve as $\omega \to 0$.

Proof

From (2.18) and (2.20) we know that $\Gamma(\omega)$ "collapses" as $\omega \to 0 \Leftrightarrow \exists$ a single-valued function $F: \mathbb{R} \to \mathbb{R} \ni$ for every $\varepsilon > 0 \;\exists\; T > 0$ and $\delta_o \in (0, \omega_o)$ satisfying:

$$|y(t) - F\circ[u(t)]| < \frac{\varepsilon}{3}, \qquad \forall\, t > T,\ \omega < \delta \tag{2.21}$$

Now from (2.16) we know that for every $\varepsilon' > 0$ and every fixed $t' > 0 \;\exists$ $\delta' \in (0, \delta_o) \ni$

$$|u(t') - \alpha| < \varepsilon', \qquad \forall\, \omega < \delta' \tag{2.22}$$

But from (2.19) we know that $F(\cdot)$ as defined in (2.21) is continuous. Hence from (2.22) $\exists \quad \varepsilon' > 0 \quad \ni$

$$\left|F\circ[u(t')] - F(\alpha)\right| < \frac{\varepsilon}{3} \qquad \forall\ \omega < \delta' \tag{2.23}$$

Since $t' > 0$ was arbitrary, let $t' > T$. Then applying the triangle inequality to (2.21) and (2.23) we get

$$\left|y(t') - F(\alpha)\right| < \frac{2}{3}\varepsilon, \quad \forall\ \omega < \delta' \tag{2.24}$$

Next let

$$\tilde{u}(t) \triangleq \alpha, \quad (t, \alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.25}$$

and let $\tilde{y}(\cdot)$ denote the corresponding response. From (2.16), (2.19), and (2.25) we know $\exists \quad \delta \in (0, \delta') \qquad \ni$

$$\left|\tilde{y}(t') - y(t')\right| < \frac{\varepsilon}{3} \qquad \forall\ \omega < \delta \tag{2.26}$$

Again applying the triangle inequality this time to (2.24) and (2.26) we get

$$\left|\tilde{y}(t') - F(\alpha)\right| < \varepsilon \qquad \forall\ \omega < \delta \tag{2.27}$$

But all the terms of the above inequality are independent of ω and $t' > T$ was arbitrary. Hence

$$\tilde{y}(t) \to F(\alpha) \qquad \text{as} \qquad t \to \infty \tag{2.28}$$

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The single-valued curve which $\Gamma(\omega)$ "collapses" to as $\omega \to 0$ is, of course, the dc input/output characteristic of S. The hysteresis loop shown in Fig. 1(b) is an example of a closed curve $\Gamma(\omega)$ which fails to "collapse" at dc [10].

Note that the link between system stability and the quasi-static criterion is again evident in Theorem 3. In this case it manifests itself in the form of assumption (2.18). It is perhaps tempting at this point to view the quasi-static criterion as essentially a stability statement. This is not entirely accurate. In insisting that a system be quasi-static we also place a requirement on the system memory. To see this more clearly let us classify systems as to the extent to which the system response depends upon the past history of the input.

We say that a (causal) system is <u>static</u> if the response is independent of the past input. Static systems are also referred to as "memoryless" systems. Clearly a static system is a special case of a quasi-static system.

Now the response of a quasi-static system will, in general, depend upon the entire past of the input. However, it is clear from Definition 1 that only the recent past has a significant effect. We, therefore, refer to a quasi-static system as being amnesic in so much as it appears to "forget" the contribution of the remote portion of the past input.

Systems which are not quasi-static, on the other hand, can exhibit a "permanent memory" capability. In fact any system with a multi-valued dc characteristic has the capacity to function as an information storage device. The most obvious example of this is the flip-flop.

The relationship between the system memory and the quasi-static criterion is perhaps best evidenced in the case of the delay line. Any delay line of finite delay is quasi-static.

We are now in a position to pose the modeling problem formally. Let S denote a nonlinear quasi-static system with input u and output y. We restrict our consideration of inputs to the following class of signals:

$$\mathcal{U} \triangleq \{u(t) = \alpha \mid (t, \alpha) \in \mathbb{R}_+ \times I\} \tag{2.29}$$

Here I denotes any compact nonempty interval in $\mathbb{R}$.

Our assumption on the past history of the input is that S be in a relaxed state that $t = 0$. Thus we assume:

$$u(t) = 0, \qquad t \in [-2T, 0) \tag{2.30}$$

where $T > 0$ is taken sufficiently large $\ni$

$$\dot{y}(t) \approx 0, \qquad t \in [-T, 0) \tag{2.31}$$

Since S is quasi-static we know $\exists$ a $T > 0$ $\ni$ (2.31) is satisfied.

We denote the relaxed state response as $y(t, u)$ where we let the dependence of y on u surface formally in the form of a second argument. To distinguish between the measured system response measured and the response predicted by the model we let $y_e(\cdot, u)$ denote the empirical waveform. Finally to avoid cumbersome mathematical formulation we abuse standard notation slightly interchanging u and α as the latter argument of the response.

We assume that $y_e(t, u)$ is continuous in both t and u with continuity of the latter argument being defined as in (2.19). Finally we assume S is both causal and time-invariant. The system is shown diagrammatically in Fig. 2. Here we have taken the output space to be L_∞, the set of all $g: \mathbb{R}_+ \to \mathbb{R}$ which are bounded in the following sense:

$$\|g\|_\infty \triangleq \sup_{t \in \mathbb{R}_+} |g(t)| < \infty \tag{2.32}$$

The problem, then, is to construct a black-box model of the system S restricted to the domain $\mathcal{U}$.

We now propose two canonic forms for such a model. Each of these forms will be burdened by an additional assumption regarding the system response, y_e. Although a theoretic necessity this assumption, in each case, will be of little practical significance.

III. A FIRST ORDER CANONIC MODEL

As a first order quasi-static model of S on $\mathcal{U}$ we propose the following algebraic-differential system:

$$y = f(x, u) \tag{3.1}$$

where $x(\cdot)$ is the solution of:

$$\dot{x} = f(x, u) + k\,, \quad x(0) = 0 \tag{3.2}$$

Here $k \in \mathbb{R}$ and $f: \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}$. A block diagram of the models is shown in Fig. 3. We see that the dynamics of the first order model manifest themselves in the form of an implicit state variable x.

A. Parameter Identification

The constant k and the function $f(\cdot, \cdot)$ are identified with the system S as follows. Let

$$k \triangleq \varepsilon - \inf_{(t,u) \in \mathbb{R}_+ \times \mathcal{U}} \{y_e(t,u)\} \tag{3.3}$$

Here ε is any positive constant. For convenience we often take $\varepsilon = 1$. Note that k is always well-defined since S is continuous and quasi-static and I (of $\mathcal{U}$) is compact.

Next we introduce the function $\phi: \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}_+$ defined as follows:

$$\phi(t, u) \triangleq \int_0^t [y_e(\tau, u) + k]\, d\tau \tag{3.4}$$

Now (3.3) and (3.4) $\Rightarrow$

$$\dot{\phi}(t, u) \geq \varepsilon > 0, \qquad (t, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.5}$$

Let

$$x \triangleq \phi(t, u) \tag{3.6}$$

Then (3.5) and the global Implicit function theorem [15]

$$\Rightarrow \exists \quad \xi: \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}_+ \quad \ni \quad (x, u) \in \mathbb{R}_+ \times \mathcal{U}$$

$$t = \xi(x, u), \tag{3.7}$$

As the function f: $\mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}$ we then take the following composition:

$$f(x, u) \triangleq y_e \circ [\xi(x, u), u] \tag{3.8}$$

B. Qualitative Properties

We can attribute the following properties to the first-order quasi-static model.

Property 1 (Existance and Uniqueness)

If $\exists \quad M > 0 \quad \ni$

$$|\dot{y}_e(t, u)| \leq M \qquad \forall \quad (t, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.9}$$

then $\forall \quad u \in \mathcal{U}$, the first-order model has a unique solution.

Proof

From (3.7) and (3.8) we have

$$\left|\frac{\partial}{\partial x} f(x, u)\right| \leq |\dot{y}_e(t,u)| \; \left|\frac{\partial}{\partial x} \xi(x, u)\right| \tag{3.10}$$

But (3.5) through (3.7) $\Rightarrow$

$$\left|\frac{\partial}{\partial x} \xi(x, u)\right| \leq \frac{1}{\varepsilon} < \infty, \qquad \forall (x, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.11}$$

Hence (3.9) through (3.11) $\Rightarrow$

$$\left|\frac{\partial}{\partial x} f(x, u)\right| \leq \frac{M}{\varepsilon} \qquad \forall \quad (x, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.12}$$

Thus $\forall \quad x_1, x_2 \in \mathbb{R}_+$, we can apply the mean-value theorem to get:

$$|f(x_1, u) - f(x_2, u)| \leq \frac{M}{\varepsilon} |x_1 - x_2| \qquad \forall \quad u \in \mathcal{U} \tag{3.13}$$

Hence $\frac{M}{\varepsilon}$ is a Lipschitz constant for $f(\cdot, \cdot)$. Therefore, a unique solution exists. ⌑

Remark

We will here-after assume that (3.9) is satisfied. Since "Mother Nature integrates" this will clearly be the case for most practical systems.

Property 2 (Step Response)

$$u \in \mathcal{U} \Rightarrow y(t, u) \equiv y_e(t, u) \tag{3.14}$$

Proof

From (3.4) we have

$$\dot{\phi}(t, u) = y_e(t, u) + k \qquad \forall \ (t, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.15}$$

and

$$\phi(0, u) = 0 \qquad \forall \ u \in \mathcal{U} \tag{3.16}$$

But (3.6) through (3.8) $\Rightarrow$

$$f\circ[\phi(t, u), u] = y_e(t, u) \qquad \forall \ (t, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{3.17}$$

Hence (3.15) through (3.17) and Property 1 $\Rightarrow$ $\phi(t, u)$ is the unique solution of (3.2).

Then (3.1) and (3.17) $\Rightarrow$

$$y(t, u) \equiv y_e(t, u) \qquad \forall \ u \in \mathcal{U} \tag{3.18}$$

Property 3 (Integral Form)

The first order model can be represented by the following integral equation:

$$y(t, u) = f\circ \left[\int_0^t [y(\tau, u) + k] \, d\tau, \ u \right] \tag{3.19}$$

Proof

This follows directly from (3.1) and (3.2) ¤

Property 4 (Static Case)

If S is memoryless, then the first order model gives rise to the exact system.

Proof

Since S is static, we have

$$y_e(t, u) \equiv F\circ[u(t)], \qquad u \in L_\infty \tag{3.20}$$

Then (3.20) and Property 2 $\Rightarrow$

$$y(t, u) \equiv F\circ[u(t)] , \qquad u \in L_\infty \tag{3.21}$$

$\square$

C. <u>An Example</u>

As an example consider the nonlinear differential system below:

$$\dot{y}_e = -u[y_e + 1], \quad y_e(0) = 0 \tag{3.22}$$

This example was chosen since it gives rise to a closed form solution. Namely for

$$u(t) \triangleq \alpha, \qquad (t, \alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{3.23}$$

we have

$$y_e(t, \alpha) = e^{-\alpha t} - 1 \tag{3.24}$$

Now from (3.3) and (3.24) we have

$$k = 1 + \varepsilon , \qquad \varepsilon > o \tag{3.25}$$

Referring to (3.4) we then have

$$\phi(t, \alpha) = \varepsilon t + \frac{1-e^{-\alpha t}}{\alpha} \tag{3.26}$$

In order to obtain an explicit analytical expression for the implicit function ξ of (3.7) we let $\varepsilon \to 0$. Then

$$t \to \xi(x, \alpha) \triangleq \frac{-1}{\alpha} \ell n(1-\alpha x) \tag{3.27}$$

Combining (3.8), (3.24) and (3.27) we then get:

$$f(x, \alpha) = -\alpha x \tag{3.28}$$

Thus the resulting model is:

$$\dot{x} = - ux + 1, \quad x(0) = 0 \tag{3.29}$$

$$y = - ux \tag{3.30}$$

To verify that this model will indeed mimic the system step response exactly for any amplitude α, we solve (3.29) with $u \equiv \alpha$ to obtain:

$$x(t, \alpha) = \frac{1-e^{-\alpha t}}{\alpha} \tag{3.31}$$

Substituting $x(t,\alpha)$ into (3.30) then yields:

$$y(t, \alpha) = e^{-\alpha t} - 1 \tag{3.32}$$

Recalling (3.24) we thus have

$$y(t, \alpha) \equiv y_e(t, \alpha) , \qquad \alpha \in \mathbb{R} \tag{3.33}$$

IV. AN NTH ORDER CANONIC MODEL

As an n^{th} order quasi-static model of S on $\mathcal{U}$ we propose the following functional sum:

$$y = F(u) + \sum_{k=1}^{n} h_k * \phi_k(u) \tag{4.1}$$

Here "*" denotes convolution, $F: \mathbb{R} \to \mathbb{R}$ is the dc input/output characteristic, $\underset{\sim}{\phi}: \mathbb{R} \to \mathbb{R}^n$ is a static nonlinearity, and

$$H_k(s) = \frac{s}{s + k} , \qquad k = 1, 2, \ldots, n \tag{4.2}$$

where $H_k(s)$ is the Laplace transform of $h_k(t)$.

A block diagram of the model is shown in Fig. 4. We see that the n^{th} order model is a parallel decomposition of n linear dynamic systems $h_k(\cdot)$ each preceded by a static nonlinearity $\phi_k(\cdot)$ together with a static subsystem $F(\cdot)$.

A. Parameter Identification

The integer "n" and the static nonlinearity $\underset{\sim}{\phi}: R \to R^n$ are identified with the system S as follows.

Let

$$\tilde{y}_e(t, u) \triangleq y_e(t, u) - F(u) , \qquad (t, u) \in \mathbb{R}_+ \times \mathcal{U} \tag{4.3}$$

Since $F(\cdot)$ is the dc characteristic, we know that if S is static then

$$\tilde{y}_e(t, u) \equiv o , \qquad \forall \; u \in \mathcal{U} \tag{4.4}$$

Hence we regard $\tilde{y}_e$ as the system response minus its <u>memoryless component</u>. We can also view (4.3) as a signal-dependent change of coordinates since $F(\cdot)$ is an instantanious transformation.

Consider the sequence of negative exponentials:

$$\mathcal{B} \triangleq \left\{ e^{-kt} \right\}_{k=1}^{\infty} \tag{4.5}$$

Now $\mathcal{B}$ constitutes a basis for L_2, the space of all $g: \mathbb{R}_+ \to \mathbb{R}$ which are bounded in the following sense [16]:

$$\|g\|_2 \triangleq \left[\int_0^{\infty} g^2(t)dt \right]^{1/2} < \infty \tag{4.6}$$

We can associate an inner product with this linear space; namely,

$$<g, h>_2 \triangleq \int_0^{\infty} g(t)\, h(t)dt, \quad g, h \in L_2 \tag{4.7}$$

Suppose we now orthonormalize the negative exponentials in L_2.
Let

$$\mathcal{E} \triangleq \left\{ e_k(t) \right\}_{k=1}^{\infty} \tag{4.8}$$

denote the resulting sequence. We then get

$$E_k(s) = \sqrt{2k}\, \frac{(s-1)(s-2)\ \ldots\ (s-k+1)}{(s+1)(s+2)\ \ldots\ (s+k)}\ , \quad k=1, 2, \ldots \tag{4.9}$$

where $E_k(s)$ denotes the Laplace transform of $e_k(t)$ [17].

Applying the inverse Laplace transformation to $E_k(s)$ we have

$$e_k(t) = \sum_{\ell=1}^{k} \gamma_{k\ell}\, e^{-\ell t} \tag{4.10}$$

where the residue vector $\underset{\sim}{\gamma}_k \in \mathbb{R}^k$ is given by:

$$\gamma_{k\ell} \triangleq \lim_{s \to -\ell} \{(s+\ell)\, E_k(s)\} \qquad \begin{array}{l} k = 1, 2, \ldots, n \\ \ell = 1, 2, \ldots, k \end{array} \tag{4.11}$$

As the static nonlinearity $\underset{\sim}{\phi}: \mathbb{R} \to \mathbb{R}^n$ we then take

$$\phi_k(u) \triangleq \sum_{\ell=k}^{n} \gamma_{\ell k} <\tilde{y}_e(\cdot, u), e_\ell>_2 \tag{4.12}$$

B. <u>Qualitative Properties</u>

We can attribute the following properties to the n^{th} order quasi-static model.

Property 1 (Step Response)

If $y_e(\cdot, u) \in L_2 \quad \forall\, u \in \mathcal{U}$, then for every $\varepsilon > o$ and every $u \in \mathcal{U}$, $\exists \quad N > 0 \quad \ni$

$$\|y(\cdot, u) - y_e(\cdot, u)\|_2 < \varepsilon \qquad \forall \quad n > N \tag{4.13}$$

Proof

Let $\varepsilon > 0$, $u \in \mathcal{U}$ be arbitrary. Now $\tilde{y}_e(\cdot, u) \in L_2 \Rightarrow$ we can represent $\tilde{y}_e(\cdot, u)$ in terms of a generalized Fourier series expansion [16] relative to the orthonormal basis $\mathcal{E}$ of (4.8). Hence

$$\tilde{y}_e(t, u) = \sum_{k=1}^{\infty} \beta_k(u)\ e_k(t) \tag{4.14}$$

where

$$\beta_k(u) \triangleq \langle y_e(\cdot, u), e_k \rangle_2 \tag{4.15}$$

is the projection of $\tilde{y}_e(\cdot, u)$ onto the k^{th} basis function, e_k.

Let

$$\tilde{y}_{en}(t, u) \triangleq \sum_{k=1}^{n} \beta_k(u)\ e_k(t) \tag{4.16}$$

From (4.14) and (4.16) we know $\exists \quad N > 0 \quad \ni$

$$\|\tilde{y}_e(\cdot, u) - \tilde{y}_{en}(\cdot, u)\|_2 < \varepsilon \qquad \forall \quad n > N \tag{4.17}$$

Now (4.10) and (4.16) $\Rightarrow$

$$\tilde{y}_{en}(t, u) = \sum_{k=1}^{n} \sum_{\ell=1}^{k} \beta_k(u)\ \gamma_{k\ell}\ e^{-\ell t} \tag{4.18}$$

Next consider the response of the model. Since $u \in \mathcal{U}$ we can write

$$\phi_k \circ [u(t)] = \phi_k(\alpha) 1(t) \tag{4.19}$$

where $1(\cdot)$ denotes the unit step function.

Since the convolution transformation is linear we can commute h_k

and $\phi_k(\alpha)$ in (4.1) to get

$$y(t, \alpha) = F(\alpha) + \sum_{k=1}^{n} \phi_k(\alpha)\; h_k(t) * 1(t) \tag{4.20}$$

But from (4.2) we have

$$h_k(t) * 1(t) = e^{-kt} \qquad\qquad k = 1, 2, \ldots \tag{4.21}$$

Thus

$$y(t, \alpha) = F(\alpha) + \sum_{k=1}^{n} \phi_k(\alpha) e^{-kt} \tag{4.22}$$

Comparing (4.12), (4.15) and (4.22) we then have

$$y(t, \alpha) = F(\alpha) + \sum_{k=1}^{n} \sum_{\ell=k}^{n} \beta_\ell(\alpha)\gamma_{\ell k} e^{-kt} \tag{4.23}$$

Rearranging the order of summation and replacing α by u we then have

$$y(t, u) = F(u) + \sum_{k=1}^{n} \sum_{\ell=1}^{k} \beta_k(u)\gamma_{k\ell} e^{-\ell t} \tag{4.24}$$

Comparing (4.18) and (4.24) then yields

$$y(t, u) = F(u) + \tilde{y}_{en}(t, u) \tag{4.25}$$

Combining (4.3), (4.17) and (4.25) we finally get

$$\|y(\cdot, u) - y_e(\cdot, u)\|_2 < \varepsilon \qquad \forall \quad n > N \tag{4.26}$$

$\square$

Remark

We will hereafter assume that the condition of Property 1 is satisfied; namely,

$$\tilde{y}(\cdot, u) \in L_2 \qquad\qquad \forall \quad u \in \mathcal{U} \tag{4.27}$$

Since S is quasi-static we note from Definition 1 and (4.3) that

$$\tilde{y}(t, u) \to 0 \text{ as } t \to \infty \qquad\qquad \forall \qquad u \in \mathcal{U} \tag{4.28}$$

Hence it is clear that for most practical systems (4.28) is indeed satisfied.

In particular any system which is exponentially stable falls into this category.

Property 2 (Measuring $\tilde{\phi}$)

For each $u \in \mathcal{U}$, the point $\tilde{\phi}(u) \in \mathbb{R}^n$ can be measured directly.

Proof

From (4.2) we know that $H_k(s)$ can be realized as the voltage transfer function of the RC 2-port shown in Fig. 5a.

Furthermore, it is clear from (4.21) that e^{-kt} is the step response of this 2-port.

Referring to (4.10) we then see that the k^{th} basis function, $e_k(\cdot)$, is the step response of the network N_k shown in Fig. 5b.

Finally consider the network shown in Fig. 6. We have

$$\eta_k(t, u) = \int_0^t \tilde{y}(\tau, u)\left[\sum_{\ell=k}^{n} \gamma_{\ell k}\, e_\ell(\tau)\right] d\tau \tag{4.29}$$

or

$$\eta_k(t, u) = \sum_{\ell=k}^{n} \gamma_{\ell k} \int_0^t \tilde{y}(\tau, u)\, e_\ell(\tau)d\tau \tag{4.30}$$

But from (4.7) and (4.12) we have

$$\phi_k(u) = \sum_{\ell=k}^{n} \int_0^\infty \tilde{y}(\tau, u)\, e_\ell(\tau)d\tau \tag{4.31}$$

Comparing (4.30) and (4.31) it is clear that

$$\eta_k(t, u) \to \phi_k(u) \qquad \text{as} \qquad t \to \infty \tag{4.32}$$

¤

The synthesis of an n^{th} order system generally requires at least n dynamic elements. The following realization is minimal in the sense that exactly n (linear) capacitors are required.

Property 3 (Realization)

The n^{th} order model can be realized with n linear resistors,

n linear capacitors, $n+1$ nonlinear resistors[3] and an $(n+1)$-terminal summer.

Proof

From Fig. 5a it is clear that $H_k(s)$, $k=1, 2, \ldots, n$ can be realized with n linear resistors and n linear capacitors. The static nonlinearities $F(\cdot)$ and $\phi_k(\cdot)$, $k=1, 2, \ldots, n$ can then be realized with $n+1$ nonlinear resistors.

Finally an $n+1$ terminal summing junction is needed to combine the outputs of $F(\cdot)$ and $H_k(s)$, $k=1, 2, \ldots, n$.

¤

Property 4 (Orthogonality)

Let $\hat{\tilde{\phi}}: \mathbb{R} \to \mathbb{R}^n$ be arbitrary and let $\hat{y}(\cdot, \cdot)$ denote the response of the n^{th} order model for $\tilde{\phi} \equiv \hat{\tilde{\phi}}$. Then $\forall \quad u \in \mathcal{U}$

$$\|y(\cdot, u) - y_e(\cdot, u)\|_2 \leq \|\hat{y}(\cdot, n) - y_e(\cdot, u)\|_2 \tag{4.33}$$

Proof

This follows directly from the orthogonality of $\mathcal{E}$ in L_2. Thus for "n" fixed, $\tilde{\phi}$ as given in (4.12) (and measured in Property 2) is the optimal choice for a static nonlinearity in the mean square sense.

¤

Property 5 (Static Case)

In the event that S is memoryless, the n^{th} order model converges to the exact system for $n = 0$.

Proof

Since S is static, we have

$$y_e(t, u) \equiv F(u) \qquad \forall \quad u \in \mathcal{U} \tag{4.34}$$

Thus (4.3) and (4.34) $\Rightarrow$

$$\tilde{y}_e(t, u) \equiv 0 \qquad \forall \quad u \in \mathcal{U} \tag{4.35}$$

[3]The nonlinear resistor is used here in a generic sense to mean a memoryless nonlinearity. In practice buffers are needed to minimize loading effects.

Hence (4.12) $\Rightarrow$

$$\phi_k(u) \equiv 0 \qquad \forall \quad u \in \mathcal{U}, \; k=1, 2, \ldots \tag{4.36}$$

Finally (4.22) $\Rightarrow$

$$y(t, u) \equiv F(u) \equiv y_e(t, u) \qquad \forall \quad u \in \mathcal{U} \tag{4.37}$$

$\square$

C. An Example

As an example, consider the following nonlinear differential system:

$$\dot{y}_e = -3y_e + 3(y_e)^{2/3}u, \quad y_e(0) = 0 \tag{4.38}$$

We have selected this particular system as an example since an analytical expression for the solution can be easily obtained; namely,

$$y_e(t, u) = \left[\int_0^t e^{-(t-\tau)}u(\tau)d\tau \right]^3 \tag{4.39}$$

Now for

$$u(t) \triangleq \alpha, \qquad (t, \alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{4.40}$$

$y_e(t, u)$ reduces to

$$y_e(t, \alpha) = \alpha^3[1 - 3e^{-t} + 3e^{-2t} - e^{-3t}] \tag{4.41}$$

Hence the dc characteristic $F(\cdot)$ is

$$y_e(t, \alpha) \to F(\alpha) \triangleq \alpha^3 \qquad \text{as} \qquad t \to \infty \tag{4.42}$$

Thus from (4.3) we have

$$\tilde{y}_e(t, \alpha) = -\alpha^3[3e^{-t} - 3e^{-2t} + e^{-3t}] \tag{4.43}$$

Clearly

$$\tilde{y}_e(\cdot, \alpha) \in L_2 \qquad \forall \quad \alpha \in \mathbb{R} \tag{4.44}$$

At this point we would normally compute the static nonlinearity $\tilde{\phi}: \mathbb{R} \to \mathbb{R}^n$ for successively increasing values of n. However, we can refer to the analytic expression for $\tilde{y}_e$ in this case and note that $\tilde{y}_e(t, \alpha)$ does not contain a power of α beyond the third. Hence let us try n = 3.

If we let $\beta_k(\alpha)$ be defined as in (4.15) then from (4.7), (4.10) and

(4.43) we have

$$\beta_k(\alpha) = -\alpha^3 \sum_{\ell=1}^{k} \gamma_{k\ell} \frac{(\ell^2 + b\ell + 11)}{(\ell+1)(\ell+2)(\ell+3)} \tag{4.45}$$

Computing the numerical values of $\gamma_{k\ell}$ from (4.11) and substituting them in to (4.45) we then get

$$\beta_1(\alpha) = \frac{-3\sqrt{2}}{4}\ \alpha^3,\ \beta_2(\alpha) = \frac{3\alpha^3}{10}\ ,\ \beta_3(\alpha) = \frac{\sqrt{6}\ \alpha^3}{60} \tag{4.46}$$

Then from (4.12) we have

$$\underset{\sim}{\phi}(\alpha) = [-3\alpha^3,\ 3\alpha^3,\ -\alpha^3]^T \in \mathbb{R}^3 \tag{4.47}$$

Hence our model is given by:

$$y(t,u) = [u(t)]^3 - \int_0^t \{[3e^{-(t-\tau)} - 3e^{-2(t-\tau)} + e^{-3(t-\tau)}] [u(\tau)]^3\}d\tau \tag{4.48}$$

To check the model's validity we set $u \equiv \alpha$ in (4.48) to get

$$y(t,\alpha) = \alpha^3[1 - 3e^{-t} + 3e^{-2t} - e^{-3t}] \tag{4.49}$$

Comparing (4.43) and (4.49) we thus have:

$$y(t,\alpha) \equiv y_e(t,\alpha), \qquad \alpha \in \mathbb{R} \tag{4.50}$$

Note that since the n^{th} order model is orthogonal, any choice of $n \geq 3$ would have given rise to the above result; that is,

$$\phi_k(\alpha) \equiv 0 \qquad \forall \quad k > 3 \tag{4.51}$$

V. DISCUSSION

We have proposed two canonic forms for a model of the nonlinear quasi-static system S restricted to the domain $\mathcal{U}$. In order for the first order model to have a unique solution we require that the time rate of change of the system response be bounded. Alternatively, for the n^{th} order model to converge we must insist that the system response minus its memoryless component be Lebesque integrable.

These two assumptions together with those of continuity, causality, and to a lesser extent time-invariance are not overly restrictive in the practical realm. Perhaps the most severe assumption, apart from the quasi-static requirement itself, is our restriction of the domain to step inputs. The relationship between a model and the domain over which that model remains valid is indeed a fundamental one. Newton's model relating force, mass, and acceleration was so immensely successful that it soon became known as a "law". Yet even this model has a valid domain, the boundary of which was first proposed by Einstein [18].

Now the black-box models we have proposed are, by construction, valid on the domain $\mathcal{U}$. Let us now attempt to extend the boundaries of this domain until we see the integrity of our models begin to falter.

Due to the lack of a practical identification theorem for nonlinear systems no direct statement can be put forth regarding the performance of our models for signals other than steps. Until the class of "not necessarily linear" systems is narrowed, it is not likely that such a result will be found.

With the above limitations in mind suppose we proceed qualitatively. Let us drive the system with a sinusoidal excitation. Now both models are exact at dc. Hence invoking continuity arguments we would expect that there exists a band of frequencies about dc for which the models are "close to exact."

To demonstrate this phenomenon consider the nonlinear differential system below:

$$\dot{y}_e = -3y_e + 3(y_e)^{2/3}u, \qquad y_e(o) = o \tag{5.1}$$

This example was previously discussed in Section IV for $u \in \mathcal{U}$. In this case suppose

$$u(t) \triangleq \alpha \operatorname{Cos}(\omega t), \quad (t,\alpha) \in \mathbb{R}_+ \times \mathbb{R} \tag{5.2}$$

From (4.39) we find the system response in this case to be

$$y_e(t,u) = \alpha^3 \left[\frac{\operatorname{Cos}(\omega t) + \omega \operatorname{Sin}(\omega t) - e^{-t}}{1 + \omega^2} \right]^3 \tag{5.3}$$

If we let y_{es} denote the steady-state component, then

$$y_{es}(t, u) = \alpha^3 \left[\frac{\text{Cos}^3(\omega t) + 3\omega \,\text{Cos}^2(\omega t)\, \text{Sin}(\omega t)}{1 + 3\omega^2} \right.$$

$$\left. \frac{+3\omega^2\, \text{Cos}(\omega t)\, \text{Sin}^2(\omega t) + \omega^3\, \text{Sin}^3(\omega t)}{+ 3\omega^4 + \omega^6} \right] \tag{5.4}$$

Next consider the response of the model. From (4.50) we have

$$y(t, u) = \alpha^3 \left[\text{Cos}^3(\omega t) - \frac{1}{4} L^{-1} \left\{ \left(\frac{3s}{s+1} - \frac{3s}{s+2} + \frac{s}{s+3} \right) \right. \right.$$

$$\left. \left. \left(\frac{3s}{s^2 + \omega^2} + \frac{s}{s^2 + 9\omega^2} \right) \right\} \right] \tag{5.5}$$

Taking inverse Laplace transforms and letting y_s denote the steady-state response we get:

$$y_s(t, u) = \alpha^3 \left[\text{Cos}^3(\omega t) - \frac{3\omega^2}{4} \left\{ \left[\frac{4}{1+\omega^2} - \frac{4}{4+\omega^2} + \frac{1}{9+\omega^2} \right] \text{Cos}(\omega t) \right. \right.$$

$$- \left[\frac{4}{1+\omega^2} - \frac{8}{4+\omega^2} + \frac{9}{9+\omega^2} \right] \text{Sin}(\omega t)$$

$$+ \left[\frac{9}{1+\ell\omega^2} - \frac{9}{4+\omega^2} - \frac{1}{3+3\omega^2} \right] \text{Cos}(3\omega t)$$

$$\left. \left. - \left[\frac{18}{1+9\omega^2} - \frac{27}{4+9\omega^2} + \frac{1}{1+\omega^2} \right] \text{Sin}(3\omega t) \right\} \right] \tag{5.6}$$

Upon comparing (5.4) and (5.6) and letting $\omega \to 0$ we get

$$y_{es}(t, u) \to [\alpha\, \text{Cos}(\omega t)]^3 \qquad \text{as} \qquad \omega \to 0 \tag{5.7}$$

and

$$y_s(t, u) \to [\alpha\, \text{Cos}(\omega t)]^3 \qquad \text{as} \qquad \omega \to 0 \tag{5.8}$$

Referring to (4.42), (5.7) and (5.8) we thus see that both the system and the model "collapse" to the dc characteristic $F(\cdot)$, as $\omega \to 0$.

This is of course, no surprise in view of Theorem 3.

VI. CONCLUSIONS

The notion of a quasi-static system has been introduced. It was shown that the quasi-static criterion involves requirements on both the system stability and the system memory. Two canonic forms were then proposed for models of nonlinear quasi-static systems. In each case the domain of the model was restricted to step inputs and the system was assumed to be continuous causal and time-invariant. The first order model was burdened by the additional assumption that the time rate of change of the system response be bounded, while the n^{th} order model had the requirement that the system response minus it's memoryless component be Lebesque integrable.

The principal virtues of the first order model are its relatively simple form and its ability to mimic the system step response exactly. It's limitations include the need to identify a function of two variables. This poses a practical interpolation problem. Furthermore it confines the first order model to a mathematical realization, at least until hardware becomes available for generating functions of two variables.

The n^{th} order model, on the other hand, can be realized directly with circuit elements. Although the step response of this model is never, in general, exact; the orthogonality provides a means for controlling the overall accuracy. In addition, the parameters of the n^{th} order model are all directly measurable. This is in contrast to the first order model which is characterized by an implicit function. Finally, all the nonlinearities of the n^{th} order model are memoryless. This is analogous to Weiner's nonlinear model which separates the dynamics from nonlinearity [2].

Collectively the principal feature of the two models can be found in the assumptions upon which they are based. The class of nonlinear quasi-static systems is indeed a large one. Perhaps the most limiting assumption in the practical realm is our restriction of the domain to step inputs. In the absence of superposition, an extension of this domain to other signals is not possible. The one claim that can be made in this regard is a qualitative one; namely, the models should perform reasonably well for smooth periodic inputs of a sufficiently low frequency. This follows from continuity arguments and the fact that both models are exact at dc.

REFERENCES

1. V. Volterra, Theory of Functionals and of Integral and Integro-Differential Equations. New York: Dover Publications, Inc., 1959.
2. N. Wiener, Nonlinear Problems in Random Theory. New York, Wiley, 1958.
3. Y. W. Lee and M. Schetzen, "Measurement of the Wiener Kernels of a Non-linear System by Cross-Correlation," International Journal of Control, Vol. 2, No. 3, pp. 237 - 254, September 1965.
4. A. Sandberg and L. Stark, "Wiener G-Function Analysis as an Approach to Nonlinear Characteristics of Human Light Reflex," Brain Research, Vol. 11, pp. 194 - 211, 1968.
5. L. O. Chua, Introduction to Nonlinear Network Theory. New York: McGraw-Hill, 1969.
6. H. E. Koenig and W. A. Blackwell, Electromechanical System Theory. New York: McGraw-Hill, 1961.
7. H. M. Paynter, Analysis and Design of Engineering Systems. Mass.: M.I.T. Press, 1961.
8. A. G. J. MacFarlane, Engineering Systems Analysis. Mass: Addison-Wesley, 1964.
9. R. E. Keller, Statics and Dynamics of Components and Systems. New York: Wiley, 1971.
10. L. O. Chua and S. C. Bass, "A Generalized Hysteresis Model," IEEE Trans. Circuit Theory, Vol. CT-19, pp. 36 - 48, January, 1972.
11. L. O. Hill, D. O. Pederson, and R. S. Pepper, "Synthesis of Electronic Bistable Circuits," IEEE Trans. Circuit Theory, Vol. CT-10, pp. 25 - 35, March 1963.

12. L. O. Chua, "Synthesis of Nonlinear Systems with Prescribed Singularities," IEEE Trans. Circuit Theory, Vol. CT-18, pp. 375 - 382, May 1971.

13. S. Kumagai and S. Kawamoto, "Multistable Circuits Using Nonlinear Reactances," IEEE Trans. Circuit Theory, Vol. CT-7, pp. 432 - 439, December 1960.

14. N. N. Krasovskii, Stability of Motion, California: Stanford University Press, 1963.

15. E. S. Kuh and I. Hajj, "Nonlinear Circuit Theory: Resistive Networks," Proc. IEEE, Vol. 59, pp. 340 - 355, March 1971.

16. G. Szego, Orthogonal Polynomials, Rhode Island: American Math. Society, Vol. XXIII, 3rd Edition, 1967.

17. W. H. Kautz, "Transient Synthesis in the Time Domain," IRE Trans. Circuit Theory, Vol. CT-1, pp. 29 - 39, Sept. 1954.

18. A. Einstein, The Meaning of Relativity, 5th Edition, New Jersey: Princeton University Press, 1955.

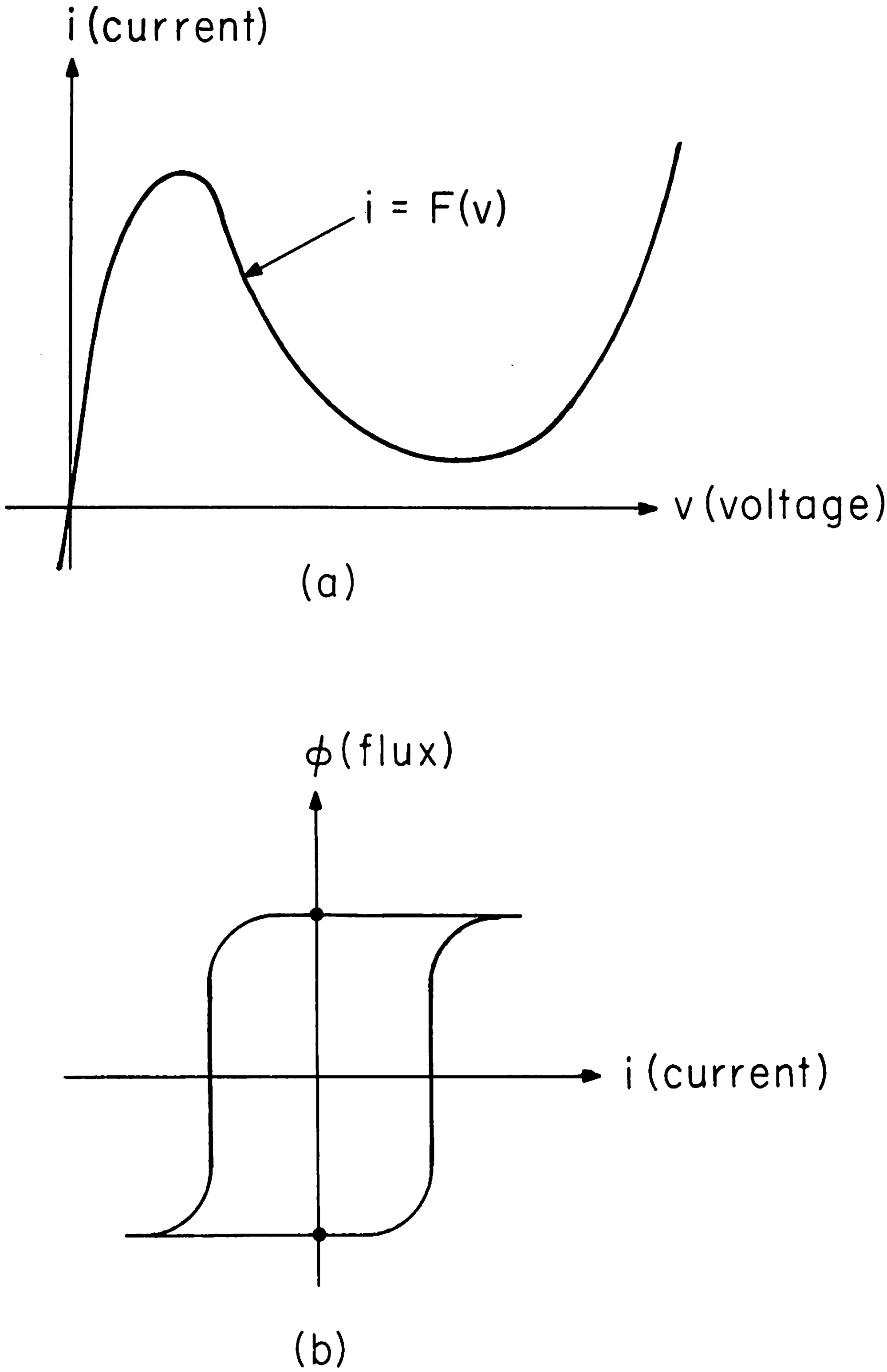

Fig. 1. DC Characteristics of Tunnel Diode (a) and Iron-Core Inductor (b).

Fig. 1

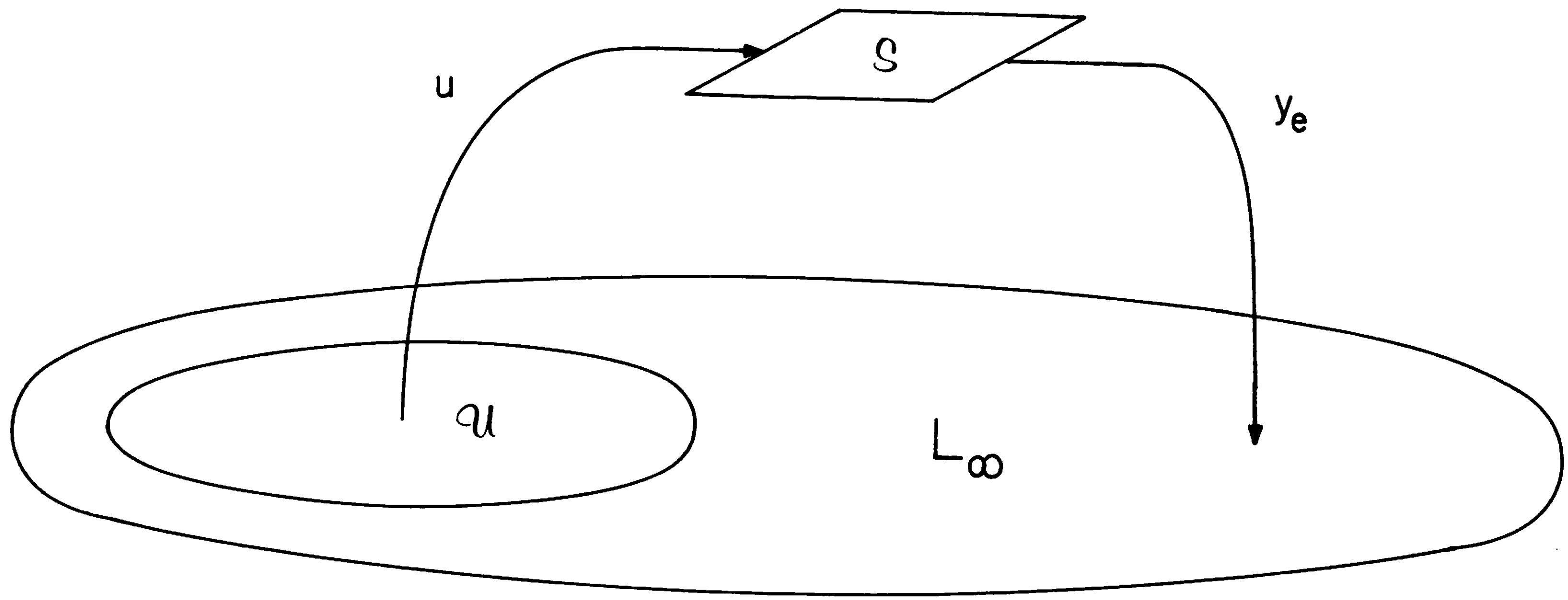

Fig. 2. The System Under Consideration

Fig. 2

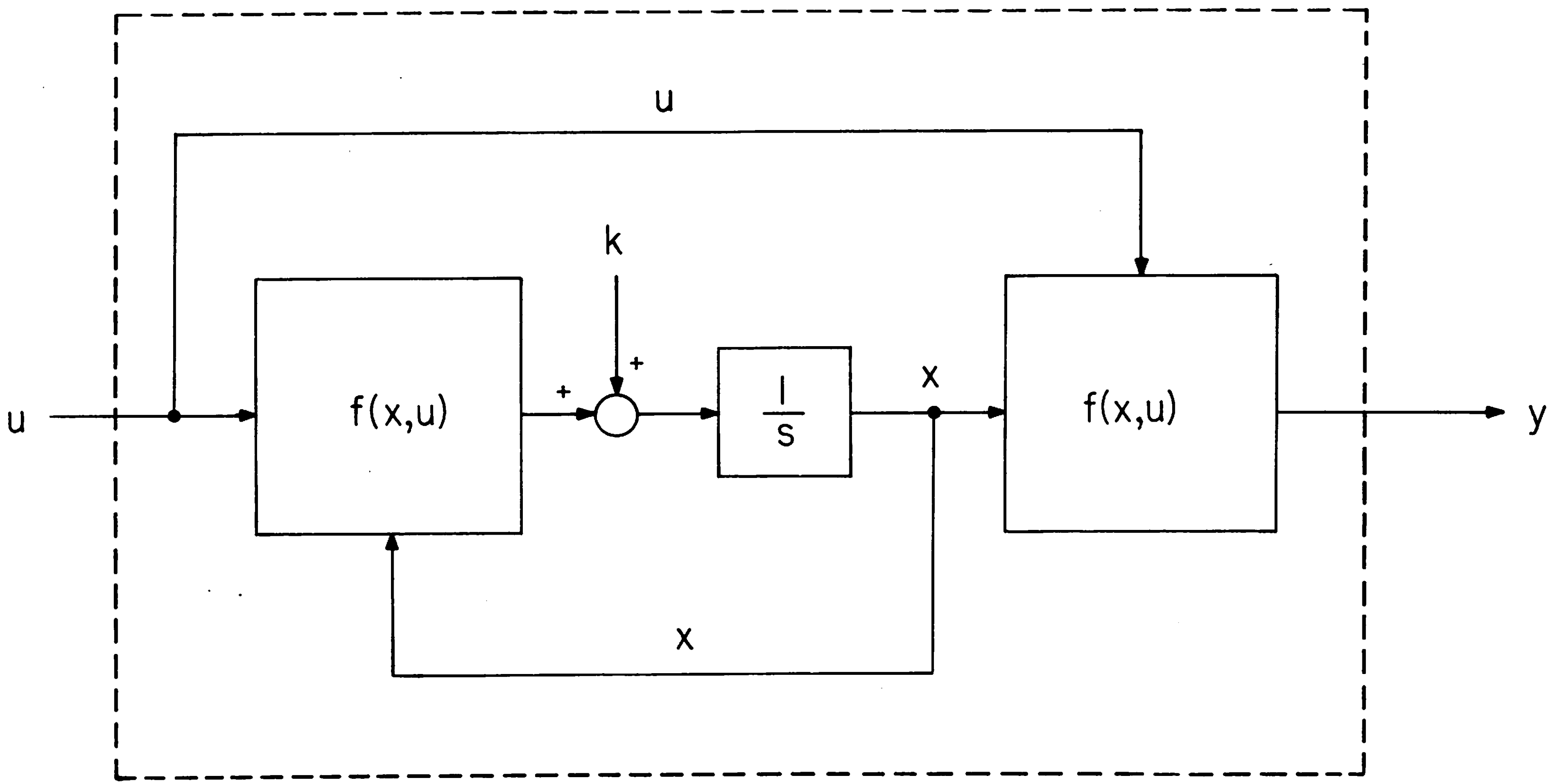

Fig. 3. A First Order Canonic Model

Fig. 3

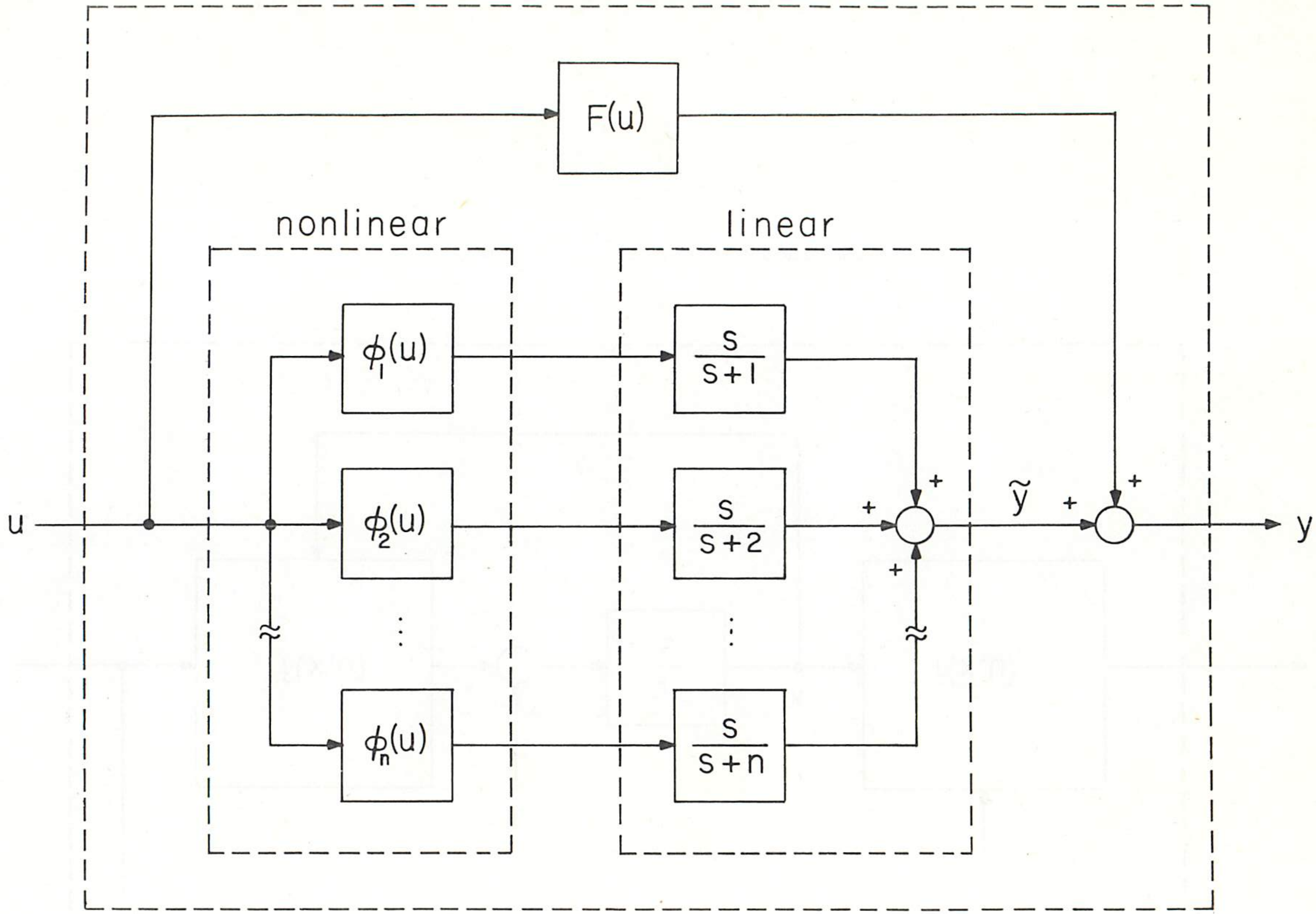

Fig. 4. An Nth Order Canonic Model

Fig. 4

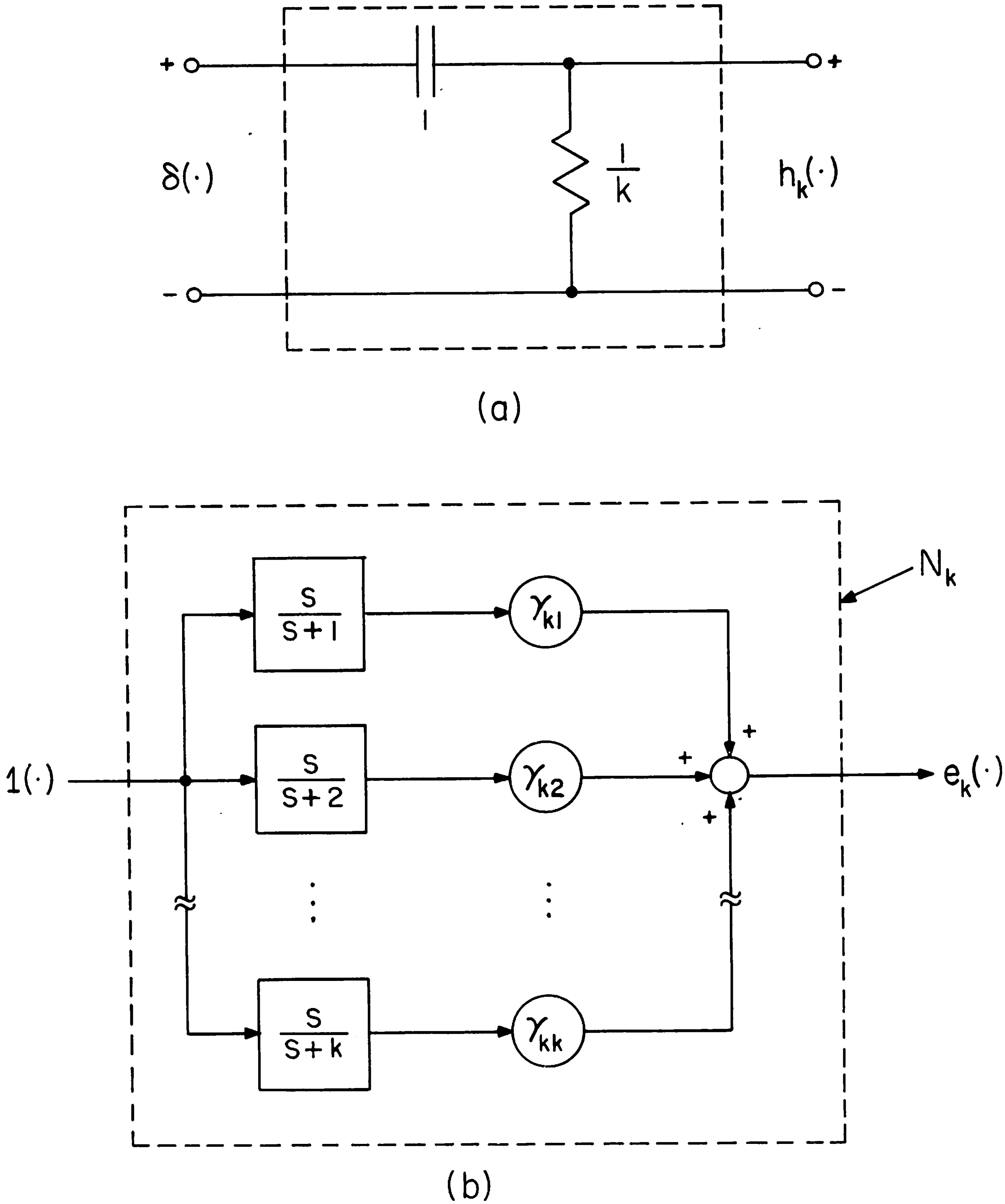

Fig. 5. Realization of $H_k(s)$; (a), Generation of $e_k(\cdot)$; (b)

Fig. 5

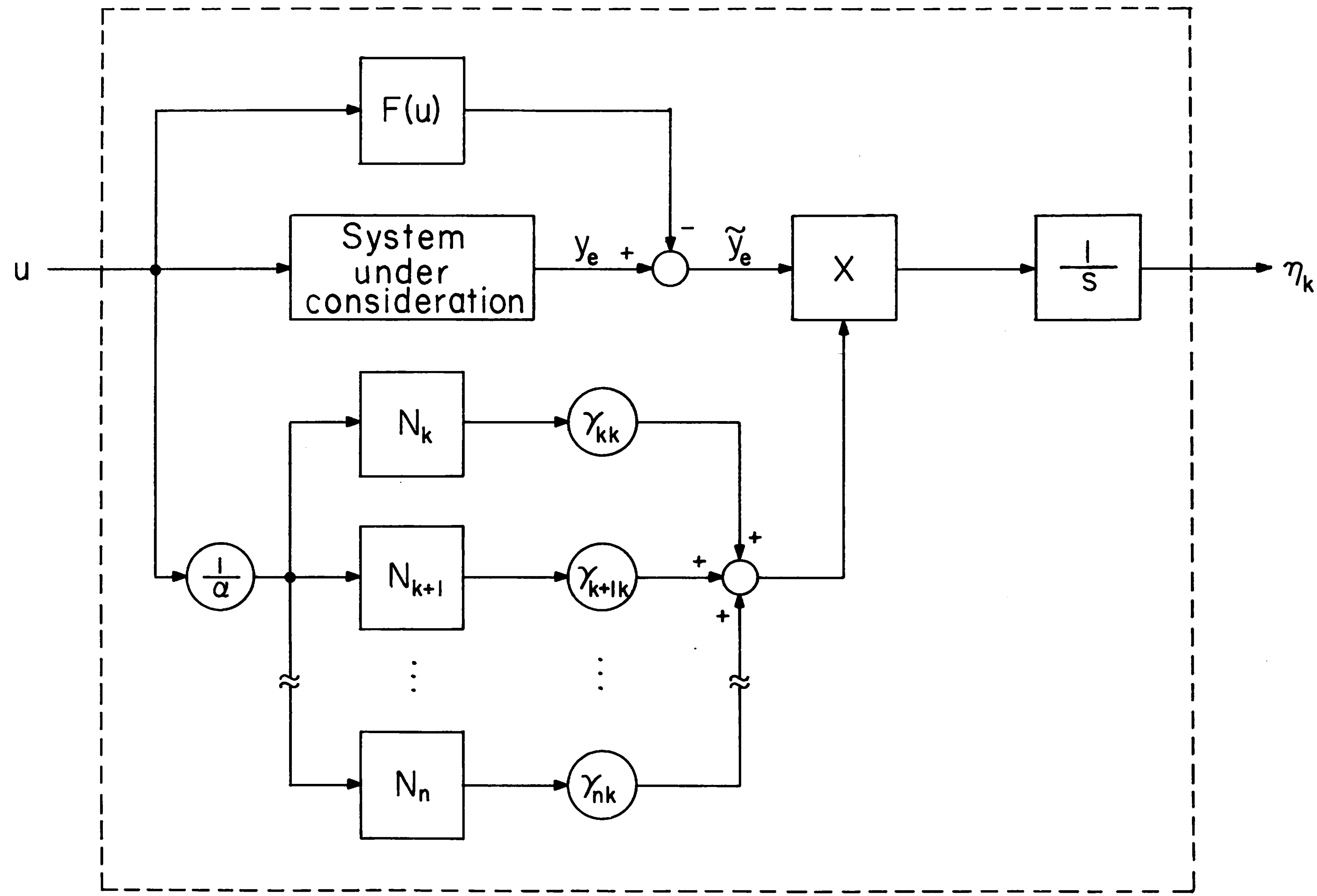

Fig. 6. Measuring $\hat{\phi}(u)$

Fig. 6