OPTIMIZING MODELS FOR URBAN DEVELOPMENT

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ABSTRACT

The paper presents an approach to the planning of urban development using an activity analysis framework in contrast with the more prevalent simulation models. Using input-output relations for production of commodities and labor, and a specific geometry for the transportation network, the model determines an "efficient" land-use pattern and transportation system by solving a linear programming problem. The model incorporates the "sunk" nature of urban capital and shows how this affects the pattern of development in a growing city. From the viewpoint of data requirements and computational complexity it appears that this approach is worth pursuing in conjunction with, or as an alternative to, the efforts devoted to simulation models.

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1. Introduction:

The relatively poor state of modeling of urban economics is usually attributed to the paucity of data. While this obviously is the case, it is equally true that the consequences of the lack of data depend upon the choice of model.

The most favoured form of urban econometric models is that of the "simulation" model, originally based upon the work of Lowry (Lowry (1964), Batty(1972),Goldner(1971)) which in turn has strong affinities with empirical studies in urban geography. The data requirements for such efforts are enormous. This is due primarily to the fact that the equations which represent locational decisions of various agents, such as households, firms, etc, are posited a priori, leading to a very large number of unknown parameters. Thus these equations are similar to the assumed "demand equations" in econometric studies.

An alternative modeling strategy would be to model the locational decision by demand equations derived from some "optimizing" principle such as minimization of transport cost or some more inclusive criterion. Two advantages would flow from such a strategy. First of all, this approach would be more in line with economic theory which usually treats demands for a particular location, by households say, as a derived demand in the sense that locations are not usually considered as directly entering utility functions. Secondly, the data requirements are considerably reduced since the locational demands are derived from the usual demand equations for non-locational commodities (which have to be present in the 'simulation models also). Of course, the derivation of locational demand from the other demand equations increases the computational requirements. The work reported here follows this alternate strategy.

There is an analogy that can be made which may further help situate this work in relation to the simulation models. Roughly speaking, macro-economic models can be dichotomized into two classes. The first class consists of the simulation models associated with the names Klein, Goldberger, etc. The second class consists of the "planning" models associated with Leontieff and the field of activity analysis. The purposes of the two sets of models are different. Whereas the simulation models are used primarily for short-term forecasting, the latter appear to be more useful for long-range planning. Similarly the simulation models for urban economies were originally constructed for forecasting. The model(s) presented here are intended to serve ultimately as a "planning" tool. The qualification ultimately is made because the data used here is fairly arbitrary, and the model is naive in some respects; hence the models should be considered as an illustration of a planning method.

What sorts of planning issues can be formulated in these models? The primary planning "variables" considered are those associated with transportation. Specifically the model choses "optimal" values of the capacities of the links in a transportation network, once the geometry (graph) of the network has been selected. Thus different network geometries can be tested. Since the model also determines an efficient land-use pattern resulting from the choice of the network, the feedback effects resulting from the impact of the transportation system upon land-use, and in turn the effect of the latter upon the demand for transportation, is automatically included. This is in contrast to many transportation planning models where demand is forecast exogenously. The determination of land-use patterns also gives an indication of the areas which will probably be more intensively developed, and hence these areas may serve as a focus for a more detailed attention of planners. Finally, the models give estimates of the "savings" arising from planning for the future. As shown in Section 4 in a dynamic context it is more efficient to anticipate future growth.

This effort has been inspired by some work reported in the pioneering book by Mills,(1972a), as well as by his later efforts in (Mills,1972b). The authors are greatly influenced by Mills' work and would like to acknowledge this debt.

2. The Basic Model

a. Model Description

The basic model is a linear programming model with a structure similar to that of (Mills,1972b), but differing from it in many important aspects. For example, the assumption that all imports and exports of the urban area pass through the center of the city is dropped, it being possible to use transportation terminals at both the center and the periphery for these purposes. Another difference is that in this model only part of the land use is considered. These and other differences will be clarified in the description of the model.

The basic framework is the same as in several other models (Mills and Ferranti,1972,Mirrlees,1972,Mills,1967,Legey et al,1973), a city in a featureless plane, inserted in an infinite environment. Angular symmetry is assumed, but is not essential in the model. The land surface of the city is divided into a rectangular grid, as shown in Fig. 1. The coordinates of the grid are measured from the city center (0,0). To simplify the mathematical structure of the model it is assumed that all traffic has to move through the center of the squares and perpendicular to its sides. The distance between a square (i,j) and the city center (0,0) is then $u = d_{ij} = |i| + |j|$. There are then S_u squares at distance u from the city center with

$$S_u = 4u \qquad u = 1,2,\ldots$$

The assumed symmetry implies that each square can be univocally represented by a single number u, the distance from the center.

Three types of commodities are produced in the city: centralized commodities, decentralized commodities and housing. The centralized goods, indexed from r=1 to $\bar{r}$-1, can be produced only in the central square, and are needed in this model in order to pin down the city at a particular point in space. The decentralized goods, indexed from r=$\bar{r}$ to $\bar{\bar{r}}$-1, and housing, indexed $\bar{\bar{r}}$, can be produced at any point in the city, except the central square.

An exogenously given amount $\bar{x}_r \geq 0$ of the good r, r=1...$\bar{\bar{r}}$, has to be exported. This export demand can be met by local production or by imports. $\bar{x}_{cr}$ is the amount of good r exported through the city center while $\bar{x}_{pr}$ is the amount exported through the periphery. These variables then must satisfy the relations:

$$\begin{aligned} &\bar{x}_{cr} + \bar{x}_{pr} = \bar{x}_r \\ &\bar{x}_{cr} \geq 0 \\ &\bar{x}_{pr} \geq 0 \qquad\qquad r = 1\ldots\bar{\bar{r}} \end{aligned} \tag{1}$$

Since this export demand drives the model it is reminiscent of the export base theory. However the model structure is considerably more rich than the one usually associated with this theory.

The mathematical specification of the housing industry in this model is identical with all other industries but, as described later, it has a broader interpretation. If, for example, no export of housing (people) is considered reasonable, then it is enough to make $\bar{x}_{\bar{r}} = 0$. Imports of any of the r commodities can similarly occur through the center or through the periphery and these variables are denoted by $x_{cr} \geq 0$ and $x_{pr} \geq 0$ respectively. In a typical situation, some of the commodities would be imported or produced for local consumption, and some would be imported or produced both for local consumption as well as to satisfy the export requirements.

The possibility of permitting flows of goods between the suburban locations of the city and the 'outside' world presumably by road, is a more realistic assumption for many modern cities where this type of transportation accounts for the major part of the intercity commodity flow, than the more usual one in which all goods are forced to be exported or imported through the city center, presumably via a harbour or railhead.

Two types of input are considered in the production technology of the commodities. First, all produced (or imported) goods can be inputs to the production of other goods. Secondly, labor, land and capital are also inputs, supplied competitively to the city. Thus the production technologies are more general than those considered in (Mills,1972b), where only labor, land and capital were considered. From the way the housing industry will be formulated it will follow that all goods can be locally consumed besides being exported. In this way the internal flow of commodities (and services) is well represented and yields, as consequence, a more 'stable' solution for the model. The following assumptions are made concerning the supply of labor, land and capital inputs. Labor can be attracted in any amount to the city at a wage w, plus the cost of housing and commuting. Capital can also be obtained in unlimited amounts at a fixed rental rate I. The opportunity cost for land is fixed at an agricultural rate of R_a per square.

Each <u>production technique</u> is represented by a set of input coefficients a_{qrs}, where a_{qrs} is the amount of input q required per unit of output r using activity s. The indexes q,r and s have the following ranges:

$$q \in Q = \{\underbrace{1,2,\ldots,\bar{\bar{r}}-1}_{\text{input goods}},\ \underbrace{\bar{\bar{r}}}_{\text{labor }(\ell)},\ \underbrace{\bar{\bar{r}}+1}_{\text{land (t)}},\ \underbrace{\bar{\bar{r}}+2}_{\text{capital (c)}}\}$$

$$r \in R = \{\underbrace{1,2,\ldots,\bar{r}-1}_{\text{centralized goods}},\ \underbrace{\bar{r},\bar{r}+1,\ldots,\bar{\bar{r}}-1}_{\text{decentralized goods}},\ \underbrace{\bar{\bar{r}}}_{\text{housing}}\}$$

$$s \in S = \{\underbrace{1,2,\ldots,\bar{s}}_{\text{activities}}\}$$

Using a similar but broader interpretation than in (Mills,1972b), the <u>s</u>th activity can be thought of as the production of one unit of output in an $f_r(s)$-story building, where $f_r(s)$ for each r is a specified integer function.

Indivisibilities in production and consumption are not permitted in this formulation of the model.

Housing production is directly associated with labor by the assumption that each household has an inelastic demand for one unit of housing and that this household provides one unit of labor. This assumption in the model is specified by considering two sets of inputs for the housing industry. The first set directly associated with housing as a physical structure consists of land, capital and labor needed for its construction and maintenance. The second set associated with the household that occupies the dwelling consists of the remaining goods consumed by the household. The housing industry has then a broader interpretation than the other industries in this model. It is similar to the treatment of labor in von-Neuman models. This formulation is very natural and is easily extended, as in Section 3, to encompass an elastic demand of housing.

The diameter $\bar{u}$, used in the model, is exogenously specified and should be chosen large enough so as to contain the total city and such that even if a bigger value were assumed no activities at a positive level would take place in squares at a distance beyond $\bar{u}$ from the center.

The model does not take into consideration all the land use in the city. The land use in the central square (0,0) is not taken into consideration at all because of its unrealistic characteristics as an export and import center. In all other squares only an exogenously given fixed part, $L(u)$, of the total surface available, is taken into account. This land is used as input either for the different industries or for the main transportation network (freeways). All other public or private land uses are not taken into consideration.

The production variables or activity levels of the model are $x_{rs}(u)$, the output per square of commodity r in a $f_r(s)$ story building u squares from the center.

By the definition of centralized and decentralized commodities, the following constraints must hold:

$$x_{rs}(0) \begin{cases} \geq 0 & , 1 \leq r < \bar{r} \\ = 0 & , \bar{r} \leq r \leq \bar{\bar{r}} \end{cases}$$

$$x_{rs}(u) \begin{cases} = 0 & , 1 \leq r < \bar{r} \\ \geq 0 & , \bar{r} \leq r \leq \bar{\bar{r}} \end{cases} \tag{2}$$

In order to satisfy the export requirements of the city, the following inequalities must be satisfied

$$\sum_s \sum_u 4u x_{rs}(u) + \sum_s x_{rs}(0) + x_{cr} + x_{pr} - \sum_{r'} \sum_s \sum_u 4u a_{rr's} x_{r's}$$
$$- \sum_{r'} \sum_s a_{rr's} x_{r's}(0) \geq \bar{x}_r, \; r \in R \tag{3}$$

These inequalities imply, for each commodity r, that the total production in the city, plus what is imported both through the center and the periphery, minus what is used of the commodity both in the production of other goods or as a consumption good, must exceed the exogenously specified final "export" demand $\bar{x}_r$.

Unless indicated otherwise, the following limits are assumed for indices both in the summation signs and in the range of equations.

$$r, r' \in R; \; u \in U = \{1, \ldots, \bar{u}\}; \; s \in S, \; k \in K = \{1, \ldots, \bar{k}\}$$

Here $\bar{s}$ the largest value of s is such that $f_r(\bar{s})$ is equal to or larger than the highest building to be constructed in the city for the production of commodity r. The choice of maximum building height $f_r(\bar{s})$ is made in such a way that even if other activities corresponding to higher buildings were available they would not be used in the solution of the model. The congestion levels k will be defined later.

There is, in this model, a possible flow of commodities between every pair of squares in the city as well as between the city and the 'outside' world in the case of exports and imports. Let $T_r(u)$, $r \in R$, be the number of units of commodities r that are transported through the center of each square at distance u from the center. $T_r(u)$ is an unconstrained variable, which will be taken as positive when the net traffic is in the direction of the city center and negative otherwise. $T_{\bar{r}}(u)$ then represents movement of labor.

Using the symmetry of the model, the principle of conservation of flow applied to each commodity, and the definition of the variable $T_r(u)$, an equation can be written for each u which states that the traffic at this u for each commodity r must equal the traffic at u-1 for this same commodity, plus half of the production less the consumption of r at this particular u. This half is due to the assumption that production and consumption are distributed uniformly in each square u. The appropriate equation then is

$$\begin{aligned} T_r(u) = \frac{1}{4u}\Big[\bar{x}_{cr} - x_{cr} - \sum_{u'=1}^{u-1} 4u'\Big(\sum_s x_{rs}(u') - \sum_{r'}\sum_s a_{rr's}x_{r's}(u')\Big) \\ - \Big(\sum_s x_{rs}(0) - \sum_{r'}\sum_s a_{rr's}x_{r's}(0)\Big) - 2u\Big(\sum_s x_{rs}(u) \\ - \sum_{r'}\sum_s a_{rr's}x_{r's}(u)\Big)\Big], \quad r \in R,\ u \in U \end{aligned} \tag{4}$$

An Eqn. (4) was written taking the center as reference a boundary condition must be met at the periphery; it states that the traffic of commodity r at the periphery is equal to what is imported minus what is exported.

$$T_r(\bar{u}) = \frac{1}{4\bar{u}} (x_{pr} - \bar{x}_{pr}), \quad r \in R \tag{5}$$

The traffic in square (0,0) is not considered in this model.

An important and obvious consequence of an efficient allocation of resources is that in such an allocation the net traffic $T_r(u)$ of commodity r equals the total traffic. It would be inefficient to have at the same u a unit of commodity r being shipped in the direction of the city center, and another one being shipped in the opposite direction. This implies that all units of r that are being transported at u are all flowing in either one or the opposite direction.

An immediate consequence of allowing the traffic to flow only in the direction of the center, as in (Mills,1972b) for example, is that in an efficient allocation, housing is automatically more "suburbanized" than production. Even if this appears to be reasonable it would be better to have a model which has a degree of freedom in this respect. Such freedom is provided in this model by (6).

$$\begin{aligned} &T_r(u) = T_{cr}(u) - T_{pr}(u) \\ &T_{cr}(u) \geq 0, \; T_{pr}(u) \geq 0 \;, \quad u \in U, \; r \in R, \end{aligned} \tag{6}$$

where $T_{cr}(u)$ is the number of units of commodity r that are transported in the direction of the center of the city at a distance u from the center. $T_{pr}(u)$ has the corresponding definition with transportation directed towards the periphery. An efficient allocation will then automatically satisfy the relationship

$$T_{pr}(u) \cdot T_{cr}(u) = 0, \; r \in R, \; u \in U$$

In order to calculate the total demand for transportation at each u different demands placed by different commodities have to be compared. Normalizing coefficients t_r are introduced for this purpose.

$$T_c(u) = \sum_r t_r T_{cr}(u)$$

$$T_p(u) = \sum_r t_r T_{pr}(u) \quad , \quad u \in U \tag{7}$$

$T_c(u)$ is then the total traffic at u in the center direction, $T_p(u)$ being the equivalent in the opposite direction. $T_c(u)$ and $T_p(u)$ are in units of vehicle-miles. The numbers t_r can be chosen in order to represent several characteristics of the transportation flow. They should represent the different demands that commodities and commuters impose on the transportation system considered. As travel time is not considered in the model and urban transportation is very unequal in travel time, with the possibility for example of a single peak hour accounting for more than 10% of total daily travel (see e.g. (Meyer *et al*,1965)), therefore another consideration in the choice of the coefficients t_r might be to give a higher weight to those commodities (or people) which use transportation facilities during peak hours. Another consideration that might affect t_r is modal choice.

The assumptions on *congestion* are those of (Mills,1972b) and only for the sake of completeness a brief description will be given here. A single production possibility is assumed for transportation using as inputs land and capital (with input-output coefficients b_t and b_c respectively). These inputs determine $T^{(1)}(u)$, which is the transportation capacity available at u if no congestion occurs.

The dot is used here, instead of the index p or c, due to the symmetric treatment of center-bound or periphery-bound traffic. If $T_{\cdot}(u)$ is larger than $T_{\cdot}^{(1)}(u)$, then congestion will occur at a level $k \geq 2$ which is determined by the following two sets of equations:

$$
\begin{aligned}
&\sum_k T_p^{(k)}(u) \geq T_p(u) \\
&\sum_k T_c^{(k)}(u) \geq T_c(u) \\
&\qquad T_p^{(k)}(u) \geq 0 \quad , T_c^{(k)}(u) \geq 0, \ u \in U,
\end{aligned}
\tag{8}
$$

and,

$$
\begin{aligned}
&T_p^{(1)}(u) \geq T_p^{(k)}(u) \\
&T_c^{(1)}(u) \geq T_c^{(k)}(u), \qquad u \in U, \ k \in K
\end{aligned}
\tag{9}
$$

Equation (8) states that is is necessary to use enough congestion levels to meet total transportation demand. As the marginal transportation cost is assumed to be an increasing function, Eqn. (9) will force in an optimal allocation the lowest conjestion levels to be filled first, i.e., at an optimum

$$
T_{\cdot}^{(k')}(u) = \begin{cases} T_{\cdot}^{(1)}(u) & 1 \leq k' \leq k \\ 0 & k < k' \leq \bar{k} \end{cases}
$$

The total congestion cost, including the operating cost of the vehicle and the time cost of travel, is then the discrete integral of the marginal cost, i.e.,

$$c(u) = \sum_k c_k (T_p^{(k)}(u) + T_c^{(k)}(u)), \; u \in U \tag{10}$$

The amount of land and capital allocated to the transportation system will then be decided by two conflicting factors: the actual cost of those inputs that grow with the size of the transportation system, and the amount of congestion, with its associated costs, which is reduced with the size of the transportation system.

Only part of the land is available for the uses described in the model. This can be written as:

$$\sum_r \sum_s a_{trs} x_{rs}(u) + b_t (T_c^{(1)}(u) + T_p^{(1)}(u)) \leq L(u), \; u \in U \tag{11}$$

An <u>efficient allocation</u> is one which minimizes production, transportation and import costs, satisfying the specified consumption levels of the households and the export requirements. The total cost is

$$\begin{aligned} Z = I \Big[& \sum_u \sum_r \sum_s 4ua_{crs} x_{rs}(u) + \sum_r \sum_s a_{crs} x_{rs}(0) \\ & + b_c \sum_u 4u (T_c^{(1)}(u) + T_p^{(1)}(u)) \Big] + R_a \Big[\sum_u \sum_r \sum_s 4ua_{trs} x_{rs}(u) \\ & + b_t \sum_u (T_c^{(1)} + T_p^{(1)}(u)) \Big] + w \Big[\sum_u \sum_r \sum_s 4ua_{\ell rs} x_{rs}(u) \\ & + \sum_r \sum_s a_{\ell rs} x_{rs}(0) \Big] + \sum_u 4uc(u) + \sum_r (p_{cr} x_{cr} + p_{pr} x_{pr}), \end{aligned} \tag{12}$$

where p_{cr} and p_{pr} are respectively the commodity r import prices at the center and at the periphery of the city.

The problem to be solved is then to minimize (12) subject to (1)-(11). This is a linear programming problem.

b. Numerical Results

Some numerical runs were performed with the model, using a particular set of parameter values. The solution of the linear programming model provides a description of the kinds, amounts and techniques of production, and the amount of transportation in each direction in each square. The dual variables or shadow prices at the optimum also provide valuable information as explained later.

Only one centralized and one decentralized industry were considered. The centralized industry, which might be thought as chiefly representing services, is used both in the manufacturing of the decentralized commodity and as consumption goods for the households. The decentralized good is used for export only and is not used either as an input or as a consumption good. All industries (including housing) use labor and capital. Land is used as input in the production of housing and the decentralized commodity and for transportation. These are probably the simplest relationships that could be devised while maintaining the main characteristics of the model. They are indicated in Fig. 2.

Due to the high level of aggregation the data used reflects the order of magnitude of the phenomena rather than any specific reality. The choice of the parameters for the model is discussed in the Appendix. A small urban area was initially considered, with a labor force of 150,000, or a population of about 500,000. In one extension, that will be discussed in section 4, this population was made to grow up to 1,000,000.

Only the main features of the solution and some sensitivity studies will be reported. A typical solution is presented in Fig. 3. For each u, as L(u) is taken to be constant, the same fraction of the total land is assumed available for use. The land-use and production levels are represented for each u by a bar diagram. The width of each bar represents the actual surface area used in the production, and its height is a measure of the number of floors of the buildings where production takes place. Hence the area of the bar is proportional to the total production of a particular activity at a particular u. Housing is represented by a white bar, decentralized industry by a black bar and transportation by a line indicating the area used. The total length of the horizontal line represents then the total area used at each u.

The typical solution then is composed of two concentric rings - one, near the center, with high-density housing and intensive transportation and the other, further away from the center, with one-story housing and decentralized industry. This structure reflects the simple relationships assumed in Fig. 2, and presumably would be more complex for a larger number of decentralized goods or a richer input-output relationship.

The solution is stable and does not present the instability reported by (Mills,1972b), where relatively small shifts in the capital input-output coefficients and in the transportation coefficients t_r caused the solution to jump from one involving no commuting, since the labor required in each square is housed in the same square, to another with relatively little goods shipment and where all the land close to the center is used only for the production of the export good and transportation. The greater stability here is due to produced goods being inputs to the production of other goods.

An interesting comparison can be made with Fig. 4 which also represents a typical solution, but with the additional constraint that all exports have to be sent through the center. The structure of the city changes considerably, even though the miximum total cost differs only by 1%. This solution also proved to be stable under different parameter variations.

Using the duality theorem for linear programming it can be proved that the marginal cost of producing one unit of the export commodity equals the average cost plus the total land rent divided by the total number of units of exports. The rent of land at distance u is just the shadow price of land at u i.e., the value of the dual variables associated with the constraint equations (11). An alternative and more direct way of computing the marginal cost would be to treat $\bar{x}_r$ in (1) and (3) as variable and to add the following additional constraint,

$$\bar{x}_r \geqq \text{exogenous export.}$$

The marginal cost would then be equal to the dual variables associated with this constraint. However this increases the number of variables and constraints in the linear program. Some results are given in Table 1.

Both conclusions presented in (Mills,1972b) - that a considerable amount of congestion is efficient, at least near the centers of big cities, and that rental values are considerably high at the center of the city, falling rapidly near the center, and finally leveling off to R_a near the edge of the city - continue to hold for this model. The rents for the city with exports through the center and periphery are considerably lower than the ones with exports only through the center, reflecting, once more, the more efficient organization of the first scheme. Figures 3 and 4 also illustrate this fact since the height of the buildings indicate that considerably more capital - land substitution occurs in the second exercise.

Finally, it is seen that there is a considerable suburbanization of production in both cases.

3. The Labour Model:

There are several ways of disaggregating the Basic model. The production sector can be treated in greater detail simply by increasing the number of decentralized and centralized industries and by making them represent different types of business such as retailing, selected services, wholesaling and manufacturing. This is a straightforward exercise and is not reported here. A more interesting extension of the model which is reported in this section is an attempt to permit more than one kind of labor and to allow each kind to be indifferent between alternative consumption bundles of housing and other goods.

To accommodate different types of labor the production functions of industries must be enlarged to include the different types of labor. This can be done within the scope of the Basic model simply by renaming the definitions of the input (q) and output (r) indexes, by writing the appropriate input-output coefficients a_{qrs} for each activity, and by changing the objective function.

The new ranges for q and r then are:

$$q \in Q = \underbrace{1,\ldots,\bar{\bar{r}}-\bar{\ell}}_{\text{input goods}},\ \underbrace{\bar{\bar{r}}-\bar{\ell}+1,\ldots,\bar{\bar{r}}}_{\text{labor}},\ \underbrace{r+1}_{\text{land}},\ \underbrace{r+2}_{\text{capital}}$$

$$r \in R = \{ \underbrace{1,\ldots,\bar{r}-1}_{\text{centralized goods}}, \underbrace{\bar{r},\ldots,\bar{\bar{r}}-\bar{\ell}}_{\text{decentralized goods}}, \underbrace{\bar{\bar{r}}-\bar{\ell}+1,\ldots,\bar{\bar{r}}}_{\text{housing}} \}$$

There are then $(\bar{r}-1)$ centralized commodities, $(\bar{\bar{r}}-r+1-\bar{\ell})$ decentralized commodities, and $(\bar{\ell})$ types of housing in this model.

The new objective function replacing (12) is:

$$Z = I[\sum_u \sum_r \sum_s 4ua_{crs}x_{rs}(u) + \sum_r \sum_s a_{crs}x_{rs}(0)$$

$$+ b_c \sum_u 4u(T_c^{(1)}(u) + T_p^{(1)}(u))] + R_a[\sum_u \sum_r \sum_s 4ua_{trs}x_{rs}(u)$$

$$+ b_t \sum_u 4u(T_c^{(1)} + T_p^{(1)}(u))] + \sum_\ell w(\ell)[\sum_u \sum_r \sum_s 4ua_{\ell rs}x_{rs}(u)$$

$$+ \sum \sum a_{\ell rs}x_{rs}(0)] + \sum_u 4uc(u) + \sum_r (p_{cr}x_{cr} + p_{pr}x_{pr})$$

where $w(\ell)$ is the wage for labor of type ℓ.

A typical solution for this model is presented in Fig. 5. The same parameters used in the Basic model were used here, with the additional assumption that the labor needs of the industries are divided equally between white and blue collar workers, and that the consumption and wage of white and blue collar workers is 20% larger and smaller, respectively, than that of labor in the Basic model. The typical solution is then composed of 3 concentric rings - one near the center with intensive transportation and high buildings with small dwelling units for the blue collar workers; another, farther away from the center, with one floor housing, for the white collar workers, and finally the third one, near the periphery of the city, with the decentralized industry and housing for both the white and blue collar workers that serve it.

Due to the different incomes and consumption patterns of the two types of workers an efficient distribution in space segregates, in a large part of the urban area, the white collar from the blue collar workers, the white collar workers being more suburbanized.

It is also seen that buildings of different height can coexist at the same point in space, indicating that a mixed capital-land substitution rate, at the same distance from the center, can be optimal.

Up to now, only different types of labor were considered. Different consumption bundles can also be offered to each household group. The optimal solution in the linear programming problem can then be thought of, for the households, as the outcome of a method of choice of a particular bundle among the "indifferent" bundles available to that household which maximizes its "savings." This is similar to the interpretation in (Herbert and Stevens, 1960).

There are several methods of implementing this modification in the model. For example, one can introduce another index in the housing industry to take into account all these new consumption bundles. Another method which is suitable for a small number of bundles, and requiring no modification in the model, is to use a higher $\bar{s}$ and take part of the range of the index s to have the same interpretation as before (height of building) and the other part as consisting of different bundles. If no new activities correspond to these additional values of s for a particular industry, then a dummy activity, equal to some of the existing ones, is introduced. An increase in $\bar{s}$ increases only the number of variables in the model by $\bar{s}\,[\bar{r}-1+\bar{u}(\bar{\bar{r}}-\bar{r}+1)]$.

Figure 6 displays a solution for this extension of the model, where the white collar workers have two new bundles from which to chose. In the first one, it consumes less of the centralized good and instead has a one floor house with a bigger surface area. In the second bundle, more goods are consumed and the household occupies less expensive housing. The optimal allocation was to provide for all white collar households with the first of these new bundles. An interesting feature of this allocation is that part of the decentralized industry moves to the ring of the blue collar workers.

4. The Dynamic Model:

The Basic model and its extension described above are static. The land-use pattern generated by successive runs of the models, using dif- derent exogenous demands, are the best in an *ex ante* sense but unrealistic for any growing city where, because much of urban capital is "sunk," the *ex post* possibilities for substitution are less than *ex ante* possibilities. In this section a model is described which takes these differences into account. At each instant of time an existing structure is removed only when the operation of a new structure is cheaper, taking into account not only the cost of the new structure, but also the capital costs of the old one, including the scrapping cost.

Three vector variables are defined, $\underline{a}^T$, a'^{T+1}, a^{T+1}, where $\underline{a}^T$ is the state vector of the city at a time T. $\underline{a}^T$ summarizes the activities of the city at time T. In terms of the Basic model of section 2,

$$\underline{a}^T = \begin{bmatrix} x_{rs}\ (0) \\ x_{rs}\ (u) \\ T_c^{(1)}(u) \\ T_p^{(1)}(u) \end{bmatrix} \qquad , \ r \in R,\ s \in S,\ u \in U$$

a^{T+1} represents the new construction that takes place between T and T+1, and a'^{T+1} represent those structures built up to T that continue to exist at T+1. Thus

$$a'^{T+1} \leq \underline{a}^{T} ,$$

$$a'^{T+1} + a^{T+1} = \underline{a}^{T+1} ,$$

so that for example, $x_{rs}^{T+1}(u)$ would be the output per square of commodity r in a $f_r(s)$-story building built between T and T+1 at distance u from the center.

The initial conditions given to the dyanmic model for time T+1 are $\underline{a}^{T}$, and the exogenous data consists of the new expected demand $\bar{x}_r^{T+1}$. The model calculates a'^{T+1} and a^{T+1}, thereby generating the initial conditions for the next period T+2. In any particular run the intervening variables are $\underline{a}^{T}$, a'^{T+1} and a^{T+1}, hence the index T can then be dropped without ambiguity.

The constraint equations for the Dynamic model are:

- without modifications, equations (1),(5),(6), and (7) of the Basic model,
- equations 3, 4, 8, 9, 10, and 11, obtained respectively from (3), (4), (8), (9), (10) and (11), by substituting

$$\begin{bmatrix} x_{rs}(0) \\ x_{rs}(u) \\ T_c^{(1)}(u) \\ T_p^{(1)}(u) \end{bmatrix} + \begin{bmatrix} z_{rs}(0) \\ z_{rs}(u) \\ P_c^{(1)}(u) \\ P_p^{(1)}(u) \end{bmatrix} = a + a' \text{ for } \begin{bmatrix} x_{rs}(0) \\ x_{rs}(u) \\ T_c^{(1)}(u) \\ T_p^{(1)}(u) \end{bmatrix} \qquad (13)$$

where a and a' have the new interpretation given in the above equation,

- equation 2, composed of Eqn. (2) and in addition the following,

$$0 \leq z_{rs}(0) \leq \bar{x}_{rs}(0) \ , \qquad 1 \leq r < \bar{r}, \ s \in S$$

$$z_{rs}(0) = 0 \ , \qquad \bar{r} \leq r \leq \bar{\bar{r}}, \ s \in S$$

$$z_{rs}(u) = 0 \ , \qquad 1 \leq r < \bar{r}, \ s \in S, \ u \in U$$

$$0 \leq z_{rs}(u) \leq \bar{x}_{rs}(u) \ , \qquad r \leq r \leq \bar{\bar{r}}, \ s \in S, \ u \in U$$

$$0 \leq P_c^{(1)}(u) \leq \bar{T}_c^{(1)}(u) \ , \qquad u \in U$$

$$0 \leq P_p^{(1)}(u) \leq \bar{T}_p^{(1)}(u) \ , \qquad u \in U$$

which is equivalent to the condition $a'^{T+1} \leq \underline{a}^{T}$.

The objective funtion (12') is equal to that of (12), with the substitutions listed in (13), denoted Z(a+a'), and with one more term,

$$Z = Z(a'+a) + I[\sum_u \sum_r \sum_s 4ud_{rs} \cdot a_{crs}(\bar{x}_{rs}(u) - z_{rs}(u))$$

$$+ \sum_r \sum_s a_{crs} d_{rs}(\bar{x}_{rs}(0) - \bar{z}_{rs}(0)) + b_c \cdot d \cdot \sum_u 4u(\bar{T}_c^{(1)}(u)$$

$$+ \bar{T}_p^{(1)}(u) - P_c^{(1)}(u) - P_p^{(1)}(u))] \qquad (12)$$

where d_{rs} is a constant representing the fraction of the capital used in the manufacturing of x_{rs}, that is of a "sunk" nature and is lost once the structure is destroyed. It includes any capital that is not mobile, and the cost of scrapping. The constant d has a similar interpretation. This new term included in (12) then represents the cost that is incurred by destroying existing structures.

In Fig. 7, a comparison of the Basic model, the Basic model with all export going through the center, and the Dynamic model is presented. All the models use the same parameters as presented in the Appendix, and an exogenous export demand for $\bar{x}_2$ equal to 5.5×10^6. The Dynamic model uses as initial conditions the ones displayed in Fig. 3 (Basic model, $\bar{x}_2 = 4.5 \times 10^6$). It is assumed that the destruction of any structure produces a loss of half the capital initially employed.

The white and black bars still represent housing and decentralized industry. The bars with dots and those with crosses indicate, in the Dynamic model, new housing and industry built in the last time period.

There is a considerable difference between the land use patterns in these models even though the difference in total costs (1.2%) between the Dynamic and the Basic models may be considered small. In the Dynamic model, some of the one floor (at u = 3,4,5, and 6) and two floor housing (at u=2) were destroyed, and the new housing was constructed as high buildings in the middle of the city and as one floor houses in the periphery. Thus new construction does not take place in a contiguous region. None of the old industry was destroyed, and all new industrial construction occurs at the periphery. Finally, some reshuffling in the transportation system occurred, consisting mainly in a reduction of the center-bound roads and an increase in the periphery-bound roads as more and more of the export goes through the periphery. Both rents and congestion levels in the Dynamic model are considerably higher than those in the Basic model.

Figure 8 represents the outcomes of growth in the Dynamic model over three periods from the initial state shown in Fig. 7 ($\bar{x}_2 = 5.5 \times 10^6$) to a state corresponding to $\bar{x}_2 = 6.5 \times 10^6$, and from there to $\bar{x}_2 = 7.5 \times 10^6$.

5. Further Extensions

There are several ways in which the earlier models can be extended.

Some of these have been proposed by (Mills,1972b), such as the elimination of the circular symmetry hypothesis, by treating each square as different from all others. This increases very much the dimension of the linear program since $\bar{u}$ is essentially replaced by $2\bar{u}(\bar{u}+1)$, the number of squares in the city. One of the possibilities of this modification well within the scope of the Basic model, is to study the influence on the structure of the city of the appearance of subcenters. Another interesting proposal is the study of different transportation modes. The impact of alternate transportation modes, like rapid transit systems, could be studied both in terms of the structure of the city and through its influence on the total cost.

Two extensions of the Dynamic model can be implemented without too much change in model structure. One is the introduction of technological change, by means of new production techniques that can be used at T, affecting $x_{rs}(u)$, but were not available at T-1 and so do not affect $z_{rs}(u)$. Different sectors may have different rates of technological change so that one can study the effect of "unbalanced" growth. This modification increases the number of constraints in the Dynamic model, but does not change the number of variables.

The Dynamic model of section 4 did not take into account the age of the different existing structures. Essentially no depreciation or obsolescence is assumed.. These effects can be incorporated by introducing a new index t for each production activity representing the age group of capital used in these activity. For example, $x_{rs}^{t_0}(u)$ might mean the output per square of commodity r, in a $f_r(s)$ story building belonging to age group t_0, at distance u from the center. If this addition is incorporated in the model it is straightforward to include depreciated values in making decisions about new construction.

Several modifications to incorporate indivisibilities could be made by introducing integer variables, while this is simple on a conceptual level it greatly increases the computational complexity due to the size of the linear program already considered, and the relative inefficiency of the available integer programming algorithms. The size of the linear programs considered and the computing effort resources needed to run them are discussed in the Appendix.

Appendix:

a. Parameters used in the models:

No statistical studies were performed to estimate the parameters but an attempt was made to obtain reasonable values. The model needs two types of parameters. One set relates to the size, basic characteristics of the city, and various economic data. The second set is related to the production technology used (the input-output coefficients) and here it was difficult to make satisfactory estimates, or to find data by other authors, due to the level of aggregation.

Distances are in miles and each square has an area one square mile. Market values of input and output flows were measured in dollars per day. The parameters whose values change through time, were chosen in dollars of the end of the 1960s. All parameters were chosen to reflect typical conditions for the United States.

A description of how the parameters were chosen follows:

- $\bar{r}$, $\bar{\bar{r}}$: As it was desired to have initially the simplest possible structure for the model, only one type of centralized industry, decentralized industry and housing were considered, giving $\bar{r} = 2$ and $\bar{\bar{r}} = 3$.

- $\bar{u}$ (outside limit of model) : It was desired to accommodate up to 1,000,000 people. Taking an average population density of 4,000 per square mile gives an average of 250 square miles. $\bar{u}$ was chosen as 10, which yields an area of 180 square miles.

- w (income per worker per day), R_a (daily rent per square mile of land used for nonurban purposes), I (daily rate of return per dollar's worth of capital): the same values taken by (Mills,1972b) were chosen. w was taken to be \$25; R_a was calculated assuming that land at the edge of a medium sized city was worth \$3,000 per acre (see, e.g. Hoch,1969), a 10% capitalization per year and 240 working days per year then give the daily opportunity cost per square mile of land as \$800; the value of I is taken to be .0005 for all uses.

- $p_c(r)$, $p_p(r)$ (import price of good r at the center and at the periphery of the urban area): initially it was not desired to take imports into consideration, so that the prices of importing goods and labor were taken sufficiently high so that in an optimal solution no imports were used. At the same time, they were not made too large, in order to reduce the numerical problems. The choice made is $p_c(1) = p_c(2) = p_p(1) = p_p(2) = 50.$ and $p_c(3) = p_p(3) = 5000.$

- c(k) (congestion cost per mile at level k): the congestion cost at level 1 is taken to be twice (for a round-trip) the cost per mile of transportation in the absence of congestion, and is taken as the same as the one for commuters in (Mills,1972b). It inlcudes the cost of operating an automobile (\$.10 per vehicle mile) and the foregone earnings of the time spent in commuting (with an average speed of 25 miles per hour and the assumed wage rate, this also gives \$.10 per vehicle mile). c(1) was taken then as \$.40. The congestion costs at other levels were calculated using c(1) as a base, and the studies of (Walters,1961 and Vickery,1965). This gives c(2) = \$1.8, c(3) = \$4.0 and c(4) = \$7.1.

- $\bar{k}$ (number of congestion levels) this was chosen experimentally as 4.

- $\bar{s}$ (number of possible activities): it was taken a priori to be 4.

- $f_r(s)$ (integer functions for height of buildings): as no land use is considered for the centralized industry, $f_1(s)$ was made equal to 1 for all s. For decentralized industry, more or less associated with manufacturing and retailing, low buildings are the rule: so, $f_2(s)$ was defined to be equal to s, and this proved experimentally to be sufficient. For housing, in cities ranging from 500,000 to 1,000,000 people, high apartment buildings must be considered. The assumed values were: $f_3(1) = 1$, $f_3(2) = 2$, $f_3(3) = 4$ and $f_3(4) = 8$.

- $t(r)$ (number of units of r that corresponds to one vehicle): These coefficients were basically interpreted as reflecting the inverse of the number of units of the commodity or people that uses one vehicle. People use transportation either for commuting or to consume commodities. A somewhat larger value was taken for the activities that produce more traffic at the peak hours. The assumed values were: $t(1) = .04$, $t(2) = .015$ and $t(3) = 1$.

- $\bar{x}(r)$ (amount of good r that has to be exported): In the simple structure considered, the centralized goods and labor are not exported so that $\bar{x}(1) = \bar{x}(3) = 0$.. It was decided to begin the study with a city of 500,000 people. Knowledge of the input-output coefficients (see below) and assuming that the labor force is a third of the population leads to $\bar{x}(2) = \$4,500,000$ per day.

- $L(u)$ (amount of land per square considered in the model): $L(u)$ was taken to be a constant, independent of u. As in the United States the average population density is about 4,000 people per square mile, L was chosen so as to provide this density in the model. This is done by calculating, using the input - output coefficients, the total area used for all activities in the city. L is then taken as .07.

- b_c, b_t (input-output coefficients for the transportation system): These coefficients represent, respectively, the amount of capital and land needed per vehicle per mile, for building a freeway operating uncongested. From (Berry _et al_,1963) we have the following data for typical freeway rights-of-way: 76 feet - 4 lanes, double deck structure without ramps, 500 feet - 8 lanes depressed freeways with collector distributor roads. A common width for an 8 lanes depressed freeway, with one-way frontage roads, is presented as 350 feet. 50 feet per lane was assumed, which means that the area occupied by a freeway lane is $.947 \times 10^{-2}$ square mile in each square. Typical design capacities in vehicles per day (see Chicago Area Transportation Study,1959) at 85% of maximum capacity, assuming a peak hour volume of 11% of the 24 hours volume, are: 4 lanes - 54,000 vehicles, 6 lanes - 81,000 vehicles, 8 lanes - 108,000 vehicles. This corresponds to 7950 vehicles per lane per day at 50% maximum capacity which is considered the uncongested level. So, the amount of land required per vehicle, per mile, in an uncongested road (b_t) is taken as 1.2×10^{-6} square miles. The cost of freeways is very much influenced by the topography of the region. Typical variation for a 4 lane freeway in each direction is from $1,000,000 to $14,000,000. Taking $4,000,000 as a reasonable average the cost per lane is $500,000 and b_c, the cost in capital per vehicle and per mile, is $62.

- a_{rls} (input-output coefficients for the centralized industry): The centralized industry is assumed to be in the central square and so the land input is not taken into account. Only one production possibility is considered. It is assumed that this industry does not use as input the product of the decentralized industry, so that two coefficients need to be specified, a_{311} and a_{411}, respectively the amount of labor per day and of capital needed to produce one dollar of centralized industry output per day. a_{311} was taken as .015, and a_{411} as 350., which implies that 36% of the output value goes to labor and 17.5% to capital.

- a_{r2s} (input-output coefficients for the decentralized industry): For each acitivty s, 4 parameters need to be specified: a_{12s}, a_{32s}, a_{42s} and a_{52s}, the input coefficients for the commodities of the centralized industry, labor, capital and land. The capital input includes the cost of the plant and equipment. In 1969, in the United States, the real net value of structures and equipment in the manufacturing industry was 129 billion dollars (62% equipment, 38% structure); the value added by manufacture was 305 billion dollars. This corresponds to 570 dollars in capital to produce one dollar a day of output. Also in 1969, the value added per dollar of wage was 3.28 dollars, which implies 1.22×10^{-2} worker-days to produce one dollar of output a day. The input costs are considered to make up 85% of the total selling price. So, .26 unit of the centralized product is used for each unit produced. The cost for construction or major alteration, in industrial and commercial buildings, in 1970 in the US, was 17.1 dollars per square foot Using 38% of 570 dollars as the structure cost for one dollar of output, it is possible to infer the average surface used to produce this output as 12.7 square feet or $.46 \times 10^{-6}$ square mile. It is assumed that the technology makes production in a multifloor building much more expensive (vertical transportation, reinforced physical structure, etc) than one in a single floor. Also, more labor and service inputs are needed, due to the vertical transportation. The parameter values used are displayed in Table 2.

- a_{r3s} (input-output coefficients for housing): The average expenditure of the disposable income of an American household in 1969 was 13.5% for transportation, 15% in housing and 14.1% in household operations (gas, furniture, water, etc). Adjusting the wage rate to include transportation, and using I, the capital investment in housing and household operations can then be calculated as \$16,750. All other consumption of the households is supposed to consist of centralized goods. Data from (Needleman,1965) suggest that it is reasonable to assume 1,000 square feet as the average surface of a housing unit (3.59×10^{-5} square miles). Using (Hoch,1969) and (Needleman,1965) the following parameter value for multi-story structures seem plausible (see Table 3).

b. Size of the linear program and computation time

The Basic Model using auxilary equations for obtaining some interesting dual variables has $\bar{\bar{r}}(3+\bar{u}) + \bar{u}[3+2(\bar{k}-1)]$ constraints and $2\bar{\bar{r}}(2+\bar{u}) + 2\bar{k}\bar{u} + \bar{s}[\bar{r}-1 + \bar{u}(\bar{\bar{r}}-\bar{r}+1)]$ variables. Typical dimensions for the tableau are 132 rows and 236 columns with a density of non-zero terms of .045. Using a CDC 6400 and the AIPHAC linear programming system available for that machine, this typical program uses 62 records of CPU time. The Dynamic model tends to be somewhat bigger, typical values for the tableau are 216 rows and 340 columns needing 150 seconds of CPU time for its solution.

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export of $\bar{x}_2$ per day (x \$$10^6$)		4.5	5.5	6.5	7.5
Basic Model	Average cost	1.451	1.46	1.474	1.488
	Marginal cost	1.489	1.539	1.551	1.56
Basic Model	Average cost	1.471	1.49	1.51	1.535
exports through center only	Marginal cost	1.597	1.61	1.65	1.661

Table 1: Cost comparisons for Basic model and Basic model with exports through center only.

s	centralized input	labor	capital	land
1	.260	.0122	570.	$.46\text{x}10^{-6}$
2	.262	.0124	630.	$.25\text{x}10^{-6}$
3	.265	.0126	680.	$.17\text{x}10^{-6}$
4	.270	.0128	720.	$.14\text{x}10^{-6}$

Table 2: Input coefficients for decentralized industry.

# of floors	centralized input	capital	land
1	14.0	16,000.	$.359\text{x}10^{-4}$
2	14.1	17,000.	$.202\text{x}10^{-4}$
4	14.2	18,200.	$.98\text{x}10^{-5}$
8	14.3	19,400.	$.50\text{x}10^{-5}$

Table 3: Input coefficients for housing.

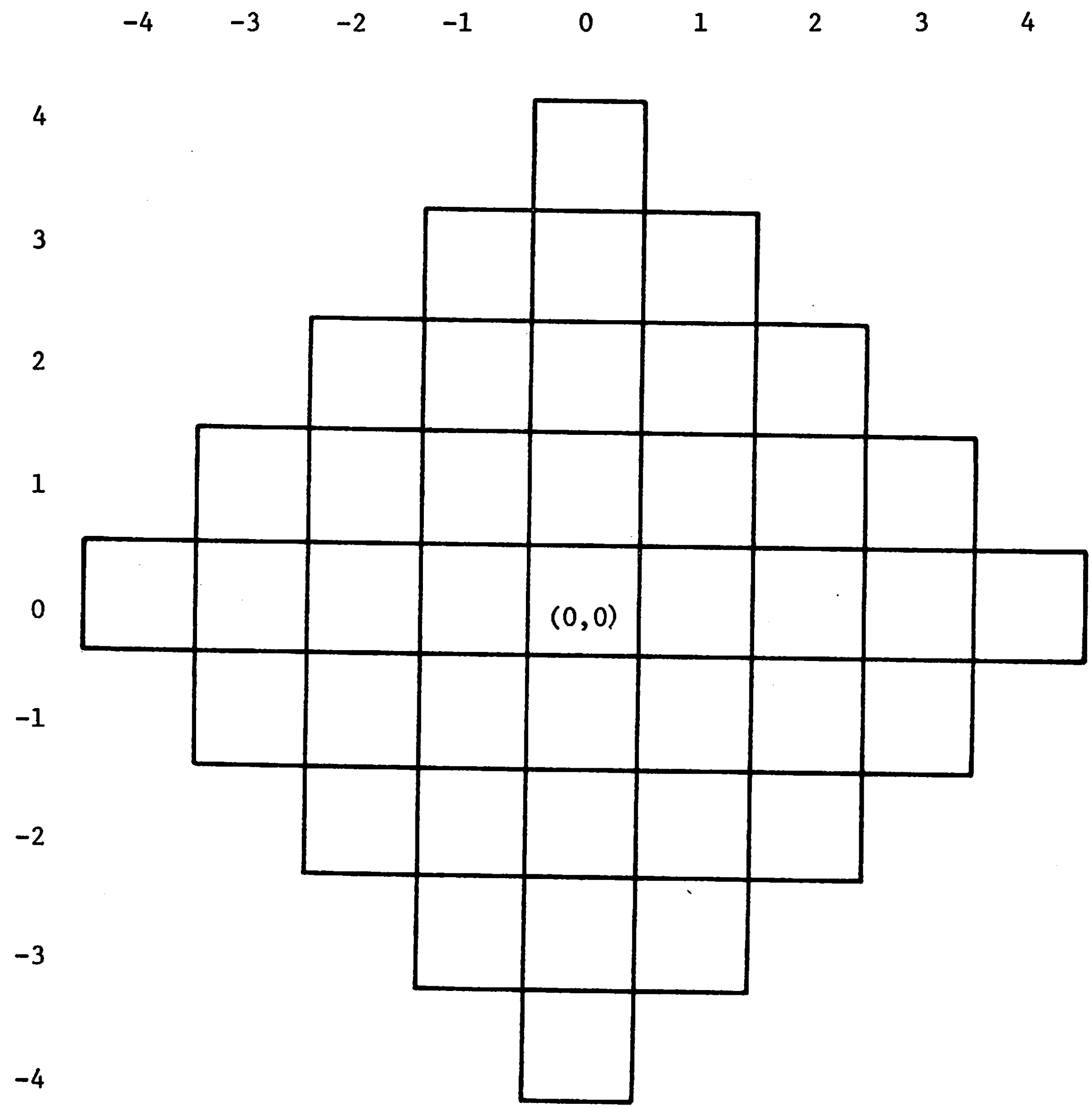

Fig. 1. City geometry

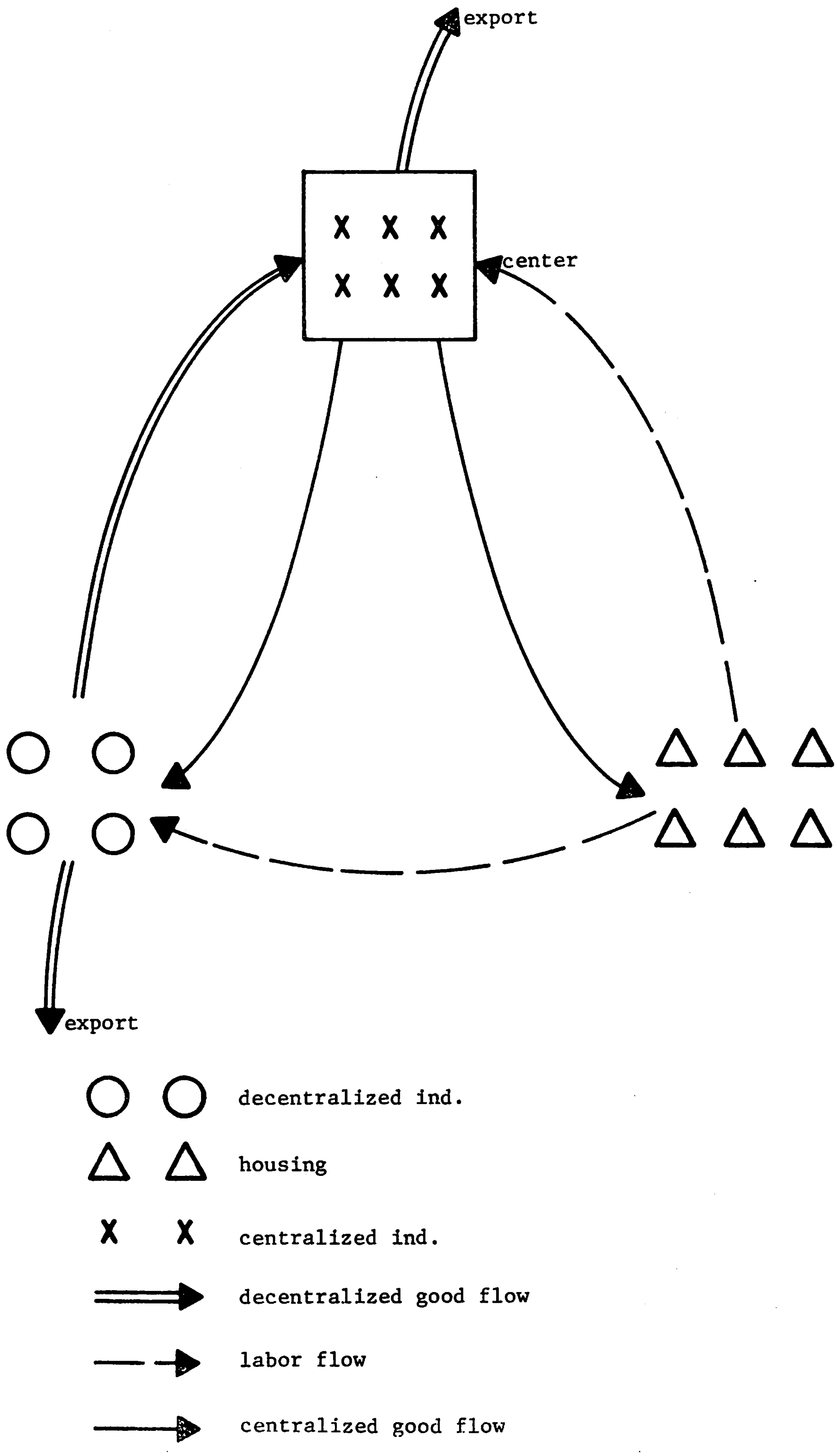

Fig. 2. Flows in the Basic model

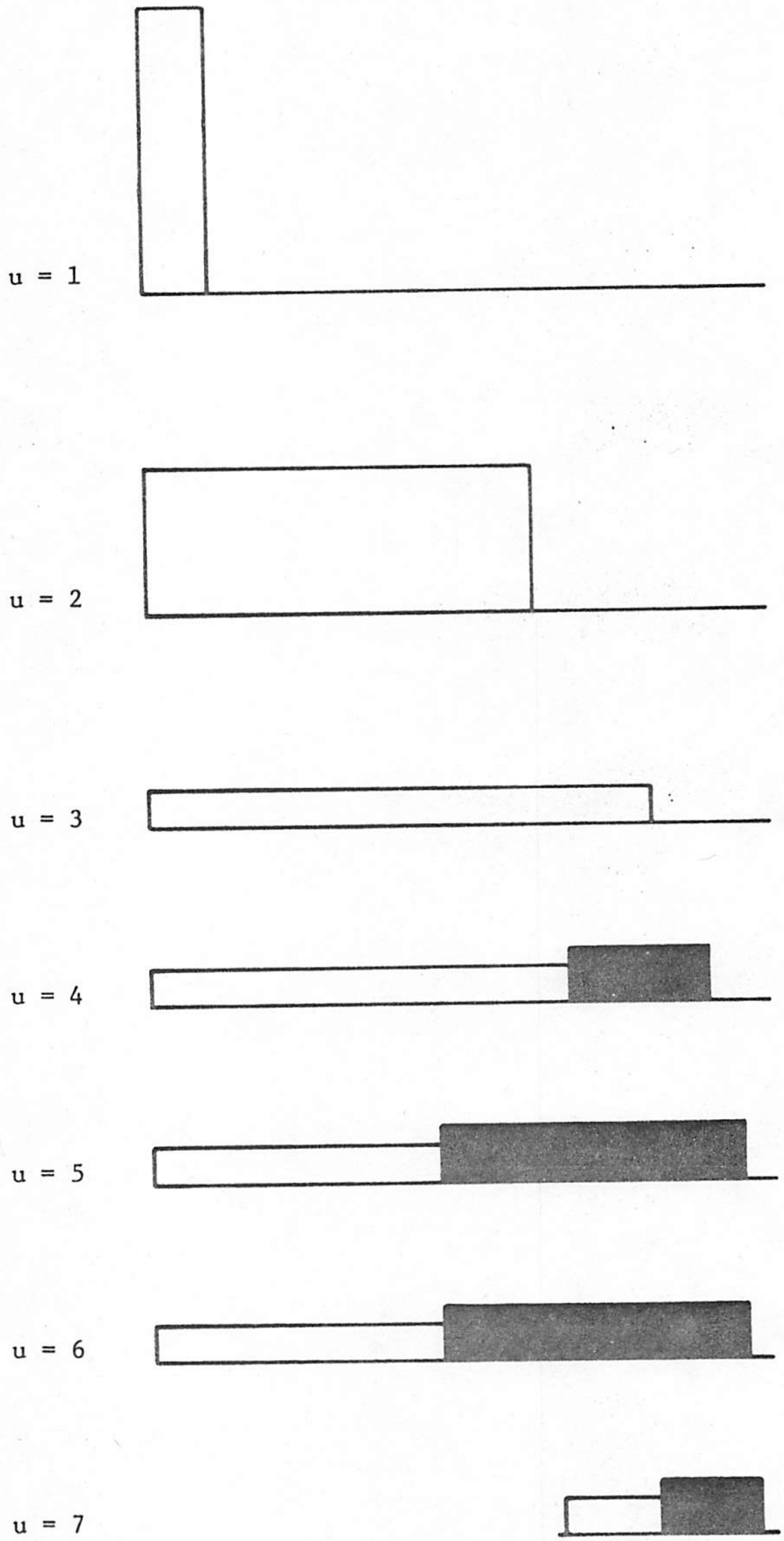

Fig. 3. Basic model allocation, $\bar{x}_2 = 4.5 \times 10^6$

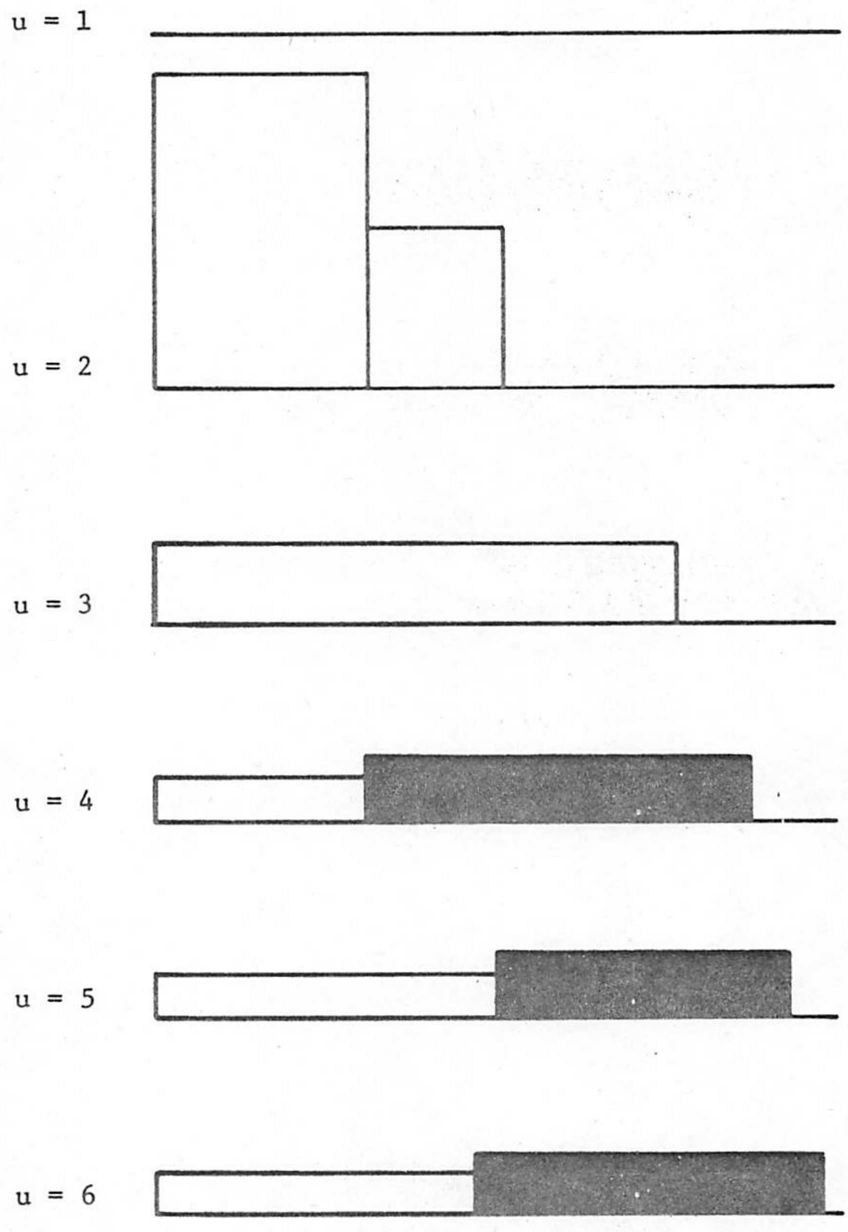

Fig. 4. Basic model allocation, export through center, $\bar{x}_2 = 4.5 \times 10^6$

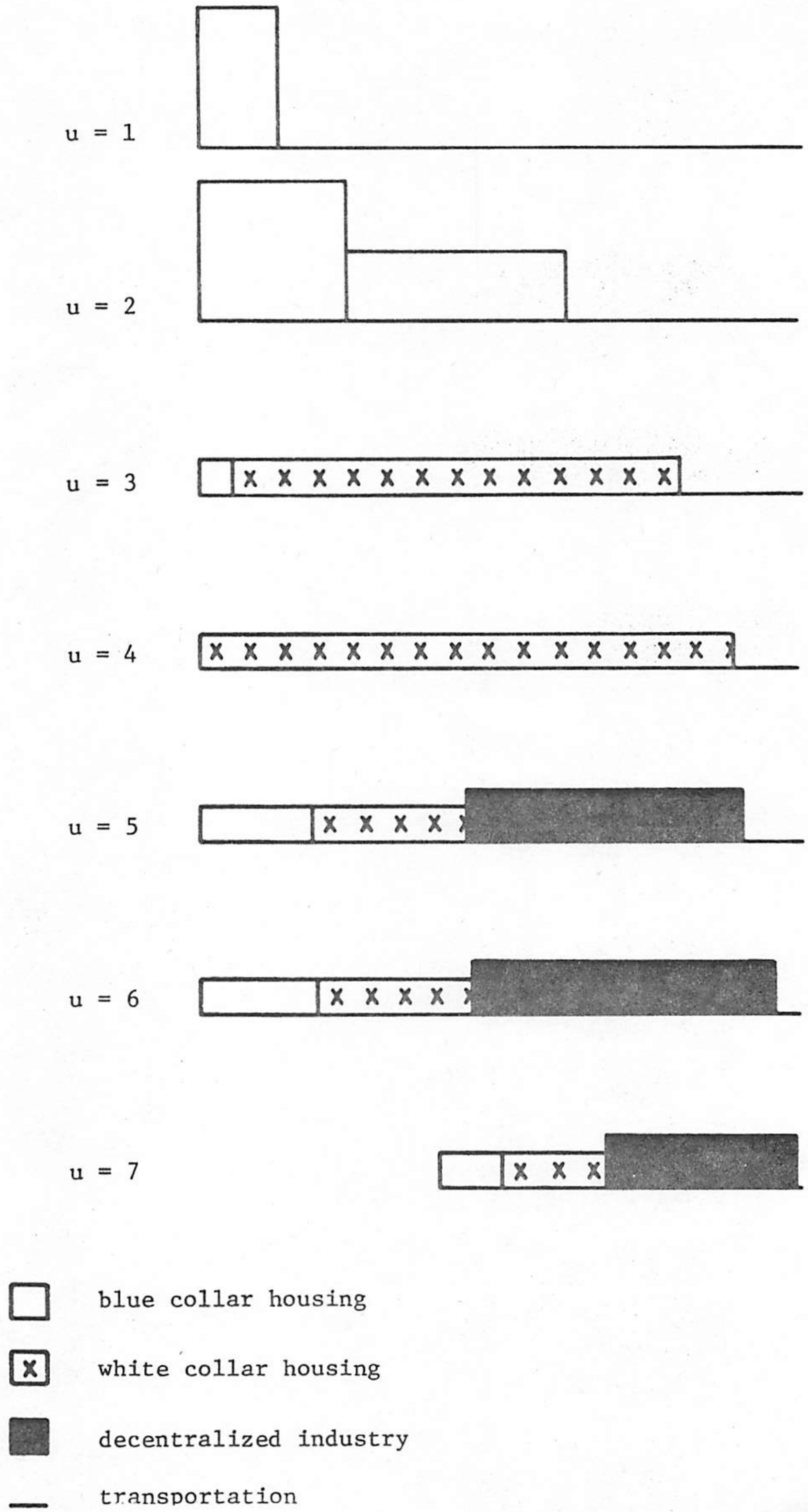

Fig. 5. Labor model allocation, $\bar{x}_2 = 4.5 \times 10^6$

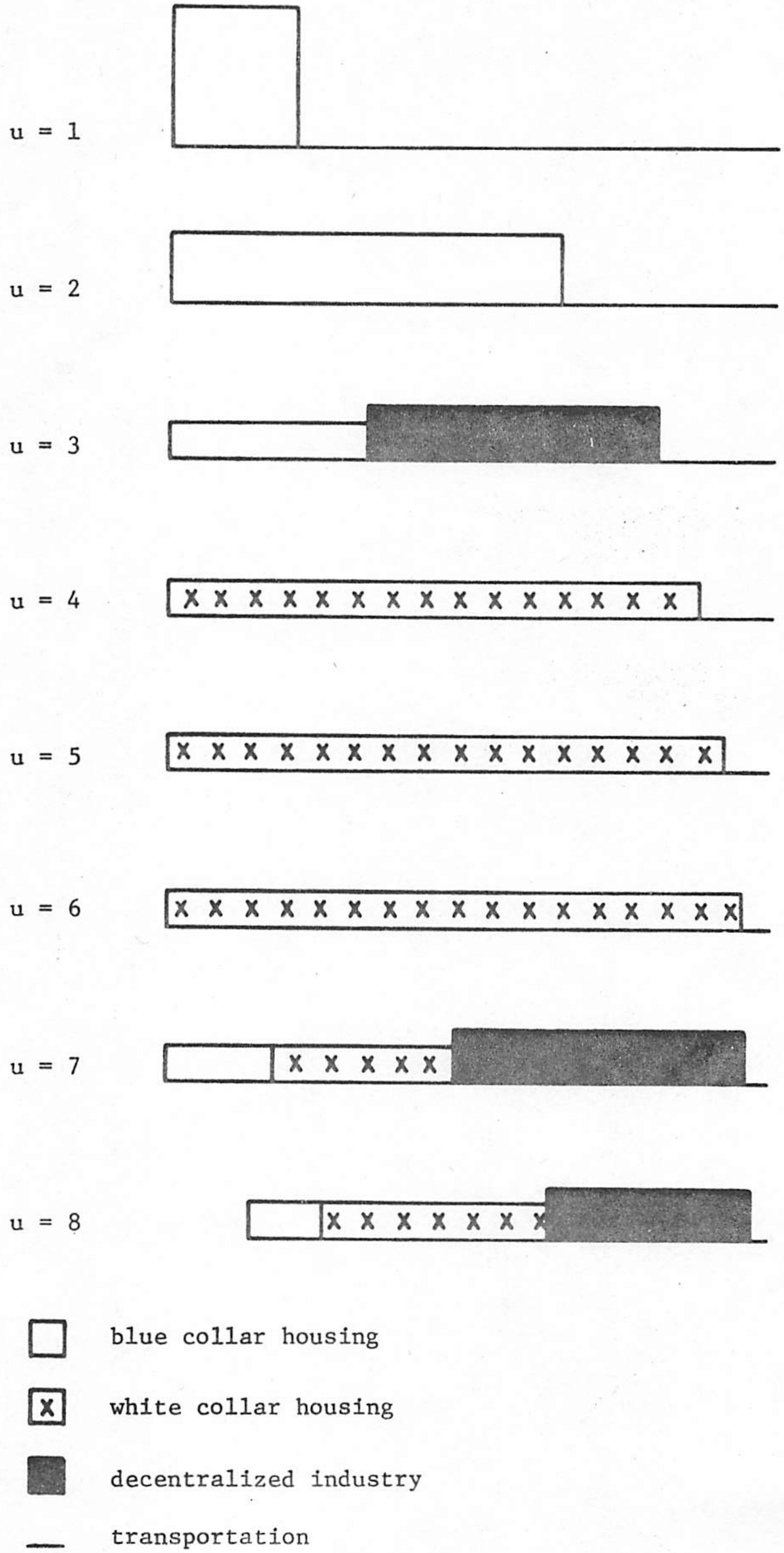

Fig. 6. Labor model allocation with elastic demand, $\bar{x}_2 = 4.5 \times 10^6$

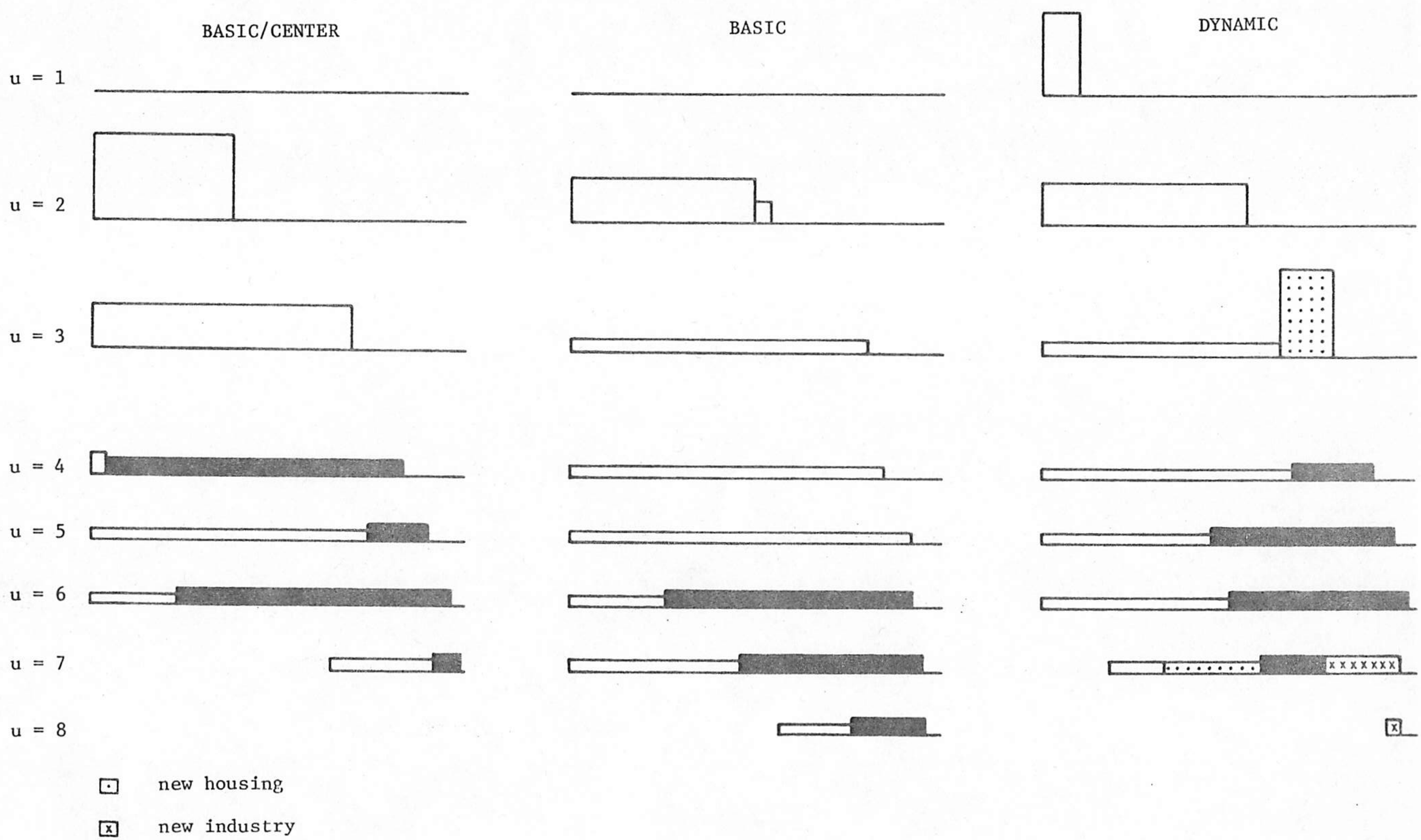

Fig. 7. Basic model. Basic model with export through center only, Dynamic model allocation, $\bar{x}_2 = 5.5 \times 10^6$

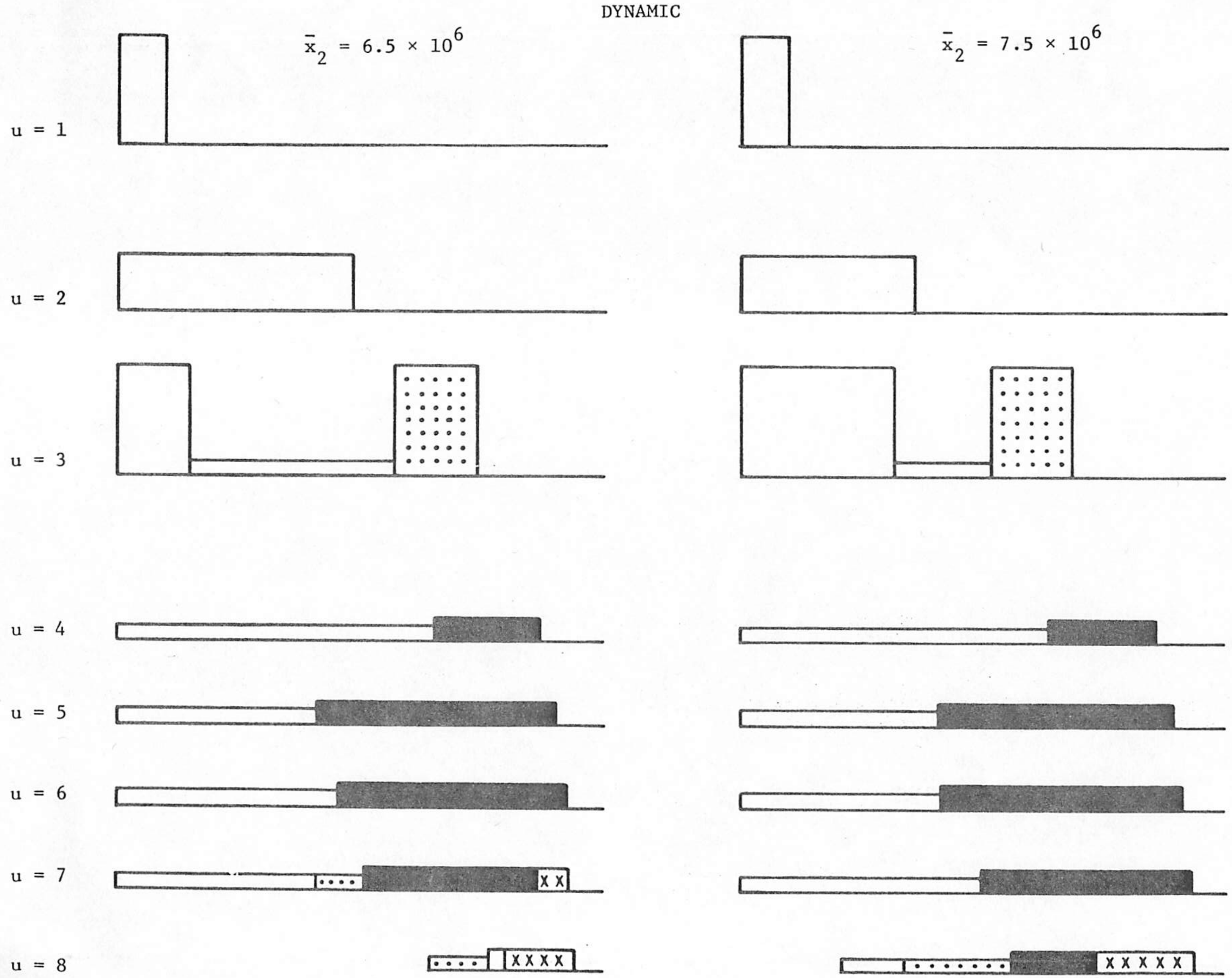

Fig. 8. Dynamic model allocations, $\bar{x}_2 = 6.5 \times 10^6$, $\bar{x}_2 = 7.5 \times 10^6$