# N-PERSON STOCHASTIC DIFFERENTIAL GAMES

by

Pravin Varaiya

Memorandum No. ERL-M447

12 June 1974

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

# N-PERSON STOCHASTIC DIFFERENTIAL GAMES*

Pravin Varaiya

Department of Electrical Engineering and Computer Sciences, and Electronics Research Laboratory, University of California, Berkeley, California 94720 U.S.A.

ABSTRACT. Necessary and sufficient conditions are given for the non-cooperative equilibrium policies of N players when they are simultaneously controlling the evolution of a stochastic system described by an Ito equation. In the case of perfect information, these conditions are generalizations of the well-known Hamilton-Jacobi equations. Conditions are also indicated for the case when the players have only partial information. Sufficient conditions are derived which guarantee that an equilibrium is also Pareto-efficient.

## 1. INTRODUCTION AND SUMMARY

We apply the results obtained in [1] to study the equilibrium policies of N players when they are simultaneously controlling the evolution of a system described by the stochastic functional differential equation

$$dz_t = f(t,z,u_t^1,\ldots,u_t^N) + dB_t \quad , \quad t \in [0,1] \tag{1.1}$$

Here $\{z_t\}$ is the "state" process and $\{B_t\}$ is a vector of independent Brownian movements. The "drift" f depends at any time t on the past $\{z_s, s \leq t\}$ of the state and also on the controls $u_t^i$ of the ith player, i=1,...,N. $u_t^i$ takes values in a fixed metric space $U_i$ and depends on the past $\{y_s^i, s \leq t\}$ of the observations made by i. The cost incurred by i is

*Research sponsored by the Joint Services Electronics Program, Contract F44620-71-C-0087.

$$J^i(u) = E\left[\int_0^1 h^i(t,z,u_t)dt\right] \tag{1.2}$$

where $\{u_t\} = \{u_t^1,\ldots,u_t^N\}$.

A set of policies $\{u_t^*\} = \{u_t^{1*},\ldots,u_t^{N*}\}$ is a (non-cooperative) <u>equilibrium</u> if for all i

$$J^i(u^*) \leq J^i({}^iu^*,u^i) \text{ for all } u^i \tag{1.3}$$ [1]

Thus $u^*$ is an equilibrium iff $u^{*i}$ is a policy which minimizes (1.2) when for all $j \neq i$ player j adopts the policy $u^{j*}$. This trivial fact allows us to use the results of [1] to obtain the equilibrium conditions. $u^*$ is <u>efficient</u> if for all $v = \{v^1,\ldots, v^N\}$

$$J^i(v) \leq J^i(u^*) \text{ for all } i$$

implies $J^i(v) = J^i(u^*)$ for all i. Evidently, if there exist numbers $\mu_1 > 0,\ldots,\mu_N > 0$ such that

$$\sum_i \mu_i J^i(u^*) \leq \sum_i \mu_i J^i(v) \text{ for all } v, \tag{1.4}$$

then $u^*$ is efficient. But (1.4) means that u* is an optimal control for the cost $\sum \mu_i J^i$ and so we can once again apply the results of [1] to obtain efficiency conditions.

These conditions are straightforward extensions of the well-known Hamilton-Jacobi equations when the game is of complete information i.e., when $y_t^i \equiv z_t$ for all i. When the information is incomplete the conditions are much more complex.

The paper is organized as follows. The next section introduces some background material dealing with the interpretation of the Ito equation (1.1), after which the relevant results of [1] are displayed. Sections 3 and 4 treat respectively the case of complete and incomplete information. Section 5 discusses some difficulties connected with the notion of efficiency in the case of incomplete information.

---

[1] We adopt the notation $({}^iu,v^i) = (u^1,\ldots,u^{i-1},v^i,u^{i+1},\ldots,u^N)$

## 2. RESULTS FROM OPTIMAL CONTROL THEORY

### 2.1 Specification of the dynamics

Let $C^k$ be the set of all continuous functions from [0,1] into $R^k$. Let $\xi^k$ be the evaluation functional on $C^k$, and, for $t \in [0,1]$, let $\mathcal{F}_t^k$ be the $\sigma$-field of subsets of $C^k$ generated by $\{\xi_s^k, s \leq t\}$. $\mathcal{A}^k$ is the $\sigma$-field of subsets of $[0,1] \times C^k$ such that a function g on $[0,1] \times C^k$ is $\mathcal{A}^k$ measurable iff $g(t,\cdot)$ is $\mathcal{F}_t^k$ measurable for all t and $g(\cdot,x)$ is Lebesgue measurable for each $x \in C^k$; thus $\mathcal{A}^k$ measurable functions are <u>non-anticipative</u>.

The state process $\{z_t\}$ is n-dimensional. The <u>i</u>th player's observation process $\{y_t^i\}$ is a $n_i$-dimensional subvector of $\{z_t\}$. The components of the drift f corresponding to $y^i$ are denoted by the vector $f^i$, The sample paths of $\{z_t\}$ are continuous, hence they lie in $C^n$ whereas those of $\{y_t^i\}$ lie in $C^{n_i}$. We can now define the admissible control policies.

$U_i$ is a separable metric space and its Borel field is $\mathcal{V}_i$. A <u>policy for player i</u> is a measurable function $u^i$: $([0,1] \times C^{n_i}, \mathcal{A}^{n_i}) \to (U_i, \mathcal{V}_i)$. The set of such policies is denoted $\mathcal{U}_i$. Let $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_N$, similarly for U, $\mathcal{V}$. The following conditions are imposed on f:

(i) $f: [0,1] \times C^n \times U \to R^n$ is measurable with respect to $\mathcal{A}^n \times \mathcal{V}$,

(ii) there exists K such that $|f(t,z,u)| \leq K(1 + \|x\|)$ for all $(t,z,u)$.

Here $|\cdot|$ is the norm in $R^n$ and $\|\cdot\|$ is the sup norm in $C^n$. The functions $h^i$ in (1.2) are assumed to satisfy the condition corresponding to (i) above and in addition the $h^i$ are assumed non-negative and uniformly bounded.

### 2.2 Solutions of (1.1)

Let P be Wiener measure on $(C^n, \mathcal{F}_1^n)$. Let z be the evaluation functional on $C^n$ so that $\{z_t, \mathcal{F}_t^n, P\}$ is a standard, n-dimensional, Brownian movement. For $u \in \mathcal{U}$ define the corresponding drift $\{\phi_t^u, \mathcal{F}_t, P\}$ by

$$\phi_t^u(z) = f(t,z,u^1(t,y^1),\ldots,u^N(t,y^N))$$

Recall that $y^i$ is a subvector of z. For future reference let $\mathcal{Y}_t^i$ be the sub $\sigma$-field of $\mathcal{F}_t^n$ generated by $\{y_s^i, s \leq t\}$. Also for each u define the non-negative random variable $\rho^u$ by

$$\rho^u = \exp\left[\int_0^1 \phi_t^u \, dz_t - \frac{1}{2}\int_0^1 |\phi_t^u|^2 dt\right]$$

**Theorem 2.1** [2,3,4] Under the above-stated assumptions on $f$, $\int_{C^n} \rho^u(z)\, P(dz) = 1$. Hence $P^u$ is a probability measure on $(C^n, \mathcal{F}_1^n)$ where

$$P^u(F) = \int_F \rho^u(z)\, P(dz)\ , \ F \in \mathcal{F}^n$$

Furthermore, the process $\{w_t^u, \mathcal{F}_t^n, P^u\}$ defined by

$$w_t^u = z_t - \int_0^t \phi_s^u\, ds$$

is a Brownian movement.

This result justifies the following definition. The <u>solution</u> of (1.1) corresponding to a policy $u \in \mathcal{U}$ is the process $\{z_t, \mathcal{F}_t^n, P^u\}$. Thus the impact on the system of a policy u is summarized by the probability distribution $P^u$.

## 2.3 Optimality conditions

Suppose N = 1. We can then drop the index i. For $u \in \mathcal{U}$, define the process $\{W_t^u, \mathcal{Y}_t, P^u\}$ by

$$W_t^u = \underset{v \in \mathcal{U}}{\text{infimum}}\ E^u\left[\int_t^1 h(s,z,v_s)ds \,\middle|\, \mathcal{Y}_t\right].$$

u is <u>value decreasing</u> if $\{W_t^u\}$ is a supermartingale i.e., if

$$E^u[W_{t+\delta}^u | \mathcal{Y}_t] \le W_t^u \quad \text{a.s. for all } t,\ \delta > 0.$$

$u^*$ is <u>optimal</u> if $J(u^*) \le J(u)$ for all $u \in \mathcal{U}$. It is known that an optimal policy is value decreasing [1, p. 242].

**Theorem 2.2** [1] $u^*$ is optimal iff there exists a constant $J^*$ and for each value decreasing u there exist processes $\{\Lambda V_t^u\}$, $\{\nabla V_t^u\}$, taking values in R and $R^m$ respectively (where m is the dimension of the observation process y), adapted to $\mathcal{Y}_t$, and satisfying the following conditions:

(i) $x_1^u = 0$, where

$$x_t^u = J^* + \int_0^t \Lambda V_s^u\, ds + \int_0^t \nabla V_s^u\, dy_s$$

(ii) $\Lambda V_t^u + \nabla V_t^u \hat{f}^y(t,z,u_t) + \hat{h}(t,z,u_t) \geq 0 = \Lambda V_t^{u^*} + \nabla V_t^{u^*} \hat{f}^y(t,z,u_t^*) + \hat{h}(t,z,u_t^*)$ for all $t,z,u_t$.

Then $x_t^{u^*} = W_t^{u^*}$ and $J^* = J(u^*)$ is the minimum cost. (Here $f^y$ is the subvector of f corresponding to y. $\hat{f}^y(t,z,u_t) = E^u[f^y(t,z,u_t) | \mathcal{Y}_t]$, and $\hat{h}$ is defined similarly).

Theorem 2.3 [1] Suppose $y_t \equiv z_t$. $u_t^*$ is optimal iff there exists $J^*$ and processes $\{\Lambda V_t\}$, $\{\nabla V_t\}$, taking values in R and $R^n$ respectively, adapted to $\mathcal{Y}_t = \mathcal{F}_t^n$, and satisfying the following conditions:

(i) $x_1 = 0$, where

$$x_t = J^* + \int_0^t \Lambda V_s \, ds + \int_0^t \nabla V_s \, dz_s$$

(ii) $\Lambda V_t + \nabla V_t \, f(t,z,u) + h(t,z,u) \geq 0 = \Lambda V_t + \nabla V_t \, f(t,z,u_t^*) + h(t,z,u_t^*)$ for all $t,z,u$.

Then $x_t = W_t^{u^*}$ and $J^*$ is the minimum cost.

## 3. EQUILIBRIUM CONDITIONS: COMPLETE INFORMATION

### 3.1 Equilibrium conditions

The next result is then an immediate consequence of Theorem 2.2.

Theorem 3.1 (Equilibrium condition) $\{u_t^*\} = \{u_t^{1*},\ldots,u_t^{N*}\}$ is an equilibrium iff for each i there exist $J^{i*}$ and process $\{\Lambda V_t^i\}$, $\{\nabla V_t^i\}$ adapted to $\mathcal{F}_t^n$ satisfying the following conditions:

(i) $x_1^i = 0$, where

$$x_t^i = J^{i*} + \int_0^t \Lambda V_s^i \, ds + \int_0^t \nabla V_s^i \, dz_s \tag{3.1}$$

(ii) $\Lambda V_t^i + \nabla V_t^i \, f(t,z,u_t^{1*},\ldots,u^i,\ldots,u_t^{N*}) + h^i(t,z,u_t^{1*},\ldots,u^i,\ldots,u_t^{N*}) \geq 0 = \Lambda V_t^i + \nabla V_t^i \, f(t,z,u_t^*) + h^i(t,z,u_t^*)$, (3.2)
for all $t,z,u^i$.

Then $J^{i*} = J^i(u^*)$. Furthermore

$$x_t^i = \underset{u^i \in \mathcal{U}_i}{\text{infimum}} \; E^{u^*}\left[\int_t^1 h^i(s,z,u_s^{1*},\ldots,u_s^i,\ldots,u_s^{N*}) \, ds \,|\, \mathcal{F}_t^n\right] \tag{3.3}$$

As a special case of this result we can deduce the conditions for a saddle point policy in a 2-player, zero-sum game. So suppose $N=2$ and $h^2=-h^1$. A policy $u_t^* = u_t^{1*}, u_t^{2*}$ is a saddle point if

$$J^1(u^1,u^{2*}) \geq J^1(u^{1*},u^{2*}) \geq J^1(u^{1*},u^2) \text{ for all } u^1,\ u^2. \tag{3.4}$$

Theorem 3.2 (Saddle point condition) $\{u_t^*\} = \{u_t^{1*}, u_t^{2*}\}$ is a saddle point iff there exists $J^{1*}$ and processes $\{\Lambda V_t^1\}$, $\{\nabla V_t^1\}$ adapted to $\mathcal{F}_t^n$ satisfying the following conditions:

(i) $x_1^1 = 0$ where

$$x_t^1 = J^{1*} + \int_0^t \Lambda V_s^1\, ds + \int_0^t \nabla V_s^1\, dz_s \tag{3.5}$$

(ii) $\Lambda V_t^1 + \nabla V_t^1 f(t,z,u^1,u_t^{2*}) + h^1(t,z,u^1,u_t^{2*}) \geq 0 = \Lambda V_t^1 + \nabla V_t^1$
$f(t,z,u_t^*) + h^1(t,z,u_t^*) = 0 \geq \Lambda V_t^1 + \nabla V_t^1\, f(t,z,u_t^{1*},u^2)$
$+ h^1(t,z,u_t^{1*},u^2)$ for all $t,z,u^1,u^2$ (3.6)

Then $J^{1*} = J^1(u^*)$ is the value of the game and

$$x_t^1 = \inf_{u^1} E^{u^*}\Big[\int_t^1 h^1(x,z,u_s^1,u_s^{2*})ds\,\Big|\,\mathcal{F}_t^n\Big] = \sup_{u^2} E^{u^*}\Big[\int_t^1 h^1(s,z,u_s^{1*}, u_s^2)ds\,\Big|\,\mathcal{F}_t^n\Big] \tag{3.7}$$

is the value function.

We give this result a form similar to that which has already appeared in the literature [5-8]. Define the Hamiltonian functional $H: [0,1] \times C^n \times U \times R^n \to R$ by $H(t,z,u^1,u^2,p) = pf(t,z,u^1,u^2) + h^1(t,z,u^1,u^2)$.

Then (3.6) is the Isaacs condition,

$$H(t,z,u_t^{1*},u_t^{2*},\nabla V_t) = \operatorname*{Max}_{u^2} \operatorname*{Min}_{u^1} H(t,z,u^1,u^2,\nabla V_t) = \operatorname*{Min}_{u^1} \operatorname*{Max}_{u^2} H(t,z,u^1,u^2,\nabla V_t)$$

Next, suppose that (1.1) is a diffusion equation i.e., the dependence at time t of f on z is through $z_t$:

$$dz_t = f(t,z_t,u_t^1,u_t^2)\, dt + dB_t,$$

and suppose further that $u_t^*$ has the same property i.e., $u^{1*}(t,z) = u^{1*}(t,z_t)$. Then the solution $\{z_t, \mathcal{F}_t^n, P^{u^*}\}$ is a diffusion, and hence from (3.7) it follows that the value function at time t depends only on $z_t$, i.e., there is a function V on $[0,1] \times R^n$ such that $x_t^1 \equiv V(t,z_t)$. Secondly, if this function is sufficiently smooth then by Ito's differential rule we can identify the processes $\{\Lambda V_t^1\}$ and $\{\nabla V_t^1\}$ as $\Lambda V_t^1 = \frac{\partial V}{\partial t}(t,z_t) + \frac{1}{2}\sum_{i,j} \frac{\partial^2 V}{\partial z_i \partial z_j}(t,z_t)$, $\nabla V_t^1 = \frac{\partial V}{\partial z}(t,z_t)$. Combining these two observations yields the well-known Hamilton-Jacobi partial differential equation for the value function,

$$\frac{\partial V}{\partial t}(t,z) + \frac{1}{2}\sum_{i,j} \frac{\partial^2 V(t,z)}{\partial z_i \partial z_j} + \underset{u^2}{\text{Max}}\ \underset{u^1}{\text{Min}}\ H(t,z,u^1,u^2,\frac{\partial V}{\partial z}(t,z)) = 0.$$

for $(t,z) \in [0,1] \times R^n$; and (3.5) yields the boundary condition $V(1,z) \equiv 0$.

## 3.2 Efficiency conditions

We return to the N-player game of complete information. The next result is immediate from our earlier remarks.

Theorem 3.3 (Sufficiency conditions) $\{u_t^*\} = \{u_t^{1*},\ldots,u_t^{N*}\}$ is an efficient equilibrium if for each i there exist $\mu_i > 0$, $J^{i*}$ and processes $\{\Lambda V_t^i\}$, $\{\nabla V_t^i\}$ adapted to $\mathcal{F}_t^n$, and satisfying conditions (3.1), (3.2) and

$$\sum_i \mu_i\{\Lambda V_t^i + \nabla V_t^i f(t,z,u_t^*) + h^i(t,z,u_t^*)\} = 0$$

$$= \underset{u \in U}{\text{Min}} \sum_i \mu_i\{\Lambda V_t^i + \nabla V_t^i f(t,z,u) + h^i(t,z,u)\} \tag{3.8}$$

Condition (3.8) appears to be a very stringent condition. It turns out, however, that if a certain convexity condition is satisfied, then this condition is also necessary for efficiency. We say that the convexity assumption holds if for all t, z the (N+n) - dimensional set

$$\{(h^1(t,z,u),\ldots,h^N(t,z,u),\ f(t,z,u))\,|\,u \in U\}$$

is convex. Now replace the original game by the following one. The dynamics of this game are given by a (N+n)-dimensional Ito equation

$$
\begin{aligned}
dq_t^1 &= h^1(t,z,u_t)\,dt + d\beta_t^1 \\
&\vdots \\
dq_t^N &= h^N(t,z,u_t)\,dt + d\beta_t^N \\
dz_t &= f(t,z,u_t)\,dt + dB_t
\end{aligned}
\tag{3.9}
$$

where $(\beta, B)$ is an $(N+n)$-dimensional Brownian movement. The cost incurred by the $i$th player is

$$
J^i(u) = E\, q_1^i \tag{3.10}
$$

It is evident that the two games are equivalent. What we have achieved by this transformation is to remove from (3.9) the explicit dependence on the policy $\{u_t\}$. Next, as in Section 2.2, for $u \in \mathcal{U}$ define

$$
\rho^u = \exp\Big[\sum_i \int_0^1 h^i(t,z,u_t(z))\,dq_t^i + \int_0^1 f(t,z,u_t(z))\,dz_t - \frac{1}{2}\sum_i \int_0^1 |h_t^i|^2\,dt - \frac{1}{2}\int_0^1 |f_t|^2\,dt\Big]
$$

and let the set of all such random variables be

$$
\mathcal{R} = \{\rho^u \mid u \in \mathcal{U}\}
$$

Then

$$
J^i(u) = \int_{C^{N+n}} q_1^i\, \rho^u(q,z)\, P(dq,dz) = \mathcal{J}^i(\rho^u) \text{ say}
$$

where $P(dq,dz)$ is Wiener measure on $(C^{N+n}, \mathcal{A}^{N+n})$. Note that the map $\mathcal{J} : \mathcal{R} \to R^N$ defined by $\mathcal{J}(\rho) = (\mathcal{J}^1(\rho), \ldots, \mathcal{J}^N(\rho))$ is <u>linear</u>. The next result is proved in [3] and [4].

<u>Lemma 3.1</u> If the convexity assumption holds then $\mathcal{R}$ is a convex subset of $L^1(C^{n+N}, \mathcal{F}_1^{n+N}, P)$

<u>Theorem 3.4</u> (Efficiency conditions) Suppose the convexity assumption holds. Then $\{u_t^*\} = \{u_t^{1*}, \ldots, u_t^{N*}\}$ is an efficient equilibrium iff for each $i$ there exist $\mu_i > 0$, $J^{i*}$ and processes $\{\Lambda v_t^i\}$, $\{\nabla v_t^i\}$ adapted to $\mathcal{F}_t^n$, and satisfying conditions (3.1), (3.2), (3.8).

Proof The sufficiency follows from Theorem 3.3. To prove the necessity let $u_t^*$ be an efficient equilibrium and suppose $J^{i*}$, $\Lambda V_t^i, \nabla V_t^i$ satisfy (3.1), (3.2), and (3.3). By lemma 3.1 the set $\Gamma = \{(J^1(u),\ldots,J^N(u)),\ u \in \mathcal{U}\}$ is a convex subset of $R^N$. By efficiency $\Gamma$ is disjoint from the convex set $\{(J^{1*}+x_1,\ldots,J^{N*}+x_N),\ x_i \le 0$ all $i$ and $\{x_i \ne 0\}$. By the separation theorem for convex sets there exist $\mu_i > 0$ such that

$$\sum_i \mu_i J^{i*} \le \sum_i \mu_i J^i(u) \quad \text{for all } u \in \mathcal{U},$$

i.e., $u^*$ is an optimal control for the cost functional $\sum_i \mu_i J^i(u)$. Hence (3.8) must hold by Theorem 2.2. ¤

It turns out in fact that this result is true without the convexity assumtpion. The proof is much more involved unfortunately. The result implies that an efficient equilibrium is more stable than may appear at first sight. Recall the following definition. A policy $\{u_t^*\}$ is in the core if for every subset of players $S \subset \{1,\ldots,N\}$ the following property holds: for all policies $\{v_t^S\}$, whenever

$$J^i(v_t^S, u_t^{S'*}) \le J^i(u_t^*) \quad , \quad i \in S.$$

then

$$J^i(v_t^S, u_t^{S'*}) = J^i(u_t^*) \quad , \quad i \in S.$$

Here $(v_t^S, u_t^{S'*})$ is the policy $\{u_t\}$ where $u_t^i = v_t^i$, $i \in S$ and $u_t^i = u_t^{i*}$ for $i \notin S$. Theorem 3.4 immediately yields the following remarkable corollary.

Corollary 3.1 Suppose the convexity assumption holds. Then the set of efficient equilibrium policies coincides with the core.

## 4. EQUILIBRIUM CONDITIONS: INCOMPLETE INFORMATION

We return to the game of incomplete information. The following result is immediate from Theorem 2.2.

Theorem 4.1 $\{u_t^*\} = \{u_t^{1*}(y^1),\ldots,u_t^{N*}(y^N)\}$ is an equilibrium policy iff for each $i$ there exists a constant $J^{i*}$ and for every value decreasing $\{u_t^i(y^i)\}$ there exist processes $\{\Lambda V_t^{iu^i}\}$, $\{\nabla V_t^{iu^i}\}$ taking values in $R$, $R^{n_i}$ respectively and adapted to $\mathcal{Y}_t^i$, such that the following conditions hold:

(i) $x_1^{iu^i} = 0$ where

$$x_t^{iu^i} = J^* + \int_0^t \Lambda V_s^{iu^i} ds + \int_0^t \nabla V_s^{iu^i} dy_s^i$$

(ii) $\Lambda V_t^{iu^i} + \nabla V_t^{iu^i} \hat{f}^i(t,z,{}^i u^*, u_t^i) + \hat{h}^i(t,z,{}^i u_t^*, u_t^i) \geq 0$

$$= \Lambda V_t^{iu^{*i}} + \nabla V_t^{iu^{*i}} \hat{f}^i(t,z,u_t^*) + \hat{h}^i(t,z,u_t^*), \text{ for all } t,z,\{u_t^i\}$$

Furthermore $J^{i*} = J^i(u^*)$ and

$$x_t^{iu^{i*}} = \underset{u^i \in \mathcal{U}^i}{\text{infimum}} \; E^{u^*}\left[\int_t^1 h^i(s,z,{}^i u_s^*, u_s^i)\, ds \,\middle|\, \mathcal{Y}_t^i\right] \tag{4.1}$$

This result is not of great interest. However some interesting observations can be deduced. We give one instance. Suppose the game is <u>constant-sum</u> i.e., suppose

$\int_t^1 \sum_i h^i(s,z,u) \equiv K_t$, a non-random function. (In particular this includes 2-person 0-sum games). But this does <u>not</u> imply that

$$\sum_i x_t^{iu^{i*}} \equiv K_t \tag{4.2}$$

This negative conclusion raises the question whether such a game should be called a constant-sum game. As is clear from (4.1), there is one special case where (4.2) holds and that is the "equal" information case, $y_t^1 = y_t^2 = \dots = y_t^N$

## 5. NOTION OF EFFICIENCY IN CASE OF INCOMPLETE INFORMATION

The definition of an equilibrium policy is an attempt to embody the concept of individual rationality, whereas efficiency is a precise criterion for group rationality. Thus an efficient equilibrium is stable against group action in the sense that the players will not derive any additional individual benefits from "cooperating" as a group. Now, in the situation of incomplete information where the information available to different players is substantially different and where "cooperation" means sharing of information as well as coordination of policies, it seems quite unlikely that a (non-cooperative) equilibrium will be efficient. This may appear puzzling since on a priori grounds one would expect equilibrium policies in the "real world" to be efficient. It is evident that one way this apparent paradox can be resolved is if we can expand our framework to include costs of cooperation, especially of sharing of information.

REFERENCES


1. M. H. A. Davis and P. Varaiya, Dynamic Programming Conditions for Partially Observable Stochastic Systems, SIAM J. Control 11(2), (1970), 226-261.
2. I. V. Girsanov, On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures, Theor. Prob. Appl. 5 (1962), 285-301.
3. V. E. Beneš, Existence of Optimal Stochastic Control Laws, SIAM J. Control 9(1971), 446-475.
4. T. E. Duncan and P. Varaiya, On the Solutions of a Stochastic Control System, SIAM J. Control 9 (1971), 354-371.
5. W. H. Fleming, The Convergence Problem for Differential Games, II. Advances in Game Theory, Annals of Math. Studies, No. 52 (1964), 195-210.
6. W. H. Fleming, The Cauchy Problem for Degenerate Parabolic Equations, J. Math. Mech. 13 (6), (1964), 987-1008.
7. A. Friedman, Stochastic Differential Games, J. Diff. Eq. 11 (1), (1972), 79-108.
8. J. Danskin, Randomized Games, (1974), unpublished.