ON THE EXISTENCE OF POSITIVE RENT GRADIENTS IN THÜNEN MODELS

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ABSTRACT

It is generally believed that land rent is a monotonically decreasing function of distance from the central Town, or from the Central Business District, in so-called Thünen models. In the present paper, we use Thünen's original assumptions to show that over an interval there may exist a positive rent gradient. We explain how this seemingly paradoxical result can arise, and we show that only by violating one of Thünen's assumptions (about the real wage) can the usual resultant of a negative rent gradient be guaranteed.

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1. Problem

The past 15 to 20 years have seen a revival of interest in Johann Heinrich von Thünen's "Der Isolierte Staat" (1826).[1] With great originality and insight, Thünen made numerous calculations about the optimal use of agricultural land, basing his work on a number of simplifying assumptions. Thünen stressed repeatedly that his work should be judged as a way of thinking -- "ein Gedankenschema" -- to aid in the understanding of locational problems. And it is precisely his conceptual scheme which has attracted modern authors. They have applied his model in the study of urban land use. Thünen's single Town has been replaced by a Central Business District (CBD), his famous "Rings" of different crops and of varying intensity of agricultural land use have been replaced in these modern studies by concentric zones of housing and other uses of urban land.[2]

Without exception, these studies of urban land use have concluded that rent decreases with increasing distance from the CBD. The form of the rent function is often illustrated as in Figure 1.

At any given distance from the center, land will be allocated to that usage which can afford the highest rent. It will dominate all other uses, and rent, as a function of distance, will be represented by the upper contour. It is generally believed that the resulting form of the rent gradient is a "Thünen effect", in other words that it invariably follows from Thünen's assumptions. A number of authors have felt the need to prove formally that rent is indeed a decreasing function of distance; some have gone to considerable length to show that the rent gradient has a specific form, usually negative exponential. In textbooks, authors are content to make a brief reference to Thünen in their assertion that the rent gradient is negatively inclined.[3]

None of the many authors on the subject appears to be aware of the fact that Thünen did not prove that under his assumptions a decreasing rent function would inevitably follow. It is the purpose of this note to disprove this wide-spread contention of a negative rent gradient. To do so it suffices to provide a counter example, and we shall choose one which in every respect uses the standard assumptions of economic theory.

2. Analysis

To cast the problem in the simplest possible form, we shall pursue the analysis in terms of two crops only. Following Thünen, we assume that the unit prices of the two crops, delivered in the Town, are fixed and known to all agents. With Thünen, we also assume that the farm worker's wage has two components, namely a money wage part and a real-wage part which is expressed in units of the first product (crop). Thünen considered these two components to be the same everywhere, in equilibrium. (He assumed labor to be mobile; hence it would migrate until such an equilibrium had been achieved.)

Thünen's transportation cost formulation included feed for the horses and food for the drivers. Merely in order to be faithful to his framework, we shall preserve his form of the transportation cost. However, we shall show that the final result of our analysis is not dependent on this particular form of transportation cost, but will hold also when we simplify the transportation cost formulation to a form which is more customary in the literature.

We use the following notations:

p_i = delivered price in the Town, per unit of the i^{th} crop; $i = 1,2$

r = distance between the Town and the location of a particular acre of land.

$\frac{a_i}{b_i + r}$ = transportation cost in money per ton-mile of crop i, at a distance r from the Town (a_i and b_i are positive constants).

$x_i(r)$ = amount of labor employed, per acre, in the cultivation of the i^{th} crop at a distance r from the Town.

$q_i(x_i,r)$ = quantity of the i^{th} crop produced per acre, at a distance r

w = real-wage part (expressed in the units of the first crop)

m = money-wage part

The net profit obtainable per acre from the growing of the first crop at a distance r from the Town can then be expressed as follows:

$$(1) \qquad \pi_1 = p_1(q_1 - wx_1) - mx_1 - (q_1 - wx_1)\frac{a_1 r}{b_1 + r}$$

Hence, and in accordance with Thünen's framework, maximization of (1) will yield the highest rent that can be paid in the cultivation of crop 1, at a distance r from the Town. Differentiating with respect to x_1, we obtain as a necessary condition for profit maximization:

$$(2) \qquad p_1 \frac{\partial q_1}{\partial x_1} - \frac{a_1 r}{b_1 + r}\frac{\partial q_1}{\partial x_1} - p_1 w + \frac{a_1 r}{b_1 + r} w - m = 0$$

For the second product, or crop, the corresponding profit can be expressed as follows:

$$(3) \qquad \pi_2 = p_2 q_2 - (p_1 - \frac{a_1 r}{b_1 + r})\, wx_2 - mx_2 - q_2 \frac{a_2 r}{b_2 + r}$$

The difference between the two profit expressions is due to the fact that the real-wage, in both sectors, is measured (and paid) in units of the first product. Again we obtain as a necessary condition for profit maximization:

$$(4) \qquad (p_2 - \frac{a_2 r}{b_2 + r}) \frac{\partial q_2}{\partial x_2} - (p_1 - \frac{a_1 r}{b_1 + r}) w - m = 0$$

Transposing the last two terms, we obtain the standard result that the marginal value productivity of labor is equated to the wage. To proceed further, we introduce the simplest possible type of production function with diminishing marginal returns to labor:[4]

$$(5) \qquad q_2 = x_2^{\beta} \qquad\qquad 0 < \beta < 1$$

This enables us to solve for x_2 in (4):

$$(6) \qquad x_2 = \left[\frac{(p_1 - \frac{a_1 r}{b_1 + r}) w + m}{\beta (p_2 - \frac{a_2 r}{b_2 + r})} \right]^{1/\beta - 1}$$

Setting $(p_1 - \frac{a_1 r}{b_1 + r}) w + m = y_1$ and

$(p_2 - \frac{a_2 r}{b_2 + r}) = y_2$, (6) reduces to:

$$(7) \qquad x_2 = \left[\frac{y_1}{\beta\, y_2} \right]^{1/\beta - 1}$$

Substituting for x_2 in (3):

$$(8) \qquad \pi_2 = y_1^{\beta/\beta - 1}\ y_2^{1/1 - \beta}\ (1/\beta - 1)$$

Taking logarithms in (8) and differentiating with respect to r:

(9) $$(d\pi_2/dr)/\pi_2 = [(dy_2/dr)/y_2 - (dy_1/dr)\ \beta/y_1]/(1-\beta)$$

The sign of (9) is determined by the expression inside the bracket. As can be seen from the definitions of y_1 and y_2, both dy_1/dr and dy_2/dr are negative. These two derivatives can be interpreted as the marginal fall in the money value of production cost per acre, and the marginal fall in the f.o.b. price, respectively, as the distance increases. Hence, if the money value of production costs falls substantially faster, over some interval of r, than the price of the second crop does, rent will be increasing over this interval.

To summarize, the right-hand side of equation (9) contains one positive and one negative term. It cannot be ruled out on any plausibility grounds that the positive term will be larger than the absolute value of the negative term over some distance interval. Thus, we conclude that the rent gradient (for the second crop) can very well be positive, over an interval of r.

In the analysis leading to expression (9) above, we have adhered strictly to Thünen's original assumption about the shape of the transportation cost function. In the relevant modern literature, however, a much simpler assumption is commonly used, namely that the transportation cost per ton-mile is a constant. By introducing this simplification here, we can carry the analysis a step further -- and in the process shed more light on the possibility of a positive rent gradient.

Using t_1 and t_2 to denote the transportation costs per ton-mile in the two sectors, we can rewrite the profit expression in (8) as follows:

$$(10) \quad \pi_2 = [(p_1 - t_1 r)\, w + m]^{\beta/\beta-1} \, (p_2 - t_2 r)^{1/1-\beta} \, (1/\beta-1)$$

Thus, like the analogous previous expression, (10) states the highest rent that can be paid for an acre at distance r. Obviously, this expression has a counterpart in the urban economics literature, and it is instructive to explore wherein the difference lies. An inspection reveals that the second factor, involving $(p_2 - t_2 r)$, appears in almost identically the same form in the literature.[5] However, the first (bracketed) factor in the expression above appears as a constant in the literature, whereas in (10) it is a function of distance r.

Now, the interesting feature about equation (10) is that within the relevant range of distance the gradients of the two bracketed expressions have opposite signs. Upon reflection, it is seen that distance from the center is bounded by p_2/t_2. Also, the expression $(p_1 - t_1 r)w$ must be defined to vanish, as distance increases beyond the region in which the first crop can economically be delivered to the center. Formally, we have:

$$(11) \quad (p_1 - t_1 r)w = 0 \text{ for } r > p_1/t_1$$

$$(12) \quad r < p_2/t_2$$

Hence, given (11) and (12), the first bracketed expression in (10) is always increasing, and the second bracketed expression in (10) is always decreasing. In order to get a firmer grasp of the net effect of these opposing factors upon rent, we once again take logarithms and differentiate to obtain:

$$(13) \quad (d\pi_2/dr)/\pi_2 = [\frac{\beta t_1 w}{(p_1 - t_1 r)w+m} - \frac{t_2}{p_2 - t_2 r}] /(1-\beta)$$

From (13), we can easily find the distance, at which the rent gradient is horizontal (assuming that (11) and (12) are satisfied):

$$(14) \quad r = (p_1 t_2 w - \beta p_2 t_1 w + t_2 m)/t_1 t_2 w(1-\beta)$$

Furthermore, an investigation of the derivative of (13) reveals that its sign is indetermine; in other words, the rent function can have a maximum or a minimum, or possibly neither, depending on the values of the constants.

3. Conclusions

We are now ready to frame our conclusions, and we shall choose to do so by reference to a set of diagrams. We repeat that -- except for a simplification of his transport cost formulation -- we have stayed within Thünen's original framework. Among other things, this means that we have accepted his assumptions of monocentricity, of complete information for all agents, of profit-maximizing behavior on the part of landlords, and of the prevalence of a long-run equilibrium situation.

Our purpose has been to analyze the possible form of the rent gradient in a Thünen framework. In particular, our aim has been to subject to some scrutiny the widely-held belief that under the Thünen assumptions rent is a decreasing function of distance from the central Town. Concentrating attention, for the moment, on one sector, or crop, which we have identified in the previous analysis by subscript "2", and assuming substitutability between the inputs (land and labor), an illustration of such a decreasing rent function is given in Figure 2.

By reference to equation (10), above, we can see that such an everywhere negative rent gradient, as illustrated in Figure 2, could be guaranteed by assigning to the constant "w" a value of zero. However, this would violate one of Thünen's fundamental assumptions, namely that the real wage is everywhere the same. It is by virtue of this last-mentioned assumption that the possibility of a positively inclined rent gradient arises. Our analysis in the previous section suggests that two such cases are worthwhile distinguishing, and they are illustrated in Figures 3 and 4.

Figure 3 illustrates the possibility of a positive rent gradient in the vicinity of the central Town, which arises because a steep (negative) wage gradient dominates the effect of decreasing transport cost. Since the wage gradient effect will gradually taper off, the rent function reaches an interior maximum point; it then decreases until it reaches the point (of zero rent) beyond which the Town ceases to be a market for the (second) crop. Figure 4 shows an analytically more interesting case with two turning-points for the rent function. Here, the rent function has a (local) minimum. Between that point and the point which marks the boundary of profitable shipments of the first crop to the town, the rent gradient turns positive.

In our analysis, we have considered only the simplest possible case, namely that of two crops, or products. But the analysis extends easily to any number of crops. Suppose, for example, that the number of distinguishable crops or products is indefinitely large, and that all but one of these products has a rent gradient which is everywhere negative. Then, Figure 2 can be taken to represent and illustrate the upper contour of the resulting rent functions. Suppose then that the one remaining sector has a rent function of the type illustrated in Figure 4. There exists of course then the possibility that, when the last-mentioned sector is combined with all the others, it is dominated -- in terms of rent-paying ability -- by the other industries. However, the possibility cannot be excluded that the end result of all the crops combined will be as illustrated in Figure 5 -- with a positive rent gradient over the interval marked a to b.

A few authors[6] have noted that a negative wage gradient may be significant also in the urban context. As far as we know, none of them have explored the possibility that their observation might be related to the existence of a positive rent gradient. Perhaps they have felt that the connection is too tenuous. If so, they may be right, in the context of the urban space, on inductive grounds. Our claim is simply that a negative rent gradient cannot be deduced from the original Thünen assumptions.

FOOTNOTES

[1]Noteworthy is a recent (abridged) translation into English. It contains an excellent introduction. See P. Hall (1966).

[2]W. Isard (1956) was apparently the first to see clearly the applicability of Thünen models to the urban space. Such models have been developed, by R. Muth (1961), L. Wingo (1961), W. Alonso (1964), E. Mills (1972) and M. Beckmann (1972), among others.

[3]See for example, W. Hirsch (1973), E. Hoover (1971) and H. Richardson (1971).

[4]We are here following M. Beckmann (1972)

[5]One example is E. Mills (1972), p. 86

[6]See L. Moses (1962), L. Moses and J. Williamson (1967), and R. Muth (1969), pp. 43-44.

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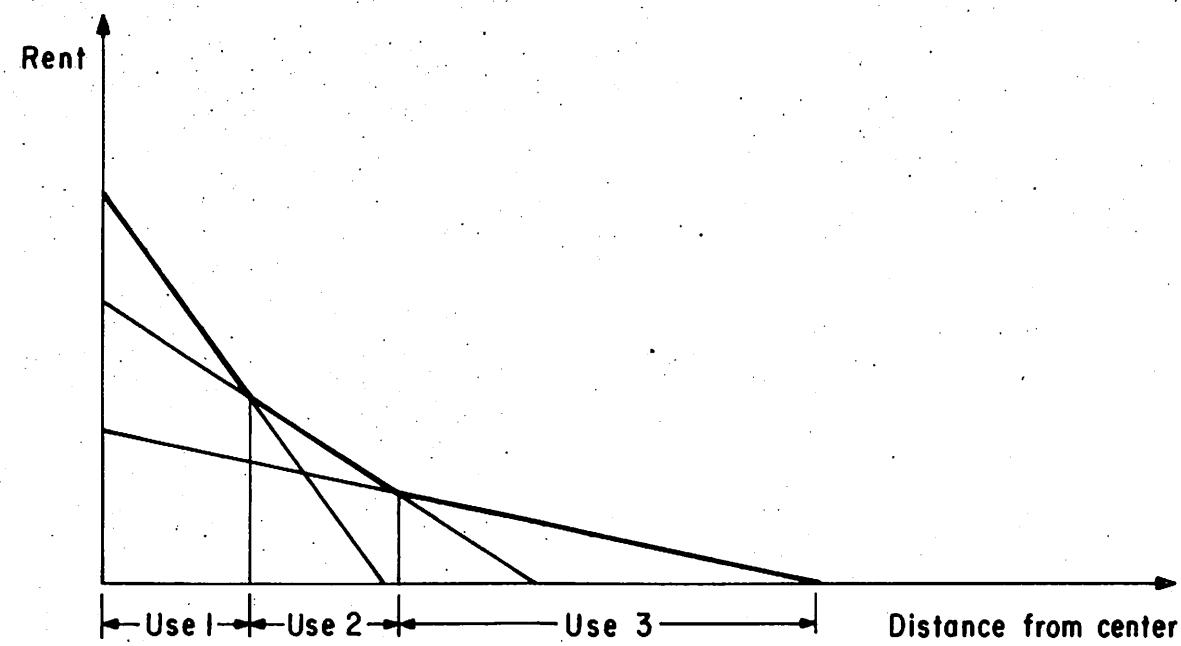
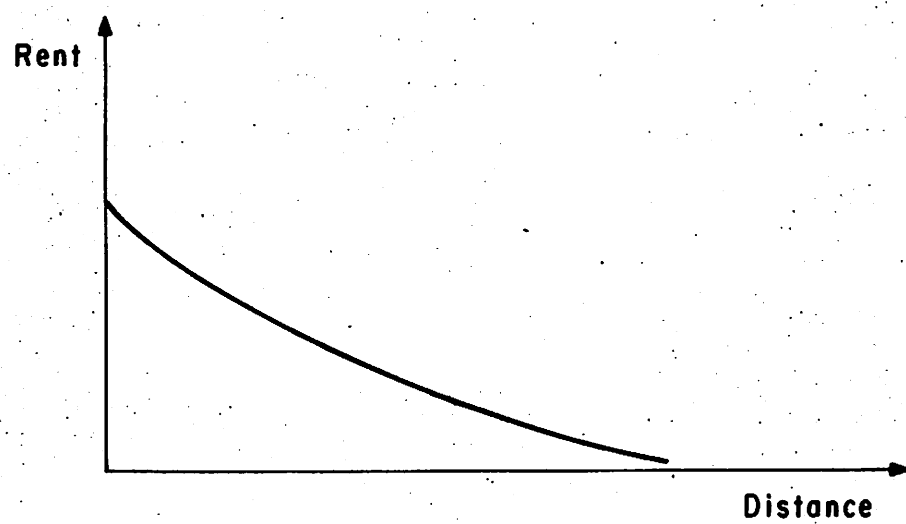
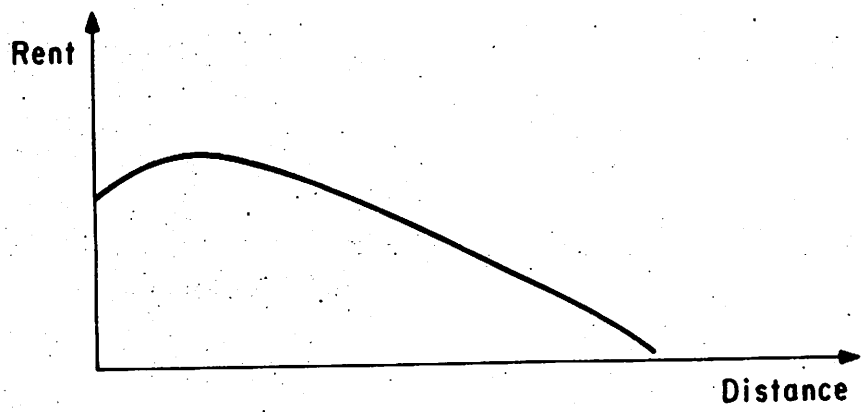
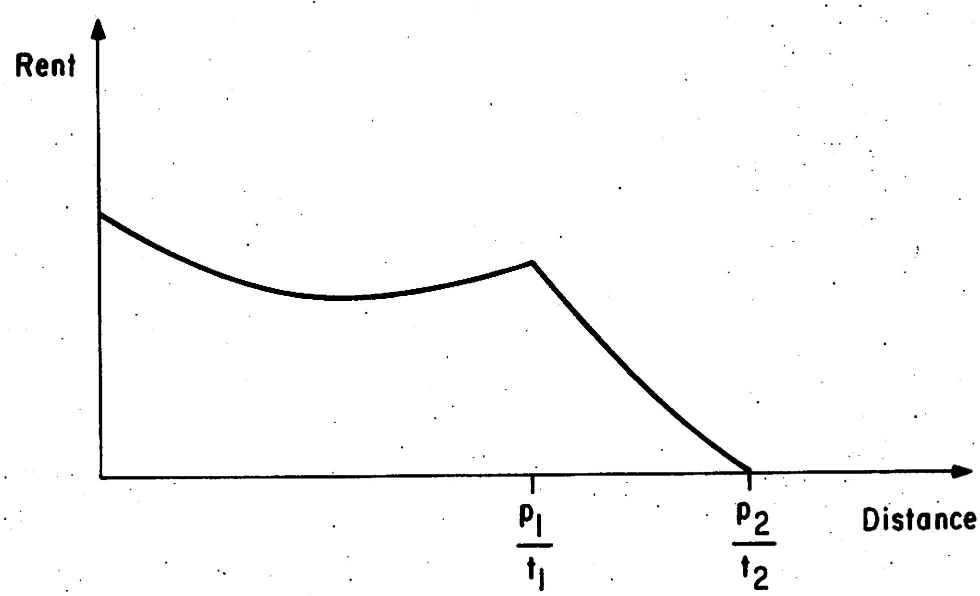
FIGURE CAPTIONS

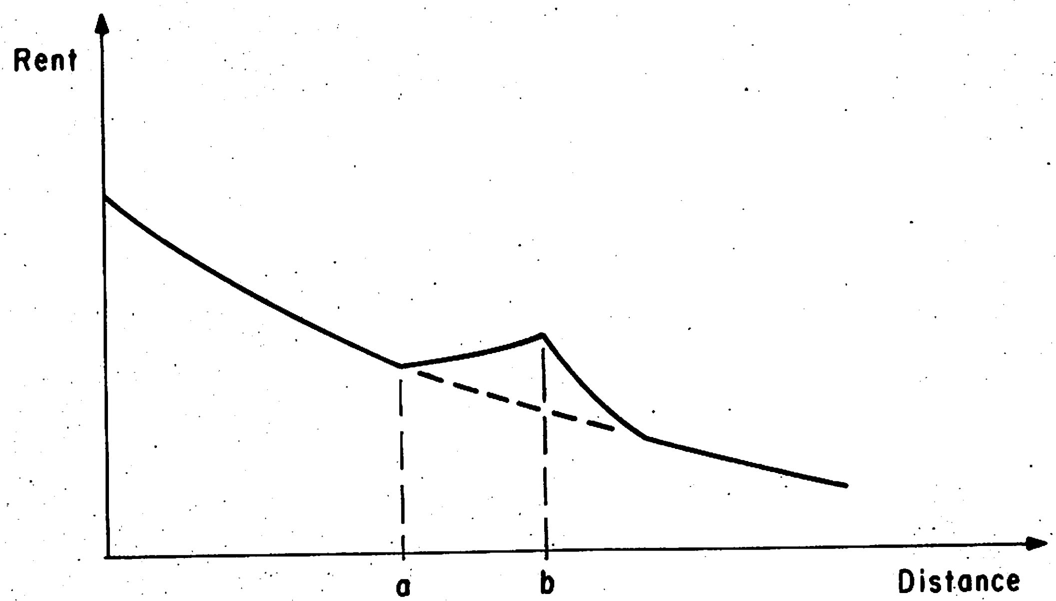

Rent
Use 1
Use 2
Use 3
Distance from center



Rent
Distance


Rent
Distance


Rent
Distance
$\frac{p_1}{t_1}$
$\frac{p_2}{t_2}$


Rent
a
b
Distance