A FUZZY-ALGORITHMIC APPROACH TO THE DEFINITION OF COMPLEX OR IMPRECISE CONCEPTS

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

To Richard Bellman

A Fuzzy-Algorithmic Approach to the Definition of Complex or Imprecise Concepts

L. A. Zadeh

Computer Science Division
Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

It may be argued, rather persuasively, that most of the concepts encountered in various domains of human knowledge are, in reality, much too complex to admit of simple or precise definition. This is true, for example, of the concepts of recession and utility in economics; schizophrenia and arthritis in medicine; stability and adaptivity in system theory; sparseness and stiffness in numerical analysis; grammaticality and meaning in linguistics; performance measurement and correctness in computer science; truth and causality in philosophy; intelligence and creativity in psychology; and obscenity and insanity in law.

The approach described in this paper provides a framework for the definition of such concepts through the use of fuzzy algorithms which have the structure of a branching questionnaire. The starting point is a relational representation of the <u>definiéndum</u> as a composite question whose constituent questions are either attributional or classificational in nature. The constituent questions as well as the answers to them are allowed to be fuzzy, e.g., the response to: "How large is x?" might

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be _not very large_, and the response to "Is x large?" might be _quite true_.

By putting the relational representation into an algebraic form, one can derive a fuzzy relation which defines the meaning of the definiendum. This fuzzy relation, then, provides a basis for an interpolation of the relational representation.

To transform a relational representation into an efficient branching questionnaire, the tableau of the relation is subjected to a process of compactification which identifies the conditionally redundant questions. From a maximally compact representation, various efficient realizations which have the structure of a branching questionnaire, with each realization corresponding to a prescribed order of asking the constituent questions, can readily be determined. Then, given the cost of constituent questions as well as the conditional probabilities of answers to them, one can compute the average cost of deducing the answer to the composite question. In this way, a relational representation of a concept leads to an efficient branching questionnaire which may serve as its operational definition.

To Richard Bellman

A Fuzzy-Algorithmic Approach to the Definition of Complex or Imprecise Concepts

L. A. Zadeh*

1. Introduction

The high standards of precision which prevail in mathematics, physics, chemistry, engineering and other "hard" sciences stand in sharp contrast to the imprecision which pervades much of sociology, psychology, political science, history, philosophy, linguistics, anthropology, literature, art and related fields. This marked difference in the standards of precision is due, of course, to the fact that the "hard" sciences are concerned in the main with the relatively simple mechanistic systems whose behavior can be described in quantitative terms, whereas the "soft" sciences deal primarily with the much more complex non-mechanistic systems in which human judgment, perception and emotions play the dominant role.

Although the conventional mathematical techniques have been and will continue to be applied to the analysis of humanistic[1] systems, it is clear

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*Computer Science Division, Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720. This work was supported in part by the Naval Electronics Systems Command under Contract N00039-71-C-0255, the Army Research Office, Durham, N.C., under Grant DA-ARO-D-31-124-71-G174, and the National Science Foundation under Grant GK-10656X3. Some of the results described in this paper were obtained while the author was a visiting member of the International Institute for Applied Systems Analysis in Vienna, Austria.

[1] By a humanistic system we mean a non-mechanistic system in which human behavior plays a major role. Examples of humanistic systems are political systems, economic systems, social systems, religious systems, etc. A single individual and his thought processes may also be viewed as a humanistic system.

that the great complexity of such systems calls for approaches that are significantly different in spirit as well as in substance from the traditional methods - methods which are highly effective when applied to mechanistic systems, but are far too precise in relation to systems in which human behavior plays an important role.

In the linguistic approach [1],[2]- which represents one such departure from conventional methods - words or sentences are used in place of numbers to describe phenomena which are too complex or too ill-defined to be susceptible of characterization in quantitative terms. For example, if the probability of an event is not known with precision, then it may be characterized linguistically as, say, quite likely, not very unlikely, highly unlikely, etc., where quite likely, not very unlikely and highly unlikely are interpreted as labels of fuzzy subsets of the unit interval.[2] Such subsets may be likened to ball-parks without sharply defined boundaries which serve to provide an approximate rather than exact characterization of the value of a variable.

The use of the linguistic approach in the case of humanistic systems is dictated by the fact that as the complexity of a system increases, our ability to make precise and yet significant statements abouts its behavior diminishes until a threshold is reached beyond which complexity, precision and significance can no longer coexist. The essence of the linguistic approach, then, is that it sacrifices precision to gain significance, thereby making it possible to analyze in an approximate manner those

[2] As a fuzzy subset of the unit interval, quite likely would be characterized by its compatibility or, equivalently, membership function $\mu_{\text{quite likely}}$: $[0,1] \to [0,1]$. Thus, $\mu_{\text{quite likely}}(0.8) = 0.9$ means that if the probability of an event is 0.8, then the degree to which 0.8 is compatible with quite likely is 0.9. Additional details may be found in the Appendix.

humanistic as well as mechanistic systems which are too complex for the application of classical techniques.

A key feature of the linguistic approach has to do with its use of the notion of a primary fuzzy set as a substitute for the basic notion of a unit of measurement.[3] More specifically, much of the power of mathematical techniques for dealing with mechanistic systems derives from the existence of a set of units for such basic parameters as length, area, weight, force, current, heat, etc. In general, such units do not exist in the case of humanistic systems, and it is this fact that contributes significantly to the difficulty of analyzing humanistic systems through the use of techniques which depend so essentially on the existence of units of measurement.

In the linguistic approach, a role comparable to that of a unit of measurement is played by one or more primary fuzzy sets from which other sets can be generated through the use of linguistic modifiers such as very, quite, more or less, extremely, essentially, completely, etc. To illustrate, consider a property,[4] say beautiful, for which we have neither a unit nor a numerical scale. The meaning of this property may be defined via exemplification by associating with each member, u, of a subset of objects in a given universe of discourse, U, the grade of membership of u in the fuzzy subset labeled beautiful. For example, the grade of membership of Fay in the class of beautiful women might be 0.9, that of Jillian 0.85, of Helen 0.8, etc. This set of women, then, would constitute

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[3] A thorough discussion of the concept of a unit of measurement may be found in [3].

[4] At this point we do not differentiate between a property (intension) and the set which it defines (extension). (See [4]-[19] for a discussion of this and other issues relating to concepts, meaning and denotation.)

a primary fuzzy set which serves as a reference for defining the meaning of <u>very beautiful</u>, <u>quite beautiful</u>, <u>more or less beautiful</u>, <u>extremely beautiful</u>, etc. as fuzzy subsets of U.[5] Thus, in terms of these subsets, an assertion of the form "<u>Nora is very beautiful</u>," may be interpreted as the assignment of a linguistic rather than a numerical value to the beauty of Nora. In this way, the linguistic values <u>beautiful</u>, <u>very beautiful</u>, <u>quite beautiful</u>, etc. which are generated from the primary fuzzy set <u>beautiful</u>, play a role which is roughly similar to that of the multiples of a unit of measurement, when such a unit exists.

Our main purpose in the present paper is to apply the linguistic approach to the definition of concepts which are too complex or too imprecise to be susceptible of exact definition. In general, such concepts are fuzzy in the sense that they correspond to classes of objects or constructs which do not have sharply defined boundaries. For example, the concepts of <u>oval</u>, <u>in love</u>, <u>young</u> and <u>masculine</u> are fuzzy whereas those of <u>straight line</u>, <u>married</u>, <u>brother</u> and <u>male</u> are not. Note that <u>oval</u> is a more complex concept than <u>straight line</u>, <u>in love</u> is more complex than <u>married</u>, <u>friend</u> is more complex than <u>brother</u>, and <u>masculine</u> is more complex than <u>male</u>. Indeed, most complex concepts tend to be fuzzy, and it is in this sense that fuzziness may be regarded as a concomitant of complexity.

<u>Note 1.1</u> In most cases, the question of whether a concept is fuzzy or not may be resolved by examining the applicability of a simple modifier such as <u>very</u> to the concept in question. Thus, for example, <u>very</u> is

[5]The computation of the meaning of a term of the form m u, where m is a modifier and u is a primary term (i.e., a label for a primary fuzzy set), is discussed in [20]-[22] and, more briefly, in the Appendix.

applicable to _masculine_ but not to _male_. Similarly, _very ill_, where _ill_ is a fuzzy concept, is acceptable whereas _very dead_ is not. Also, _very much greater_ is acceptable (_much greater_ is fuzzy), while _very greater_ (_greater_ is nonfuzzy) is not.

How can a fuzzy concept be defined? The conventional approaches are: (1) Giving a dictionary type of definition; (2) Writing an essay; and (3) Approximating to a fuzzy concept by a nonfuzzy concept and giving a precise definition for the latter. To illustrate, a typical dictionary definition of a fuzzy concept such as _democracy_ might read, "A form of government in which the supreme power is vested in the people and exercised by them or by their elected agents under a free electoral system," while a more detailed definition might occupy a chapter in a text on political science. A typical example of (3) is the definition of a _recession_ [23] as a condition which obtains when the gross national product declines in two successive quarters. In this case, what is in reality a fuzzy concept is defined as one which is both nonfuzzy and simple to understand. The price, of course, is a definition that is oversimplified to a point of uselessness.

An alternative and more systematic approach which is described in the sequel is based on the notion of a fuzzy algorithm [25]-[27], that is, an algorithm (or a program or a decision table) in which some of the steps involve the execution of fuzzy instructions, which in turn may require the verification of fuzzy conditions. More specifically, in the _fuzzy-algorithmic_ approach the definition of a fuzzy concept F is expressed as a fuzzy recognition algorithm[6] which acts on a given object u and upon

[6] A recognition algorithm is essentially an algorithmic representation of the membership function of a fuzzy set.

execution yields the degree to which u is compatible with F or, equivalently, the grade of membership of u in the fuzzy set labeled F.

As an illustration, suppose that the concept of an economic recession is defined by a fuzzy algorithm labeled RECESSION. Then, acting on relevant economic data, RECESSION would yield the degree - expressed numerically, e.g., 0.8, or linguistically, e.g., _very true_ - to which the data in question are compatible with the concept of _recession_ as defined by the algorithm. Similary, a fuzzy-algorithmic definition of a disease, say _arthritis_, would yield the degree to which a given patient belongs to the class of arthritics. Similarly, a fuzzy-algorithmic definition of the concept of _sparseness_ would yield the degree to which a given matrix is sparse. And so on.

As will be seen in the following sections, a fuzzy-algorithmic definition has the form of a branching questionnaire, Q, in which both the questions and the answers are allowed to be fuzzy in nature. For example, to a question such as "Is Valentina _tall_?" (which will be abbreviated as _tall_?) the answer might be "_quite tall_," which may be viewed as being equivalent to the assignment of the linguistic value _quite high_ to the grade of membership of Valentina in the class of tall people.

A question, Q_i, in Q may be either _classificational_ or _attributional_. In the case of classificational questions, Q_i is concerned with the grade of membership of the subject in a fuzzy set F_i, or, equivalently, with the truth-value of the predicate[7] which corresponds to F_i. For example, Q_i

[7]The term _predicate_ (or, more generally, _fuzzy predicate_) as used here is essentially synonymous with the _membership_ (or _compatibility_) _function_. To simplify the notation, the label of a predicate and the label of the set which it defines will be used interchangeably.

may be "Is Rahim honest?" An answer such as very high would mean that the grade of membership of the subject in the class of honest people is very high. Equivalently, an answer of the form very true would be interpreted as the assignment of the truth-value very true to the predicate labeled honest evaluated at $x \triangleq$ Rahim.[8]

In the case of attributional questions, Q_i relates to the value of an attribute of the subject. For example, an instance of Q_i may be "How old is Norman?" with the answer being either numerical, e.g., 24 or linguistic, e.g., quite young. Thus, in this case the answer may be viewed as the assignment of either a numerical or a linguistic value to an attribute of the subject.

The totality of the questions in Q constitues a basis for Q, or, more specifically, the fuzzy concept defined by Q. If all of the questions in Q are classificational in nature, then the basis for Q defines a collection of fuzzy sets each of which corresponds to a question in Q. In this case, the questionnaire may be viewed as a way of defining the fuzzy set corresponding to Q in terms of the fuzzy sets corresponding to the questions in Q. As a simple illustration, if the predicate big is defined as the conjunction of the predicates long, wide, and tall, i.e.,

$$\underline{\text{big}} = \underline{\text{long}} \text{ and } \underline{\text{wide}} \text{ and } \underline{\text{tall}} \tag{1.2}$$

then Q_1, Q_2 and Q_3 may be expressed (in abbreviated form) as

$$Q_1 \triangleq \underline{\text{long}}? \tag{1.3}$$

$$Q_2 \triangleq \underline{\text{wide}}? \tag{1.4}$$

$$Q_3 \triangleq \underline{\text{tall}}? \tag{1.5}$$

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[8]The symbol $\triangleq$ stands for denotes or is defined to be or is equal by definition.

and (1.2) is equivalent to

$$\underline{\text{big}} = \underline{\text{long}} \cap \underline{\text{wide}} \cap \underline{\text{tall}} \tag{1.6}$$

where big, long, wide and tall are interpreted as the fuzzy sets corresponding to Q, Q_1, Q_2 and Q_3, respectively, and the intersection is defined in the fuzzy-set-theoretic sense. Thus, (1.6) expresses the fuzzy set big as a function of the fuzzy sets long, wide and tall, which implies that from the knowledge of the answers to Q_1, Q_2 and Q_3 one can determine the grade of membership of the object under test in the fuzzy set big. For example, if the answers to specific instances of Q_1, Q_2 and Q_3 are true, very true and very true, respectively, then from (1.6) it follows that the answer to the question big? is true. A more detailed discussion of this aspect of fuzzy-algorithmic definitions will be presented in Sec. 3.

By their nature, fuzzy-algorithmic definitions are best suited for the characterization of concepts which are intrinsically fuzzy, that is, fuzzy to a degree which makes it unrealistic to approximate to them by nonfuzzy concepts. For example, in law, insanity and obscenity are intrinsically fuzzy concepts whereas perjury is not. Similarly, in system theory the concepts of large-scale, reliable and adaptive are intrinsically fuzzy, whereas those of observability and controllability are not. In numerical analysis, the concept of a sparse matrix is intrinsically fuzzy while that of a bounded error is not. In medicine, most degenerative diseases are intrinsically fuzzy while the infectious diseases, for the most part, are not.

In addition to the intrinsically fuzzy concepts, there are many concepts in various fields which though fuzzy in nature are at present defined in nonfuzzy terms, largely because of a lack of alternative modes of definition. This is true, for example, of the concepts of _recession_ and _equilibrium_ in economics; _complexity_ and _approximation_ in mathematics; _structured programming_ and _correctness_ in computer science; _stability_ and _linearity_ in system theory; _arthritis_ and _hypertension_ in medicine, etc. It is very likely that, in time, the use of fuzzy-algorithmic techniques for the characterization of such concepts will become a fairly common practice.

In what follows, our discussion of fuzzy-algorithmic definitions will begin with the notion of an _atomic_ question. This notion will serve as a basis for the definition of a _composite_ question, which in turn will lead to the concept of a fuzzy-algorithmic _branching questionnaire_. In order to make the discussion self-contained, a brief summary of the relevant aspects of the linguistic approach is presented in the Appendix.

2. Atomic Questions

Our focus of attention in this section is the concept of what might be called an atomic question, that is, a question which has no constituents other than itself. By contrast, a composite question - as its name implies - is composed of a collection of constituent questions. The manner in which the constituent questions are combined to form a composite question as well as other issues relating to the concept of a composite question will be discussed in Sec. 3.

Example 2.1 The question $Q \triangleq$ Is Ruth tall? is an atomic question if no other questions have to be asked in order to answer Q.

The question $Q \triangleq$ Is x big? where x is some object, is a composite question if big is defined as the conjunction of long, wide and high (as in (1.2)), and the answer to Q is deduced from the answers to the constituent questions $Q_1 \triangleq$ Is x long?, $Q_2 \triangleq$ Is x wide?, and $Q_3 \triangleq$ Is x high?

A questionnaire is, in effect, a representation of a composite question, and a branching questionnaire is a representation in which the order in which the constituent questions are asked is determined by the answers to the previous questions.

In what follows, we shall examine the concept of an atomic question in greater detail with a view to providing a basis for a systematic representation of fuzzy-algorithmic definitions in the form of branching questionnaires.

Notation and Terminology

Definition 2.2 An atomic question, Q, is characterized by a triple $Q \triangleq (X,B,A)$, where X, the object-set, is a set of objects to which Q applies; B, the body of Q, is a label of either a class or an attribute;

and A, the answer-set, is a set of admissible answers to the question. Where necessary, specific instances of Q, X and A will be denoted generically by q, x and a, respectively.[1] When X and A are implied, Q will be written in an abbreviated form as

$$Q \triangleq B\ ?$$

and a specific question together with an admissible answer to it will be expressed as

$$Q/A \triangleq B\ ?\ a \tag{2.3}$$

or equivalently

$$q/a \triangleq B\ ?\ a$$

The pair Q/A will be referred to as a question/answer pair (or simply Q/A pair). Graphically, an atomic question (with implied x) will be represented in the form of a fan as shown in Fig. 2.1.

Example 2.4 Consider a specific instance of a question Q, e.g., "Is Nancy well-dressed?" In this case, with the subject $x \triangleq$ Nancy implied, the specific question may be expressed as

$$q \triangleq \underline{\text{well}}\text{-}\underline{\text{dressed}}? \tag{2.5}$$

where well-dressed is the body of Q. Correspondingly, a specific Q/A pair might be

$$q/a \triangleq \underline{\text{well}}\text{-}\underline{\text{dressed}}?\ \underline{\text{true}} \tag{2.6}$$

in which true, as an admissible answer, is an element of the answer-set A.

[1] To avoid a proliferation of symbols, Q and q will be used interchangeably when no confusion is likely to arise.

If the other elements of the answer-set are false and borderline, then A may be expressed as

$$A = \underline{\text{true}} + \underline{\text{borderline}} + \underline{\text{false}} \tag{2.7}$$

where + denotes the union rather than the arithmetic sum.

The linguistic truth-values in (2.8) are, in effect, names of fuzzy subsets of the unit interval. In terms of their respective membership functions, these subsets may be expressed as (see the Appendix)

$$\underline{\text{true}} = \int_0^1 \mu_t(v)/v \tag{2.8}$$

$$\underline{\text{borderline}} = \int_0^1 \mu_b(v)/v \tag{2.9}$$

and

$$\underline{\text{false}} = \int_0^1 \mu_f(v)/v \tag{2.10}$$

where μ_t, μ_b and μ_f are the membership functions of true, borderline and false, respectively, and an expression such as (2.8) means that the fuzzy set labeled true is the union of fuzzy singletons $\mu_t(v)/v$ in which the point v in [0,1] has the grade of membership $\mu_t(v)$ in true. Typical forms of μ_t, μ_b and μ_f are shown in Fig. 2.2

Note 2.11 For the representation of μ_t, μ_b and μ_f it is frequently convenient to employ standardized functions with adjustable parameters, e.g., the S and Π functions which are defined below (see Figs. 2.3a and 2.3b).

$$S(v;\ \alpha,\beta,\gamma) = 0 \qquad \text{for} \qquad v \leq \alpha \tag{2.12}$$

$$= 2\left(\frac{v - \alpha}{\gamma - \alpha}\right)^2 \qquad \text{for} \quad \alpha \leq v \leq \beta$$

$$= 1 - 2\left(\frac{v - \gamma}{\gamma - \alpha}\right)^2 \qquad \beta \leq v \leq \gamma$$

$$= 1 \qquad \text{for} \quad v \geq \gamma$$

$$\Pi(v;\ \beta,\gamma) = S(v;\ \gamma-\beta,\ \gamma - \frac{\beta}{2},\ \gamma) \qquad \text{for} \quad v \leq \gamma \tag{2.13}$$

$$= 1 - S(v;\ \gamma,\ \gamma + \frac{\beta}{2},\ \gamma+\beta) \ \text{ for } v \geq \gamma$$

In $S(v;\ \alpha,\beta,\gamma)$, the parameter β, $\beta = (\alpha + \gamma)/2$, is the crossover point, that is, the value of v at which S takes the value 0.5. In $\Pi(v;\ \beta,\ \gamma)$, β is the bandwidth, that is, the distance between the crossover points of Π, while γ is the point at which Π is unity.

In terms of S and Π, μ_t, μ_b and μ_f may be expressed as (suppressing the argument v)

$$\mu_t = S(\alpha,\ \beta,\ 1) \tag{2.14}$$

$$\mu_b = \Pi(\beta',\ 0.5) \tag{2.15}$$

$$\mu_f = 1 - S(0,\ \beta,\ \gamma) \tag{2.16}$$

where the use of the symbol β' in (2.15) signifies that the bandwidth of b need not be equal to the value of β in (2.14).

Note 2.17 In cases in which the three linguistic truth-values true, borderline and false do not offer a sufficiently wide choice, it may be convenient to use, in addition, the truth-values rather true and rather false, abbreviated as rt and rf, respectively.

As a fuzzy subset of [0,1], rather true may be defined as

rather true $\triangleq$ not very true and not (false or borderline) and its membership function may be approximated by a Π function with γ at, say, the crossover point of <u>very true</u>. <u>Rather false</u> may be defined similarly in terms of <u>false</u> and <u>borderline</u>.

<u>Classificational and attributional questions</u>

A question, Q, is <u>classificational</u> if its body, B, is the label of a fuzzy or nonfuzzy set.

A question, Q, is <u>attributional</u> if B is the label of an attribute.

In the case of a classificational question, an answer, a, represents the grade of membership of x in the fuzzy set B. The answer might be numerical, e.g., a $\triangleq$ 0.8, or linguistic, e. g., a $\triangleq$ <u>high</u>. Equivalently, the answer may be expressed as the truth-value of the predicate B(x),[2] e.g., <u>true</u>, <u>borderline</u>, <u>false</u>, <u>very true</u>, etc.

In the case of an attributional question, Q = B?, an answer, a, represents the value of the attribute, B, of an object x, e.g., B $\triangleq$ <u>age</u> and x $\triangleq$ Haydee. Again, a may be numerical, e.g., a $\triangleq$ 35, or linguistic, e.g., a $\triangleq$ <u>young</u>, a $\triangleq$ <u>very young</u>, etc.

<u>Comment 2.18</u> As defined above, a question Q = (X,B,A) may be viewed as a collection of variables {B(x)}, $x \in X$. From this point of view, answering a classificational question addressed to an x in X corresponds to assigning a value, at x, to the membership function of the fuzzy set B (or, equivalently, assigning a truth-value to the fuzzy predicate B(x)). Similarly, answering an attributional question may be interpreted as the assignment of a value to the attribute B(x). In either case,

[2]Depending on the circumstances, the arguments of a predicate may be displayed, as in B(x), or suppressed, as in B.

answering a question with body B may be represented as an assignment equation

$$B(x) = a$$

in which a numerical or a linguistic value a is assigned to the variable B(x).

Example 2.19 Suppose that X is the set of objects in a room and Q = red? is a fuzzy classificational question. Furthermore, suppose that the set of admissible answers is the interval [0,1], representing the grades of membership of objects in X in the fuzzy subset red of X. In this case, an answer such as true 0.8 to the question "Is the vase red?" may be represented as the assignment equation

$$\underline{\text{red}}\ (\text{vase}) = 0.8$$

which implies that the truth-value of the predicate red (x) evaluated at x $\triangleq$ vase is 0.8 or, equivalently, that the grade of membership of the object x $\triangleq$ vase in the fuzzy set labeled red is 0.8.

Example 2.20 Same as Example 2.19 except that the set of admissible answers, A, is assumed to be expressed by

$$A = \underline{\text{low}} + \underline{\text{low}}^2 + \underline{\text{low}}^{1/2} + \underline{\text{medium}} + \underline{\text{medium}}^2 + \underline{\text{medium}}^{1/2} + \underline{\text{high}} + \underline{\text{high}}^2 + \underline{\text{high}}^{1/2} \tag{2.21}$$

where high and medium and low are primary fuzzy subsets of the unit interval which are defined in terms of the S and Π functions by (2.14), (2.15) and (2.16), and w^2 and $w^{1/2}$ are abbreviations for very w and more or less w, respectively. Thus, if w is a subset of a universe of discourse U, then

$$w^2 = \int_V (\mu_w(u))^2/u \tag{2.22}$$

and

$$w^{1/2} = \int_V (\mu_w(u))^{1/2}/u, \tag{2.23}$$

which means that the membership functions of w^2 and $w^{1/2}$ are equal respectively, to the square and square root of the membership function of w.

Example 2.24 Same as Example 2.19, but with the question assumed to be worded as, "Is it true that x is red?" and the set of admissible answers expressed by

$$A = \underline{\text{true}} + \underline{\text{true}}^2 + \underline{\text{true}}^{1/2} + \underline{\text{false}} + \underline{\text{false}}^2 + \underline{\text{false}}^{1/2} + \underline{\text{borderline}} + \underline{\text{borderline}}^2 + \underline{\text{borderline}}^{1/2} \tag{2.25}$$

where true, false and borderline are defined in the same way as high, low and medium and may be used in the same manner. Thus, for example, if the answer to the question "Is it true that the vase is red?" is $\underline{\text{true}}^2$ ($\triangleq$ very true), then the grade of membership of the vase in the class of red objects is given by the assignment equation.

$$\mu_{\underline{red}}\ (\text{vase}) = \underline{\text{true}}^2 \tag{2.26}$$

where the right-hand member of (2.26) represents a linguistic truth-value whose meaning is defined by (2.22), and the left-hand member is the membership function of the fuzzy set red evaluated at x $\triangleq$ vase.

Example 2.27 As an illustration of an attributional question, suppose that X is the set of employees in a company and Q $\triangleq$ age? is an attributional question (e.g., "What is the age of Elizabeth?"). If the set of admissible answers is the set of integers

$$A = 20 + 21 + \ldots + 60 \tag{2.27}$$

then the answer to the question "What is the age of Elizabeth?" might be

$$\underline{age}\ (\text{Elizabeth}) = 32$$

On the other hand, if the admissible answers are linguistic in nature, e.g.,

$$A = \underline{young} + \underline{not}\ \underline{young} + \underline{very}\ \underline{young} + \underline{not}\ \underline{very}\ \underline{young} + \underline{old} + \underline{very}\ \underline{old} + \ldots \tag{2.28}$$

then an answer might have the form

$$\underline{age}\ (\text{Elizabeth}) = \underline{very}\ \underline{young}$$

with the understanding that very young is a linguistic value which is assigned to the linguistic variable age (Elizabeth). It should be noted that in (2.28) young and old play the role of primary fuzzy sets which have a specified meaning, e.g.,

$$\mu_{\underline{young}} = 1 - S(20,30,40) \tag{2.29}$$

$$\mu_{\underline{old}} = S(50,60,70) \tag{2.30}$$

where the S and Π functions are defined by (2.12) and (2.13), and $\mu_{\underline{young}}$ and $\mu_{\underline{old}}$ denote the membership functions of young and old, respectively. The meaning of the other terms in (2.28) may be computed from the definitions of the modifiers not and very. Thus,

$$\mu_{\underline{not}\ \underline{young}} = 1 - \mu_{\underline{young}} \tag{2.31}$$

$$\mu_{\underline{very}\ \underline{young}} = (\mu_{\underline{young}})^2 \tag{2.32}$$

$$\mu_{\underline{not}\ \underline{very}\ \underline{young}} = 1 - (\mu_{\underline{young}})^2 \tag{2.33}$$

and so on. Note that A may be viewed, in effect, as a minilanguage with its own syntax and semantics.

Nested questions

Consider an attributional question of the form "How old is Francoise?" to which a linguistic answer might be, "Francoise is young," with young defined by (2.29).

At this point, one could ask a classificational question concerning the answer "Francoise is young," namely, "Is it true that (Francoise is young)?" to which a linguistic answer might be very true. Continuing this process, one could ask the question "Is it true that ((Francoise is young) is very true)?" to which a linguistic answer might be more or less true. On further repetition, we are led to a nested question which, in general terms, may be expressed as

$$\text{Is it true that } (..(((x \text{ is } w) \text{ is } \tau_1) \text{ is } \tau_2) \ldots \text{ is } \tau_n) \text{ ?} \tag{2.34}$$

in which w is an attribute-value and $\tau_1, \tau_2, \ldots, \tau_n$ are numerical or linguistic truth-values.

How should the meaning of an answer of the form

$$a \triangleq (\ldots(((x \text{ is } w) \text{ is } \tau_1) \text{ is } \tau_2) \ldots \text{ is } \tau_n) \tag{2.35}$$

be interpreted? A clue is furnished by the following example. Suppose that the answer to the question "Is it true that (Francoise is young)?" is a numerical truth-value, say 0.5. As stated earlier, this implies that the grade of membership of Francoise in the class of young women is 0.5, which in turn implies (by (2.29)) that Francoise is 30 years old. Thus, we have

(Francoise is <u>young</u>) is 0.5 <u>true</u> $\Rightarrow$ Francoise is 30 years old. (2.36)

More generally, let u be a base variable for an attribute B and let $\mu_{\underline{young}}$ denote the membership function which defines the answer a $\triangleq$ <u>young</u> as a fuzzy subset of the universe of discourse, U, which is associated with the attribute B (e.g., if B $\triangleq$ <u>age</u>, then u is a number in the interval [0,100] and U = [0,100] is the universe of discourse associated with <u>age</u>). Now suppose that v is a numerical truth-value of the answer Francoise is <u>young</u>. Then, the age of Francoise is given by

$$B(\text{Francoise}) = \mu_B^{-1}(v) \tag{2.37}$$

where μ_B^{-1} is the function inverse to the function μ_B.[3] Thus, in the particular case where v = 0.5, (2.29) gives

$$\begin{aligned} B(\text{Francoise}) &= \mu_B^{-1}(0.5) \\ &= 30 \end{aligned} \tag{2.38}$$

At this juncture, we can employ the extension principle (see the Appendix) to compute the meaning of the answer a $\triangleq$ (Francoise is <u>young</u>) is τ, where τ is a linguistic truth-value which is characterized by a membership function μ_τ. (E.g., if τ is <u>true</u>, then μ_τ is given by (2.14).) Thus, substituting τ in (2.37), we obtain

$$\begin{aligned} B(\text{Francoise}) &= \mu_B^{-1}(\tau) \\ &= \mu_B^{-1} \circ \tau \end{aligned} \tag{2.39}$$

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[3] If the mapping μ_B: U $\rightarrow$ [0,1] is not 1-1, then μ_B^{-1} is the relation -rather than the function - that is inverse to μ_B. In any case, the graph of μ_B^{-1} is the same as that of μ_B, but with the abscissae of μ_B^{-1} being the ordinates of μ_B and vice-versa.

which should be interpreted as the composition[4] of the binary relation μ_B^{-1} and the unary relation τ. In more general terms, this result may be stated as the following proposition.

<u>Proposition 2.40</u> An answer of the form

$$a \triangleq (x \text{ is } w_1) \text{ is } \tau \tag{2.41}$$

where x is an object in X, w_1 is a fuzzy subset of U, and τ is a truth-value (numerical or linguistic), implies the answer

$$a^* \triangleq x \text{ is } w_2 \tag{2.42}$$

where w_2 is related to w_1 and τ by

$$w_2 = \mu_{w_1}^{-1} \circ \tau \tag{2.43}$$

In (2.43), $\mu_{w_1}^{-1}$ is the relation inverse to μ_{w_1}, where μ_{w_1} is the membership function of w_1, and the right-hand member of (2.43) represents the composition of $\mu_{w_1}^{-1}$ with the unary relation (fuzzy set) τ. (See Appendix.)

Repeated application of Proposition 2.40 to an answer of the form (2.35) leads to the general result

$$a \triangleq (\ldots(((x \text{ is } w_1) \text{ is } \tau_1) \text{ is } \tau_2) \ldots \text{ is } \tau_n \Rightarrow a^* \triangleq x \text{ is } w_{n+1} \tag{2.44}$$

where

$$w_{n+1} = \mu_{w_n}^{-1} \circ \tau_n \tag{2.45}$$

$$w_n = \mu_{w_{n-1}}^{-1} \circ \tau_{n-1}$$

- - - - - - - - -

[4]The composition of a binary relation R in $U_1 \times U_2$ with a unary relation S in U_2 is a unary relation RoS in U_1 whose membership function is given by $\mu_{RoS}(u_1) = \vee_{u_2} \mu_R(u_1, u_2) \wedge \mu_S(u_2)$, where $\vee \triangleq$ max and $\wedge \triangleq$ min.

$$w_2 = \mu_{w_1}^{-1} \circ \tau_1$$

and μ_{w_1}, i = 1,..., n is the membership function of w_1.

As a simple illustration of (2.43), a graphical representation of the composition $\mu_{w_1}^{-1} \circ \tau_1$, is shown in Fig. 2.4. Here $\mu_{\underline{young}}$ is the membership function of $w_1 = \underline{young}_1$, with the base variable being the numerical age u. τ_1 is assumed to be <u>very true</u>, whose membership function is plotted as shown, with v playing the role of abscissa. The point, α, on $\mu_{\underline{very\ true}}$ which has the abscissa v has the ordinate $\mu_{\underline{very\ true}}(v)$, and, correspondingly, the point, β, on $\mu_{\underline{young}_1}^{-1}$ which has the abscissa v has the ordinate $\mu_{\underline{young}_1}^{-1}(v)$. Now, from α and β we can construct a point γ on $\mu_{\underline{young}_2}$ with abscissa $\mu_{\underline{young}_1}^{-1}(v)$ and ordinate $\mu_{\underline{very\ true}}(v)$. In this way, by varying v from 0 to 1, we can generate the plot of $\mu_{\underline{young}_2}$, which is the membership function of w_2 as defined by (2.43).

An important conclusion which is implicit in (2.44) is that any nested assertion of the form

$$((x \text{ is } w_1) \text{ is } \tau_1) \ldots \text{ is } \tau_n) \tag{2.46}$$

may be replaced by an equivalent assertion of the form

$$x \text{ is } w_{n+1} \tag{2.47}$$

which does not contain any truth-values. Thus, the use of truth-values in (2.46) serves indirectly the same function as a linguistic modifier m which transforms w_1 into mw_1.

<u>The relation between classificational and attributional questions</u>

In the case of a nonfuzzy classificational question, the answer-set, A, has only two elements which are usually designated as {YES, NO}, {TRUE, FALSE} or {0,1}. By contrast, the answer-set of an attributional question is usually a continuum U or a countable set of linguistic values defined over U. Thus, in general, an answer to an attributional question conveys considerably more information than an answer to a nonfuzzy classificational question.

In the case of fuzzy classificational questions, however, the answer-set may be the unit interval [0,1] or a countable set of linguistic values defined over [0,1]. In such cases, the distinction between classificational and attributional questions is much less pronounced and, in fact, there may be equivalence between them.

To be more specific, let us assume for concreteness that U is the real line and F is a fuzzy subset of U. F will be said to be <u>amodal</u> if its membership function μ_F is strictly monotone, which implies that the mapping $\mu_F: u \rightarrow [0,1]$ is one-one. If F is not amodal but is convex[5] or concave, then F will be said to be <u>modal.</u> Typically, the membership function of an amodal fuzzy set has the form shown in Fig. 2.5, whereas that of a modal set has the appearance of a peak or a valley (Fig. 2.6).

Let $Q_c \triangleq F?$ be a classificational question which has the same body as an attributional question $Q_a \triangleq F?$ For example, a specific question q_c may be worded as "Is Jeanne <u>young</u>?" while the wording of q_a might be "How <u>young</u> is Jeanne?" Clearly, if <u>young</u> is an amodal fuzzy set, then from an answer to q_c such as "Jeanne is 0.9 <u>young</u>" we can deduce the age of Jeanne and, conversely, from the age of Jeanne, say age $\triangleq$ 32, we can

---

[5]A fuzzy set F in U is convex if μ_F satisfies the inequality $\mu_F(\lambda\, u_1 + (1-\lambda)u_2) \geq \min(\mu_F(u_1), \mu_F(u_2))$ for all u_1, u_2 in U and all λ in [0,1]. A fuzzy set F is <u>concave</u> if its complement is convex. Additional details may be found in [28].

deduce her grade of membership in the fuzzy set young. Thus, when F is an amodal fuzzy set or, more generally, a fuzzy set whose membership function is a one-one mapping, the answer to a classificational question conveys the same information as the answer to an attributional question.

Now suppose that F is a modal fuzzy set, e.g., F $\triangleq$ middle-aged, whose membership function has the form shown in Fig. 2.7. In this case, from the specification of the grade of membership in middle-aged, one cannot deduce the value of the attribute age uniquely. Thus, if F is modal, an answer to the classificational question "Is x F?" e.g., "Is Frieda middle-aged?" is less informative that an answer to the attributional question "What is the age of Frieda?"

It should be noted that Comment 2.18 implies that a classificational question Q $\triangleq$ B? may always be regarded as an attributional question whose body is the label of the membership function of B. Thus, what the above discussion indicates is that although it is not true in general that an attributional question is equivalent to a classificational question with the same body, this is the case when B is a modal fuzzy set.

3. Composite Questions and Their Representations

The concept of an atomic question which we discussed in the preceding section provides a basis for the definition of the more general concept of a composite question. This concept and its representations will be the focus of our attention in the sequel.

Stated informally, an n-adic composite question Q, with body B, is a question composed of n constituent questions $Q_1,\ldots,Q_n$ with bodies $B_1,\ldots,B_n$, respectively, such that the answer to Q is dependent upon the answers to $Q_1,\ldots,Q_n$. Thus, a monadic question has a single constituent, a dyadic question has two constituents, a triadic question has three constituents, etc. A constituent question may be atomic or composite.

An n-adic composite question or, simply, an n-adic question, Q, is characterized by its relational representation, $B(B_1,\ldots,B_n)$(or simply B, when no confusion with the body, B, of Q can arise), whose tableau has the form shown in Table 3.1. In this tableau, r_i^j and r_i range over the admissible answers to Q_j and Q, respectively, with A_j and A representing the answer-sets associated with Q_j and Q, and a^j and a denoting their generic elements. Thus, if Q is an n-adic question, then B is a nonfuzzy (n+1)-ary relation from the cartesian product $A_1\times \ldots \times A_n$ to A. In particular, if Q is a monadic question, then B is a binary relation, and if Q is atomic then B is a unary relation

Q_1	Q_2	Q_j	Q_n	Q
r_1^1	r_1^2	r_1^j	r_1^n	r_1
r_2^1	r_2^2	r_2^j	r_2^n	r_2
r_i^1	r_i^2	r_i^j	r_i^n	r_i
r_m^1	r_m^2	r_m^j	r_m^n	r_m

Table 3.1. Relational representation of Q. (Depending on the circumstances, the columns of B may be labeled $Q_1,\ldots,Q_n$, Q, or $B_1,\ldots,B_b$, B.)

Generally, we shall assume that the entries in B are linguistic in nature, i.e., are linguistic attribute-values and/or linguistic truth-values and/or linguistic grades of membership. Thus, if U_j is a universe of discourse associated with A_j, then an answer $a_i^j \in A_j$ will, in general, be a label of a fuzzy subset of U_j. The generic elements of U_j and U will be denoted by u_j and u, respectively, and will be referred to as the <u>base variables</u> for A_j and A. When it is necessary to differentiate between attributional and classificational questions, the universes of discourse for the latter will be denoted by V instead of U.

<u>Example 3.1</u> Consider a composite classificational question Q $\triangleq$ <u>big</u>? which is composed of two classificational atomic questions $Q_1 \triangleq$ <u>wide</u>? and $Q_2 \triangleq$ <u>long</u>?, and one attributional atomic question $Q_3 \triangleq$ <u>height</u>? The answer-sets associated with Q_1, Q_2, Q_3 and Q are assumed to be given by (f, b, t, ℓ, m, h are abbreviations for <u>false</u>, <u>borderline</u>, <u>true</u>, <u>low</u>, <u>medium</u> and <u>high</u>, respectively)

$$A_1 = A_2 = A = f + b + t \tag{3.2}$$

$$A_3 = \ell + m + h \tag{3.3}$$

where f, b and t are fuzzy subsets of the unit interval defined by (2.8), (2.9) and (2.10), and ℓ, m and h are fuzzy subsets of the real line defined by expressions of the form (2.16), (2.15) and (2.14) with parameters α, β, and γ.

The relational tableau for $B(B_1,\ldots,B_n)$ is assumed to be given by (in partially tabulated form) by Table 3.2.

wide?	long?	height?	big?
t	t	h	t
t	t	m	t
t	t	ℓ	b
t	t	ℓ	f
t	b	h	b
t	f	h	b
t	f	h	f
b	f	ℓ	b
.	.	.	.
f	f	ℓ	f

Table 3.2. Relational representation of big (wide, long, height)

There are two important observations to be made concerning $B(B_1,...,B_n)$. First, in general $B(B_1,...,B_n)$ is a relation rather than a function. In Table 3.2, this manifests itself by the fact that the entries in the column labeled big? are not uniquely determined by the entries in the columns wide?, long? and height?. For example, corresponding to $a^1 = t$, $a^2 = t$ and $a^3 = \ell$, we have both $a = b$ and $a = f$. This implies that if the answer to wide? is true, to long? is true and to height? is low, then the answer to big? could be either borderline or false.

Second, the tableau may not be complete, that is, certain combinations of the admissible answers to constituent questions may be missing from the table. For example, $a^1 = f$, $a^2 = b$ and $a^3 = b$ may not be in the table. This may imply that (i) the particular combination of answers cannot occur, or (ii) the answer to Q corresponding to the missing entries is not known - which is equivalent to assuming that the answer is the union of all admissible answers, i.e., is the answer-set A.

Case (i) implies that there is some interdependence between the constituent questions in the sense that the knowledge of answers to some of the constituent questions restricts the possible answers to others. If the Q_i are viewed as variables as in (2.18), then (i) implies that the Q_i are λ-interactive in the sense defined in [1]. Unless stated to the contrary, we shall assume that the missing rows imply (i) rather than (ii). A more detailed discussion of this issue will be presented in Sec. 4.

Alternative representations of B: Algebraic representation

The relational representation, B, of a composite question Q may in turn be represented in a variety of ways of which the most useful ones are: (a) The tabular representation, which we have described already; (b) the algebraic representation, which we shall discuss presently; the analytic representation, which we shall discuss following (b); and the branching questionnaire representation, which will be discussed in Sec. 4.

In the algebraic representation, the i^{th} row, $i = 1,2,\ldots,m$ of the tableau of B is expressed as a Q/A sequence of the form

$$Q_1 \, r_i^1 \, Q_2 \, r_i^2 \ldots Q_n \, r_i^n \, /\!\!/ \, Q \, r_i \qquad (3.4)$$

or, more simply as a Q/A string

$$r_i^1 \, r_i^2 \ldots r_i^n \, /\!\!/ \, r_i \qquad (3.5)$$

where it is understood that r_i^j, $j = 1, \ldots, n$ is an admissible answer to the constituent question Q_j, and r_i is an admissible answer to the composite question Q. B as a whole, then, may be expressed algebraically as the summation (i.e., the union) of the Q/A strings corresponding to the rows of the tableau of B. Thus, we may write

$$B = r_1^1 r_1^2 \ldots r_1^n /\!\!/ r_1 + r_2^1 r_2^2 \ldots r_2^n /\!\!/ r_2 + \ldots + r_m^1 r_m^2 \ldots r_m^n /\!\!/ r_m \quad (3.6)$$

or, more compactly,

$$B = \sum_i r_i^1 r_i^2 \ldots r_i^n /\!\!/ r_i \quad (3.7)$$

Example 3.8 In the algebraic form, the tableau of the relational representation defined by Table 3.2 may be expressed as

$$\begin{aligned} B = \; & tth /\!\!/ t + ttm /\!\!/ t + tt\ell /\!\!/ b \\ & + tt\ell /\!\!/ f + tbh /\!\!/ b + tfh /\!\!/ b \\ & + tfh /\!\!/ f + bf\ell /\!\!/ b + \ldots + ff\ell /\!\!/ f \end{aligned} \quad (3.9)$$

As in the case of regular expressions, an important advantage of representations of the form (3.9) is that the operations of union (+) and string concatenation may be treated in much the same manner as addition and multiplication. Thus, the terms in (3.9) may be combined or expanded in accordance with the replacement rules which are illustrated below by examples.

$$ttf /\!\!/ f + tff /\!\!/ t = t(tf /\!\!/ t + ff /\!\!/ t) \quad (3.10)$$

$$ttf /\!\!/ t + ftf /\!\!/ t = (t + f)tf /\!\!/ t \quad (3.11)$$

$$tfb /\!\!/ t + ttb /\!\!/ t = t(f + t)b /\!\!/ t \quad (3.12)$$

$$tfb /\!\!/ t + tfb /\!\!/ b = tfb /\!\!/ (t + b) \quad (3.13)$$

$$(t + f)(f + b)t /\!\!/ t = tft /\!\!/ t + fft /\!\!/ t + tbt /\!\!/ t + fbt /\!\!/ t \quad (3.14)$$

For example, using the above identities in (3.9), we can write B in a partially factored form as

$$B = tt(h + m) /\!\!/ t + tt\ell /\!\!/ (b + f) + t(b + f)h /\!\!/ b + \quad (3.15)$$

$+ (tfh + ff\ell) /\!\!/ f + \ldots + bf\ell /\!\!/ b$

It should be noted that the replacement of the left-hand member by the right-hand member involves a factorization in (3.10), (3.11), (3.12) and (3.13), and an expansion in (3.14). In general, factorization has the effect of raising the level of an expression (in the sense of decreasing the number of operations that have to be performed for its evaluation), while an expansion has the opposite effect. For example, the evaluation of the arithmetic expression $xy + xz$ requires three operations, while that of the factored form $x(y+z)$ requires only two. In this case, the representation of B in the normal form[1] (3.9) has the lowest possible level among all algebraic representations involving the admissible answers to the Q_j and Q.

The meaning of B

The question of what constitutes the meaning of B may be viewed as a special case of the following problem in semantics.[2] Suppose that we are given a string of terms (words) $W_1W_2\ldots W_n$ with the meaning of each term defined as a subset of a universe of discourse U. What is the meaning of the composite term $W_1W_2\ldots W_n$ - that is, what is the subset of U whose label is $W_1W_2\ldots W_n$?

As a special instance of this problem consider two finite nonfuzzy sets G and H whose elements are $g_1,\ldots,g_m$ and $h_1,\ldots, h_n$, respectively.

[1]This usage of the term normal form is consistent with that of E.F. Codd in his work on relational models of data [29]. A related concept is that of a characteristic set in the Vienna definition language [30]-[31].

[2]A more detailed discussion of this problem may be found in [22] and [32].

When we write

$$G = g_1 + \ldots + g_m \tag{3.16}$$

$$H = h_1 + \ldots + h_n \tag{3.17}$$

the right-hand side of the equation defines the meaning[3] of the label on the left-hand side. Now, if we write the cartesian product $G \times H$ as a string GH, then the meaning of $G \times H$ may be obtained very simply by expanding the algebraic product of G and H. Thus,

$$\begin{aligned} G \times H &= GH \\ &= (g_1 + \ldots + g_m)(h_1 + \ldots + h_n) \\ &= g_1h_1 + \ldots + g_mh_n \end{aligned} \tag{3.18}$$

where g_ih_j should be interpreted as the ordered pair (g_i,h_j).

Now suppose that G and H are finite fuzzy sets defined by

$$G = \mu_1/g_1 + \ldots + \mu_m/g_m \tag{3.19}$$

$$H = \nu_1/h_1 + \ldots + \nu_n/h_n \tag{3.20}$$

where μ_i/g_i means that the grade of membership of g_i in G is μ_i, and likewise for H. Then, for the cartesian product of G and H we obtain

$$\begin{aligned} G \times H &= (\mu_1/g_1 + \ldots + \mu_m/g_m)(\nu_1/h_1 + \ldots + \nu_n/h_n) \\ &= (\mu_1 \wedge h_1)/g_1h_1 + \ldots + (\mu_m \wedge \nu_n)/g_mh_n \end{aligned} \tag{3.21}$$

where

$$\mu_i \wedge \nu_j \triangleq \min(\mu_i,\nu_j) \tag{3.22}$$

---

[3] The term <u>meaning</u> is used here in the sense of denotational semantics [4]-[9].

More generally, let $G_1,\ldots,G_n$ be fuzzy subsets of $U_1,\ldots,U_n$ defined by

$$G_j = \sum_{i=1}^{m_j} \mu_i^j/u_i^j \tag{3.23}$$

Then

$$G_1 \times\ldots\times G_n = G_1\ldots G_n \tag{3.24}$$

$$= \sum_i (\mu_{i_1}^1 \wedge\ldots\wedge \mu_{i_n}^n)/u_{i_1}^1 \ldots u_{i_n}^n$$

which implies that the right-hand member of (3.24) constitutes the meaning of the string $G_1\ldots G_n$ (or, equivalently, $G_1 \times\ldots\times G_n$).

Returning to the question of what constitutes the meaning of B, let us focus our attention on the algebraic representation of B as expressed by (3.6). If the r_i^j and r_i in (3.6) are assumed to be fuzzy subsets of $U_1,\ldots,U_n$, U, then each term in (3.6) is a cartesian product of fuzzy sets in the sense of (3.24), and B as a whole is the union of such cartesian products. Thus, upon the expansion of each term in accordance with (3.24) and summing the results, we obtain the expression for a fuzzy (n+1)-ary relation from $U_1 \times\ldots\times U_n$ to U which may be viewed as the <u>denotational meaning</u> of B.[4] This fuzzy relation will be denoted by B_β and will be referred to as the <u>β-representation</u> of B, with β - standing for <u>base variable</u> - serving to signify that B_β is a fuzzy relation from $U_1\times\ldots\times U_n$ to U whereas B is a nonfuzzy relation from $A_1\times\ldots\times A_n$ to A.

In summary, the main points of the foregoing discussion may be stated

[4]In performing the expansion and summation of terms in B, we are tacitly assuming that the constituent questions $Q_1,\ldots,Q_n$ are <u>β-noninteractive</u> [1] in the sense that the base variables $u_1,\ldots,u_n$ are jointly unconstrained.

as follows.

Proposition 3.25 Let B be an (n+1)-ary nonfuzzy relation from $A_1 \times \ldots \times A_n$ to A which constitutes a relational representation of a composite question Q. If the answers to Q and the constituent questions in Q are fuzzy subsets of their respective universes of discourse U, $U_1, \ldots, U_n$, then B induces an (n+1)-ary fuzzy relation B_β which may be derived from B by the process of expansion. The fuzzy relation B_β constitutes the denotational meaning of B in the universe of discourse $U_1 \times \ldots \times U_n \times U$.[5]

Example 3.26 As a very simple illustration of (3.25), consider a B whose algebraic representation reads

$$B = tt \,/\!\!/ f^2 + ff \,/\!\!/ t \tag{3.27}$$

where t($\triangleq$ true), f($\triangleq$ false) and f^2($\triangleq$ very false) are fuzzy subsets of the universe of discourse

$$V = 0 + 0.2 + 0.4 + 0.6 + 0.8 + 1 \tag{3.28}$$

and are defined by

$$t = 0.6/0.8 + 1/1 \tag{3.29}$$

$$f = 1/0 + 0.6/0.2 \tag{3.30}$$

and

$$f^2 = 1/0 + 0.36/0.2 \tag{3.31}$$

On substituting (3.29)-(3.31) into $tt \,/\!\!/ f^2$ and expanding, we have

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[5] In cases in which the body, B_i, of a classification question, Q_i, is a fuzzy subset of a universe of discourse which does not possess a numerically-valued base variable (e.g., $Q_i \triangleq$ beautiful), it may be necessary to define B_i by exemplification [22],[65]. In general, exemplificational (or ostensive) definitions are human - rather than machine-oriented.

$$tt /\!\!/ f^2 = (0.6/0.8 + 1/1)(0.6/0.8 + 1/1) /\!\!/ (1/0 + 0.36/0.2) \tag{3.32}$$
$$= 0.6/(0.8,0.8,0) + 0.6/(0.8,1,0)$$
$$+ 0.6/(1,0.8,0) + 1/(1,1,0)$$
$$+ 0.36/(0.8,0.8,0.2) + 0.36/(0.8,1,0.2)$$
$$+ 0.36/(1,0.8,0.2) + 0.36/(1,1,0.2)$$

Performing the same operation on the other term in (3.27) and summing the results, we obtain the desired expression for B_β

$$B_\beta = 0.36/((0.8,0.8,0.2) + (0.8,1,0.2) + (1,0.8,0.2) + (1,1,0.2)) \tag{3.33}$$
$$+ 0.6/((0,0,0.8) + (0,0.2,0.8) + (0.2,0,0.8) + (0.2,0.2,0.8)$$
$$+ (0,0.2,1) + (0.2,0.2,1)) + 1/((0,0,1) + (1,1,0))$$

as a ternary fuzzy relation in $[0,1]\times[0,1]\times[0,1]$.

Interpolation of B

Knowledge of B_β is of importance in that it provides a basis for an interpolation of B, that is, an approximate way of deducing answers to Q corresponding to entries in B which are not elements of the answer-sets $A_1,\ldots,A_n$.

To illustrate, suppose that Q is a dyadic classificational question whose constituent classificational questions Q_1 and Q_2 have the answer-sets

$$A_1 = A_2 = A = t + b + f$$

Let B be a relational representation of Q and assume that we wish to find the answer to Q when the answers to Q_1 and Q_2 are, respectively

$$a^1 = \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}} \tag{3.34}$$

and

$$a^2 = \underline{\text{rather}}\ \underline{\text{true}} \tag{3.35}$$

Since a^1 and a^2 are not among the entries in the Q_1 and Q_2 columns of the tableau of B, we cannot use B to find the corresponding entry in the Q column. On the other hand, if we have B_β as a fuzzy ternary relation in $V_1 \times V_2 \times V$ (which is $[0,1]\times[0,1]\times[0,1]$ in the case under consideration), then by interpolating B we can obtain an approximation to the answer to Q which corresponds to the answers a^1 = not very true and a^2 = rather true.

Specifically, the desired approximation is given by the composition of B_β with the fuzzy sets a^1 and a^2, treating a^1 and a^2 as unary fuzzy relations in [0,1]. Thus,[6]

$$\text{Answer to } Q \cong B_\beta \circ a^1 \circ a^2 \tag{3.36}$$

The significance of (3.36) becomes somewhat clearer if the right-hand member of (3.36) is interpreted as the projection on V of the intersection of B_β with the cylindrical extensions of a^1 and a^2.[7] Thus, if B_β is visualized as a fuzzy surface in $V_1 \times V_2 \times V$, then a^1 and a^2 may be likened to fuzzy points on the coordinate axes V_1 and V_2, and their cylindrical extensions play the role of fuzzy planes passing through these points. The intersection of these planes with the fuzzy surface is a fuzzy point in $V_1 \times V_2 \times V$ which upon projection on V becomes a fuzzy subset of V expressed by the right-hand member of (3.36). A two-dimensional version of this process is shown in Fig. 3.3.

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[6] It is understood that the right-hand member of (3.36) should be approximated to by an admissible answer to Q.

[7] The cylindrical extensions of a^1 and a^2 are, respectively, the ternary fuzzy relations $a^1\times V\times V$ and $V\times a^2\times V$. The definition of the projection of a fuzzy relation is given in the Appendix. (Additional details may be found in [1] and [28].)

Analytic representation of $B(B_1,\ldots,B_n)$

Consider a composite classificational question $Q = B?$ whose constituents are classificational questions $Q_1 = B_1\ ?$, $Q_2 = B_2\ ?$, ..., $Q_n = B_n\ ?$ in which the body, B_i, of Q_i, $i = 1,\ldots,n$, is a specified fuzzy subset of the universe of discourse V_i. Furthermore, assume that the relation $B(B_1,\ldots,B_n)$ is a function from $A_1 \times\ldots\times A_n$ to A. This implies that an answer to Q - which may be interpreted as a specification of the grade of membership of a given object x in B - is a function of the grades of membership of x in $Q_1,\ldots,Q_n$. In this sense, the B_i form a basis for Q.

When a collection of fuzzy sets $B_1,\ldots,B_n$ forms a basis for Q, it may be convenient to express B, the body of Q, as an explicit function of $B_1,\ldots,B_n$. Such a function may involve such standard operations as the union, B_1+B_2; intersection, $B_1 \cap B_2$; complement, B_1'; product, B_1B_2; cartesian product, $B_1 \times B_2$; etc. In addition, it may involve other specified operations - in particular, the interactive versions of $+$ and $\cap$, which will be denoted by $<+>$ and $<\cap>$, respectively.[8] The expression for B as a function of $B_1,\ldots,B_n$ will be referred to as an <u>analytic representation</u> of B.

<u>Example 3.37</u> Suppose that we wish to define the concept of HIPPIE. To this end, we form the classificational question Q = HIPPIE? and assume that the basis for HIPPIE is the collection of fuzzy sets $B_1 \triangleq$ LONG HAIR, $B_2 \triangleq$ BALD, $B_3 \triangleq$ DRUGS and $B_4 \triangleq$ EMPLOYED, which will be abbreviated as LH, B, D and EMP, respectively.

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[8] In general, the angular brackets are used to identify an interactive version of an operation, e.g., <<u>and</u>> is an interactive version of <u>and</u>. A brief discussion of interactive operations is given in the Appendix.

An analytic representation for B which constitutes the definition of HIPPIE in terms of B_1, B_2, B_3 and B_4 might be[9]

$$\text{HIPPIE} = (\text{LH} + \text{B}) \cap \text{DRUGS} \cap \text{EMP}' \tag{3.38}$$

or equivalently

$$\text{HIPPIE} = (\text{LH } \underline{\text{or}} \text{ B}) \ \underline{\text{and}} \ \text{DRUGS} \ \underline{\text{and}} \ \underline{\text{not}} \ \text{EMP} \tag{3.39}$$

which implies that the grade of membership of a subject x in the fuzzy set HIPPIE is related to the grades of membership of x in the fuzzy set of LONG HAIR subjects, BALD subjects, DRUG TAKING subjects and EMPLOYED subjects by the expression

$$\mu_H(x) = (\mu_{LH}(x) \vee \mu_B(x)) \wedge \mu_D(x) \wedge (1 - \mu_{EMP}(x)) \tag{3.40}$$

where $\vee \triangleq$ max and $\wedge \triangleq$ min. A representation of (3.39) in the form of a flowchart is shown in Fig. 3.4, with the understanding that YES and NO are answers of the form YES μ and NO $(1-\mu)$, where μ is the grade of membership of x in the fuzzy set which labels the question.

<u>Note 3.41</u> If (3.40) does not constitute an acceptable approximation to the expression for $\mu_H(x)$ as a function of $\mu_{LH}(x)$, $\mu_B(x)$, $\mu_D(x)$ and $\mu_{EMP}(x)$, it may be possible to improve on the approximation by employing interactive versions of <u>and</u> and/or <u>or</u>. For example, we may write

$$\text{HIPPIE} = ((\text{LH } \underline{\text{or}} \text{ B}) \ \langle\underline{\text{and}}\rangle \ \text{DRUGS}) \ \underline{\text{and}} \ \underline{\text{not}} \ \text{EMP} \tag{3.41}$$

where <<u>and</u>> is defined by a linguistic relation of the form

---

9 This definition is used only for illustrative purposes and has no pretense at being a realistic definition of the concept of HIPPIE.

u	v	u <and> v
t	t	t^2
t	b	b
t	f	f
·	·	·
f	f	f^2

in which t, b, f, t^2 and f^2 are abbreviations for the linguistic truth-values true, borderline, false, very true, and very false.

Basically, the interactive versions of and and or serve to extend the usefulness of these connectives by providing a means of taking into account the trade-offs that might exist between their operands. However, it should be noted that, in general, <and> and <or> will not possess such properties as associativity, distributivity, etc., and hence could not be manipulated as conveniently as their noninteractive counterparts.

We turn next to the representation of B by means of branching questionnaires.

4. Branching Questionnaires

In one form or another, the concept of a branching questionnaire plays an important role in many fields, especially in taxonomy, pattern recognition, diagnostics and, more particularly, the identification of sequential machines [34]-[48]. In what follows, the term branching questionnaire will be used in a more specific sense to refer to a representation of a composite question, $Q \triangleq B?$, in which the constituent questions $Q_1,\ldots,Q_n$ are asked in an order determined by the answers to the previous questions. A branching questionnaire representation of $Q \triangleq B?$ will be denoted by Q^* or, more explicitly, by B^*.

A branching questionnaire, Q^*, may be conveniently represented in the form of a tree as shown in Fig. 4.1 (or, alternatively, in the form of a block diagram, as in Fig. 4.2). The root of this tree is labeled with the name of the composite question, Q, or with the name of the body of Q; the leaves are labeled with the admissible answers to Q; and the internal nodes are labeled with the names of the constituent questions or the names of their bodies. Thus, each fan[1] of the tree represents a constituent question, with each branch of the fan corresponding to an admissible answer to that question. If a branch such as a_1^2 of question Q_2 terminates on Q_1, it means that if the answer to question Q_2 is a_1^2, then the next question to be asked is Q_1. This implies that if the answer to Q_2 is a_1^2, the answer to Q_1 is a_3^1 and the answer to Q_3 is a_1^3, then the answer to Q is a_2.

Each path from the root of the tree to a leaf represents a particular Q/A sequence, e.g.,

[1] By the fan of a tree we mean a node of the tree together with the branches connected to it.

$$Q_2\ a_1^2\ Q_1\ a_1^1\ Q_3\ a_1^3\ /\!\!/ Q\ a_1 \tag{4.1}$$

which may be written more simply as

$$a_1^2\ a_1^1\ a_1^3\ /\!\!/ a_1 \tag{4.2}$$

if the names of the answers to the constituent questions are labeled in a way that makes it possible to associate each answer in the sequence with a unique constituent question.

It is important to note that the only condition on the structure of a branching questionnaire is that on any path from the root to a leaf each constituent question is encountered at most once. A prescription of the order in which the constituent questions are to be asked (without regard to the answers to Q) is characterized in the manner shown in Fig. 4.3.

The summation (union) of all Q/A sequences of the form (4.2) constitutes an algebraic representation of Q*. For example, for the branching questionnaire of Fig. 4.1, we have the representation (using Q* in place of B*)

$$\begin{aligned} Q^* = {} & a_1^2\ a_1^1\ a_1^3\ /\!\!/ a_1 + a_1^2\ a_1^1\ a_2^3\ /\!\!/ a_2 + a_1^2\ a_2^1\ /\!\!/ a_2 \\ & + a_1^2\ a_3^1\ a_1^3\ /\!\!/ a_2 + a_1^2\ a_3^1\ a_2^3\ /\!\!/ a_1 + a_2^2\ /\!\!/ a_1 \end{aligned} \tag{4.3}$$

A Q/A sequence which terminates on an internal node of the tree defines an access path to that node and thereby uniquely identifies it. For example, the Q/A sequence a_1^2 identifies the node Q_1 in the tree of Fig. 4.1. Similarly, the leftmost Q_3 in Fig. 4.1 is identified by the Q/A sequence $a_1^2\ a_1^1$.[2]

[2]It should be noted that such Q/A sequences serve a role similar to that of composite selectors in the case of a Vienna definition language object [31].

Each internal node of the tree may be viewed as the root of a subtree which corresponds to a subquestionnaire of Q*. Thus, on factoring a_1^2 in (4.3), we obtain

$$Q^* = a_1^2(a_1^1\ a_1^3 \mathbin{/\!/} a_1 + a_1^1\ a_2^3 \mathbin{/\!/} a_2 + a_2^1 \mathbin{/\!/} a_2 + a_3^1\ a_1^3 \mathbin{/\!/} a_2 + a_3^1\ a_2^3 \mathbin{/\!/} a_1) + a_2^2 \mathbin{/\!/} a_1 \tag{4.4}$$

in which the expression within the parentheses may be regarded as an algebraic representation of a subquestionnaire which has Q_1 and Q_3 as its constituents.

Comment 4.5 By analogy with the concept of a derivative in the case of regular expressions [49]-[51], the coefficients of a_1^2 and a_2^2 in (4.4) may be defined to be the derivatives of Q* with respect to a_1^2 and a_2^2, respectively. Thus, on denoting these derivatives by $D_{a_1^2}Q^*$ and $D_{a_2^2}Q^*$, the expression for Q* may be rewritten as

$$Q^* = a_1^2\ D_{a_1^2}\ Q^* + a_2^2\ D_{a_2^2}\ Q^* \tag{4.6}$$

More generally, let w denote a Q/A sequence (e.g., $w \triangleq a_1^2\ a_1^1$), and let S_w denote the subtree of Q* which is uniquely determined by w. Then, we may write

$$D_w Q^* = S_w \tag{4.7}$$

Now let $N_1,\ldots,N_r$ be the nodes in a cut[3] of Q* and let $Q/A_1,\ldots,Q/A_r$

[3]The cut of a tree is a set of nodes with the following properties: (a) No two nodes in the cut are on the same path from the root to a leaf; and (b) No other node of the tree can be added to the cut without violating (a) [40],[52].

denote the Q/A sequences which lead from the root of Q* to $N_1,\ldots,N_r$, respectively. Then in consequence of (4.7), we can assert the identity

$$Q^* = \sum_{i=1}^{r} Q/A_i \; D_{Q/A_i} \; Q^* \tag{4.8}$$

of which (4.6) may be viewed as a special case.

Note 4.9 It should be observed that the constituent questions in Q* may be λ-interactive in the sense defined in [1], that is, the answers to, say, $Q_{i_1},\ldots,Q_{i_k}$, where $(i_1,\ldots,i_k)$ is a subsequence of the index sequence $(1,2,\ldots,n)$, may restrict the possible answers to $Q_{j_1},\ldots,Q_{j_\ell}$, where $(j_1,\ldots,j_\ell)$ is a subsequence complementary to $(i_1,\ldots,i_k)$. (E.g., for $n = 5$, $(i_1,i_2,i_3) \triangleq (2,4,5)$ and $(j_1,j_2) \triangleq (1,3)$.) For example, if the answer to an attributional question $Q_1 \triangleq$ mother of Julie? is Frances, then the answer to $Q_2 \triangleq$ sister of Julie? cannot be Frances if there is just one Frances in the universes U_1 and U_2. Thus, the answer a^2 = Frances is conditionally impossible given a^1 = Frances.

In the tree representation of a branching questionnaire, the conditional impossibility of an answer to a single question is indicated by associating θ (empty set) with the leaf of the corresponding branch (Fig. 4.4). Thus, in the example under consideration, a_2^1 is conditionally impossible given a_1^2. Note that any conditionally impossible answer must of necessity be a leaf of the tree since a Q/A sequence is aborted when a conditionally impossible answer is encountered.

The set of all possible answers to $Q_1,\ldots,Q_n$ constitutes a restriction on $Q_1,\ldots,Q_n$. Correspondingly, the conditionally possible answers to $Q_{j_1},\ldots,Q_{j_\ell}$ given the answers to $Q_{i_1},\ldots,Q_{i_k}$ constitute a conditioned restriction on $Q_{j_1},\ldots,Q_{j_\ell}$ given $Q_{i_1},\ldots,Q_{i_k}$.[4] In terms of restrictions, the constituent questions $Q_1,\ldots,Q_n$ are λ-noninteractive iff the restriction on $Q_1,\ldots,Q_n$ is the cartesian product of the answer-sets $A_1,\ldots,A_n$. Stated more simply, the noninteraction of $Q_1,\ldots,Q_n$ means that the answers to any subset of constituent questions, say $Q_{i_1},\ldots,Q_{i_k}$, do not affect the possible answers to the complementary questions $Q_{j_1},\ldots,Q_{j_\ell}$. In what follows, we shall assume, unless stated to the contrary, that the constituent questions in Q are λ-noninteractive.

Conditional redundance

In comparing the algebraic representations of B and Q*, (3.6) and (4.3), we observe that every term in B involves the answers to all of the constituent questions in Q, whereas a term in Q* involves, in general, a subset of the answers to $Q_1,\ldots,Q_n$.

More specifically, a term such as $a_2^2 /\!\!/ a_1$ in Q* implies that if the answer to Q_2 is a_2^2, then regardless of the answers to Q_1 and Q_3, the answer to Q is a_1. Thus, in this instance we may say that Q_1 and Q_3 are conditionally redundant given a_2^2. Similarly, Q_3 is conditionally redundant given $a_1^2\, a_2^1$. By implication, then, a constituent question, Q_i, is unconditionally redundant iff the answers to Q are independent of the answers to Q_i.

A constituent question Q_i will be said to be conditionally redundant given $Q_{i_1},\ldots,Q_{i_k}$ iff for every set of possible answers $a_{\lambda_1}^{i_1},\ldots,a_{\lambda_k}^{i_k}$, Q_i is conditionally redundant given $a_{\lambda_1}^{i_1},\ldots,a_{\lambda_k}^{i_k}$. As we shall see in

[4] A more detailed discussion of conditioned restrictions may be found in [1].

Sec. 5, the detection of conditional redundancies plays an important role in the construction of efficient branching questionnaires.

Comment 4.10 It should be noted that if the answer to Q_i is uniquely determined by the answers to $Q_{i_1},\ldots,Q_{i_k}$, then Q_i is conditionally redundant given $Q_{i_1},\ldots,Q_{i_k}$. However, in general, conditional redundance of Q_i given $Q_{i_1},\ldots,Q_{i_k}$ is weaker than the dependence of Q_i on $Q_{i_1},\ldots,Q_{i_k}$.

Tabular representation of a branching questionnaire

As was pointed out already, a term such as $a_2^2 /\!\!/ a_1$ in (4.3) signifies that if the answer to Q_2 is a_2^2, then the answer to Q is a_1, no matter what the answers to Q_1 and Q_2 might be. Now, "no matter what" or, equivalently, "don't care" may be interpreted as the answers $a^1 \triangleq a_1^1 + a_2^1 + a_3^1$ to Q_1 and $a^3 \triangleq a_1^3 + a_2^3$ to Q_3. Thus, more generally, a "don't care" answer to Q_i may be interpreted as the answer $a^i \triangleq A_i \triangleq$ answer-set of Q_i.

For simplicity, it is convenient to represent an answer of the form $a^i \triangleq A_i$ by * or, if necessary, by $*_i$. With this notation, the tableau of Q* (see (4.3)) assumes the form shown in Table 4.5. (The dotted line(s) in this tableau serves to identify the groups of rows which have the same entry in the Q column.)

Q_1	Q_2	Q_3	Q
a_1^1	a_1^2	a_1^3	a_1
a_3^1	*	a_2^3	a_1
*	a_2^2	*	a_1
a_3^1	a_1^2	a_1^3	a_2
a_1^1	a_1^2	a_2^3	a_2
a_2^1	a_1^2	*	a_2

Table 4.5. Tableau of Q*

A row such as $*a_2^2*$ in this tableau may be presented algebraically as[5]

$$
\begin{aligned}
a_2^2 &= (a_1^1 + a_2^1 + a_3^1)\, a_2^2 (a_1^3 + a_2^3) \qquad (4.11)\\
&= a_1^1\, a_2^2\, a_1^3 + a_1^1\, a_2^2\, a_2^3 + a_2^1\, a_2^2\, a_1^3\\
&+ a_2^1\, a_2^2\, a_2^3 + a_3^1\, a_2^2\, a_1^3 + a_3^1\, a_2^2\, a_2^3
\end{aligned}
$$

On performing similar expansions for all rows in Table 4.5 which contain stars, we obtain the complete tableau of Q*, as shown in Table 4.6.

Q	Q	Q	Q
a_1^1	a_1^2	a_1^3	a_1
a_3^1	a_1^2	a_2^3	a_1
a_3^1	a_2^2	a_2^3	a_1
a_1^1	a_2^2	a_1^3	a_1
a_1^1	a_2^2	a_2^3	a_1
a_2^1	a_2^2	a_2^3	a_1
a_2^1	a_2^2	a_2^3	a_1
a_3^1	a_2^2	a_1^3	a_1
a_3^1	a_1^2	a_1^3	a_2
a_1^1	a_1^2	a_2^3	a_2
a_2^1	a_1^2	a_1^3	a_2
a_2^1	a_1^2	a_2^3	a_2

Table 4.6. Complete tableau of Q*.

[5] In the terminology of switching theory, the terms on the right-hand side of (4.11) are <u>covered</u> by $*a_2^2*$, and $*a_2^2*$ constitutes a <u>prime implicant</u> of Q*[53],[54],[51].

The preceding discussion indicates that the tableau of Table 4.5 may be derived from that of Table 4.6 by a factorization of terms in the algebraic representation of Q*, (4.3), and replacing by *'s those factors which have the form of the sum of all admissible answers to a constituent question. A systematic procedure for carrying out such factorizations will be described in the following section.

Note 4.12 If the constituent questions in Q are λ-interactive, then in a term such as $a_3^1 * a_2^3$, the star would represent the conditioned restriction on Q_2 given $a_3^1\ a_2^3$. More generally, in a term of the form $a_{\lambda_1}^{i_1} \ldots a_{\lambda_k}^{i_k}$ $*_{j_1} \ldots *_{j_\ell}$, the sequence $*_{j_1} \ldots *_{j_\ell}$ would represent the conditioned restriction on $Q_{j_1}, \ldots, Q_{j_\ell}$ given the Q/A sequence $a_{\lambda_1}^{i_1} \ldots a_{\lambda_k}^{i_k}$.

5. Construction of Branching Questionnaires

In constructing a fuzzy-algorithmic definition of a concept B, the first step would normally involve a tabulation of the relational representation, $B(Q_1,\ldots,Q_n)$, of a composite question, Q = B?, which has B as its body. The second step, then, would involve the construction of a branching questionnaire realization of Q which is efficient in the sense of minimizing a cost function whose components are the costs of answering the constituent questions in Q. In practice, such a cost function would usually be prescribed in a highly approximate fashion.

As an illustration of the first step, suppose that we wish to construct a fuzzy-algorithmic definition of the concept of recession. Using our intuitive knowledge of the factors which enter into this concept and the interrelations between them, we construct in an approximate fashion a linguistic relational representation for RECESSION which might have the form shown in Table 5.1.[1] In this table, the observation interval is assumed to be a two-quarter period; GNP↓ denotes the decline in the gross national product; UNEMP denotes unemployment; BANKR↑ represents the increase in bankruptcies; and DJ↓ denotes the decline in the Dow Jones stock average in relation to its maximum value over the observation interval.

GNP↓	UNEMP	BANKR↑	DJ↓	RECESSION
<u>small</u>	<u>low</u>	<u>small</u>	<u>small</u>	<u>false</u>
<u>moderate</u>	<u>low</u>	<u>small</u>	<u>small</u>	<u>not</u> <u>true</u>
<u>high</u>	<u>low</u>	<u>small</u>	<u>small</u>	<u>borderline</u>
.	.	.	.	.
<u>high</u>	<u>moderate</u>	<u>moderate</u>	<u>large</u>	<u>rather</u> <u>true</u>
<u>high</u>	<u>high</u>	<u>large</u>	<u>large</u>	<u>very</u> <u>true</u>

Table 5.1. Tableau of relational representation of RECESSION

[1] This representation is used merely for illustrative purposes and should not be taken as a realistic definition of the concept of recession. A brief but informative discussion of recessions may be found in [23] and [24].

It should be noted that the composite question Q $\triangleq$ RECESSION? is treated as a classificational question in Table 5.1, although all of the constituent questions in RECESSION are attributional in nature. Normally, the meaning of the linguistic values of the attributes would be defined in terms of their compatibility functions, which can be computed from the knowledge of the compatibility functions of the primary fuzzy sets. For example, in the case of unemployment, the compatibility functions of the primary fuzzy sets labeled low and high might be of the form shown in Fig. 5.2. From these, one can compute, if needed, the compatibility functions of very low, more or less high, etc., by the use of (2.22) and (2.23).

There are several basic problems which are ancillary to the transformation of a relational representation of the definiendum (i.e., the concept under definition) into an efficient branching questionnaire. Of these, one is that of determining the conditional redundancies and/or restrictions which may be present in the relational representation. Another is that of determining the order in which the constituent questions must be asked in order to minimize the average cost of finding the answer to Q.

These and related problems have many features in common with the minimization of switching circuits [53]-[55], optimal encoding [56], feature selection in pattern recognition [57]-[60], and the optimization of decision tables [61]-[64]. However, the construction of an efficient branching questionnaire for the purpose of defining a concept presents some special problems relating to the fact that the efficiency of a branching questionnaire is influenced not only by the conditional redundancies but also by the cost of the constituent questions as well as by the conditional probabilities of the admissible answers - probabilities which are conditioned on the answers to the preceding questions in the questionnaire.

In what follows, our discussion of the construction of efficient branching questionnaire will be quite restricted in scope. Thus, we shall focus our attention mainly on the determination of the conditional redundancies in a relational representation and an illustration of the computation of the average cost of finding an answer to Q for a given branching questionnaire realization of B.

Compactification of Q

By the compactification of Q (or B) we mean the process of putting the representation of Q (tabular, algebraic or graphical) into a form that places in evidence the conditional redundancies and/or restrictions in the relational representation of Q, and thereby achieves a greater degree of compactness in its mode of representation. Thus, the transition from the tableau of Table 4.6 to that of Table 4.5 is an instance of compactification of a tabular representation of a composite question.

If the initial representation has the form of a graph or, more specifically, a tree, than the following rule - which is both general and simple to apply - may be employed to compactify the representation.

Rule 5.1 (merger rule) Let Q* be a tree representation of a branching questionnaire, and let $S_1, S_2, \dots, S_k$ be subtrees of Q* which are identical (i.e., have the same structure as well as branch and node labels)

$$S_1 \equiv S_2 \equiv \dots \equiv S_\ell \equiv S \tag{5.2}$$

Then $S_1, \dots, S_\ell$ may be merged into a single subtree S, as shown in Figs. 5.3 and 5.4.

Comment 5.3 It should be noted that the structure resulting from a merger is not a tree but an acyclic graph (with the branches oriented downward) which, for convenience, may be referred to as a semitree. More generally, then, in the statement of Rule 5.1 the term tree should be replaced throughout by semitree.

The basis for the merger rule is provided by the following observation. Let $Q/A_1,\ldots,Q/A_\ell$ be the Q/A sequences which terminate on the roots of $S_1,\ldots,S_\ell$ and let Q^* be an algebraic representation of Q (see (4.3)). Then by (4.7)

$$S_1 = D_{Q/A_1}Q^* \tag{5.4}$$

. . . . . .

$$S_\ell = D_{Q/A_\ell}Q^* \tag{5.5}$$

where D_{Q/A_λ} denotes the derivative of Q^* with respect to Q/A_λ, $\lambda = 1,\ldots,\ell$.

Now let $N_1,\ldots,N_r$, $r \geq \ell$, be a set of nodes in Q^* which form a cut, with the roots of $S_1,\ldots,S_\ell$ identified with $N_1,\ldots,N_\ell$, respectively. Then by (4.8), we can express Q^* as

$$Q^* = Q/A_1\ D_{Q/A_1}Q^* +\ldots+ Q/A_\ell\ D_{Q/A_\ell}Q^* +\ldots+ Q/A_r\ D_{Q/A_r}Q^* \tag{5.6}$$

From (5.6) and the assumption that $S_1 \equiv \ldots \equiv S_\ell \equiv S$, it follows that the common factor $D_{Q/A_1}Q^*$ may be factored from the first ℓ terms in (5.6), yielding the simpler expression

$$Q^* = (Q/A_1 +\ldots+ Q/A_\ell)\ D_{Q/A_1}Q^* +\ldots+ Q/A_r\ D_{Q/A_r}Q^* \tag{5.7}$$

The conclusion that follows from (5.7), then, is that the result of application of Rule 5.1 is a semitree whose algebraic representation is expressed by (5.7).

The conditionally redundant questions in Q* may readily be deduced by a straightforward application of the merger rule, as illustrated in Fig. 5.5. Thus, assume that the roots of $S_1,\ldots,S_\ell$, where $S_1 \equiv \ldots \equiv S_\ell \equiv S$, are the leaves of a fan which represents a constituent question, say Q_3. Then from (5.7) it follows at once that Q_3 is conditionally redundant given Q/A_3. More generally, if some of the answers to a constituent question are conditionally impossible (e.g., as in Q_2 in Fig. 5.5), then the condition $S_1 \equiv \ldots \equiv S_\ell$ need hold only for the conditionally possible answers to Q_2. Thus, in Fig. 5.5, Q_2 is conditionally redundant given Q/A_2.

It is helpful to summarize the foregoing discussion in the form of a proposition.

<u>Proposition 5.8</u> Let Q_i be a constituent question in Q* whose conditionally possible leaves (i.e., the leaves corresponding to conditionally possible answers) are $N_{i_1},\ldots,N_{i_k}$, and let Q/A_i denote the Q/A sequence terminating on Q_i. Then Q_i is conditionally redundant given Q/A_i if the subtrees (or, more precisely, the semitrees) with roots at $N_{i_1},\ldots,N_{i_k}$ are identical.

<u>Compactification of a tabular representation</u>

Like most graphical procedures, the merger rule discussed above serves to provide a visual and hence more readily comprehensible idea of how it works. For computational purposes, however, it is preferable to employ compactification techniques which operate on tables rather than graphs.

A technique of this type which is described below[2] is a straight-

[2]For simplicity, we shall assume that the constituent questions are non-interactive in the sense of [1].

forward adaptation of the well-known Quine-McCluskey algorithm [53]-[55] for the minimization of switching functions. More specifically, suppose that we wish to compactify a given tableau $Q(Q_1,\ldots,Q_n)$, e.g., that of Table 4.6, in which the rows which have the same entry in the Q column are grouped together as shown. The steps described below, then, would be applied to each such group. (For easier comprehension, the algorithm is expressed in informal terms.)

Algorithm 5.9 The following steps are performed successively for each column in Q, starting with $j = 1$. r_i^j denotes an admissible answer in the ith row of the jth column.

1. Starting with $i=1$, check if r_1^1 can be replaced by * (i.e., by the answer-set A_1). (The answer is YES if there are rows in Q which upon addition (treating the rows as strings, as in (3.6) and factoring the common factor $r_1^2\ldots r_1^n$ yield the term $A_1 r_1^2 \ldots r_1^n$.) If the answer is YES, add the row * $r_1^2\ldots r_1^n$ to the tableau, yielding what will be referred to as an augmented tableau.

As an illustration, in the tableau of Table 5.6, the answer is NO for r_1^1 and YES for r_2^1. Consequently, * a_2^2 a_1^3 is added to the tableau as shown in Table 5.6.

2. Step 1 is applied in succession to all of the entries in column 1 of Q which fall into the group under consideration.

This conclude Pass (1) of the algorithm, yielding an augmented tableau which consists of the original rows together with rows in which the entry in column 1 is a star.

3. Steps 1 and 2 are applied successively to the entries in Columns 2,3,...,n, with the understanding that the initial tableau for Pass (i + 1)

is the augmented tableau obtained at the conclusion of Pass (i), with * treated as if it were an element of an answer-set. Furthermore, in applying Step 1 to an entry in column j, all of the rows augemented up to that point must be considered.

4. In the final augmented tableau obtained at the conclusion of Pass (n), each of the rows is checked to see if it is contained as a term in an expansion of a starred term in the final augmented tableau. If the answer is NO, the row in question is transferred to a tableau labeled Q**, with the corresponding answer to Q being the same as for the group under consideration.

As an illustration, in Table 5.6 $a_1^1\ a_2^2\ a_1^3$ is contained as a term in the expansion of $*\ a_2^2\ a_1^3$ and hence is not transferred to Q**. The row $a_1^1\ a_1^2\ a_1^3$ is not contained in the expansion of any starred term and hence is transferred to Q**, with a_1 being the entry in column Q. The row $a_1^1\ a_2^2$ * is contained in $*\ a_2^2$ * and hence is not transferred to Q**.

5. On applying Steps 1,2,3,4 to each group in the original tableau, we obtain the final form of Q**. The tableau of Q** represents the desired compactified form of Q. It can readily be verified that Q** places in evidence all of the conditionally redundant questions in Q. For this reason, it will be referred to as a <u>maximally compact</u> representation of Q.

<u>Note 5.10</u> The rows in Q** correspond to the prime implicants of a switching function. For our purposes, it is not necessary to compactify Q** still further by deleting the nonessential prime implicants, that is, those terms in Q** which are contained in sums of expansions of some of the starred terms in Q**.

<u>Example 5.11</u> Intermediate results of the application of Algorithm 5.9 to

the tableau of Table 4.6 are shown in Tables 5.6 and 5.7. The final result, Q**, is shown in Table 5.8.

Q_1	Q_2	Q_3	Q	
a_1^1	a_1^2	a_1^3	a_1	
a_1^1	a_2^2	a_1^3	a_1	
a_2^1	a_2^2	a_1^3	a_1	
a_3^1	a_2^2	a_1^3	a_1	
a_3^1	a_1^2	a_2^3	a_1	Group 1 (initial)
a_1^1	a_2^2	a_2^3	a_1	
a_2^1	a_2^2	a_2^3	a_1	
a_3^1	a_2^2	a_2^3	a_1	
*	a_2^2	a_1^3	a_1	Pass (1)
*	a_2^2	a_2^3	a_1	
a_3^1	*	a_2^3	a_1	Pass (2)
a_1^1	a_2^2	*	a_1	Pass (3)
a_2^1	a_2^2	*	a_1	
a_3^1	a_2^2	*	a_1	
*	a_2^2	*	a_1	

Table 5.6. Intermediate results of Algorithm 5.9 for group 1 of rows of Q.

Q_1	Q_2	Q_3	Q	
a_2^1	a_1^2	a_1^3	a_2	
a_3^1	a_1^2	a_1^3	a_2	Group 2 (initial)
a_1^1	a_1^2	a_2^3	a_2	
a_2^1	a_1^2	a_2^3	a_2	
a_2^1	a_1^2	*	a_2	Pass (3)

Table 5.7. Intermediate results of Algorithm 5.9 for group 2 of rows of Q.

Q_1	Q_2	Q_3	Q
a_1^1	a_1^2	a_1^3	a_1
a_3^1	*	a_2^3	a_1
*	a_2^2	*	a_1
- - - -	- - -	- - -	- - - - -
a_3^1	a_1^2	a_1^3	a_2
a_1^1	a_1^2	a_2^3	a_2
a_2^1	a_1^2	*	a_2

Table 5.8. Maximally compact representation of Q.

Computation of the average cost of finding an answer to Q

Given a branching questionnaire, Q*, together with (a) the conditional probabilities of the admissible answers to each constituent question given the answers to the preceding questions; and (b) the cost of each constituent question, it is a simple matter to compute the average cost of finding an answer to Q through the use of Q*.

Specifically, let B* be an algebraic representation for a branching questionnaire Q* (see (4.3)), and let $a_{\lambda_1}^{j_1} \ldots a_{\lambda_k}^{j_k} /\!/ a_{\lambda_{k+1}}$ be a term in B* corresponding to a path from the root to a leaf of Q*. Let $p_{\lambda_1}^{j_1},\ldots,p_{\lambda_k}^{j_k}$ be the probabilities associated with the branches $a_{\lambda_1}^{j_1},\ldots,a_{\lambda_k}^{j_k}$ along this path, and let $C_{j_1},\ldots,C_{j_k}$ be the costs associated with $Q_{j_1},\ldots,Q_{j_k}$. Then the expected cost of an answer to Q through the use of Q* is given by

$$C_{av} = \sum p_{\lambda_1}^{j_1} \ldots p_{\lambda_k}^{j_k} (C_{j_1} + \ldots + C_{j_k}) \tag{5.12}$$

where the summation is taken over all possible paths from the root to the leaves of Q*.

Example 5.13 Consider the branching questionnaire shown in Fig. 5.9, in which $C_1 = 2$, $C_2 = 3$, $C_3 = 1$ and the conditional probabilities have the indicated values. (Note that the probabilities associated with the root are not conditional.) Then using (5.12), we have

$$\begin{aligned} C_{av} &= 0.04 \times 6 + 0.01 \times 6 + 0.125 \times 5 + \\ &\quad 0.045 \times 6 + 0.03 \times 6 + 0.75 \times 3 \\ &= 3.625 \end{aligned} \tag{5.14}$$

Clearly, the determination of a realization of Q** in the form of maximally efficient branching questionnaire - that is, a realization which minimizes C_{av} - is a nontrivial problem. However, since in most situations the conditional probabilities and the costs of constituent questions are likely to be known imprecisely, if at all, highly approximate solutions which yield merely reasonably efficient realizations are likely to be adequate. This may well be the case, for example, in the construction of efficient branching questionnaires for purposes of medical diagnosis, in which both the costs and the probabilities of constituent questions are likely to be both highly variable and poorly defined.

We shall not dwell further upon this problem in the present paper.

6. Concluding Remarks

The ideas presented in this paper are merely a first step toward the development of a much more comprehensive theory of fuzzy-algorithmic definitions. We have not considered, for example, fuzzy-algorithmic definitions in which the answers to Q have the form of a probability distribution over an answer-set A. Nor have we considered more complicated types of definitions in which the object, x, is not the same for all constituent questions, or in which the order in which the questions are asked is fuzzy or probabilistic.

Although lacking in complete generality, the relatively simple types of definitions which we have discussed may find useful applications in a variety of fields. Experience with such applications may well suggest many improvements in the approach described in this paper and point to areas requiring further exploration.

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7. Appendix

For convenience of the reader, a brief summary of some of the relevant aspects of the theory of fuzzy sets and the linguistic approach is presented in this section. More detailed discussions of the topics touched upon in the sequel may be found in the appended list of references and related publications.

Notation and terminology

The symbol U denotes a universe of discourse, which may be an arbitrary collection of objects or mathematical constructs.

If A is a finite subset of U whose elements are $u_1,\ldots,u_n$, A is expressed as

$$A = u_1 + \ldots + u_n \tag{A1}$$

A finite fuzzy subset of U is expressed as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n \tag{A2}$$

or, equivalently, as

$$F = \mu_1/u_1 + \ldots + \mu_n/u_n \tag{A3}$$

where the μ_i, $i=1,\ldots,n$, represent grades of membership of the u_i in F. Unless stated to the contrary, the μ_i are assumed to lie in the interval [0,1], with 0 and 1 denoting no membership and full membership, respectively.

More generally, a fuzzy subset of U is expressed as

$$F = \int_U \mu_F(u)/u \tag{A4}$$

where μ_F: $U \rightarrow [0,1]$ is the membership (or compatibility) function of F, and $\mu_F(u)/u$ is a fuzzy singleton. In effect, (A4) expresses F as the union of its constituent fuzzy singletons.

The points in U at which $\mu_F(u) > 0$ constitute the <u>support</u> of F. The points at which $\mu_F(u) = 0.5$ are the <u>crossover</u> points of F.

<u>Example A5</u> Assume

$$U = a + b + c + d \tag{A6}$$

Then, we may have

$$A = a + b + d \tag{A7}$$

and

$$F = 0.3a + 0.9b + d \tag{A8}$$

as nonfuzzy and fuzzy subsets of U, respectively.

If

$$U = 0 + 0.1 + 0.2 + \ldots + 1 \tag{A9}$$

then a fuzzy subset of U would be expressed as, say,

$$F = 0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1 \tag{A10}$$

If U = [0,1], then F might be expressed as

$$F = \int_0^1 \frac{1}{1 + u^2} /u \tag{A11}$$

which means that F is a fuzzy subset of the unit interval [0,1] whose membership function is defined by

$$\mu_F(u) = \frac{1}{1 + u^2} \tag{A12}$$

<u>Operation on fuzzy sets</u>

If F and G are fuzzy subsets of U, their <u>union</u>, F+G, and <u>intersection</u>, $F \cap G$, are fuzzy subsets of U defined by

$$F+G \triangleq \int_U \mu_F(u) \vee \mu_G(u) \;/u \tag{A13}$$

$$F \cap G \triangleq \int_U \mu_F(u) \wedge \mu_G(u)/u \tag{A14}$$

where $\vee$ and $\wedge$ denote max and min, respectively. The complement of F is defined by

$$F' = \int_U (1 - \mu_F(u)) \;/u \tag{A15}$$

Example A16 For U defined by (A6) and

$$F = 0.4a + 0.9b + d \tag{A17}$$

$$G = 0.6A + 0.5b \tag{A18}$$

we have

$$F+G = 0.6 + 0.9b + d \tag{A19}$$

$$F \cap G = 0.4a + 0.5b \tag{A20}$$

$$F' = 0.6a + 0.1b + c \tag{A21}$$

The linguistic connectives and (conjunction) and or (disjunction) are identified with $\cap$ and +, respectively. Thus,

$$F \text{ and } G \triangleq F \cap G \tag{A22}$$

and

$$F \text{ or } G \triangleq F + G \tag{A23}$$

As defined by (A22) and (A23), and and or are implied to be noninteractive in the sense that there is no "trade-off" between their operands. When this is not the case, and and or are denoted by <and> and <or>, respectively, and are defined in a way that reflects the nature of the trade-off. For example, we may have

$$F \ \underline{<\text{and}>} \ G \triangleq \int_U \mu_F(u) \ \mu_G(u) \ /u \tag{A24}$$

$$F \ \underline{<\text{or}>} \ G \triangleq \int_U (\mu_F(u) + \mu_G(u) - \mu_F(u) \ \mu_G(u))/u \tag{A25}$$

whose + denotes the arithmetic sum. In general, the interactive versions of <u>and</u> and <u>or</u> do not possess the simplifying properties of the connectives defined by (A22) and (A23), e.g., associativity, distributivity, etc.

If α is a real number, then F^α is defined by

$$F^\alpha \triangleq \int_V (\mu_F(n))^\alpha/u \tag{A26}$$

For example, for the fuzzy set defined by (A17), we have

$$F^2 = 0.16a + 0.81b + d \tag{A27}$$

and

$$F^{1/2} = 0.63a + 0.95b + d \tag{A28}$$

These operations may be used to approximate, very roughly, to the effect of the linguistic modifiers <u>very</u> and <u>more or less</u>. Thus,

$$\text{very } F \triangleq F^2 \tag{A29}$$

and

$$\text{more or less } F \triangleq F^{1/2} \tag{A30}$$

If $F_1,\ldots,F_n$ are fuzzy subsets of $U_1,\ldots,U_n$, then the <u>cartesian product</u> of $F_1,\ldots,F_n$ is a fuzzy subset of $U_1 \times\ldots\times U_n$ defined by

$$F_1 \times\ldots\times F_n = \int_{U_1\times\ldots\times U_n} (\mu_{F_1}(u_1) \wedge\ldots\wedge \mu_{F_n}(u_n))/(u_1,\ldots,u_n) \tag{A31}$$

As an illustration, for the fuzzy sets defined by (A17) and (A18), we have

$$F\times G = (0.4a + 0.9b + d)\times(0.6a + 0.5b) \tag{A32}$$
$$= 0.4/(a,a) + 0.4/(a,b) + 0.6/(b,a)$$
$$+ 0.5/(b,b) + 0.6/(d,a) + 0.5/(d,b)$$

which is a fuzzy subset of $(a + b + c + d)\times(a + b + c + d)$.

Fuzzy relations

An n-any fuzzy relation R in $U_1 \times\ldots\times U_n$ is a fuzzy subset of $U_1 \times\ldots\times U_n$. The projection of R on $U_{i_1} \times\ldots\times U_{i_k}$, where $(i_1,\ldots i_k)$ is a subsequence of $(1,\ldots,n)$, is a relation in $U_{i_1} \times\ldots\times U_{i_k}$ defined by

$$\text{Proj } R \text{ on } U_{i_1} \times\ldots\times U_{i_k} \triangleq \int_{U_{i_1}\times\ldots\times U_{i_k}} \vee_{u_{j_1},\ldots,u_{j_\ell}} \mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n) \tag{A33}$$

where $(j_1,\ldots,j_\ell)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if n=6 then (1,3,6) is complementary to (2,4,5)), and $\vee_{u_{j_1},\ldots,u_{j_\ell}}$ denotes the supremum over $U_{j_1} \times\ldots\times U_{j_\ell}$.

If R is a fuzzy subset of $U_{i_1},\ldots,U_{i_k}$, then its cylindrical extension in $U_1 \times\ldots\times U_n$ is a fuzzy subset of $U_1 \times\ldots\times U_n$ defined by

$$\bar{R} = \int_{U_1 \times\ldots\times U_n} \mu_R(u_{i_1},\ldots,u_{i_k})/(u_1,\ldots,u_n) \tag{A34}$$

In terms of their cylindrical extensions, the composition of two binary relation R and S (in $U_1 \times U_2$ and $U_2 \times U_3$, respectively) is expressed by

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \tag{A35}$$

where $\bar{R}$ and $\bar{S}$ are the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Similarly, if R is a binary relation in $U_1 \times U_2$ and S is a unary relation in U_2, their composition is given by

$$R \circ S = \text{Proj } R \cap \bar{S} \text{ on } U_1 \tag{A36}$$

<u>Example A37</u> Let R be defined by the right-hand member of (A32) and

$$S = 0.4a + b + 0.8d \tag{A38}$$

Then

$$\text{Proj } R \text{ on } U_1 (\triangleq a + b + c + d) = 0.4a + 0.6b + 0.6d \tag{A39}$$

and

$$R \circ S = 0.4a + 0.5b + 0.5d \tag{A40}$$

<u>Linguistic variables</u>

Informally, a linguistic variable, $\mathcal{X}$, is a variable whose values are words or sentences in a natural or artificial language. For example, if age is interpreted as a linguistic variable, then its <u>term-set</u>, $T(\mathcal{X})$, that is, the set of linguistic values, might be

$$T(\underline{age}) = \underline{young} + \underline{old} + \underline{very}\ \underline{young} + \underline{not}\ \underline{young} + \underline{very}\ \underline{old} + \underline{very}\ \underline{very}\ \underline{young} + \underline{rather}\ \underline{young} + \underline{more}\ \underline{or}\ \underline{less}\ \underline{young} + \ldots \tag{A41}$$

where each of the terms in T(<u>age</u>) is a label of a fuzzy subset of a universe of discourse, say U = [0,100].

A linguistic variable is associated with two rules: (a) a <u>syntactic rule</u>, which defines the well-formed sentences in $T(\mathcal{X})$; and (b) a <u>semantic rule</u>, by which the meaning of the terms in $T(\mathcal{X})$ may be determined. If

X is a term in $T(\mathcal{X})$, then its meaning (in a denotational sense) is a subset of U. A primary term in $T(\mathcal{X})$ is a term whose meaning is a primary fuzzy set, that is, a term whose meaning must be defined a priori, and which serves as a basis for the computation of the meaning of the non-primary terms in $T(\mathcal{X})$. For example, the primary terms in (A41) are young and old, whose meaning might be defined by their respective compatibility functions $\mu_{\underline{young}}$ and $\mu_{\underline{old}}$. From these, then, the meaning - or, equivalently, the compatibility functions - of the non-primary terms in (A41) may be computed by the application of a semantic rule. For example, employing (A29) and (A30), we have

$$\mu_{\underline{very}\ \underline{young}} = (\mu_{\underline{young}})^2 \tag{A42}$$

$$\mu_{\underline{more}\ \underline{or}\ \underline{less}\ \underline{old}} = (\mu_{\underline{old}})^{1/2} \tag{A43}$$

$$\mu_{\underline{not}\ \underline{very}\ \underline{young}} = 1 - (\mu_{\underline{young}})^2 \tag{A44}$$

For illustration, plots of the compatibility functions of these terms are shown in Fig. A1.

The extension principle

Let f be a mapping from U to V. Thus,

$$v = f(u) \tag{A45}$$

where u and v are generic elements of U and V, respectively.

Let F be a fuzzy subset of U expressed as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n \tag{A46}$$

or more generally

$$F = \int_U \mu_F(u)/u \tag{A47}$$

By the extension principle [1], the image of F under f is given by

$$f(F) = \mu_1\ f(u_1) + \ldots + \mu_n\ f(u_n) \tag{A48}$$

or, more generally,

$$f(F) = \int_U \mu_F(u)/f(u) \tag{A49}$$

Similarly, if f is a mapping from U × V to W, and F and G are fuzzy subsets of U and V, respectively, then

$$f(F,G) = \int_W (\mu_F(u) \wedge \mu_G(v))/f(u,v) \tag{A50}$$

<u>Example A51</u> Assume that f is the operation of squaring. Then, for the set defined by (A10), we have

$$f(0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1) = 0.3/0.25 + 0.6/0.49 + 0.8/0.81 + 1/1 \tag{A51}$$

Similarly, for the binary operation $\vee$ ($\triangleq$ max) we have

$$\begin{aligned}(0.9/0.1 + 0.2/0.5 + 1/1) \vee (0.3/0.2 + 0.8/0.6) &= \\ &= 0.3/0.2 + 0.2/0.5 + 0.3/1 \\ &+ 0.8/0.6 + 0.2/0.6 + 0.8/0.6\end{aligned} \tag{A52}$$

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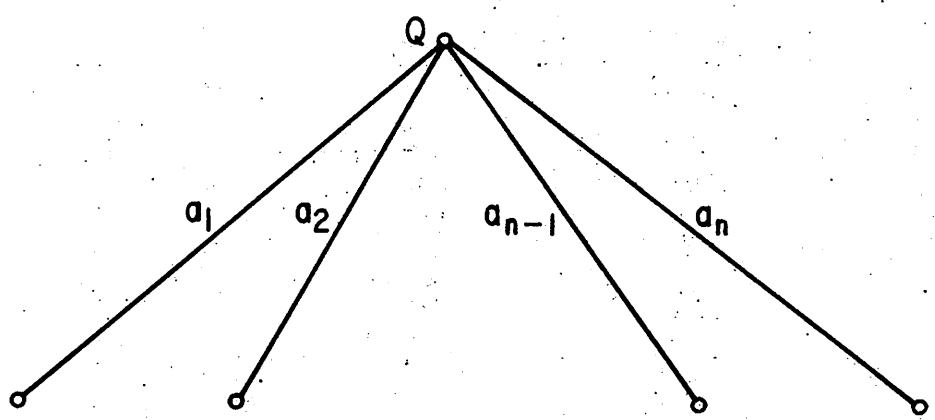

Fig. 2.1. Graphical representation of an atomic question.

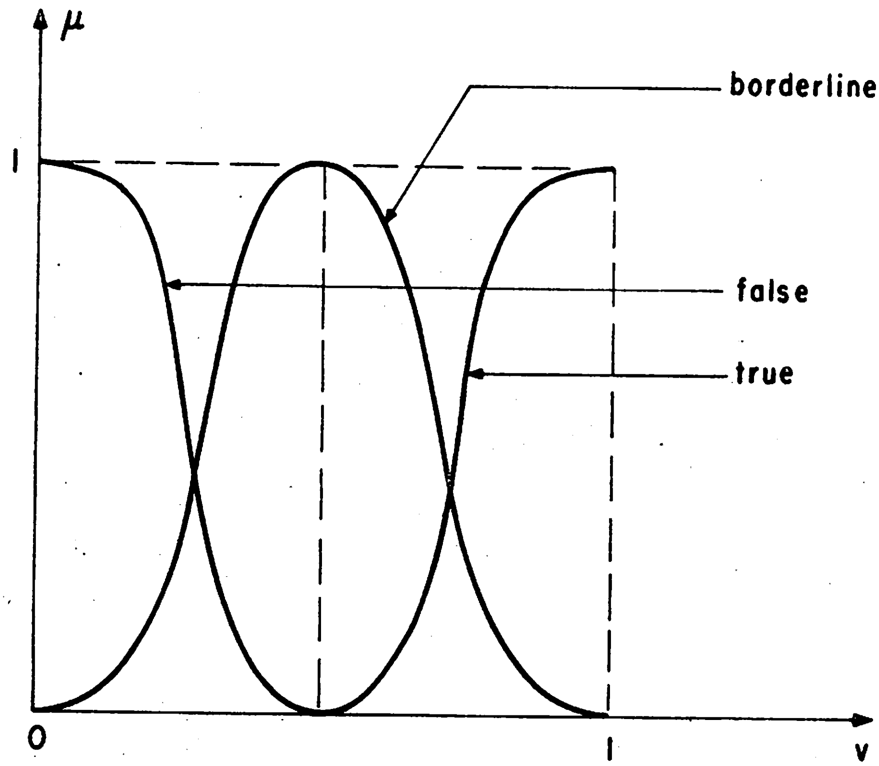

Fig. 2.2. Membership functions of true, borderline and false.

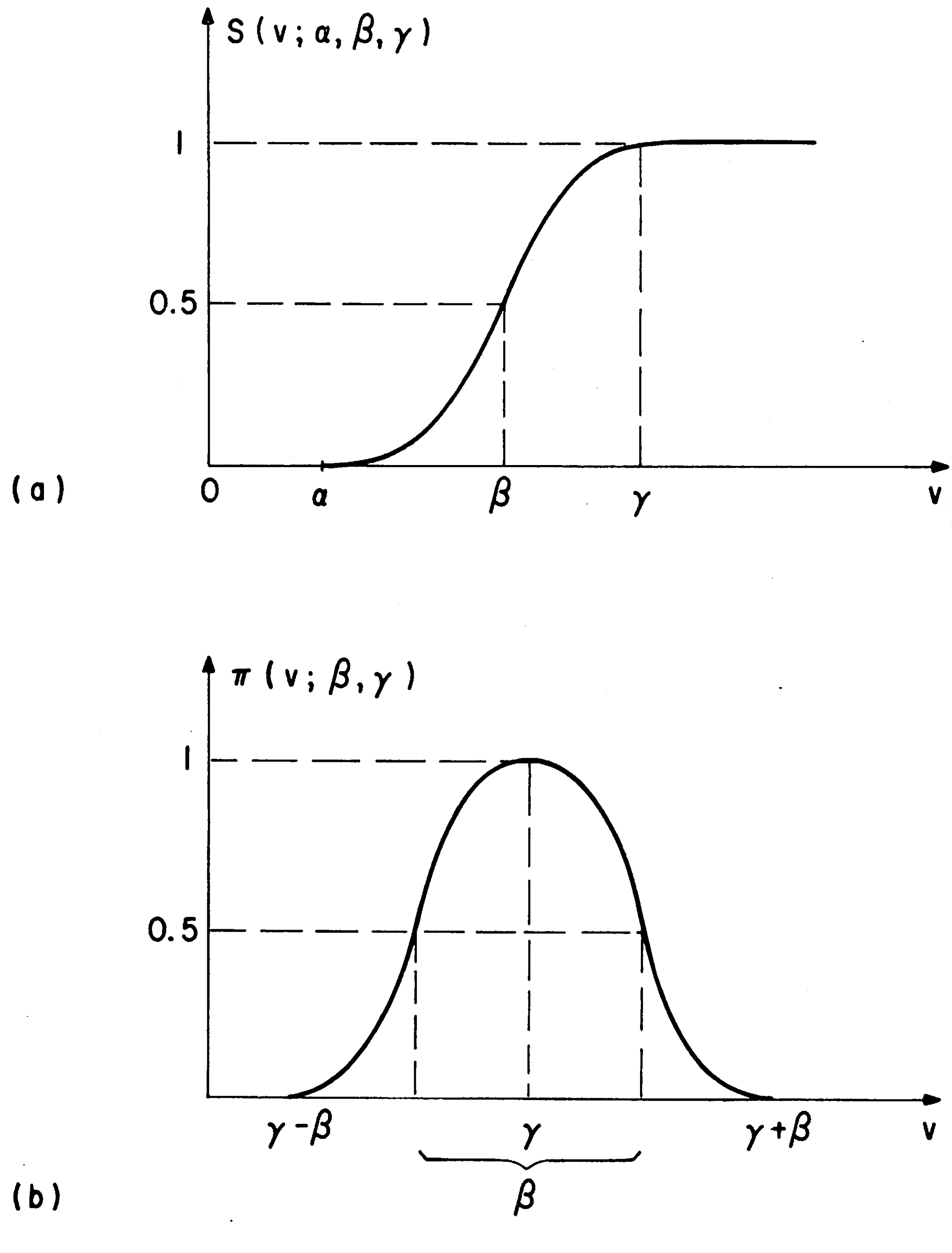

Fig. 2.3. Plots of S and Π functions.

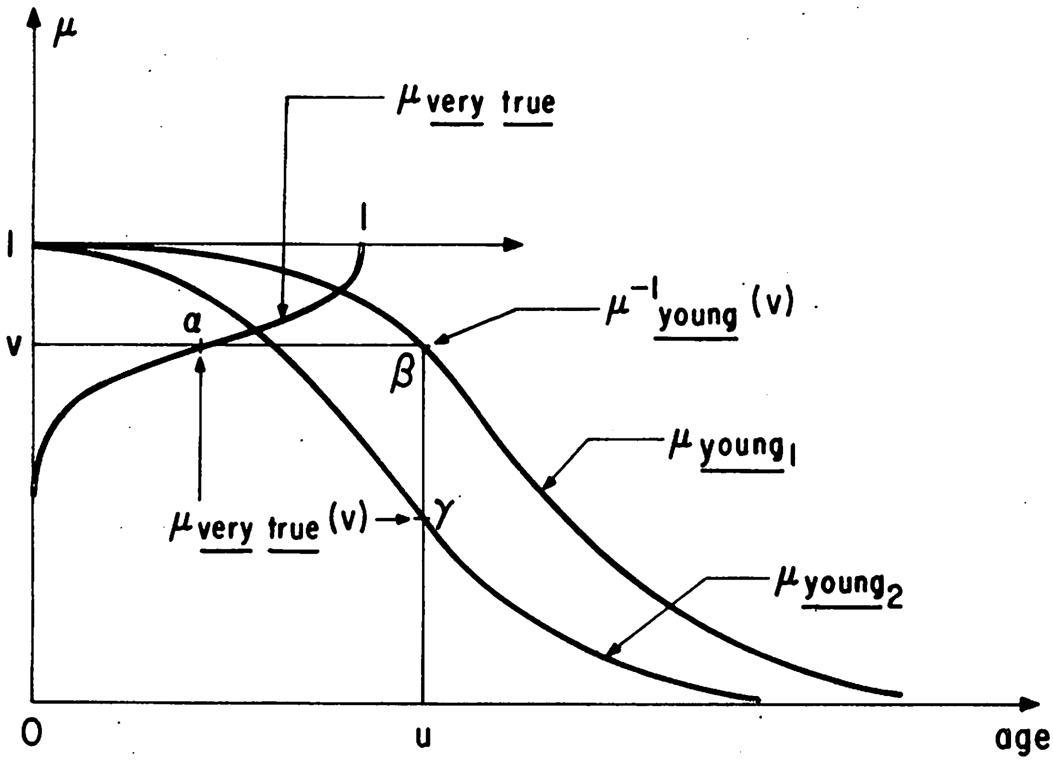

Fig. 2.4. Composition of $\mu^{-1}_{\underline{young}_1}$ with $\tau \triangleq \underline{very}\ \underline{true}$.

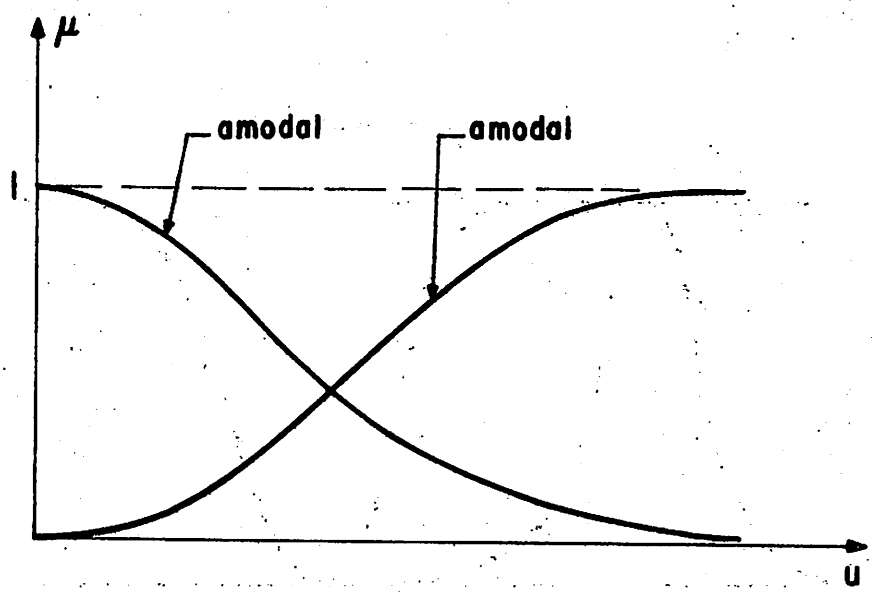

Fig. 2.5. Amodal fuzzy sets.

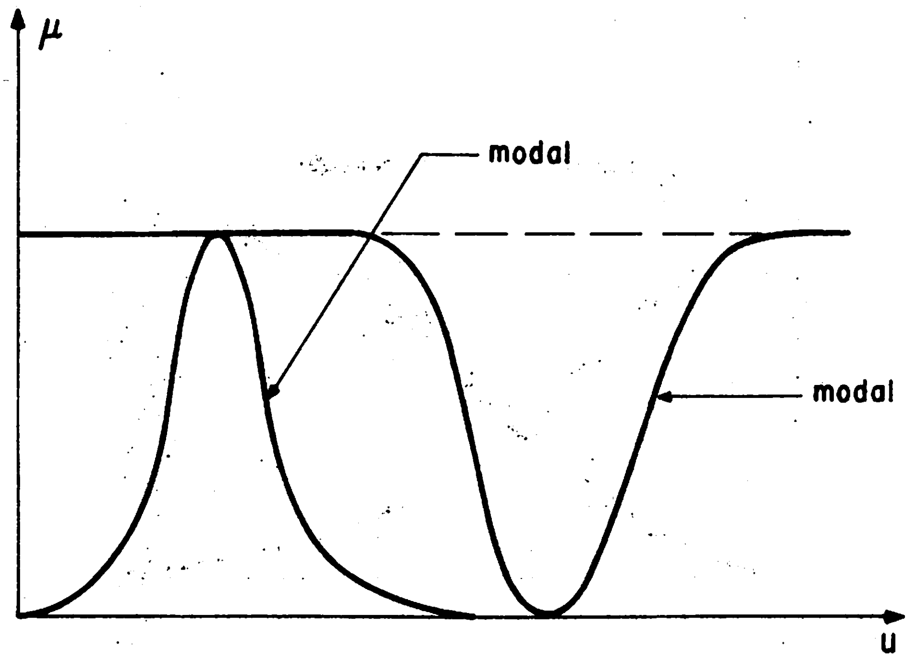

Fig. 2.6. Compatibility functions of modal fuzzy sets.

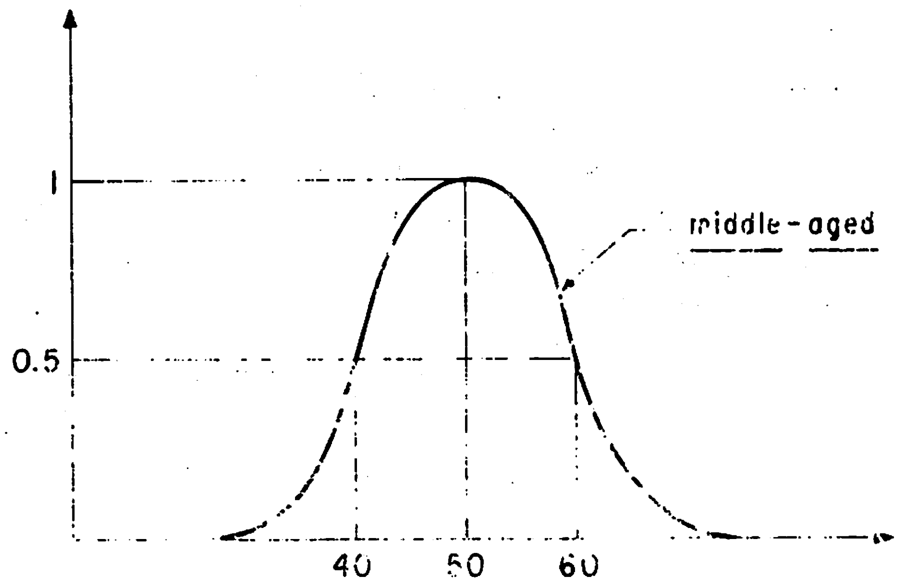

Fig. 2.7. Representation of middle-aged as a modal fuzzy set.

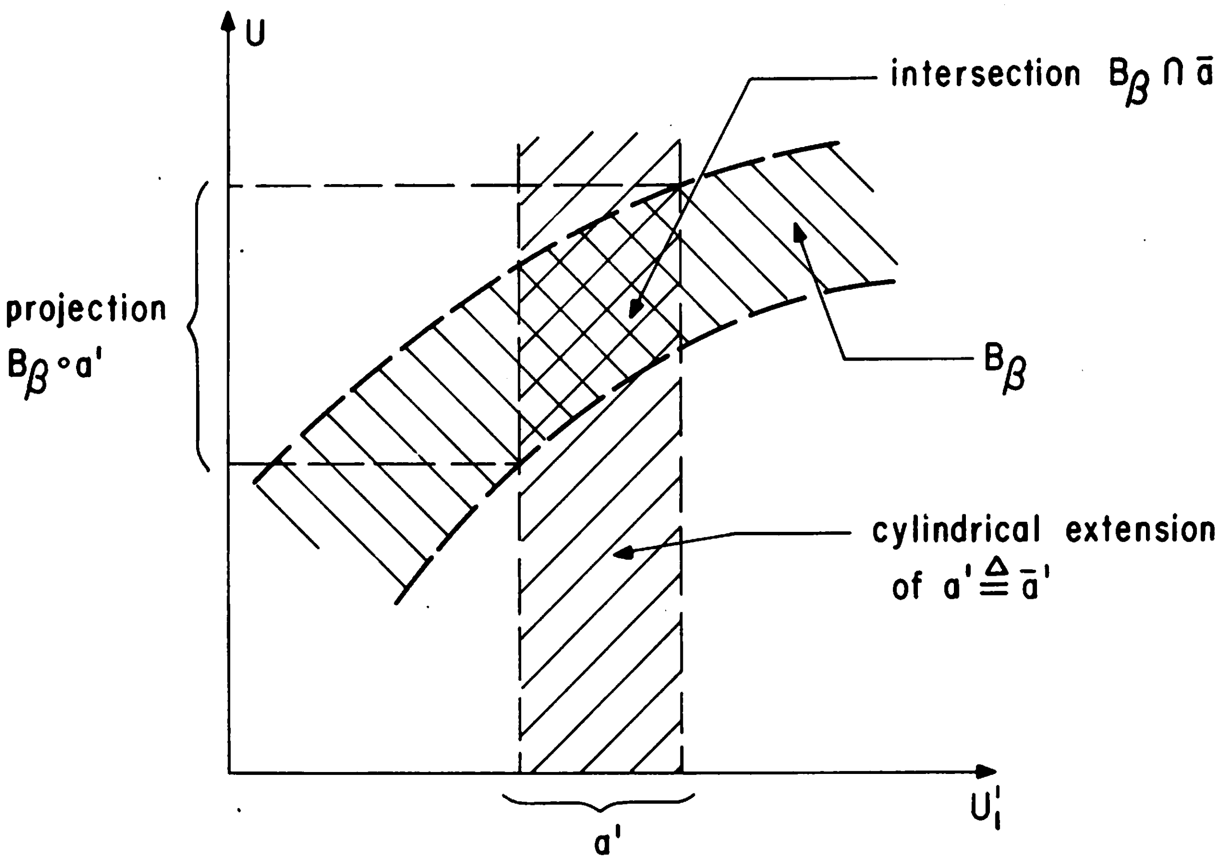

Fig. 3.3. Graphical interpretation of B_β o a^1.

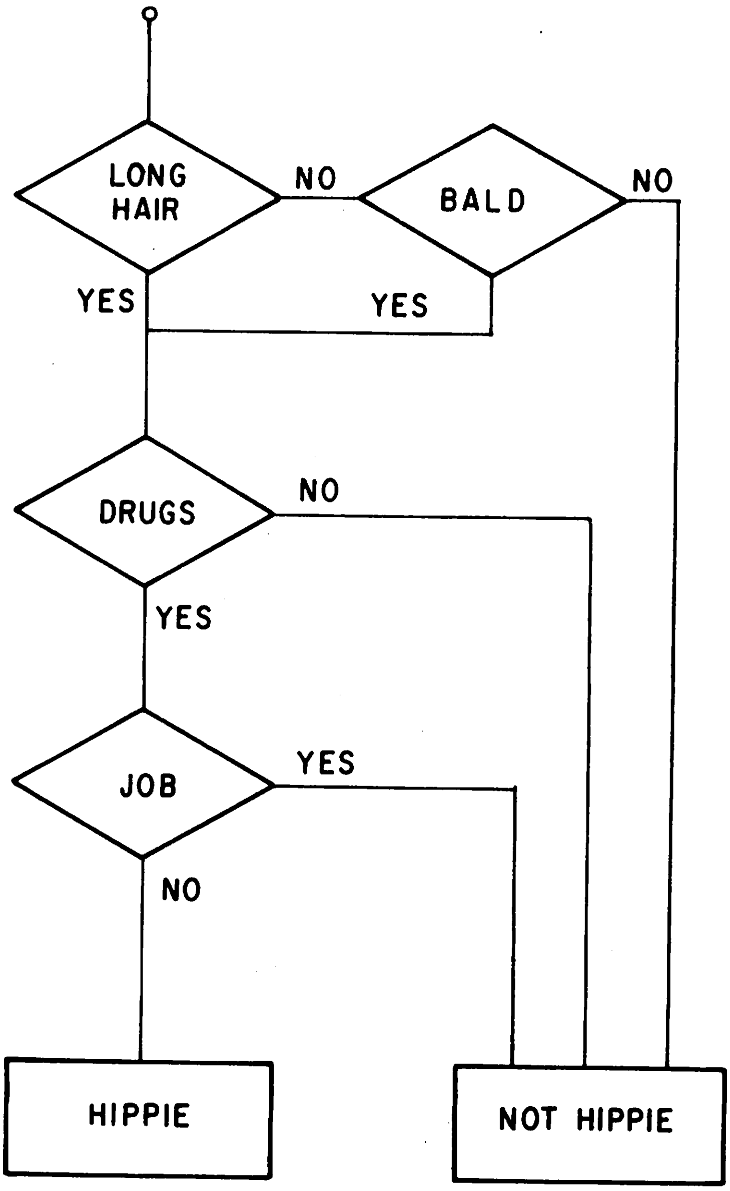

Fig. 3.4. Flowchart representation of HIPPIE.

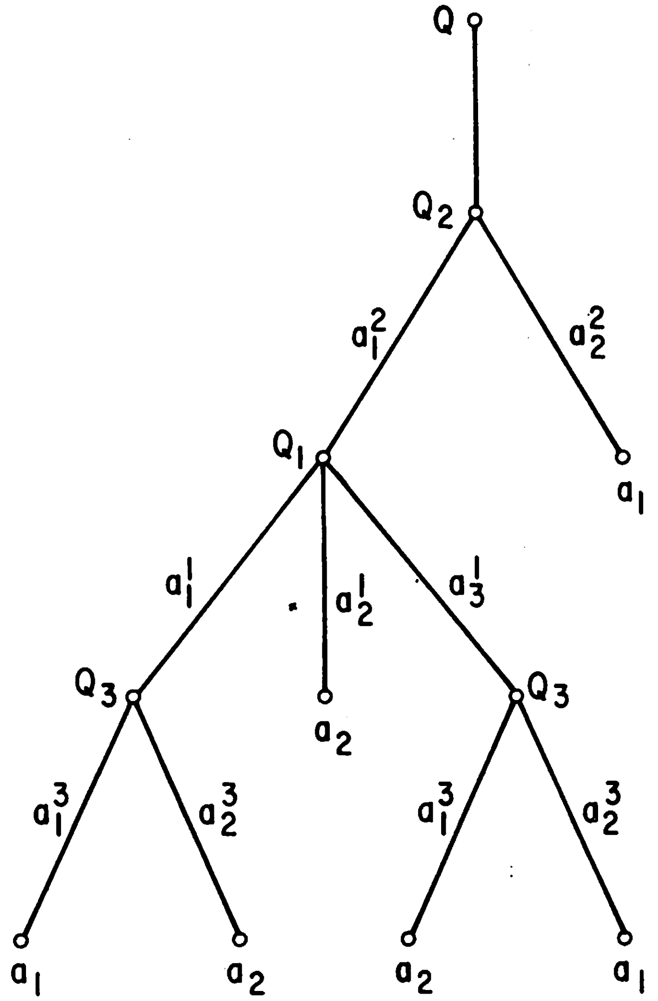

Fig. 4.1. An example of a branching questionnaire.

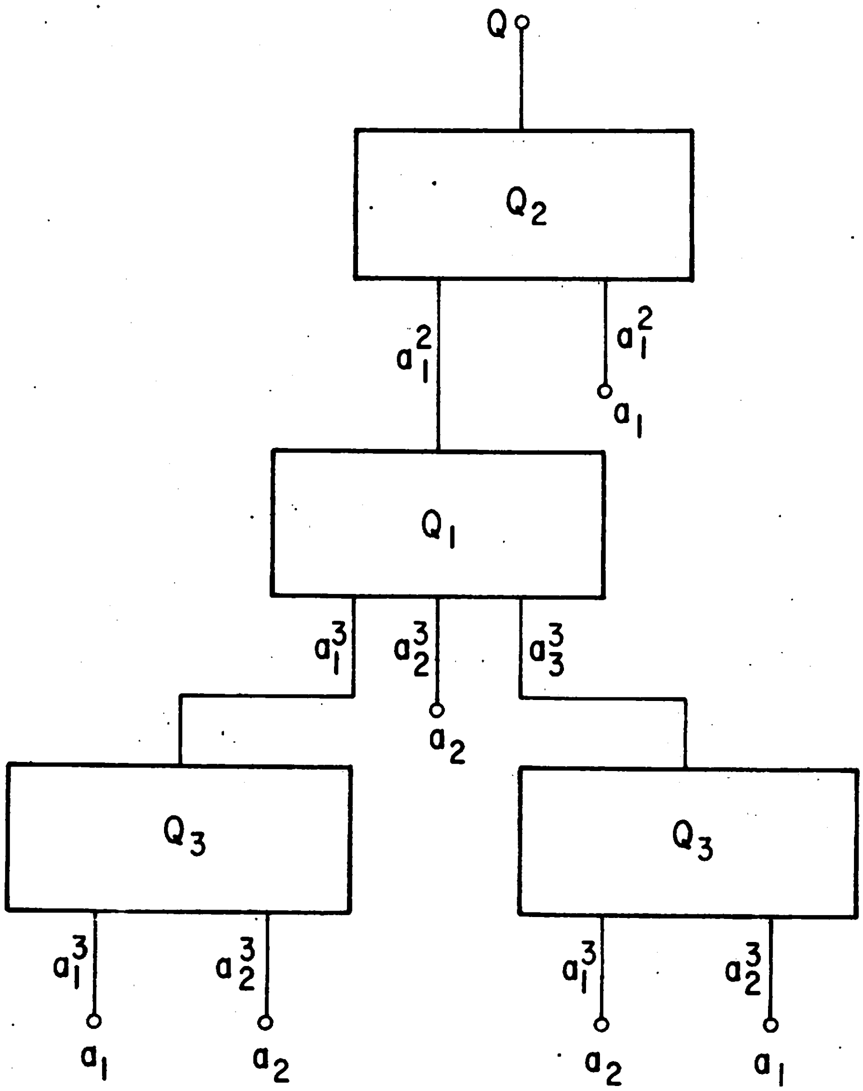

Fig. 4.2. Block diagram representation of a branching questionnaire.

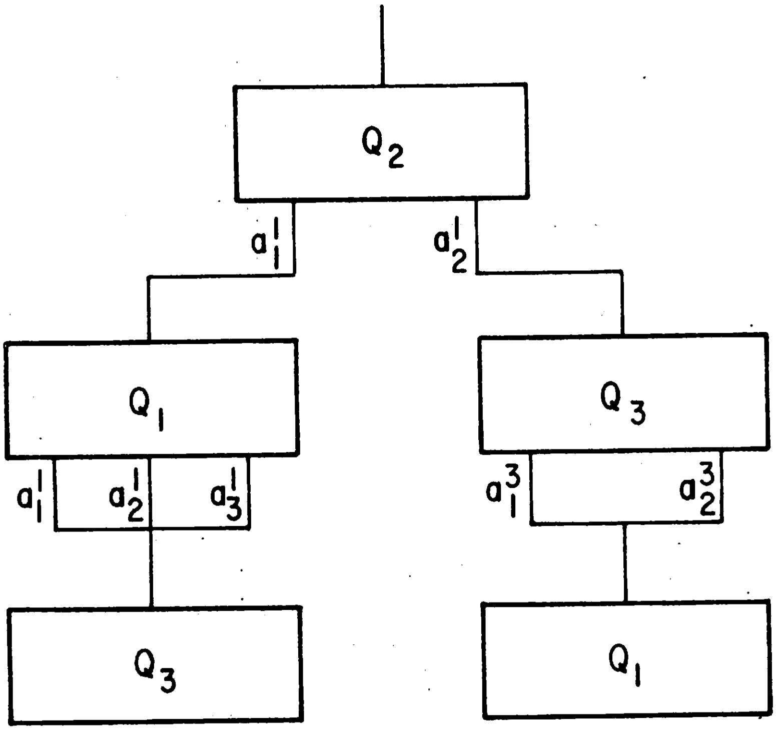

Fig. 4.3. Specification of the order of questioning.

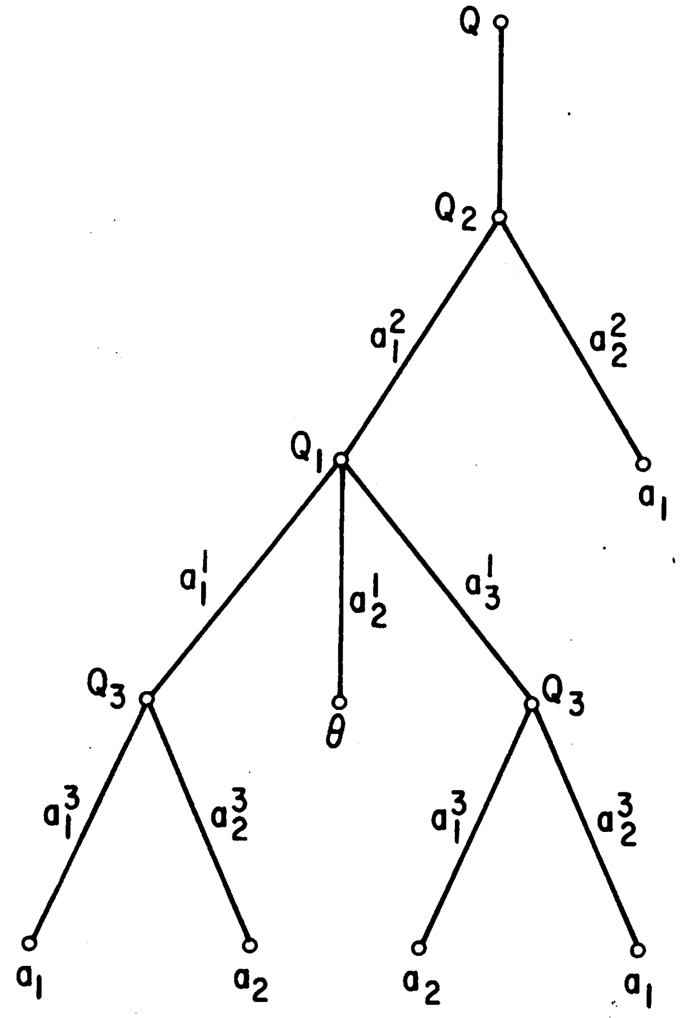

Fig. 4.4. Illustration of conditional impossibility.

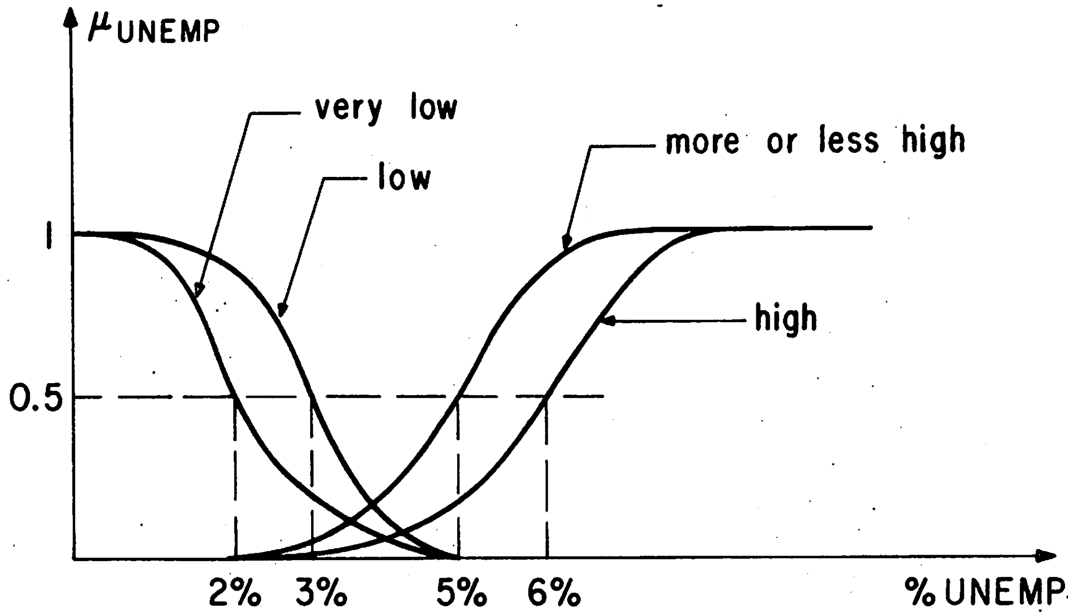

Fig. 5.2. Compatibility functions of low, very low, high and more or less high. (Not to scale.)

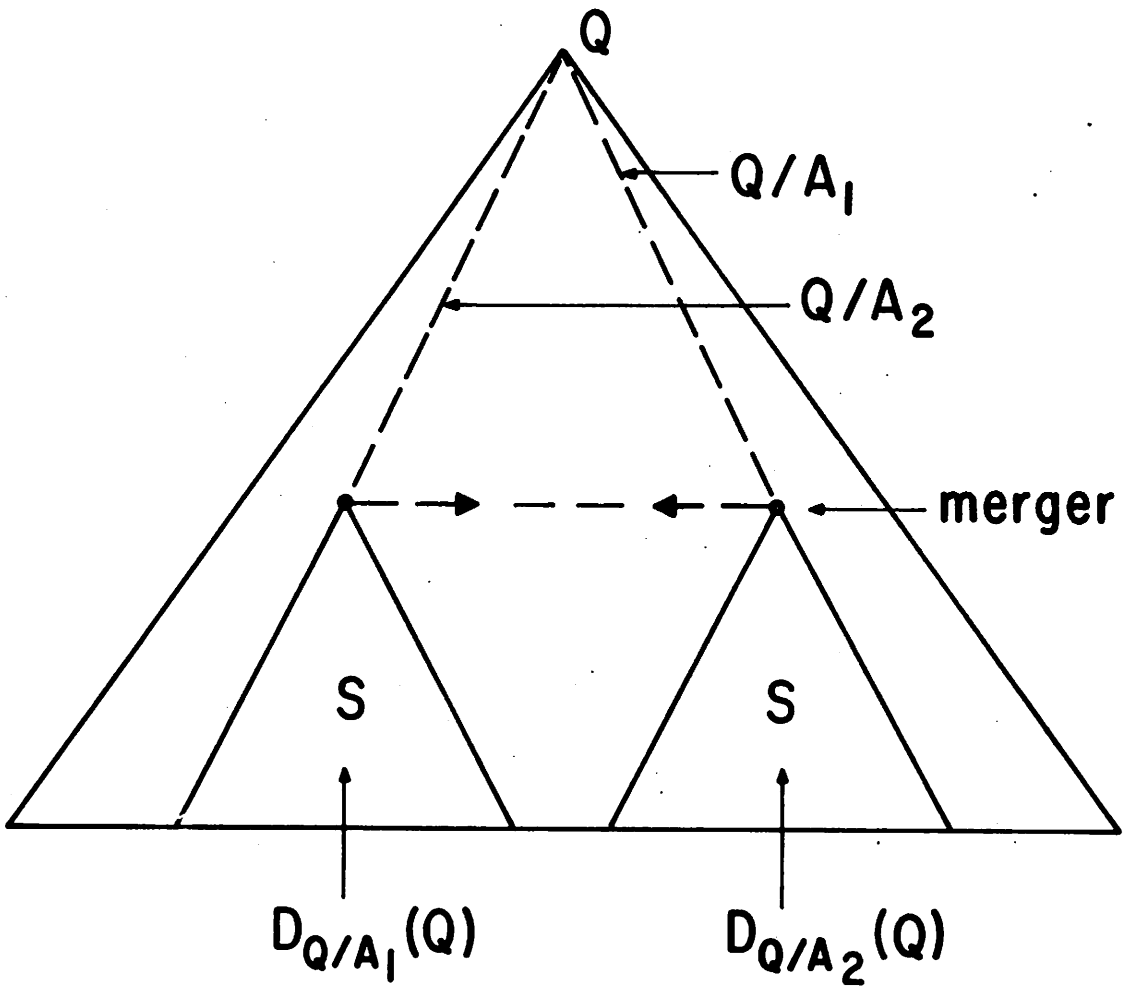

Fig. 5.3. Illustration of the merger rule.

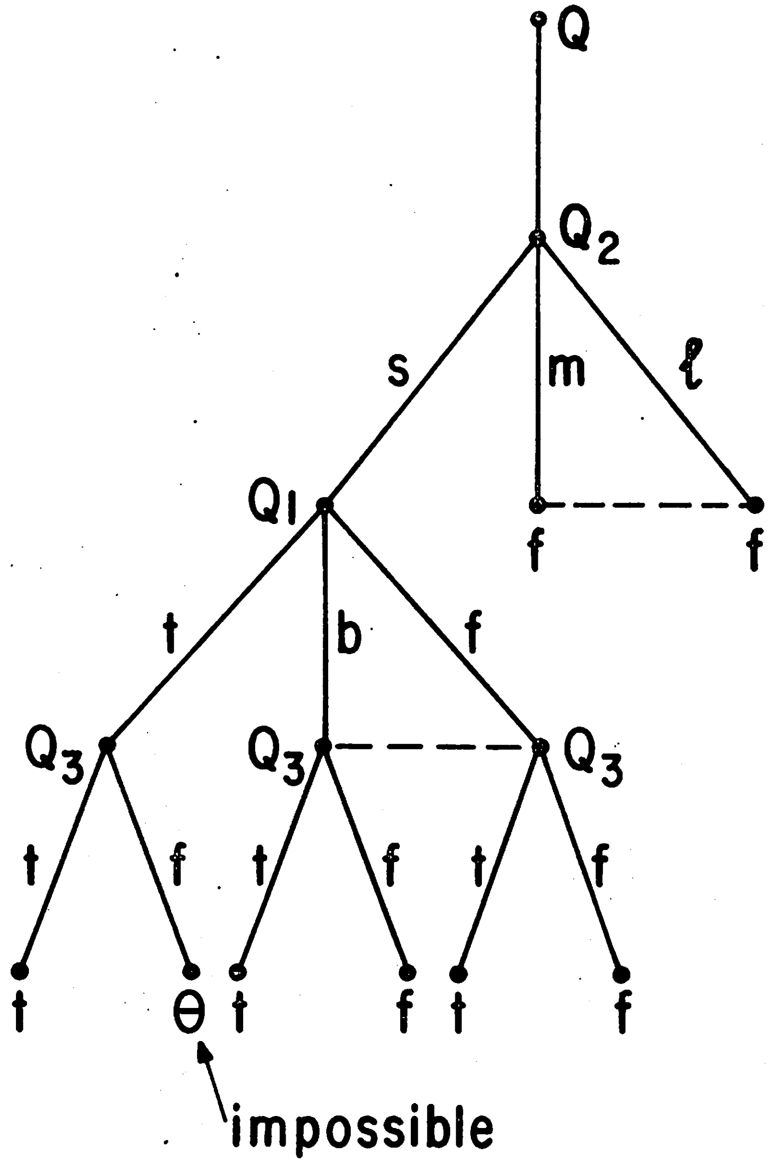

Fig. 5.4. Illustration of the merger rule. The nodes connected by dashed lines are mergeable.

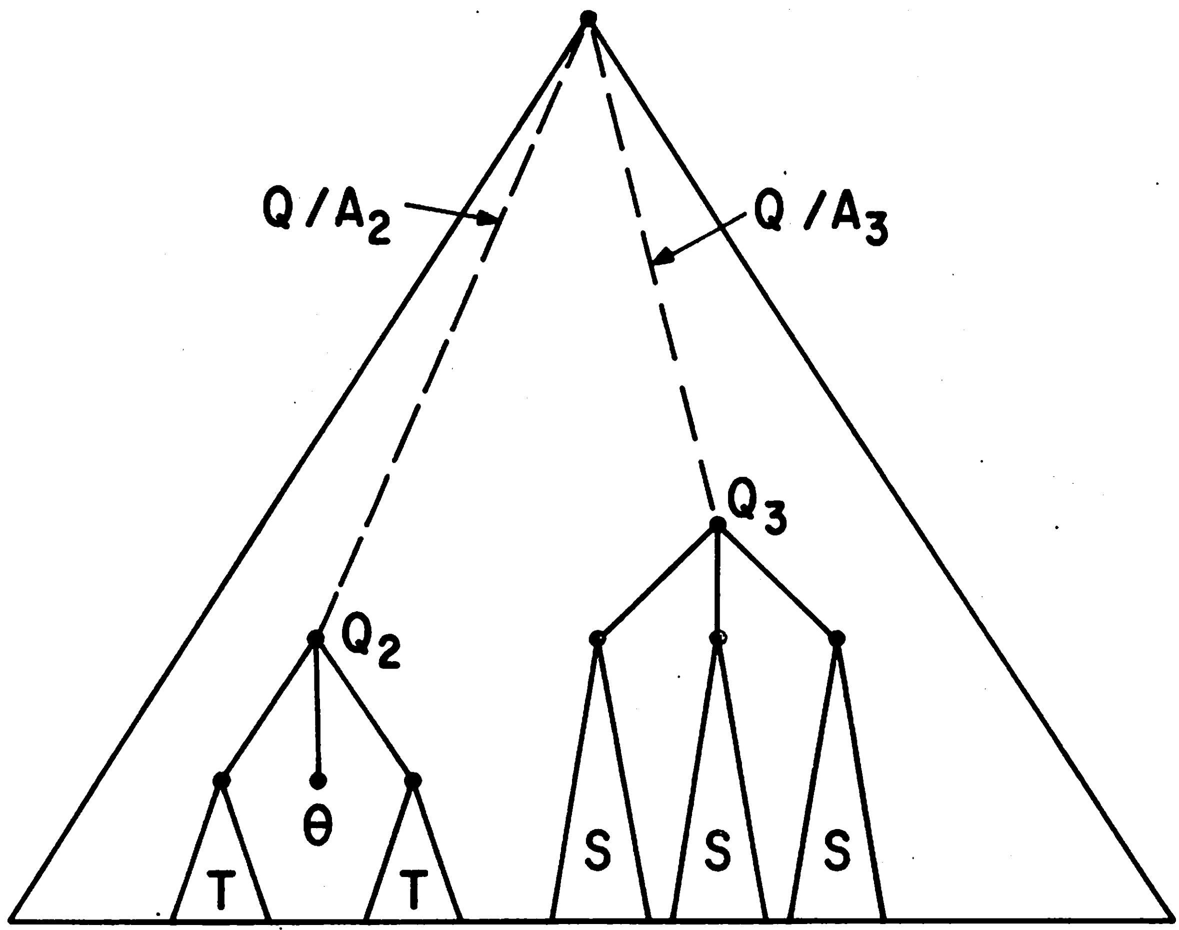

Fig. 5.5. Application of the merger rule to the identification of conditionally redundant questions.

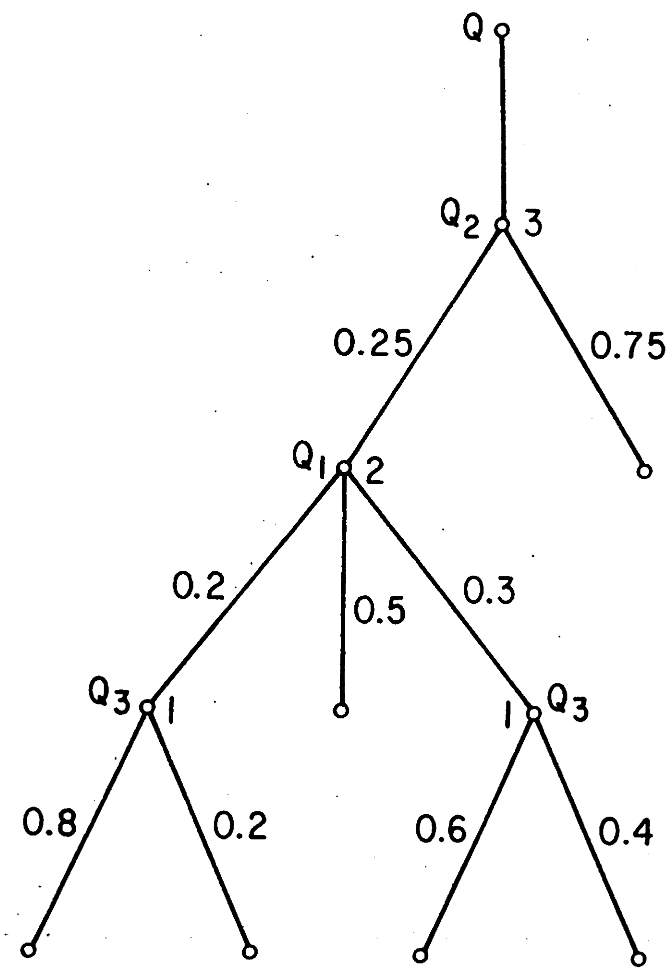

Fig. 5.9. Conditional probabilities and costs associated with constituent questions.

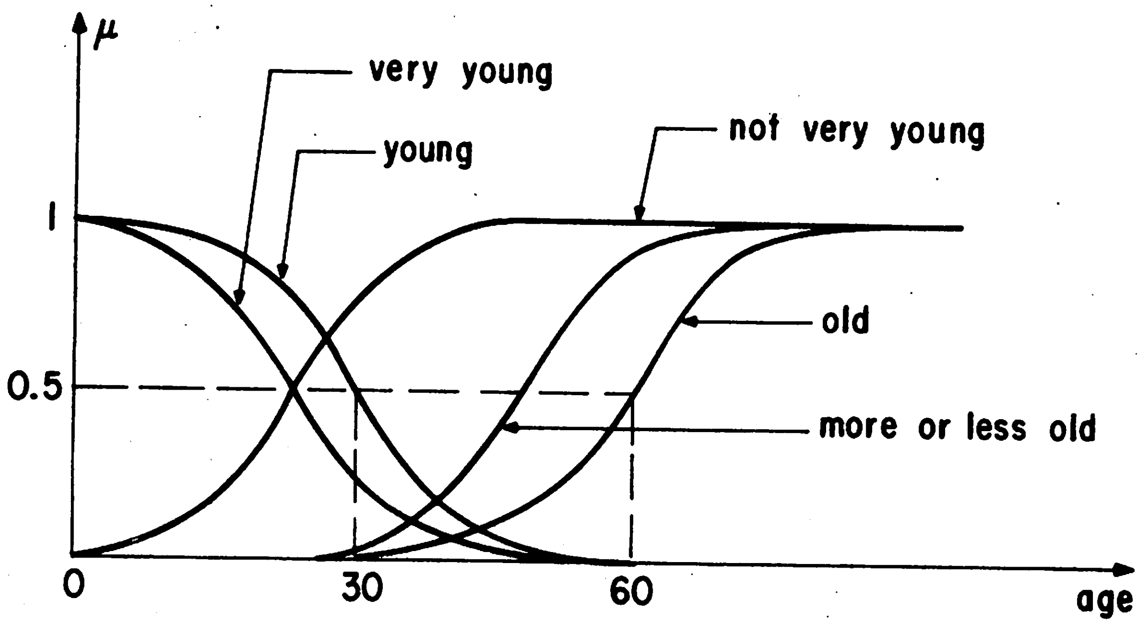

Fig. A.1. Compatibility function of young, old, very young, more or less old, not very young. (Not to scale.)