CALCULUS OF FUZZY RESTRICTIONS

by

L. A. Zadeh

Memorandum No. ERL-M502

19 February 1975

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

CALCULUS OF FUZZY RESTRICTIONS

L. A. Zadeh

Division of Computer Science
Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

A fuzzy restriction may be visualized as an elastic constraint on the values that may be assigned to a variable. In terms of such restrictions, the meaning of a proposition of the form "x is P," where x is the name of an object and P is a fuzzy set, may be expressed as a relational assignment equation of the form R(A(x)) = P, where A(x) is an implied attribute of x, R is a fuzzy restriction on x, and P is the unary fuzzy relation which is assigned to R. For example, "Stella is young," where young is a fuzzy subset of the real line, translates into R(Age(Stella)) = young.

The calculus of fuzzy restrictions is concerned, in the main, with (a) translation of propositions of various types into relational assignment equations, and (b) the study of transformations of fuzzy restrictions which are induced by linguistic modifiers, truth-functional modifiers, compositions, projections and other operations. An important application of the calculus of fuzzy restrictions relates to what might be called approximate reasoning, that is, a type of reasoning which is neither very exact nor very inexact. The main ideas behind this application are outlined and illustrated by examples.

Research sponsored by the Naval Electronics Systems Command, Contract N00039-75-0034, the Army Research Office, Grant DAHC04-75-G-0056, and the National Science Foundation, Grant GK-43024X.

1. Introduction

During the past decade, the theory of fuzzy sets has developed in a variety of directions, finding applications in such diverse fields as taxonomy, topology, linguistics, automata theory, logic, control theory, game theory, information theory, psychology, pattern recognition, medicine, law, decision analysis, system theory and information retrieval.

A common thread that funs through most of the applications of the theory of fuzzy sets relates to the concept of a <u>fuzzy restriction</u> - that is, a fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable. Such restrictions appear to play an important role in human congnition, especially in situations involving concept formation, pattern recognition and decision-making in fuzzy or uncertain environments.

As its name implies, the <u>calculus of fuzzy restrictions</u> is essentially a body of concepts and techniques for dealing with fuzzy restrictions in a systematic fashion. As such, it may be viewed as a branch of the theory of fuzzy relations, in which it plays a role somewhat analogous to that of the calculus of probabilities in probability theory. However, a more specific aim of the calculus of fuzzy restrictions is to furnish a conceptual basis for fuzzy logic and what might be called <u>approximate reasoning</u> [1], that is, a type of reasoning which is neither very exact nor very inexact. Such reasoning plays a basic role in human decision-making because it provides a way of dealing with problems which are too complex for precise solution. However, approximate reasoning is more than a method of last recourse for coping with insurmountable complexities. It is also a way of simplifying the performance of tasks in which a high degree of precision is neither needed nor required. Such tasks pervade much of what we do on both conscious and subconscious levels.

What is a fuzzy restriction? To illustrate its meaning in an informal fashion, consider the following propositions (in which italicized words represent fuzzy concepts):

Tosi is young (1.1)

Ted has gray hair (1.2)

Sakti and Kapali are approximately equal in height. (1.3)

Starting with (1.1), let Age (Tosi) denote a numerically-valued variable which ranges over the interval [0,100]. With this interval regarded as our universe of discourse U, young may be interpreted as the label of a fuzzy subset[1] of U which is characterized by a compatibility function, μ_{young}, of the form shown in Fig. 1.1. Thus, the degree to which a numerical age, say u = 28, is compatible with the concept of young is 0.7, while the compatibilies of 30 and 35 with young are 0.5 and 0.2, respectively. (The age at which the compatibility takes the value 0.5 is the crossover point of young.) Equivalently, the function μ_{young} may be viewed as the membership function of the fuzzy set young, with the value of μ_{young} at u representing the grade of membership of u in young.

Since young is a fuzzy set with no sharply defined boundaries, the conventional interpretation of the proposition "Tosi is young," namely, "Tosi is a member of the class of young men," is not meaningful if

[1] A summary of the basic properties of fuzzy sets is presented in the Appendix.

membership in a set is interpreted in its usual mathematical sense. To circumvent this difficulty, we shall view (1.1) as an assertion of a restriction on the possible values of Tosi's age rather than as an assertion concerning the membership of Tosi in a class of individuals. Thus, on denoting the restriction on the age of Tosi by R(Age(Tosi)), (1.1) may be expressed as an assignment equation

$$R(\text{Age}(\text{Tosi})) = \underline{\text{young}} \tag{1.4}$$

in which the fuzzy (or, equivalently, the unary fuzzy relation young) is assigned to the restriction on the vairable Age(Tosi). In this instance, the restriction R(Age(Tosi)) is a fuzzy restriction by virtue of the fuzziness of the set young.

Using the same point of view, (1.2) may be expressed as

$$R(\text{Color}(\text{Hair}(\text{Ted}))) = \underline{\text{gray}} \tag{1.5}$$

Thus, in this case, the fuzzy set gray is assigned as a value to the fuzzy restriction on the variable Color(Hair(Ted)).

In the case of (1.1) and (1.2), the fuzzy restriction has the form of a fuzzy set, or, equivalently, a unary fuzzy relation. In the case of (1.3), we have two variables to consider, namely, Height (Sakti) and Height(Kapali). Thus, in this instance, the assignment equation takes the form

$$R(\text{Height}(\text{Sakti})), \text{Height}(\text{Kapali})) = \underline{\text{approximately equal}} \tag{1.6}$$

in which approximately equal is a binary fuzzy relation characterized by a compatibility matrix $\mu_{\underline{\text{approximately equal}}}(u,v)$ such as shown in Table 1.2

u \ v	5'6	5'8	5'10	6	6'2	6'4
5'6	1	0.8	0.6	0.2	0	0
5'8	0.8	1	0.9	0.7	0.3	0
5'10	0.6	0.9	1	0.9	0.7	0
6	0.2	0.7	0.9	1	0.9	0.8
6'2	0	0.3	0.7	0.9	1	0.9
6'4	0	0	0	0.8	0.9	1

Table 1.2. Compatibility matrix of the fuzzy relation <u>approximately equal</u>

Thus, if Sakti's height is 5'8 and Kapali's is 5'10, then the degree to which they are approximately equal is 0.9.

The restrictions involved in (1.1), (1.2) and (1.3) are unrelated in the sense that the restriction on the age of Tosi has no bearing on the color of Ted's hair or the height of Sakti and Kapali. More generally, however, the restrictions may be interrelated, as in the following example:

u is <u>small</u> (1.7)

u and v are <u>approximately equal</u> (1.8)

In terms of the fuzzy restrictions on u and v, (1.7) and (1.8) translate into the assignment equations

$$R(u) = \underline{\text{small}} \tag{1.9}$$

$$R(u,v) = \underline{\text{approximately equal}} \tag{1.10}$$

where R(u) and R(u,v) denote the restrictions on u and (u,v), respectively.

As will be shown in Section 2, from the knowledge of a fuzzy restriction on u and a fuzzy restriction on (u,v) we can deduce a fuzzy restriction on v. Thus, in the case of (1.9) and (1.10), we can assert that

$$\begin{aligned} R(v) &= R(u) \circ R(u,v) \\ &= \underline{\text{small}} \circ \underline{\text{approximately equal}} \end{aligned} \tag{1.11}$$

where o denotes the composition[2] of fuzzy relations.

The rule by which (1.11) is inferred from (1.9) and (1.10) is called the <u>compositional rule of inference</u>. As will be seen in the sequel, this rule is a special case of a more general method for deducing a fuzzy restriction on a variable from the knowledge of fuzzy restrictions on related variables.

In what follows, we shall outline some of the main ideas which form the basis for the calculus of fuzzy restrictions and sketch its application to approximate reasoning. For convenient reference, a summary of those aspects of the theory of fuzzy sets which are relevant to the calculus of fuzzy restrictions is presented in the Appendix.

---

[2] If A is a unary fuzzy relation in U and B is a binary fuzzy relation in $U \times V$, the membership function of the composition of A and B is expressed by $\mu_{A \circ B}(v) = \vee_u (\mu_A(u) \wedge \mu_B(u,v))$, where $\vee_u$ denotes the supremum over $u \in U$. A more detailed discussion of the composition of fuzzy relations may be found in [2] and [3].

2. Calculus of Fuzzy Restrictions

The point of departure for our discussion of the calculus of fuzzy restrictions is the paradigmatic proposition[1]

$$p \triangleq x \text{ is } P \tag{2.1}$$

which is exemplified by

$$x \text{ is a positive integer} \tag{2.2}$$

$$\text{Soup is } \underline{\text{hot}} \tag{2.3}$$

$$\text{Elvira is } \underline{\text{blond}} \tag{2.4}$$

If P is a label of a nonfuzzy set, e.g., $P \triangleq$ set of positive integers, then "x is P," may be interpreted as "x belongs to P," or, equivalently, as "x is a member of P." In (2.3) and (2.4), however, P is a label of a fuzzy set, i.e., $P \triangleq$ <u>hot</u> and $P \triangleq$ <u>blond</u>. In such cases, the interpretation of "x is P," will be assumed to be characterized by what will be referred to as a <u>relational assignment equation</u>. More specifically, we have

<u>Definition 2.5</u> The meaning of the proposition

$$p \triangleq x \text{ is } P \tag{2.6}$$

where x is a name of an object (or a construct) and P is a label of a fuzzy subset of a universe of discourse U, is expressed by the <u>relational assignment equation</u>

---

[1]The symbol $\triangleq$ stands for "denotes" or "is defined to be."

$$R(A(x)) = P \tag{2.7}$$

where A is an <u>implied attribute</u> of x, i.e., an attribute which is implied by x and P; and R denotes a fuzzy restriction on A(x) to which the value P is assigned by (2.7). In other words, (2.7) implies that the attribute A(x) takes values in U and that R(A(x)) is a fuzzy restriction on the values that A(x) may take, with R(A(x)) equated to P by the relational assignment equation.

As an illustration, consider the proposition "Soup is <u>hot</u>." In this case, the implied attribute is Temperature and (2.3) becomes

$$R(\text{Temperature (Soup)}) = \underline{\text{hot}} \tag{2.8}$$

with <u>hot</u> being a subset of the interval [0,212] defined by, say, a compatibility function of the form (see Appendix)

$$\mu_{\underline{\text{hot}}}(u) = S(u;32,100,200) \tag{2.9}$$

Thus, if the temperature of the soup is u = 100°, then the degree to which it is compatible with the fuzzy restriction <u>hot</u> is 0.5, whereas the compatibility of 200° with <u>hot</u> is unity. It is in this sense that R(Temperature(Soup)) plays the role of a fuzzy restriction on the soup temperature which is assigned the value <u>hot</u>, with the compatibility function of <u>hot</u> serving to define the compatibilities of the numerical values of soup temperature with the fuzzy restriction <u>hot</u>.

In the case of (2.4), the implied attribute is Color(Hair), and the relational assignment equation takes the form

$$R(\text{Color}(\text{Hair}(\text{Elvira}))) = \underline{\text{blond}} \qquad (2.10)$$

There are two important points that are brought out by this example. First, the implied attribute of x may have a nested structure, i.e., may be of the general form

$$A_k(A_{k-1}(\ldots A_2(A_1(x)) \ldots)); \qquad (2.11)$$

and second, the fuzzy set which is assigned to the fuzzy restriction (i.e., blond) may not have a numerically-valued base variable, that is, the variable which ranges over the universe of discourse U. In such cases, we shall assume that P is defined by exemplification, that is, by pointing to the specific instances of x and indicating the degree (either numerical or linguistic) to which that instance is compatible with P. For example, we may have $\mu_{\underline{blond}}$ (June) = 0.2, $\mu_{\underline{blond}}$(Jurata) = very high, etc. In this way, the fuzzy set blond is defined in an approximate fashion as a fuzzy subset of a universe of discourse comprised of a collection of individuals U = {x}, with the restriction R(x) playing the role of a fuzzy restriction on the values of x rather than on the values of an implied attribute A(x).[2] (In the sequel, we shall write R(x) and speak of the restriction on x rather than on A(x) not only in those cases in which P is defined by exemplification, but also when the implied attribute is not identified in an explicit fashion.)

So far, we have confined our attention to fuzzy restrictions which are defined by a single proposition of the form "x is P." In a more general setting, we may have n constituent propositions of the form

[1] A more detailed discussion of this and related issues may be found in [3], [4] and [5].

$$x_i \text{ is } P_i \quad , \quad i = 1,\ldots,n \tag{2.12}$$

in which P_i is a fuzzy subset of U_i, $i = 1,\ldots,n$. In this case, the propositions "x_i is P_i," $i = 1,\ldots,n$, collectively define a fuzzy restriction on the n-ary object $(x_1,\ldots,x_n)$. The way in which this restriction depends on the P_i is discussed in the following.

The Rules of Implied Conjunction and Maximal Restriction

For simplicity we shall assume that $n = 2$, with the constituent propositions having the form

$$x \text{ is } P \tag{2.13}$$

$$y \text{ is } Q \tag{2.14}$$

where P and Q are fuzzy subsets of U and V, respectively. For example,

Georgia is <u>very warm</u> (2.15)

George is <u>highly intelligent</u> (2.16)

or, if $x = y$,

Georgia is <u>very warm</u> (2.17)

Georgia is <u>highly intelligent</u> (2.18)

The rule of *implied conjunction* asserts that, in the absence of additional information concerning the constituent propositions, (2.13) and (2.14) taken together imply the composite proposition "x is P and y is Q;" that is,

$$x \text{ is } P,\ y \text{ is } Q \Rightarrow x \text{ is } P \text{ and } y \text{ is } Q \tag{2.19}$$

Under the same assumption, the <u>rule of maximal restriction</u> asserts that

$$x \text{ is } P \text{ and } y \text{ is } Q \Rightarrow (x,y) \text{ is } P \times Q \tag{2.20}$$

and, if $x = y$,

$$x \text{ is } P \text{ and } x \text{ is } Q \Rightarrow x \text{ is } P \cap Q \tag{2.21}$$

where $P \times Q$ and $P \cap Q$ denote, respectively, the cartesian product and the intersection of P and Q.[3]

The rule of maximal restriction is an instance of a more general principle which is based on the following properties of n-ary fuzzy restrictions.

Let R by a n-ary fuzzy relation in $U_1 \times \ldots \times U_n$ which is characterized by its membership (compatibility) function $\mu_R(u_1,\ldots,u_n)$. Let $q = (i_1,\ldots,i_k)$ be a subsequence of the index sequence $(1,\ldots,n)$ and let q' denote the complementary subsequence $(j_1,\ldots,j_\ell)$. (E.g., if $n = 5$ and $q = (2,4,5)$, then $q' = (1,3)$.). Then, the <u>projection of</u> R <u>on</u> $U_{(q)} = U_{i_1} \times \ldots \times U_{i_k}$ is a fuzzy relation, R_q, in $U_{(q)}$ whose membership function is related to that of R by the expression

$$\mu_{R_q}(u_{i_1},\ldots,u_{i_k}) = \vee_{u_{(q')}} \mu_R(u_1,\ldots,u_n) \tag{2.22}$$

---

[3] <u>The cartesian product</u> of P and Q is a fuzzy subset of $U \times V$ whose membership function is expressed by $\mu_{P\times Q}(u,v) = \mu_P(u) \wedge \mu_Q(v)$. The membership function of $P \cap Q$ is given by $\mu_{P\cap Q}(u) = \mu_P(u) \wedge \mu_Q(u)$. The symbol $\wedge$ stands for min. (See the Appendix for more details.)

where the right-hand member represents the supremum of $\mu_R(u_1,\ldots,u_n)$ over the u's which are in $u_{(q')}$.

If R is interpreted as a fuzzy restriction on $(u_1,\ldots,u_n)$ in $U_1\times\ldots\times U_n$, then its projection on $U_{i_1}\times\ldots\times U_{i_k}$, R_q, constitutes a <u>marginal restriction</u> which is induced by R in $U_{(q)}$. Conversely, given a fuzzy restriction R_q in $U_{(q)}$, there exist fuzzy restrictions in $U_1\times\ldots\times U_n$ whose projection on $U_{(q)}$ is R_q. From (2.22), it follows that the largest[4] of these restrictions is the <u>cylindrical extension</u> of R_q, denoted by $\bar{R}_q$, whose membership function is given by

$$\mu_{\bar{R}_q}(u_1,\ldots,u_n) = \mu_{R_q}(u_{i_1},\ldots,u_{i_k}) \tag{2.23}$$

and whose <u>base</u> is R_q. ($\bar{R}_q$ is referred to as the cylindrical extension of R_q because the value of $\mu_{\bar{R}_q}$ at any point $(u_1',\ldots,u_n')$ is the same as at the point $(u_1,\ldots,u_n)$ so long as $u_{i_1}' = u_{i_1},\ldots,u'_{i_k} = u_{i_k}$.)

Since $\bar{R}_q$ is the largest restriction in $U_1\times\ldots\times U_n$ whose base is R_q, it follows that

$$R \subset \bar{R}_q \tag{2.24}$$

for all q, and hence that R satisfies the <u>containment relation</u>

$$R \subset \bar{R}_{q_1} \cap \bar{R}_{q_2} \cap \ldots \cap \bar{R}_{q_r} \tag{2.25}$$

which holds for arbitrary index subsequences $q_1,\ldots,q_r$. Thus, if we are given the marginal restrictions $R_{q_1},\ldots,R_{q_r}$, then the restriction

[4]A fuzzy relation R in U is <u>larger</u> than S (in U) iff $\mu_R(u) \geq \mu_S(u)$ for all u in U.

$$R_{MAX}(R_{q_1},\ldots,R_{q_r}) \triangleq \overline{R}_{q_1} \cap \ldots \cap \overline{R}_{q_r} \qquad (2.26)$$

is the <u>maximal</u> (i.e., least restrictive) restriction which is consistent with the restrictions $R_{q_1},\ldots,R_{q_r}$. It is this choice of R_{MAX} given $R_{q_1},\ldots,R_{q_r}$ that constitutes a general selection principle of which the rule of maximal restriction is a special case.[5]

By applying the same approach to the disjunction of two propositions, we are led to the rule

$$x \text{ is } P \text{ or } y \text{ is } Q \Rightarrow (x,y) \text{ is } \overline{P} + \overline{Q} \qquad (2.27)$$

or, equivalently,

$$x \text{ is } P \text{ or } y \text{ is } Q \Rightarrow (x,y) \text{ is } (P' \times Q')' \qquad (2.28)$$

where P' and Q' are the complements of P and Q, respectively, and + denotes the union.[6]

As a simple illustration of (2.27), assume that

$$U = 1 + 2 + 3 + 4$$

and that

---

[5] A somewhat analogous role in the case of probability distributions is played by the maximum entropy principle of R. Jaynes and M. Tribus [6], [7].

[6] The membership function of P' is related to that of P by $\mu_{P'}(u) = 1 - \mu_P(u)$. The membership function of the union of P and Q is expressed by $\mu_{P+Q}(u) = \mu_P(u) \vee \mu_Q(u)$, where $\vee$ denotes max.

$$P \triangleq \underline{\text{small}} \triangleq 1/1 + 0.6/2 + 0.2/3 \tag{2.29}$$

$$\underline{\text{large}} \triangleq 0.2/2 + 0.6/3 + 1/4 \tag{2.30}$$

$$Q \triangleq \underline{\text{very large}} = 0.04/2 + 0.36/3 + 1/4 \tag{2.31}$$

Then

$$\begin{aligned} P' &= 0.4/2 + 0.8/3 + 1/4 \\ Q' &= 1/1 + 0.96/2 + 0.64/3 \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} \overline{P} + \overline{Q} = (P'\times Q')' = 1/((1,1) &+ (1,2) + (1,3) + (1,4) \\ &+ (2,4) + (3.4) + (4,4)) \\ &+ 0.6/((2,1)+(2,2)+(2,3)) \\ &+ 0.3/((3,1)+(3,2))+0.36/((3,3)+(4,3)) \\ &+ 0.04/(4,2) \end{aligned} \tag{2.33}$$

Conditional propositions

In the case of conjunctions and disjunctions, our intuition provides a reasonably reliable guide for defining the form of the dependence of $R(x,y)$ on $R(x)$ and $R(y)$. This is less true, however, of conditional propositions of the form

$$p \triangleq \text{If } x \text{ is } P \text{ then } y \text{ is } Q \text{ else } y \text{ is } S \tag{2.34}$$

and

$$q \triangleq \text{If } x \text{ is } P \text{ then } y \text{ is } Q \tag{2.35}$$

where P is a fuzzy subset of U, while Q and S are fuzzy subsets of V.

With this qualification, two somewhat different definitions for the restrictions induced by p and q suggest themselves. The first, to which we shall refer as the maximin rule of conditional propositions, is expressed by

$$\text{If } x \text{ is } P \text{ then } y \text{ is } Q \text{ else } y \text{ is } S \Rightarrow (x,y) \text{ is } P \times Q + P' \times S, \tag{2.36}$$

which implies that the meaning of P is expressed by the relational assignment equation

$$R(x,y) = P \times Q + P' \times S \tag{2.37}$$

The conditional proposition (2.35) may be interpreted as a special case of (2.34) corresponding to $S = V$. Under this assumption, we have

$$\text{If } x \text{ is } P \text{ then } y \text{ is } Q \Rightarrow (x,y) \text{ is } P \times Q + P' \times V \tag{2.38}$$

As an illustration, consider the conditional proposition

$$p \triangleq \text{If Maya is } \underline{\text{tall}} \text{ then Turkan is } \underline{\text{very tall}} \tag{2.39}$$

Using (2.38), the fuzzy restriction induced by p is expressed by the relational assignment equation

$$R(\text{Height(Maya)}, \text{Height(Turkan)}) = \underline{\text{tall}} \times \underline{\text{very tall}} \times \underline{\text{not tall}} \times V$$

where V might be taken to be the interval [150,200] (in centimeters), and tall and very tall are fuzzy subsets of V defined by their respective compatibility functions (see Appendix)

$$\mu_{\underline{tall}} = S(160,170,180) \tag{2.40}$$

and

$$\mu_{\underline{very\ tall}} = S^2(160,170,180) \tag{2.41}$$

in which the argument u is suppressed for simplicity.

An alternative definition, to which we shall refer as the arithmetic rule of conditional propositions, is expressed by

$$\text{If } x \text{ is } P \text{ then } y \text{ is } Q \text{ else } y \text{ is } S \Rightarrow (x,y) \text{ is } ((P\times V \ominus U\times Q) + (P'\times V \ominus U\times S))' \tag{2.42}$$

or, equivalently and more simply,

$$\text{If } x \text{ is } P \text{ then } y \text{ is } Q \text{ else } y \text{ is } S \Rightarrow (x,y) \text{ is } (\overline{P}'\oplus\overline{Q}) \cap (\overline{P}\oplus\overline{S}) \tag{2.43}$$

where $\oplus$ and $\ominus$ denote the bounded-sum and bounded-difference operations,[7] respectively; $\overline{P}$ and $\overline{Q}$ are the cylindrical extensions of P and Q; and + is the union. This definition may be viewed as an adaptation to fuzzy sets of Lukasiewicz's definition of material implication in L_{aleph_1} logic, namely [8]

$$v(r\rightarrow s) \triangleq \min(1,1-v(r)+v(s)) \tag{2.44}$$

where v(r) and v(s) denote the truth-values of r and s, respectively, with $0 \leq v(r) \leq 1$, $0 \leq v(s) \leq 1$.

---

[7] The membership functions of the bounded-sum and-difference of P and Q are defined by $\mu_{P\oplus Q}(u) = \min(1,\mu_P(u)+\mu_Q(u))$ and $\mu_{P\ominus Q}(u) = \max(0,\mu_P(u)-\mu_Q(u)$, $u \in U$, where + denotes the arithmetic sum.

In particular, if S is equated to V, then (2.43) reduces to

$$\text{If } x \text{ is } P \text{ then } y \text{ is } Q \Rightarrow (x,y) \text{ is } (\overline{P}'\oplus\overline{Q}) \qquad (2.45)$$

Note that in (2.42), $P \times V$ and $U \times Q$ are the cylindrical extensions, $\overline{P}$ and $\overline{Q}$, of P and Q, respectively.

Of the two definitions stated above, the first is somewhat easier to manipulate but the second seems to be in closer accord with our intuition. Both yield the same result when P, Q and S are nonfuzzy sets.

As an illustration, in the special case where $x = y$ and $P = Q$, (2.45) yields

$$\begin{aligned}\text{If } x \text{ is } P \text{ then } x \text{ is } P &\Rightarrow x \text{ is } (P'\oplus P) \qquad (2.46)\\ & \quad\; x \text{ is } V\end{aligned}$$

which implies, as should be expected, that the proposition in question induces no restriction on x. The same holds true, more generally, when $P \subset Q$.

<u>Modification of Fuzzy Restrictions</u>

Basically, there are three distinct ways in which a fuzzy restriction which is induced by a proposition of the form

$$p \triangleq x \text{ is } P$$

may be modified.

First, by a combination with other restrictions, as in

$$r \triangleq x \text{ is } P \text{ and } x \text{ is } Q \qquad (2.47)$$

which transforms P into $P \cap Q$.

Second, by the application of a modifier m to P, as in

Hans is very kind (2.48)

Maribel is highly temperamental (2.49)

Lydia is more or less happy (2.50)

in which the operators very, highly and more or less modify the fuzzy restrictions represented by the fuzzy sets kind, temperamental and happy, respectively.

And third, by the use of truth-values, as in

(Sema is young) is very true (2.51)

in which very true is a fuzzy restriction on the truth-value of the proposition "Sema is young."

The effect of modifiers such as very, highly, extremely, more or less, etc., is discussed in greater detail in [9], [10] and [11]. For the purposes of the present discussion, it will suffice to observe that the effect of very and more or less may be approximated very roughly by the operations CON (standing for CONCENTRATION) and DIL (standing for DILATION) which are defined respectively by

$$CON(A) = \int_U (\mu_A(u))^2/u \tag{2.52}$$

and

$$DIL(A) = \int_U (\mu_A(u))^{0.5}/u \tag{2.53}$$

where A is a fuzzy set in U with membership function μ_A, and

$$A = \int_U \mu_A(u)/u \tag{2.54}$$

is the integral representation of A. (See the Appendix.) Thus, as an approximation, we assume that

$$\underline{\text{very}}\ A = \text{CON}(A) \tag{2.55}$$

and

$$\underline{\text{more or less}}\ A = \text{DIL}(A) \tag{2.56}$$

For example, if

$$\underline{\text{young}} = \int_0^{100} (1+(\tfrac{u}{30})^2)^{-1}/u \tag{2.57}$$

then

$$\underline{\text{very young}} = \int_0^{100} (1+(\tfrac{u}{30})^2)^{-2}/u \tag{2.58}$$

and

$$\underline{\text{more or less young}} = \int_0^{100} (1+(\tfrac{u}{30})^2)^{-0.5}/u \tag{2.59}$$

The process by which a fuzzy restriction is modified by a fuzzy truth-value is significantly different from the point-transformations expressed by (2.55) and (2.56). More specifically, the <u>rule of truth-functional modification</u>, which defines the transformation in question, may be stated in symbols as

$$(x \text{ is } Q) \text{ is } \tau \Rightarrow x \text{ is } \mu_Q^{-1} \circ \tau \tag{2.60}$$

where τ is a linguistic truth-value (e.g., _true_, _very true_, _false_, _not very true_, _more or less true_, etc.); μ_Q^{-1} is a relation inverse to the compatibility function of A, and $\mu_Q^{-1} \circ \tau$ is the composition of the non-fuzzy relation μ_Q^{-1} with the unary fuzzy relation τ. (See footnote 2 in Section 1 for the definition of composition.)

As an illustration, the application of this rule to the proposition

$$(\text{Sema is } \underline{\text{young}}) \text{ is } \underline{\text{very true}} \tag{2.61}$$

yields

$$\text{Sema is } \mu_{\underline{\text{young}}}^{-1} \circ \underline{\text{very true}} \tag{2.62}$$

Thus, if the compatibility functions of _young_ and _very true_ have the form of the curves labeled $\mu_{\underline{\text{young}}_1}$ and $\mu_{\underline{\text{very true}}}$ in Fig. 2.1, then the compatibility function of $\mu_{\underline{\text{young}}} \circ \underline{\text{very true}}$ is represented by the curve $\mu_{\underline{\text{young}}_2}$. The ordinates of $\mu_{\underline{\text{young}}_2}$ can readily be determined by the graphical procedure illustrated in Fig. 2.1.

The important point brought out by the foregoing discussion is that the association of a truth-value with a proposition does not result in a proposition of a new type; rather, it merely modifies the fuzzy restriction induced by that proposition in accordance with the rule expressed by (2.60). The same applies, more generally, to nested propositions of the form

$$(\ldots (((x \text{ is } P_1) \text{ is } \tau_1) \text{ is } \tau_2) \ldots \text{ is } \tau_n) \tag{2.63}$$

in which $\tau_1,\ldots,\tau_n$ are linguistic or numerical truth-values. It can be shown[8] that the restriction on x which is induced by a proposition of this form may be expressed as

$$x \text{ is } P_{n+1}$$

where

$$P_{k+1} = \mu_{P_k}^{-1} \circ \tau_k \qquad , \qquad k = 1,2,\ldots,n \quad (2.64)$$

[8] A more detailed discussion of this and related issues may be found in [4].

3. Approximate Reasoning (AR)

The calculus of fuzzy restrictions provides a basis for a systematic approach to approximate reasoning (or AR, for short) by interpreting such reasoning as the process of approximate solution of a system of relational assignment equations. In what follows, we shall present a brief sketch of some of the main ideas behind this interpretation.

Specifically, let us assume that we have a collection of objects $x_1,\ldots,x_n$, a collection of universes of discourse $U_1,\ldots,U_n$, and a collection $\{p_r\}$, of propositions of the form

$$p_r \triangleq (x_{r_1},x_{r_2},\ldots,x_{r_k}) \text{ is } P_r \quad , \quad r = 1,\ldots,N \tag{3.1}$$

in which P_r is a fuzzy relation in $U_{r_1} \times \ldots \times U_{r_k}$.[1] E.g.,

$$p_1 \triangleq x_1 \text{ is } \underline{\text{small}} \tag{3.2}$$

$$p_2 \triangleq x_1 \text{ and } x_2 \text{ are } \underline{\text{approximately equal}} \tag{3.3}$$

in which $U_1 \triangleq U_2 = (-\infty,\infty)$; small is a fuzzy subset of the real line $(-\infty,\infty)$; and approximately equal is a fuzzy binary relation in $(-\infty,\infty) \times (-\infty,\infty)$.

As stated in Section 2, each p_r in $\{p_r\}$ may be translated into a relational assignment equation of the form

$$R(A_{r_1}(x_{r_1}),\ldots,A_{r_k}(x_{r_k})) = P_r \quad , \quad r = 1,\ldots,N \tag{3.4}$$

---

[1] In some cases, the proposition "$(x_{r_1},\ldots,x_{r_k})$ is P_r," may be expressed more naturally in English as "x_{r_1} and ... x_{r_k} are P_r."

where A_{r_i} is an implied attribute of x_{r_i}, $i = 1,\ldots,k$, (with k dependent on r). Thus, the collection of propositions $\{p_r\}$ may be represented as a system of relational assignment equations (3.4).

Let $\overline{P}_r$ be the cylindrical extension of P_r, that is,

$$\overline{P}_r = P_r \times U_{s_1} \times \ldots \times U_{s_\ell} \tag{3.5}$$

where the index sequence $(s_1,\ldots,s_\ell)$ is the complement of the index sequence $(r_1,\ldots r_k)$ (i.e., if $n = 5$, for example, and $(r_1,r_2,r_3) = (2,4,5)$, then $(s_1,s_2) = (1,3)$).

By the rule of the implied conjunction, the collection of propositions $\{p_r\}$ induces a relational assignment equation of the form

$$R(A_1(x_1),\ldots,A_n(x_n)) = \overline{P}_1 \cap \ldots \cap \overline{P}_N \tag{3.6}$$

which subsumes the system of assignment equations (3.4). It is this equation that forms the basis for approximate inferences from the given propositions $p_1,\ldots,p_N$.

Specifically, by an <u>inference about</u> $(x_{r_1},\ldots,x_{r_k})$ <u>from</u> $\{p_r\}$, we mean the fuzzy restriction resulting from the projection of $P = \overline{P}_1 \cap \ldots \cap \overline{P}_N$ on $U_{r_1} \times \ldots \times U_{r_k}$. Such an inference will, in general, be approximate in nature because of (a) approximations in the computation of the projection of P; and/or (b) linguistic approximation to the projection of P by variables whose values are linguistic rather than numerical.[2]

[2] A linguistic variable is a variable whose values are words or sentences in a natural or artificial language. For example, Age is a linguistic variable if its values are assumed to be <u>young</u>, <u>not young</u>, <u>very young</u>, <u>more or less young</u>, etc. A more detailed discussion of linguistic variables may be found in [3], [4] and [11]. (See also Appendix.)

As a simple illustration of (3.6), consider the propositions

$$x_1 \text{ is } P_1 \tag{3.7}$$

$$x_1 \text{ and } x_2 \text{ are } P_2 \tag{3.8}$$

In this case, (3.6) becomes

$$R(A(x_1),A(x_2)) = \overline{P}_1 \cap \overline{P}_2 \tag{3.9}$$

and the projection of $\overline{P}_1 \cap \overline{P}_2$ on U_2 reduces to the composition of P_1 and P_2. In this way, we are led to the <u>compositional rule of inference</u> which may be expressed in symbols as

$$\begin{array}{l} x_1 \text{ is } P_1 \\ x_1 \text{ and } x_2 \text{ are } P_2 \\ \hline x_2 \text{ is } P_1 \circ P_2 \end{array} \tag{3.10}$$

or, more generally,

$$\begin{array}{l} x_1 \text{ and } x_2 \text{ are } P_1 \\ x_2 \text{ and } x_3 \text{ are } P_2 \\ \hline x_1 \text{ and } x_3 \text{ are } P_1 \circ P_2 \end{array} \tag{3.11}$$

in which the respective inferences are shown below the horizontal line.

As a more concrete example, consider the propositions

x_1 is small (3.12)

x_1 and x_2 are approximately equal (3.13)

where

$$U_1 \triangleq U_2 = 1 + 2 + 3 + 4 \tag{3.14}$$

$$\underline{\text{small}} \triangleq 1/1 + 0.6/2 + 0.2/3 \tag{3.15}$$

and

$$\underline{\text{approximately equal}} \triangleq 1/((1,1)+(2,2)+(3,3)+(4,4)) + 0.5/((1,2)+(2,1)+(2,3)+(3,2) +(3,4)+(4,3)) \tag{3.16}$$

In this case, the composition small o approximately equal may be expressed as the max-min product of the relation matrices of small and approximately equal. Thus

$$\underline{\text{small}} \circ \underline{\text{approximately equal}} = [1 \quad 0.6 \quad 0.2 \quad 0] \circ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

$$= [1 \quad 0.6 \quad 0.5 \quad 0.2] \tag{3.17}$$

and hence the fuzzy restriction on x_2 is given by

$$R(x_2) = 1/1 + 0.6/2 + 0.5/3 + 0.2/4 \tag{3.18}$$

Using the definition of more or less (see (2.56)), a rough linguistic approximation to (3.18) may be expressed as

$$LA(1/1+0.6/2+0.5/3+0.2/4) = \underline{\text{more or less small}}$$

where LA stands for the operation of linguistic approximation. In this way, from (3.12) and (3.13) we can deduce the approximate conclusion

$$x_2 \text{ is } \underline{\text{more or less small}} \qquad (3.20)$$

which may be regarded as a approximate solution of the relational assignment equations

$$R(x_1) = \underline{\text{small}} \qquad (3.21)$$

and

$$R(x_1,x_2) = \underline{\text{approximately equal}} \qquad (3.22)$$

Proceeding in a similar fashion in various special cases, one can readily derive one or more approximate conclusions from a given set of propositions, with the understanding that the degree of approximation in each case depends on the definition of the fuzzy restrictions which are induced by the propositions in question. Among the relatively simple examples of such approximate inferences are the following:

$$\begin{array}{l} x_1 \text{ is } \underline{\text{close}} \text{ to } x_2 \\ x_2 \text{ is } \underline{\text{close}} \text{ to } x_3 \\ \hline x_1 \text{ is } \underline{\text{more or less close}} \text{ to } x_3 \end{array} \qquad (2.23)$$

Most Swedes are tall (3.24)

Nils is a Swede

It is very likely that Nils is tall

Most Swedes are tall

Most tall Swedes are blond (3.25)

Karl is a Swede

It is very likely that Karl is tall and it is more or less (very likely) that Karl is blond

It should be noted that the last two examples involve a fuzzy quantifier, most, and fuzzy linguistic probabilities very likely and more or less (very likely). By defining most as a fuzzy subset of the unit interval, and tall as a fuzzy subset of the interval [150,200], the proposition $p \triangleq$ Most Swedes are tall induces a fuzzy restriction on the distribution of heights of Swedes, from which the conclusion "It is very likely that Nils is tall," follows as a linguistic approximation. The same applies to the last example, except that the probability very likely is dilated in the consequent proposition because of the double occurrence of the quantifier most among the antecedent propositions. The goodness of the linguistic approximation in these examples depends essentially on the degree to which very likely approximates to most.

A more general rule of inference which follows at once from (2.45) and (3.10) may be viewed as a generalization of the classical rule of modus ponens. This rule, which will be referred to as the compositional modus ponens, is expressed by

$$\begin{array}{l} x \text{ is } P \\ \text{If } x \text{ is } Q \text{ then } y \text{ is } S \\ \hline y \text{ is } P \circ (\overline{Q}' \oplus \overline{S}) \end{array} \tag{3.26}$$

where $\oplus$ is the bounded-sum operation, $\overline{Q}'$ is the cylindrical extension of the complement of Q, and $\overline{S}$ is the cylindrical extension of S. Alternatively, using the maximin rule for conditional propositions (see (2.36)), we obtain

$$\begin{array}{l} x \text{ is } P \\ \text{If } x \text{ is } Q \text{ then } y \text{ is } S \\ \hline y \text{ is } P \circ (Q \times S + \overline{Q}') \end{array} \tag{3.27}$$

where + is the union and $\overline{Q}' \triangleq Q' \times V$.

<u>Note 3.28</u> If P = Q and P and S are nonfuzzy, both (3.26) and (3.27) reduce to the classical <u>modus ponens</u>

$$\begin{array}{l} x \text{ is } P \\ \text{If } x \text{ is } P \text{ then } y \text{ is } S \\ \hline y \text{ is } S \end{array} \tag{3.29}$$

However, if P = Q and P is fuzzy, we do not obtain (3.29) because of the <u>interference effect</u> of the implied part of the conditional proposition "If x is P then y is S," namely "If x is P' then y is V." As a simple illustration of this effect, let U = 1 + 2 + 3 + 4 and assume that

$$P = 0.6/2 + 1/3 + 0.5/4 \tag{3.30}$$

and

$$S = 1/2 + 0.6/3 + 0.2/4 \tag{3.31}$$

In this case,

$$\overline{P}' \oplus \overline{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 1 & 0.6 \\ 0 & 1 & 0.6 & 0.6 \\ 0.5 & 1 & 1 & 0.7 \end{bmatrix} \tag{3.32}$$

$$P \times S + \overline{P}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 0.6 & 0.6 & 0.6 \\ 0 & 1 & 0.6 & 0.2 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{3.33}$$

and both (3.26) and (3.27) yield

$$y = 0.5/1 + 1/2 + 0.6/3 + 0.6/4 \tag{3.34}$$

which differs from S at those points at which $\mu_S(v)$ is below 0.5.

The compositional form of the modus ponens is of use in the formulation of fuzzy algorithms and the execution of fuzzy instructions [11]. The paper by S. K. Chang [12] and the recent theses by Fellinger [13] and LeFaivre [14] present a number of interesting concepts relating to such instructions and contain many illustrative examples.

4. Concluding Remarks

In the foregoing discussion, we have attempted to convey some of the main ideas behind the calculus of fuzzy restrictions and its application to approximate reasoning. Although our understanding of the processes of approximate reasoning is quite fragmentary at this juncture, it is very likely that, in time, approximate reasoning will become an important area of study and research in artificial intelligence, psychology and related fields.

Appendix

Fuzzy Sets - Notation, Terminolgy and Basic Properties

The symbols U, V, W,..., with or without subscripts, are generally used to denote specific universes of discourse, which may be arbitrary collections of objects, concepts or mathematical constructs. For example, U may denote the set of all real numbers; the set of all residents in a city; the set of all sentences in a book; the set of all colors that can be perceived by the human eye, etc.

Conventionally, if A is a fuzzy subset of U whose elements are $u_1,\ldots,u_n$, then A is expressed as

$$A = \{u_1,\ldots,u_n\} \tag{A1}$$

For our purposes, however, it is more convenient to express A as

$$A = u_1 + \ldots + u_n \tag{A2}$$

or

$$A = \sum_{i=1}^{n} u_i \tag{A3}$$

with the understanding that, for all i,j,

$$u_i + u_j = u_j + u_i \tag{A4}$$

and

$$u_i + u_i = u_i \tag{A5}$$

As an extension of this notation, a finite fuzzy subset of U is expressed as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n \tag{A6}$$

or, equivalently, as

$$F = \mu_1/u_1 + \ldots + \mu_n/u_n \tag{A7}$$

where the μ_i, $i = 1,\ldots,n$, represent the grades of membership of the u_i in F. Unless states to the contrary, the μ_i are assumed to lie in the interval [0,1], with 0 and 1 denoting no membership and full membership, respectively.

Consistent with the representation of a finite fuzzy set as a linear form in the u_i, an arbitrary fuzzy subset of U may be expressed in the form of an integral

$$F = \int_U \mu_F(u)/u \tag{A8}$$

in which μ_F: $U \to [0,1]$ is the membership or, equivalently, the compatibility function of F; and the integral $\int_U$ denotes the union (defined by (A28)) of fuzzy singletons $\mu_F(u)/u$ over the universe of discourse U.

The points in U at which $\mu_F(u) > 0$ constitute the support of F. The points at which $\mu_F(u) = 0.5$ are the crossover points of F.

Example A9 Assume

$$U = a + b + c + d \tag{A10}$$

Then, we may have

$$A = a + b + d \tag{A11}$$

and

$$F = 0.3a + 0.9b + d \tag{A12}$$

as nonfuzzy and fuzzy subsets of U, respectively.

If

$$U = 0 + 0.1 + 0.2 + \ldots + 1 \tag{A13}$$

then a fuzzy subset of U would be expressed as, say,

$$F = 0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1 \tag{A14}$$

If U = [0,1], then F might be expressed as

$$F = \int_0^1 \frac{1}{1+u^2}/u \tag{A15}$$

which means that F is a fuzzy subset of the unit interval [0,1] whose membership function is defined by

$$\mu_F(u) = \frac{1}{1+u^2} \tag{A16}$$

In many cases, it is convenient to express the membership function of a fuzzy subset of the real line in terms of a standard function whose parameters may be adjusted to fit a specified membership function in an approximate fashion. Two such functions, of the form shown in Fig. A1, are defined below.

$$S(u;\alpha,\beta,\gamma) = 0 \qquad \text{for } u \leq \alpha \tag{A17}$$

$$= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 \qquad \text{for } \alpha \leq u \leq \beta$$

$$= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 \qquad \text{for } \beta \leq u \leq \gamma$$

$$= 1 \qquad \text{for } u \geq \gamma$$

$$\pi(u;\beta,\gamma) = S(u;\gamma-\beta,\gamma-\frac{\beta}{2},\gamma) \qquad \text{for } u \leq \gamma \tag{A18}$$

$$= 1 - S(u;\gamma,\alpha+\frac{\beta}{2},\gamma+\beta) \qquad \text{for } u \geq \gamma$$

In $S(u;\alpha,\beta,\gamma)$, the parameter β, $\beta = \frac{\alpha+\gamma}{2}$, is the crossover point. In $\pi(u;\beta,\gamma)$, β is the bandwidth, that is, the separation between the crossover points of π, while γ is the point at which π is unity.

In some cases, the assumption that μ_F is a mapping from U to [0,1] may be too restrictive, and it may be desirable to allow μ_F to take values in a lattice or, more particularly, in a Boolean algebra [15], [16], [17]. For most purposes, however, it is sufficient to deal with the first two of the following hierarchy of fuzzy sets.

<u>Definition A19</u>. A fuzzy subset, F, of U is of <u>type</u> 1 if its membership function μ_F, is a mapping from U to [0,1]; and F is of type n, n = 2, 3,..., if μ_F is a mapping from U to the set of fuzzy subsets of type n - 1. For simplicity, it will always be understood that F is of type 1 if it is not specified to be of a higher type.

<u>Example A20</u>. Suppose that U is the set of all nonnegative integers and F

is a fuzzy subset of U labeled <u>small integers</u>. Then F is of type 1 if the grade of membership of a generic element u in F is a number in the interval [0,1], e.g.,

$$\mu_{\underline{\text{small integers}}}(u) = (1+(\frac{u}{5})^2)^{-1} \qquad u = 0,1,2, \tag{A21}$$

On the other hand, F is of type 2 if for each u in U $\mu_F(u)$ is a fuzzy subset of [0,1] of type 1, e.g., for u = 10,

$$\mu_{\underline{\text{small integers}}}(10) = \underline{\text{low}} \tag{A22}$$

where <u>low</u> is a fuzzy subset of [0,1] whose membership function is defined by, say,

$$\mu_{\underline{\text{low}}}(v) = 1 - S(v;0,0.25,0.5), \quad v \in [0,1] \tag{A23}$$

which implies that

$$\underline{\text{low}} = \int_0^1 (1-S(v;0,0.25,0.5))/v \tag{A24}$$

If F is a fuzzy subset of U, then its α-<u>level-set</u>, F_α, is a nonfuzzy subset of U defined by [18]

$$F_\alpha = \{u \mid \mu_F(u) \geq \alpha\} \tag{A25}$$

for $0 < \alpha \leq 1$.

If U is a linear vector space, then F is <u>convex</u> iff for all $\lambda \in [0,1]$ and all u_1, u_2 in U,

$$\mu_F(\lambda u_1+(1-\lambda)u_2) \geq \min(\mu_F(u_1),\mu_F(u_2)) \tag{A26}$$

In terms of the level-sets of F, F is convex iff the F_α are convex for all $\alpha \in (0,1]$.[1]

The relation of containment for fuzzy subsets F and G of U is defined by

$$F \subset G \Leftrightarrow \mu_F(u) \leq \mu_G(u) \quad , \quad u \in U \tag{A27}$$

Thus, F is a fuzzy subset of G if (A27) holds for all u in U.

Operations on fuzzy sets

If F and G are fuzzy subsets of U, their union, $F \cup G$, intersection, $F \cap G$, bounded-sum, $F \oplus G$, and bounded-difference, $F \ominus G$, are fuzzy subsets of U defined by

$$F \cup G \triangleq \int_U \mu_F(u) \vee \mu_G(u)/u \tag{A28}$$

$$F \cap G = \int_U \mu_F(u) \wedge \mu_G(u)/u \tag{A29}$$

$$F \oplus G = \int_U 1 \wedge (\mu_F(u)+\mu_G(u))/u \tag{A30}$$

$$F \ominus G = \int_U 0 \vee (\mu_F(u)-\mu_G(u))/u \tag{A31}$$

where $\vee$ and $\wedge$ denote max and min, respectively. The complement of F is defined by

$$F' = \int_U (1-\mu_F(u))/u \tag{A32}$$

[1]This definition of convexity can readily be extended to fuzzy sets of type 2 by applying the extension principle (see (A75)) to (A26).

or, equivalently,

$$F' = U \ominus F \tag{A33}$$

It can be readily shown that F and G satisfy the identities

$$(F \cap G)' = F' \cup G' \tag{A34}$$

$$(F \cup G)' = F' \cap G' \tag{A35}$$

$$(F \oplus G)' = F' \ominus G \tag{A36}$$

$$(F \ominus G)' = F' \oplus G \tag{A37}$$

and that F satisfies the resolution identity [2]

$$F = \int_0^1 \alpha F_\alpha \tag{A38}$$

where F_α is the α-level-set of F; αF_α is a set whose membership function is $\mu_{\alpha F_\alpha} = \alpha \mu_{F_\alpha}$, and $\int_0^1$ denotes the union of the αF, with $\alpha \in (0,1]$.

Although it is traditional to use the symbol $\cup$ to denote the union of nonfuzzy sets, in the case of fuzzy sets it is advantageous to use the symbol + in place of $\cup$ where no confusion with the arithmetic sum can result. This convention is employed in the following example, which is intended to illustrate (A28), (A29), (A30), (A31) and (A32).

Example A39. For U defined by (A10) and F and G expressed by

$$F = 0.4a + 0.9b + d \tag{A40}$$

$$G = 0.6a + 0.5b \tag{A41}$$

we have

$$F + G = 0.6a + 0.9b + d \tag{A42}$$

$$F \cap G = 0.4a + 0.5b \tag{A43}$$

$$F \oplus G = a + b + d \tag{A44}$$

$$F \ominus G = 0.4b + d \tag{A45}$$

$$F' = 0.6a + 0.1b + c \tag{A46}$$

The linguistic connectives *and* (conjunction) and *or* (disjunction) are identified with $\cap$ and +, respectively. Thus,

$$F \ \underline{\text{and}}\ G \triangleq F \cap G \tag{A47}$$

and

$$F \ \underline{\text{or}}\ G \triangleq F + G \tag{A48}$$

As defined by (A47) and (A48), *and* and *or* are implied to be *non-interactive* in the sense that there is no "trade-off" between their operands. When this is not the case, *and* and *or* are denoted by <*and*> and <*or*>, respectively, and are defined in a way that reflects the nature of the trade-off. For example, we may have

$$F \ \underline{\text{and}}\ G = \int_U \mu_F(u) \ {}_G(u)/u \tag{A49}$$

$$F \ \underline{\text{or}}\ G = \int_U (\mu_F(u)+\mu_G(u)-\mu_F(u)\mu_G(u))/u \tag{A50}$$

whose + denotes the arithmetic sum. In general, the interactive versions of and and or do not possess the simplifying properties of the connectives defined by (A47) and (A48), e.g., associativity, distributivity, etc. (See [4].)

If α is a real number, then F^α is defined by

$$F^\alpha \triangleq \int_V (\mu_F(n))^\alpha/u \tag{A51}$$

For example, for the fuzzy set defined by (A40), we have

$$F^2 = 0.16a + 0.81b + d \tag{A52}$$

and

$$F^{1/2} = 0.63a + 0.95b + d \tag{A53}$$

These operations may be used to approximate, very roughly, to the effect of the linguistic modifiers <u>very</u> and <u>more or less</u>. Thus,

$$\underline{\text{very}}\ F \triangleq F^2 \tag{A54}$$

and

$$\underline{\text{more or less}}\ F \triangleq F^{1/2} \tag{A55}$$

If $F_1,\ldots,F_n$ are fuzzy subsets of $U_1,\ldots,U_n$, then the <u>cartesian product</u> of $F_1,\ldots,F_n$ is a fuzzy subset of $U_1 \times \ldots \times U_n$ defined by

$$F_1 \times \ldots \times F_n = \int_{U_1\times\ldots\times U_n} (\mu_{F_1}(u_1) \wedge \ldots \wedge \mu_{F_n}(u_n))/(u_1,\ldots,u_n) \tag{A56}$$

As an illustration, for the fuzzy sets defined by (A40) and (A41), we have

$$F \times G = (0.4a+0.9b+d) \times (0.6a+0.5b) \tag{A57}$$
$$= 0.4/(a,a) + 0.4/(a,b) + 0.6/(b,a)$$
$$+ 0.5/(b,b) + 0.6/(d,a) + 0.5/(d,b)$$

which is a fuzzy subset of $(a+b+c+d) \times (a+b+c+d)$.

Fuzzy relations

An n-ary fuzzy relation R in $U_1 \times \ldots \times U_n$ is a fuzzy subset of $U_1 \times \ldots \times U_n$. The projection of R on $U_{i_1} \times \ldots \times U_{i_k}$, where $(i_1,\ldots,i_k)$ is a subsequence of $(1,\ldots,n)$, is a relation in $U_{i_1} \times \ldots \times U_{i_k}$ defined by

$$\text{Proj } R \text{ on } U_{i_1} \times \ldots \times U_{i_k} \stackrel{\Delta}{=} \int_{U_{i_1}\times\ldots\times U_{i_k}} \vee_{u_{j_1},\ldots,u_{j_\ell}} \mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n) \tag{A58}$$

where $(j_1,\ldots,j_\ell)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if $n = 6$ then $(1,3,6)$ is complementary to $(2,4,5)$), and $\vee_{u_{j_1},\ldots,u_{j_\ell}}$ denotes the supremum over $U_{j_1} \times \ldots \times U_{j_\ell}$.

If R is a fuzzy subset of $U_{i_1},\ldots,U_{i_k}$, then its cylindrical extension in $U_1 \times \ldots \times U_n$ is a fuzzy subset of $U_1 \times \ldots \times U_n$ defined by

$$\overline{R} = \int_{U_1\times\ldots\times U_n} \mu_R(u_{i_1},\ldots,u_{i_k})/(u_1,\ldots,u_n) \tag{A59}$$

In terms of their cylindrical extensions, the composition of two binary relations R and S (in $U_1 \times U_2$ and $U_2 \times U_3$, respectively) is expressed by

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \tag{A60}$$

where $\bar{R}$ and $\bar{S}$ are the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Similarly, if R is a binary relation in $U_1 \times U_2$ and S is a unary relation in U_2, their composition is given by

$$R \circ S = \text{Proj } R \cap \bar{S} \text{ on } U_1 \tag{A61}$$

Example A62. Let R be defined by the right-hand member of (A57) and

$$S = 0.4a + b + 0.8d \tag{A63}$$

Then

$$\text{Proj } R \text{ on } U_1 (\triangleq a+b+c+d) = 0.4a + 0.6b + 0.6d \tag{A64}$$

and

$$R \circ S = 0.4a + 0.5b + 0.5d \tag{A65}$$

Linguistic variables

Informally, a linguistic variable, $\mathcal{X}$, is a variable whose values are words or sentences in a natural or artificial language. For example, if age is interpreted as a linguistic variable, then its term-set, $T(\mathcal{X})$, that is, the set of its linguistic values, might be

$$T(\underline{age}) = \underline{young} + \underline{old} + \underline{very}\ \underline{young} + \underline{not}\ \underline{young} + \underline{very}\ \underline{old} + \underline{very}\ \underline{very}\ \underline{young} + \underline{rather}\ \underline{young} + \underline{more}\ \underline{or}\ \underline{less}\ \underline{young} + \dots \tag{A66}$$

where each of the terms in $T(\underline{age})$ is a label of a fuzzy subset of a universe of discourse, say $U = [0,100]$.

A linguistic variable is associated with two rules: (a) a <u>syntactic rule</u>, which defines the well-formed sentences in $T(\mathcal{X})$; and (b) a <u>semantic rule</u>, by which the meaning of the terms in $T(\mathcal{X})$ may be determined. If X is a term in $T(\mathcal{X})$, then its <u>meaning</u> (in a denotational sense) is a subset of U. A <u>primary term</u> in $T(\mathcal{X})$ is a term whose meaning is a <u>primary fuzzy set</u>, that is, a term whose meaning must be defined a priori, and which serves as a basis for the computation of the meaning of the non-primary terms in $T(\mathcal{X})$. For example, the primary terms in (A66) are <u>young</u> and <u>old</u>, whose meaning might be defined by their respective compatibility functions $\mu_{\underline{young}}$ and $\mu_{\underline{old}}$. From these, then, the meaning - or, equivalently, the compatibility functions - of the non-primary terms in (A66) may be computed by the application of a semantic rule. For example, employing (A54) and (A55), we have

$$\mu_{\underline{very}\ \underline{young}} = (\mu_{\underline{young}})^2 \tag{A67}$$

$$\mu_{\underline{more}\ \underline{or}\ \underline{less}\ \underline{old}} = (\mu_{\underline{old}})^{1/2} \tag{A68}$$

$$\mu_{\underline{not}\ \underline{very}\ \underline{young}} = 1 - (\mu_{\underline{young}})^2 \tag{A69}$$

For illustration, plots of the compatibility functions of these terms are shown in Fig. A2.

The extension principle

Let f be a mapping from U to V. Thus,

$$v = f(u) \tag{A70}$$

where u and v are generic elements of U and V, respectively.

Let F be a fuzzy subset of U expressed as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n \tag{A71}$$

or, more generally,

$$F = \int_U \mu_F(u)/u \tag{A72}$$

By the extension principle [3], the image of F under f is given by

$$f(F) = \mu_1 f(u_1) + \ldots + \mu_n f(u_n) \tag{A73}$$

or, more generally,

$$f(F) = \int_U \mu_F(u)/f(u) \tag{A74}$$

Similarly, if f is a mapping from U × V to W, and F and G are fuzzy subsets of U and V, respectively, then

$$f(F,G) = \int_W (\mu_F(u) \wedge \mu_G(v))/f(u,v) \tag{A75}$$

Example A76 Assume that f is the operation of squaring. Then, for the set defined by (A14), we have

$$f(0.3/0.5+0.6/0.7+0.8/0.9+1/1) = 0.3/0.25 + 0.6/0.49 + 0.8/0.81 + 1/1 \quad \text{(A77)}$$

Similarly, for the binary operation $\vee$ ($\triangleq$ max), we have

$$(0.9/0.1+0.2/0.5+1/1) \vee (0.3/0.2+0.8/0.6) = 0.3/0.2 + 0.2/0.5 + 0.8/1 + 0.8/0.6 + 0.2/0.6 \quad \text{(A78)}$$

It should be noted that the operation of squaring in (A77) is different from that defined by (A51) and (A52).

References

1. L. A. Zadeh, "Fuzzy Logic and Approximate Reasoning," ERL Memorandum M-479, November 1974. To appear in Synthese.
2. L. A. Zadeh, "Similarity Relations and Fuzzy Orderings," Inf. Sci., vol. 3, pp. 177-200, 1971.
3. L. A. Zadeh, "The Concept of a Linguistic Variable and Its Application to Approximate Reasoning," ERL Memorandum M-411, October 1973. To appear in Information Sciences.
4. L. A. Zadeh, "A Fuzzy-Algorithmic Approach to the Definition of Complex or Imprecise Concepts," ERL Memorandum M474, October 1974.
5. R. E. Bellman, R. Kalaba and L. A. Zadeh, "Abstraction and Pattern Classification," Jour. Math. Analysis and Appl., vol. 13, pp. 1-7, 1966.
6. E. T. Jaynes, "Maximum Entropy for Hypothesis Formulation," Annals of Math. Stat., vol. 34, pp. 911-930.
7. M. Tribus, Rational Descriptions, Decisions and Designs, Pergamon Press, New York and Oxford, 1969.
8. N. Rescher, Many-Valued Logic, McGraw-Hill, New York, 1969.
9. L. A. Zadeh, "A Fuzzy-Set-Theoretic Interpretation of Linguistic Hedges," Jour. of Cybernetics, vol. 2, pp. 4-34, 1972.
10. G. Lakoff, "Hedges: A Study of Meaning Criteria and the Logic of Fuzzy Concepts," Jour. of Philosophical Logic, vol. 2, pp. 458-508, 1973.
11. L. A. Zadeh, "Outline of a New Approach to the Analysis of Complex Systems and Decision Processes," IEEE Trans. on Systems, Man and Cybernetics, vol. SMC-3, pp. 28-44, January 1973.

12. S. K. Chang, "On the Execution of Fuzzy Programs Using Finite State Machines," IEEE Trans. Elec. Comp., vol. C-21, pp. 241-253, 1972.

13. L. Fellinger, "Specifications for a Fuzzy System Modeling Language," Ph.D. Thesis, Oregon State University, Corvallis, Ore., 1974.

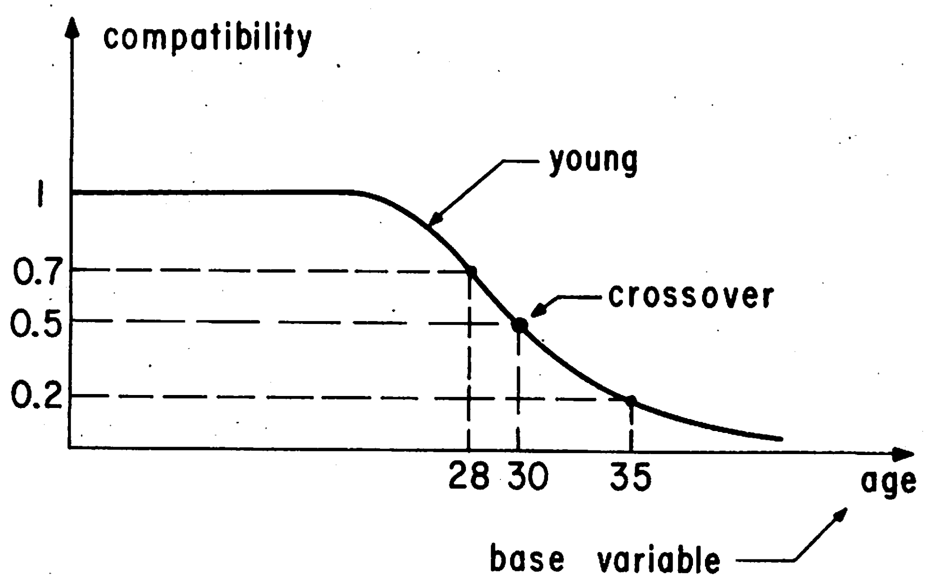
14. R. LeFaivre, "Fuzzy Problem Solving," Tech. Report 37, Madison Academic Computing Center, Univ. of Wisconsin, August 1974.

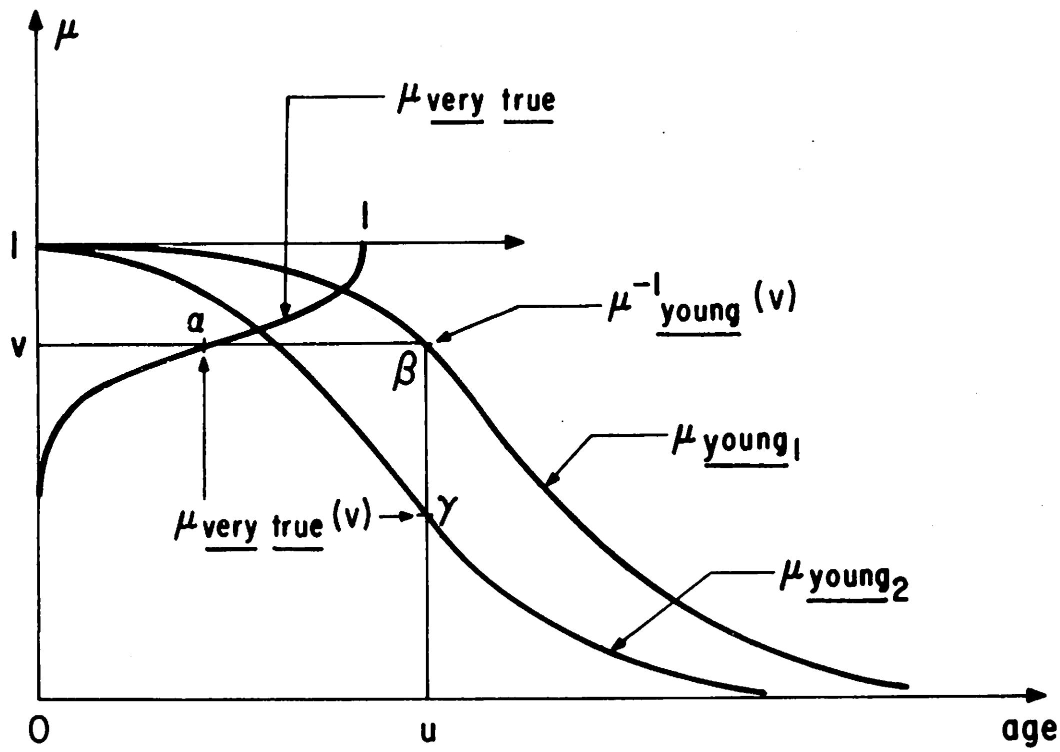
15. J. Goguen, "L-Fuzzy Sets," Jour. Math. Analysis and Appl., vol. 18, pp. 145-174, 1967.

16. J. G. Brown, "A Note on Fuzzy Sets," Inf. Control, vol. 18, pp. 32-39, 1971.

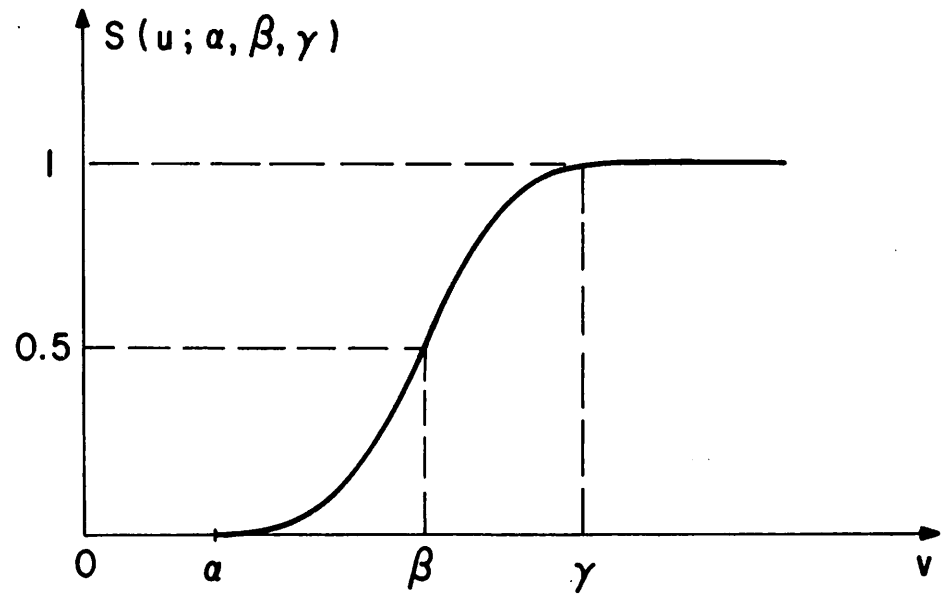
17. A. De Luca and S. Termini, "Algebraic Properties of Fuzzy Sets," Jour. Math. Analysis and Appl., vol. 40, pp. 373-386, 1972.

Figure Captions

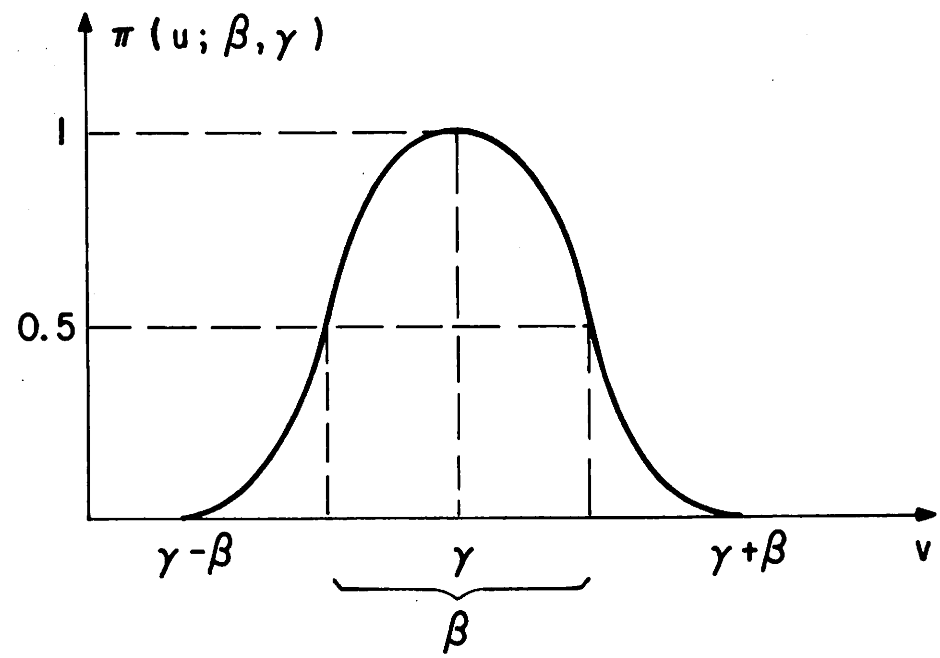

compatibility
young
1
0.7
0.5
0.2
crossover
28
30
35
age
base variable



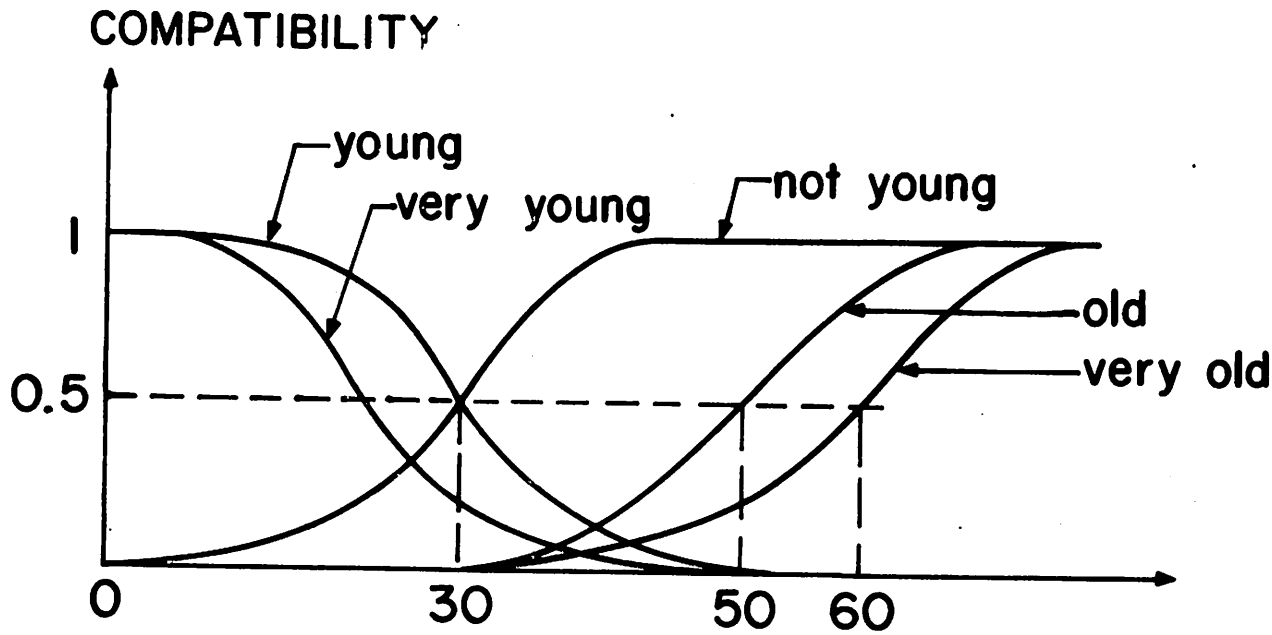
μ
μ very true
1
1
μ⁻¹ young (v)
a
v
β
μ young₁
μ very true (v)
γ
μ young₂
0
u
age



(a)



(b)


COMPATIBILITY
young
very young
not young
old
very old
1
0.5
0
30
50
60