# A QUALITATIVE ANALYSIS OF THE BEHAVIOR OF NONLINEAR DYNAMIC NETWORKS: STABILITY OF AUTONOMOUS NETWORKS

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# A QUALITATIVE ANALYSIS OF THE BEHAVIOR OF NONLINEAR DYNAMIC NETWORKS: STABILITY OF AUTONOMOUS NETWORKS

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## Abstract

Several theorems are presented which predict in a qualitative manner the behavior of dynamic nonlinear networks. In particular, conditions are given which assure that the voltage and current waveforms of a dynamic nonlinear network $\mathcal{N}$ have no finite escape-time solutions, or when $\mathcal{N}$ is autonomous, the waveforms are bounded, or eventually uniformly bounded, or converge to a globally asymptotically stable equilibrium point. An algorithm is presented which computes a maximum "transient decay" time constant when waveforms converge exponentially to the globally asymptotically stable equilibrium point. Several examples are discussed. These results are extended in [15] to nonautonomous networks.

The theorems are significant in that they apply to a large class of networks. Furthermore, their hypotheses are simple and easily verifiable. The hypotheses are of two types: First, very general conditions on the network state equations, and second, conditions on the individual element characteristics and their interconnection. The latter type of theorems use graph-theoretic results of [14] and involve solely the examination of the global nature of each network element and the verification of a topological "loop-cutset" condition.

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## I. Introduction

Much of the analysis of dynamic nonlinear networks has been in the area of the formulation of networks equations [1]-[5], and in the area of numerically solving these equations [6] - [7]. There are results concerning the behavior of networks containing specific nonlinear elements such as transistors or iron-core inductors [8]-[10] but there are relatively few results which examine in a qualitative way the behavior of general nonlinear dynamic networks [1], [4], [11], [12], [13]. This paper is the second of three papers which develop methods for predicting in a qualitative way the behavior of dynamic nonlinear networks. The other two papers are titled "Graph-Theoretic Properties of Dynamic Nonlinear Networks" [14], and "A Qualitative Analysis of the Behavior of Dynamic Nonlinear Networks: Steady-State Solutions of Nonautonomous Networks," [15]. In [14] graph-theoretic methods are used to determine properties of network equations. We combine these results with the mathematical analysis of the equations to determine the behavior of autonomous networks in this paper, and to determine the behavior of nonautonomous networks in [15]. In these papers, we answer the following types of questions: Let $\mathcal{N}$ be a dynamic nonlinear network. Under what condition may we conclude all network voltage and current waveforms are bounded, or eventually uniformly bounded?[1] If $\mathcal{N}$ contains a T-periodic source, when is there a T-periodic solution of $\mathcal{N}$, or a subharmonic solution? If $\mathcal{N}$ contains

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[1]When $\underset{\sim}{v}(t)$ and $\underset{\sim}{i}(t)$ are the voltage and current waveforms of $\mathcal{N}$, we say that they are eventually uniformly bounded if, and only if, there exists $k > 0$ such that for each $\underset{\sim}{v}(t)$ and $\underset{\sim}{i}(t)$, there exists $t_0 \geq 0$ such that $\left\| \begin{pmatrix} \underset{\sim}{v}(t) \\ \underset{\sim}{i}(t) \end{pmatrix} \right\| < k$ for all $t \geq t_0$.

constant independent voltage and current sources, when does $\mathcal{N}$ have a unique, globally asymptotically stable equilibrium point? When $\mathcal{N}$ has time-varying sources, under what conditions does $\mathcal{N}$ have a unique steady-state solution (in the same sense as with linear networks)? In this case, do the transients decay exponentially? To demonstrate the significance of these questions, let us examine the Wien Bridge Oscillator [16] of Fig. 1a. The operational amplifier together with its feedback resistor $R_F$ and source resistor $R_S$ function as a resistive two-port containing a controlled voltage source. The circuit model is shown in Fig. 1b. Its state equation is easily derived to be

$$\begin{pmatrix} \dot{v}_{C_1} \\ \dot{v}_{C_2} \end{pmatrix} = - \begin{bmatrix} \frac{1}{C_1 R} & 0 \\ 0 & \frac{1}{C_2 R} \end{bmatrix} \begin{pmatrix} v_{C_1} + v_{C_2} - f(v_{C_2}) \\ v_{C_1} + 2v_{C_2} - f(v_{C_2}) \end{pmatrix} \tag{1}$$

A common approach by which the network is analyzed by engineers proceeds as follows: Assume $C_1 = C_2$, and that the controlled voltage source function $f(\cdot)$ is as shown in Fig. 1c. Assume at $t = 0$, $\underline{V}_{sat} < f(v_{C_2}(0)) < \overline{V}_{sat}$; that is, $f(\cdot)$ is a "linear" function at $t = 0$. Then, (1) reduces to a linear state equation $\begin{pmatrix} \dot{v}_{C_1} \\ \dot{v}_{C_2} \end{pmatrix} = \underset{\sim}{M} \begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix}$ and the following conclusions are made:

(i) When $0 \leq A_v < 3$ ($A_v$ is the slope of the linear portion of $f(\cdot)$ in Fig. 1c) the eigenvalues of matrix $\underset{\sim}{M}$ have negative real parts, so $\lim_{t\to+\infty} v_{C_1}(t) = \lim_{t\to+\infty} v_{C_2}(t) = 0$.

(ii) When $A_v = 3$, the eigenvalues of $\underset{\sim}{M}$ have zero real parts, and the network oscillates.

(iii) When $A_v > 3$, the eigenvalues of $\underset{\sim}{M}$ have positive real parts, and there are unstable oscillations which grow until saturation "stabilizes" them.

This analysis is unsatisfactory from a theoretical point of view because it involves linear methods in a nonlinear network. In many cases, using linear methods in nonlinear systems has led to wrong conclusions; e.g., Aizerman's Conjecture [17], though in this case (i), (ii) and (iii) above correctly describe the circuit behavior when $f(\cdot)$ is as shown in Fig. 1c. We are interested in finding a more rigorous method for determining the behavior of the network. Moreover, we want to answer the following questions: If at $t = 0$, $f(v_{C_2}) > \overline{V}_{sat}$; i.e., we are not operating in the linear region, and $0 \leq A_v < 3$, then may we still conclude $\lim_{t\to\infty} v_{C_1}(t) = \lim_{t\to\infty} v_{C_2}(t) = 0$ as in (i) above? When $f(\cdot)$ is not precisely linear for $\underline{V}_{sat} \leq f(v) \leq \overline{V}_{sat}$, the above analysis is no longer valid; under what conditions do we obtain oscillations? In what manner does saturation "stabilize" the waveforms? We will return to this example in the following sections of this paper.

In Section II, a very general class of dynamic nonlinear networks is defined along with a characterization of the various types of resistive n-ports to be considered in the sequel. Various properties of functions such as the passivity property, the increasing property, the strictly increasing property, etc., are defined. The properties have been discussed extensively in [14]. The graph-theoretic results of [14] which are needed later are presented and discussed here.

In Section III, the mathematical results used in this paper are presented. In Theorem A, properties of a $C^1$-strictly increasing diffeomorphic state function (Defs. 1-4) are developed. The proof of Theorem A is given in the Appendix. In Theorem B, three Lyapunov-type theorems are given in which the qualitative behavior of solutions of the general differential equations (22) and (23) are analyzed. Specifically, conditions are given such that (Theorem B-1) solutions of the differential equations (22) and (23) are bounded or eventually uniformly bounded; conditions are given such that (Theorem B-2) solutions of (22) and (23) exist for all t as $t \to +\infty$ (there are no finite escape-time solutions); conditions are given such that (Theorem B-3) the solutions of (23) decay exponentially to a globally asymptotically stable equilibrium point.

In Sections IV, V and VI, theorems are given for analyzing the qualitative behavior of nonlinear dynamic networks. The hypotheses of those theorems are of two types; namely, conditions upon the network state equations, and condition on the constitutive relations of the network elements and their interconnection. The difference between these two types of hypotheses is discussed in a general way in Section III. These conditions are used in Theorems 1-8 to show (i) that the voltage and current waveforms exist for all $t \geq 0$, or to show (ii) the waveforms are bounded or eventually uniformly bounded, or (iii) the waveforms converge (possibly exponentially) to a globally asymptotically stable equilibrium point. The important aspect of our results is that the hypotheses apply to a large class of networks and that they are easily verifiable. In their final form, the hypotheses involve simply investigating the passive or increasing nature of each network element, and satisfying an easily verifiable topological "loop-cutset" condition on the interconnection of the elements. As illustrated in the examples in Sections IV, V and VI, the results may be applied to transistor networks, operational amplifier networks, etc. The general network equations need not be solved or formed.

## II. Characterization of State Equations

Consider the dynamic nonlinear network $\mathcal{N}$ shown in Fig. 2. It contains $n_C$ (possibly coupled) one-port capacitors, and $n_L$ (possibly coupled) one-port inductors.[2] Let $\underset{\sim}{v}_C$, $\underset{\sim}{i}_C$, $\underset{\sim}{q}_C \in \mathbb{R}^{n_C}$ and $\underset{\sim}{v}_L$, $\underset{\sim}{i}_L$, $\underset{\sim}{\phi}_L \in \mathbb{R}^{n_L}$ denote respectively the capacitor voltages, currents, charges, and the inductor voltages, currents and fluxes. The constitutive relations of a charge-controlled capacitor and a flux-controlled inductor are given respectively by:

$$\begin{aligned} \underset{\sim}{v}_C &= \underset{\sim}{h}_C(\underset{\sim}{q}_C) \\ \underset{\sim}{i}_L &= \underset{\sim}{h}_L(\underset{\sim}{\phi}_L) \end{aligned} \tag{2}$$

where $\underset{\sim}{h}_C$: $\mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $\underset{\sim}{h}_L$: $\mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$. Define the $n_p$-vectors ($n_p=n_C+n_L$) (the subscript "p" denotes a "port variable")

$$\underset{\sim}{v}_p = \begin{pmatrix} \underset{\sim}{v}_C \\ \underset{\sim}{v}_L \end{pmatrix}; \quad \underset{\sim}{i}_p = \begin{pmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{i}_L \end{pmatrix}; \quad \underset{\sim}{x}_p = \begin{pmatrix} \underset{\sim}{v}_C \\ \underset{\sim}{i}_L \end{pmatrix};$$

$$\underset{\sim}{y}_p = \begin{pmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{v}_L \end{pmatrix}; \quad \underset{\sim}{z}_p = \begin{pmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{\phi}_L \end{pmatrix} \tag{3}$$

then (2) becomes

$$\underset{\sim}{x}_p = \underset{\sim}{h}_p(\underset{\sim}{z}_p) \tag{4}$$

$\underset{\sim}{h}_p(\cdot) = [\underset{\sim}{h}_C^T(\cdot), \underset{\sim}{h}_L^T(\cdot)]^T$ (where the superscript "T" denotes transpose).

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[2] There is no loss of generality in our choice of this network model, since any multi-port or multi-terminal capacitor (resp., inductor) can always be modeled as a system of "coupled" one-port capacitors (resp., inductors). Observe also that an (n+1)-terminal element can always be modeled as a "grounded" n-port.

Remark: In [14], the capacitors and inductors are respectively voltage-controlled and current-controlled; i.e., instead of (4), we have $\tilde{z}_p = \tilde{f}_p(\tilde{x}_p)$. We use $\tilde{f}_p$ in [14], and we use $\tilde{h}_p$ here and in [15] purely for ease of notation in each paper. In some of the theorems in this paper and in [15], $\tilde{h}_p$ is bijective; hence $\tilde{f}_p = \tilde{h}_p^{-1}$ exists, and either $\tilde{h}_p$ or $\tilde{f}_p$ may be considered as the capacitor-inductor function. See Example 5.

We view the capacitors and inductors of $\mathcal{N}$ as attached to an $n_p$-port N which contains (nonlinear) one-port resistors, (nonlinear) multi-port resistors,[3] and independent voltage and current sources -- see Fig. 2. The vectors $\tilde{v}_p, \tilde{i}_p, \tilde{x}_p, \tilde{y}_p \in \mathbb{R}^{n_p}$ of (3) are the port variables of N as well as the capacitor and inductor variables.

Assume resistor $R_\alpha$ of N is an $n_\alpha$-port resistor. Its voltage and current are, respectively, $\tilde{v}_{R_\alpha}, \tilde{i}_{R_\alpha} \in \mathbb{R}^{n_\alpha}$. In defining its constitutive relations (when it exists) we assume that for each port of the $n_\alpha$-port resistor either the port voltage or the port current is an independent resistor variable, and the remaining port variable is a dependent resistor variable. Let $\tilde{x}_{R_\alpha}, \tilde{y}_{R_\alpha} \in \mathbb{R}^{n_\alpha}$ denote respectively the independent and dependent resistor vectors. The constitutive relation is therefore

$$\tilde{y}_{R_\alpha} = \tilde{g}_{R_\alpha}(\tilde{x}_{R_\alpha}) \tag{5}$$

Let $m_R$ be the number of resistors of N, and let $n_R$ be the number of all internal resistor ports of N ($m_R = n_R$ if, and only if, all resistors

[3] N also contains controlled voltage and current sources in the following sense: We assume every controlled source of N is represented by "coupling" within multi-port resistors. For example, although transistors, FET, and operational amplifiers are multi-terminal elements which are often modeled using controlled sources, they can also be represented as multi-port resistors. Hence, a transistor can be characterized by the constitutive relation (78) of Example 3.

are two-terminal elements). The composite resistor vectors are $\underset{\sim}{v}_R$, $\underset{\sim}{i}_R \in \mathbb{R}^{n_R}$ representing respectively all internal voltages and currents. Let the $m_R$ resistors be described by their constitutive relations $\underset{\sim}{g}_{R_1}(\cdot)$, $\underset{\sim}{g}_{R_2}(\cdot),\ldots,\underset{\sim}{g}_{R_{m_R}}(\cdot)$, and let $\underset{\sim}{x}_R$, $\underset{\sim}{y}_R \in \mathbb{R}^{n_R}$ denote, respectively, the independent and dependent resistor vectors, then

$$\underset{\sim}{y}_R = \underset{\sim}{g}_R(\underset{\sim}{x}_R) \tag{6}$$

is the composite resistor constitutive relation representing all internal resistors, where $\underset{\sim}{g}_R(\cdot) = [\underset{\sim}{g}^T_{R_1}(\cdot), \underset{\sim}{g}^T_{R_2}(\cdot),\ldots, \underset{\sim}{g}^T_{R_\alpha}(\cdot),\ldots, \underset{\sim}{g}^T_{R_{m_R}}(\cdot)]^T$.

Let $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ denote the voltages of the independent voltage sources and the currents of the independent current sources. The constitutive relation of the "overall resistor" $n_p$-port N, when it exists, is

$$\underset{\sim}{y}_p = -\underset{\sim}{g}_p(\underset{\sim}{x}_p,\underset{\sim}{u}_S) \tag{7}$$

where $\underset{\sim}{g}_p(\cdot,\cdot)$: $\mathbb{R}^{n_p+n_S} \rightarrow \mathbb{R}^{n_p}$, or if there are no independent sources

$$\underset{\sim}{y}_p = -\underset{\sim}{g}_p(\underset{\sim}{x}_p) \tag{8}$$

where $\underset{\sim}{g}_p(\cdot)$: $\mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_p}$. We will use both forms of $\underset{\sim}{g}_p$ in the sequel, and in every case we will state explicitly (if necessary) which equation is being used.

Remarks: 1. Eq. (8) can represent N containing constant sources. This is shown in [14; Theorem 8].

2. Eqs. (7) and (8) have a negative sign because the port currents (in Fig. 2) are directed away from the ports on "voltage-driven" (i.e., capacitor) ports, and the port voltages are reversed on

the "current-driven" (i.e., inductor) ports. These reference directions and polarities are chosen so that they are consistent with those assigned to capacitors and inductors.

Using (4) with (7) and (8), we can write the state equation describing $\mathcal{N}$. Note that $\frac{d}{dt} z_p(t) = \dot{z}_p(t) = y_p(t)$; we have

$$\dot{z}_p = -g_p\left(h_p(z_p), u_S\right) \tag{9a}$$

and

$$\dot{z}_p = -g_p\left(h_p(z_p)\right) \tag{9b}$$

In this paper, we are interested mainly in autonomous networks, and therefore we use (9b) in most of the theorems of Sections IV, V and VI. In [15] we examine (9a).

The following definitions which characterize various types of resistive n-ports considered in this paper have been presented and discussed in [14].

Def. 1: The function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is

(i) passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, for all $x \in \mathbb{R}^n$

$$(x-x_0)^T f(x) \geq 0 \tag{10}$$

(ii) strictly passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, (10) is true and the left side is positive for all $x \neq x_0$.

(iii) eventually passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, there exists $k_0 > 0$ so that for all $\|x\| > k_0$[4]

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[4]The norm $\|\cdot\|$ we have used in this paper is the Euclidean norm, $\|x\| = [(x_1)^2+\ldots+(x_n)^2]^{1/2}$. Of course, the following results remain valid for any choice of norm in $\mathbb{R}^n$.

$$(\underset{\sim}{x}-\underset{\sim}{x}_0)^T \underset{\sim}{f}(\underset{\sim}{x}) \geq 0 \tag{11}$$

(iv) eventually strictly passive with respect to $\underset{\sim}{x}_0 \in \mathbb{R}^n$ if, and only if, (11) is satisfied where the left side is strictly greater than zero.

Remarks: 1. If $\underset{\sim}{x}_0 = \underset{\sim}{0} \in \mathbb{R}^{n_p}$, we say simply that $\underset{\sim}{f}$ is passive, strictly passive, eventually passive, or eventually strictly passive.

2. In (i) and (ii), the domain of $\underset{\sim}{f}$ may be an arbitrary connected set $D \subseteq \mathbb{R}^n$, $\underset{\sim}{x}_0 \in D$.

Def. 2: [19] Let $D \subseteq \mathbb{R}^n$ be convex. The function $\underset{\sim}{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ is

(i) increasing on D if, and only if, for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}'-\underset{\sim}{x}'')^T \left[\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right] \geq 0 \tag{12}$$

(ii) strictly increasing on D if, and only if, the left side of (12) is positive for all $\underset{\sim}{x}' \neq \underset{\sim}{x}''$.

(iii) uniformly increasing on D if, and only if, there exists $\gamma > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}'-\underset{\sim}{x}'')^T \left[\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right] \geq \gamma \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \tag{13}$$

(iv) strongly uniformly increasing on D if, and only if, there exists $\overline{\gamma} \geq \underline{\gamma} > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$,

$$\underline{\gamma} \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \leq (\underset{\sim}{x}'-\underset{\sim}{x}'')^T \left[\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right] \leq \overline{\gamma} \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \tag{14}$$

Def. 3: [19] For any integer $\mu \geq 0$, $\underset{\sim}{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ is a $C^\mu$-diffeomorphism on $\mathbb{R}^n$ (or is a $C^\mu$-diffeomorphic function on $\mathbb{R}^n$) if, and only if, $\underset{\sim}{f}$ is injective on $\mathbb{R}^n$, and the functions $\underset{\sim}{f}$, $\underset{\sim}{f}^{-1}$ are $C^\mu$. Furthermore, $\underset{\sim}{f}$ is a $C^\mu$-diffeomorphism mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$ if, and only if, $\underset{\sim}{f}$

is a $C^{\mu}$-diffeomorphism and $\underset{\sim}{f}$ is surjective.

Def.4: [19] The $C^1$-function $\underset{\sim}{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ is a state function if, and only if, its Jacobian $\frac{\partial \underset{\sim}{f}(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ is symmetric for all $\underset{\sim}{x} \in \mathbb{R}^n$.

In Sections IV-VI, each result concerning the behavior of $\mathcal{N}$ takes two forms: First, the behavior of the solutions of the network state Eqs. (9) are analyzed using the mathematical methods of Section III, and the preceeding definitions. The hypotheses of these theorems are in the form of conditions on the function $\underset{\sim}{h}_p$ describing the capacitors and inductors, and on the function $\underset{\sim}{g}_p$ which describes the overall resistive $n_p$-port. In each of the theorems, we make the following assumption: The qualitative behavior of the voltage and current waveforms of each element of $\mathcal{N}$ may be uniquely determined from the behavior of solutions $\underset{\sim}{z}_p(t)$ of (9). In its second form, the conclusions are identical but the hypotheses are in terms of the properties of the individual network elements and the interconnection of these elements. The conditions placed upon the elements are those placed on the resistor function $\underset{\sim}{g}_{R_\alpha}$, $\alpha = 1,2,\ldots,m_R$, and upon the capacitor-inductor function $\underset{\sim}{h}_p$. We then use the graph theoretic results of [14]. At this point, it is instructive to state the interconnection assumption of the theorems of [14].

Fundamental Topological Assumption: There is no loop and no cutset formed exclusively by capacitors and/or inductors.

If this assumption is satisfied, we know for example that if each $\underset{\sim}{g}_{R_\alpha}$ is strictly increasing, then $\underset{\sim}{g}_p$ in (9b) is strictly increasing [14; Theorem 9]. This conclusion and others are used throughout the sequel.

## III. Mathematical Methods

A $C^1$-function $\underset{\sim}{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ is a state function if, and only if, there is a $C^2$-functional $F\colon \mathbb{R}^n \to \mathbb{R}^1$ (called a *potential function*) such that $\nabla F(\underset{\sim}{x}) = \underset{\sim}{f}(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in \mathbb{R}^n$ [18]. If $\underset{\sim}{f}$ in addition is a strictly-increasing diffeomorphism, the function F also has interesting properties. The following theorem is proved in the Appendix.

*Theorem A*: Let $\underset{\sim}{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-*strictly-increasing diffeomorphic state function* mapping $\mathbb{R}^n$ *onto* $\mathbb{R}^n$. Define $F\colon \mathbb{R}^n \to \mathbb{R}^1$ to be the unique $C^2$-function such that

$$\nabla F(\underset{\sim}{x}) = \underset{\sim}{f}(\underset{\sim}{x}), \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n; \quad F(\underset{\sim}{f}^{-1}(\underset{\sim}{0})) = 0 \tag{15}$$

Then the following properties hold:

*A-1* [20] $F(\cdot)$ is a *strictly-convex* function[5]

*A-2*

$$F(\underset{\sim}{x}) > 0 \qquad \forall\, \underset{\sim}{x} \neq \underset{\sim}{f}^{-1}(\underset{\sim}{0}) \tag{16}$$

*A-3*

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \frac{1}{\|\underset{\sim}{x}\|} F(\underset{\sim}{x}) = +\infty \tag{17}$$

*A-4*

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x}) = +\infty \tag{18}$$

*A-5* For each $k > 0$, the set

$$K \triangleq \{x \in \mathbb{R}^n\colon\ F(\underset{\sim}{x}) \leq k\} \tag{19}$$

is compact and convex[5] in $\mathbb{R}^n$.

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[5] A function $F\colon \mathbb{R}^n \to \mathbb{R}^1$ is *strictly convex* if, and only if, for each $\sigma \in (0,1)$, for each pair $\underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n$,

$$F((1-\sigma)\underset{\sim}{x}' + \underset{\sim}{x}'') < (1-\sigma)F(\underset{\sim}{x}') + F(\underset{\sim}{x}'')$$

A set $S \subseteq \mathbb{R}^n$ is *convex* if, and only if, for each $\sigma \in (0,1)$, for each pair $\underset{\sim}{x}', \underset{\sim}{x}'' \in S$, $\underset{\sim}{x}_\sigma \triangleq (1-\sigma)\underset{\sim}{x}' + \sigma\underset{\sim}{x}'' \in S$.

A-6. If $\underset{\sim}{f}$ is, in addition, strongly uniformly increasing, there exist constants $\bar{\gamma} \geq \underline{\gamma} > 0$ such that for each $\underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n$, (14) is true, and

$$\underline{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{x}''\| \leq \|\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\| \leq \bar{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{x}''\| \tag{20}$$

$$\frac{1}{2}\underline{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\|^2 \leq F(\underset{\sim}{x}') \leq \frac{1}{2}\bar{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\|^2 \tag{21}$$

Remark: It is possible to extend A-6 in the following way: If for some $k > 0$, (14) is true for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n$ satisfying $\|\underset{\sim}{x}'\| > k$, $\|\underset{\sim}{x}''\| > k$, then (20) is true for these $\underset{\sim}{x}', \underset{\sim}{x}''$, and an equation of the form (21) is also true. See Corollary A in the Appendix.

In order to develop results concerning the behavior of the solutions of the network state Eqs. (9a) and (9b) we examine solutions of the general differential equations;

$$\dot{\underset{\sim}{x}} = -\underset{\sim}{f}(\underset{\sim}{x},t) \tag{22}$$

where $\underset{\sim}{f}$: $\mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$ is $C^1$, and

$$\dot{\underset{\sim}{x}} = -\underset{\sim}{f}(\underset{\sim}{x}) \tag{23}$$

where $\underset{\sim}{f}$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is $C^1$. We use Lyapunov's well-known theorem [21] which gives conditions guaranteeing the existence of a globally, asymptotically stable equilibrium point of (22) or (23).[6] We will use three other results which are similar in nature; they are summarized in Theorem B below:

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[6] $\underset{\sim}{x}^* \in \mathbb{R}^n$ is a globally, asymptotically stable equilibrium point of (22) or (23) if, and only if, for any solution $\underset{\sim}{x}(t)$, $\lim_{t\to\infty} \underset{\sim}{x}(t) = \underset{\sim}{x}^*$.

Theorem B-1 [21-22]: Assume for some $k_0 \geq 0$ there is a $C^1$-function $\mathcal{V}: \mathbb{R}^n \rightarrow \mathbb{R}^1$ such that for $\underset{\sim}{f}$ in (22),

$$\lim_{\|\underset{\sim}{x}\| \to \infty} \mathcal{V}(\underset{\sim}{x}) = +\infty \tag{24}$$

$$\frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x},t) \geq 0 \qquad \forall \|\underset{\sim}{x}\| > k_0, \qquad \forall\, t \in \mathbb{R}^1 \tag{25}$$

Then every solution $\underset{\sim}{x}(t)$ of (22) is bounded. Furthermore, if

$$\frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x},t) > 0 \qquad \forall \|\underset{\sim}{x}\| > k_0, \qquad \forall\, t \in \mathbb{R}^1 \tag{26}$$

then the solutions of (22) are eventually uniformly bounded in the sense that there is a positive $k_1 > 0$, $k_1 \triangleq \sup_{\|\underset{\sim}{x}\| \leq k_0} \mathcal{V}(\underset{\sim}{x})$, and a compact set $X \subseteq \mathbb{R}^n$,

$$X \triangleq \{\underset{\sim}{x} \in \mathbb{R}^n: \mathcal{V}(\underset{\sim}{x}) \leq k_1\} \tag{27}$$

such that for every solution $\underset{\sim}{x}(t)$ of (22), there is a time $t_0 \in \mathbb{R}^1$ so that

$$\underset{\sim}{x}(t) \in X \qquad \forall\, t \geq t_0 \tag{28}$$

Furthermore, this theorem applies to the autonomous state Eq. (15b) where we conclude in addition that (15b) has an equilibrium point $\underset{\sim}{x}^* \in X$.

Remarks: 1. The bulk of this theorem is proved in both [21] and [22]. The conclusion that the autonomous differential Eq. (15b) has an equilibrium point is proved in a more general way by Pliss in [22]. See [15] for a discussion.

2. Except for the conclusion that the autonomous Eq.

(15b) has an equilibrium point, the proof of this theorem is similar to the proof of Lyapunov's Theorem [21]; it need only be outlined here. First, note that without loss of generality, we may assume $\mathcal{V}(\underset{\sim}{x}) > 0$ for all $\underset{\sim}{x} \in \mathbb{R}^n$. We make this assumption upon noting that the continuity of $\mathcal{V}(\cdot)$ and hypothesis (24) imply that for some $\hat{k} \in \mathbb{R}^1$, $\hat{\mathcal{V}}(\underset{\sim}{x}) \triangleq \mathcal{V}(\underset{\sim}{x}) + \hat{k} > 0$ for all $\underset{\sim}{x} \in \mathbb{R}^n$. Furthermore, $\hat{\mathcal{V}}(\cdot)$ satisfies hypotheses (24), (25) and (26).

Next, using (25) we see that $\frac{d}{dt}\mathcal{V}(\underset{\sim}{x}(t)) \leq 0$ for any solution $\underset{\sim}{x}(t)$ such that $\|\underset{\sim}{x}(t)\| > k_0$. Since $\mathcal{V}(\underset{\sim}{x}(t)) > 0$, this means that $\mathcal{V}(\underset{\sim}{x}(t))$ is bounded in $\mathbb{R}^1$. From condition (24), we conclude $\underset{\sim}{x}(t)$ is bounded in $\mathbb{R}^n$.

Similarly, using hypothesis (26), $\frac{d}{dt}\mathcal{V}(\underset{\sim}{x}(t)) < 0$ for any solution $\underset{\sim}{x}(t)$ such that $\|\underset{\sim}{x}(t)\| > k_0$. Now, using (24) we see that $k_1 \triangleq \sup_{\|\underset{\sim}{x}\| \leq k_0} \mathcal{V}(\underset{\sim}{x})$ exists, and that X in (16) is compact. Then, it is clear that $\lim_{t\to\infty}\mathcal{V}(\underset{\sim}{x}(t)) \leq k_1$, and (28) follows from this.

If there is no possible $\mathcal{V}$: $\mathbb{R}^n \to \mathbb{R}^1$ such that (24) and (25) of <u>Theorem B-1</u> are satisfied, then (22) may have unbounded solutions. Furthermore, there may be <u>finite escape-time</u> solutions; that is, for some initial condition $\underset{\sim}{x}(t_0)$, $t_0 \in \mathbb{R}^1$, there exists $t_1 > t_0$ so that for the corresponding solution $\underset{\sim}{x}(t)$ of (22) having this initial condition, $\lim_{t\to t_1} \|\underset{\sim}{x}(t)\| = +\infty$. In the following theorem and corollary, conditions are given under which there is no finite escape-time solution.

<u>Theorem B-2</u>: Assume there is a continuous function $\psi$: $\mathbb{R}^1 \to \mathbb{R}^1$ such that $\psi(u) > 0$ for all $u > 0$, and[7]

$$\int^{\infty} \frac{du}{\psi(u)} = +\infty \tag{29}$$

[7]Equation (29) is equivalent to $\lim_{u_1\to\infty} \int_{u_0}^{u_1} \frac{du}{\psi(u)} = +\infty \ , \ \forall\, u_0 > 0$

Assume for some $k_0 \geq 0$ there is a $C^1$-function $\mathcal{V}: \mathbb{R}^n \rightarrow \mathbb{R}^1$ such that for $\underset{\sim}{f}$ in (22)

$$\lim_{\|\underset{\sim}{x}\| \to \infty} \mathcal{V}(\underset{\sim}{x}) = +\infty \tag{30}$$

$$\frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x},t) \geq -\psi(\mathcal{V}(\underset{\sim}{x})) \qquad \forall \|\underset{\sim}{x}\| > k_0, \ \forall\, t \in \mathbb{R}^1 \tag{31}$$

Then for any initial time $t_0 \in \mathbb{R}$, for any initial condition $\underset{\sim}{x}(t_0) \in \mathbb{R}^n$, the corresponding solution $\underset{\sim}{x}(t)$ exists for all $t \geq t_0$. That is, (22) has no finite escape-time solution.

Proof: Because of (30), we see that for solution $\underset{\sim}{x}(t)$ of (22), there exists $t_1 > t_0$ such that $\lim_{t \to t_1} \|\underset{\sim}{x}(t)\| = +\infty$ if, and only if, $\lim_{t \to t_1} \mathcal{V}(\underset{\sim}{x}(t)) = +\infty$. We will show that this is not possible.

As discussed in the sketch of the proof of Theorem B-1, we may assume without loss of generality that $\mathcal{V}(\underset{\sim}{x}) > 0$ for all $\underset{\sim}{x} \in \mathbb{R}^n$. So, assume for some solution $\underset{\sim}{x}(t)$ of (22) $\lim_{t \to t_1} \mathcal{V}(\underset{\sim}{x}(t)) = +\infty$. Find time $\tau_1 < t_1$ such that $\underset{\sim}{x}(\tau_1) > k_0$. Then, for all $t \in [\tau_1, t_1)$, $\mathcal{V}(\underset{\sim}{x}(t))$ satisfies the differential inequality

$$\frac{d}{dt}\mathcal{V}(\underset{\sim}{x}(t)) \leq \psi(\mathcal{V}(\underset{\sim}{x}(t))) \tag{32}$$

Since $\mathcal{V}(\underset{\sim}{x}(t)) > 0$ for each $t \in [\tau_1, t_1)$, we have

$$\int_{\mathcal{V}(\underset{\sim}{x}(\tau_1))}^{\mathcal{V}(\underset{\sim}{x}(t))} \frac{d\mathcal{V}}{\psi(\mathcal{V})} \leq t - \tau_1 \tag{33}$$

As $t \to t_1$, the left side of (33) tends to $+\infty$ because of (29). But the right side of (33) remains bounded, and we have a contradiction. ■

Corollary 1: Assume there exists $k_0 > 0$, and a continuous function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x},t) \geq - \psi(\|\underset{\sim}{x}\|^2) \tag{34}$$

for all $\|\underset{\sim}{x}\| > k_0$ and for all $t \in \mathbb{R}^1$. Then (22) has no finite escape-time solution.

Proof: We apply Theorem B-2, with $\mathcal{V}(\underset{\sim}{x}) \triangleq \|\underset{\sim}{x}\|^2$.

Remark: Theorem B-2 and Corollary 1 are extensions of a theorem of Wintner [23]. The statement of Wintner's theorem is as follows: If there exists some continuous function $\psi_1: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int^{\infty} \frac{du}{\psi_1(u)} = + \infty \tag{35}$$

$$\|\underset{\sim}{f}(\underset{\sim}{x},t)\| \leq \psi_1(\|x\|) \qquad \forall \ \|\underset{\sim}{x}\| > k_0, \qquad \forall \ t \in \mathbb{R}^1 \tag{36}$$

where $k_0$ is a positive constant, then every solution $\underset{\sim}{x}(t)$ of (22) exists for all $t \in (-\infty,\infty)$.

The difference between our preceeding results and with Wintner's theorem is that while our results guarantee that solution $\underset{\sim}{x}(t)$ is defined as $t \to +\infty$, Wintner in addition guarantees that $\underset{\sim}{x}(t)$ also exists as $t \to -\infty$. From a physical point of view, this conclusion is not useful and, in fact, is not satisfied by solutions of many nonlinear dynamic networks of practical interest. For example, examine the network of Fig. 3a. The diode equation is given by $i_R = I_S\left(e^{v_R/v_T}-1\right)$ where the positive constant $I_S$ and $v_T$ represent respectively the saturation current and the thermal voltage. The capacitor voltage $v_C(t)$ satisfies the state equation

$$\dot{v}_C(t) = -\frac{I_S}{C}\left(e^{v_C(t)/v_T} - 1\right) \tag{37}$$

One solution of this equation is $v_C(t) \equiv 0$. For any initial condition $v_C(0) \neq 0$, the solution is

$$v_C(t) = v_C(0) \ln\left(\frac{e^{f(t)}}{e^{f(t)} - \operatorname{sgn}(v_C(0))}\right)^{v_T/v_C(0)} \tag{38a}$$

where[8]

$$f(t) \triangleq \frac{I_S}{Cv_T} t + \ln\left(\frac{\operatorname{sgn}(v_C(0))}{1-e^{-v_C(0)/v_T}}\right) \tag{38b}$$

For any $t \geq 0$ and for any $v_C(0) \neq 0$, $f(t)$ in (38b) is positive. Hence, $v_C(t)$ in (38a) is well-defined, and $\lim_{t\to+\infty} v_C(t) = 0$. However, when $v_C(0) > 0$, define time $t_1$,

$$t_1 \triangleq -\frac{Cv_T}{I_S} \ln\left(\frac{1}{1-e^{-v_C(0)/v_T}}\right) < 0 \tag{39}$$

$f(t_1) = 0$, and $|v_C(t_1)| = +\infty$. That is, the solutions of this network exhibit the finite escape-time phenomenon in negative time. Furthermore, since $v_C \cdot \frac{I_S}{C}\left(e^{v_C/v_T}-1\right) \geq 0$ for all $v_C \in \mathbb{R}^1$, equation (34) of Corollary 1 is satisfied while it may easily be seen that Wintner's condition (36) is violated. Let us investigate further the difference between condition (34) of Corollary 1 and condition (36) of Wintner's Theorem: The diode in Fig. 3a is replaced by an arbitrary voltage-controlled resistor whose constitutive relation is $i_R = g_R(v_R)$, where $g_R(\cdot)$ is $C^1$. Now, it can easily be shown that if for some $k_1 > 0$, $k_2 > 0$ and $\beta > 0$, if either

[8] By definition $\operatorname{sgn}(v_C(0)) = +1$ (resp., $-1$) if $v_C(0) > 0$ (resp., $v_C(0) < 0$).

$$g_R(v_R) \geq k_1(v_R)^{1+\beta} \qquad \forall\, v_R > k_2 \tag{40a}$$

or

$$g_R(v_R) \leq k_1(v_R)^{1+\beta} \qquad \forall\, v_R < k_2 \tag{40b}$$

then Wintner's condition (36) is violated. Indeed, if $g_R(v_R) = (v_R)^n$, where n is any positive integer other than 1, Wintner's condition is violated. Thus, a <u>reasonable</u> sufficient condition for (36) to be true is that for some $k_1 > 0$ and $k_2 > 0$,

$$|g_R(v_R)| \leq k_1|v_R| \qquad \forall\, |v_R| > k_2 \tag{41}$$

The possible range of $g_R(\cdot)$ satisfying (41) is illustrated by the shaded portion in Fig. 3c. On the other hand, (34) of <u>Corollary 1</u> can be shown to be satisfied if for some $k_1 > 0$ and $k_2 > 0$,

$$v_R g_R(v_R) \geq -k_1(v_R)^2 \qquad \forall\, |v_R| > k_2 \tag{42}$$

The possible range of $g_R(\cdot)$ satisfying (42) is illustrated by the shaded portion in Fig. 3c. This illustrates that the class of functions $\underset{\sim}{f}(\cdot,\cdot)$ satisfying the hypotheses of <u>Theorem B-2</u> or <u>Corollary 1</u> is much larger than the class of functions satisfying Wintner's condition.

As a final remark on this subject, note that Wintner's Theorem may be proved using <u>Corollary 1</u>: Assume there exists a positive continuous function $\psi_1$ such that (35) and (36) are true. Define $\psi(u) \triangleq u^{1/2}\psi_1(u^{1/2})$. Here, $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. From (36),

$$\underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x},t) \geq -\|\underset{\sim}{x}\| \cdot \|\underset{\sim}{f}(\underset{\sim}{x},t)\|$$

$$\geq -\|\underset{\sim}{x}\| \cdot \psi_1(\|\underset{\sim}{x}\|) = -\psi(\|\underset{\sim}{x}\|^2) \tag{43a}$$

and thus (34) is satisfied. To show (29), let $\hat{u} \triangleq \sqrt{u}$ for $u > 0$,

$$\int \frac{du}{\psi(u)} = 2\int \frac{d\hat{u}\cdot\hat{u}}{\psi(\hat{u}^2)} = 2\int \frac{d\hat{u}}{\psi_1(\hat{u})} \tag{43b}$$

and (29) follows from (35). Hence, for any initial time $t_0 \in \mathbb{R}^1$, all solutions of (22) exist for all $t \geq t_0$. To show that all solutions exist for $t \leq t_0$, we repeat the above analysis for the differential equation $\dot{x} = -(-f(x,t))$. This differential equation also has solutions existing for all $t \geq t_0$, where if $x(t)$ is a solution of this equation, then $x(-t)$ is a solution of (22).

For our final mathematical result, we look again at the existence of a globally asymptotically stable equilibrium point of (23). In the following theorem, conditions are given which guarantee that solutions converge <u>exponentially</u> to the equilibrium point.

<u>Theorem B-3</u>: Let $x^* \in \mathbb{R}^n$ be the globally asymptotically stable equilibrium point of (23). Let $D \subseteq \mathbb{R}^n$ be open, $x^* \in D$. Assume there exists $\beta > 0$, $\gamma_2 \geq \gamma_1 > 0$, $\gamma_4 \geq \gamma_3 > 0$ and there exists a $C^1$-function $\mathcal{V}: D \to \mathbb{R}^1$ such that for all $x \in D$,

$$\gamma_1 \|x-x^*\|^\beta \leq \mathcal{V}(x) \leq \gamma_2 \|x-x^*\|^\beta \tag{44}$$

$$\gamma_3 \|x-x^*\|^\beta \leq \frac{\partial \mathcal{V}(x)}{\partial x} f(x) \leq \gamma_4 \|x-x^*\|^\beta \tag{45}$$

Then for any solution $x(t)$ of (23) such that $x(t) \in D$ for all $t \geq 0$, we have

$$\left[\frac{\gamma_1}{\gamma_2}\right]^{1/\beta} e^{-\frac{\gamma_4}{\beta\gamma_1}t} \|x(0)-x^*\| \leq \|x(t)-x^*\| \leq \left[\frac{\gamma_2}{\gamma_1}\right]^{1/\beta} e^{-\frac{\gamma_3}{\beta\gamma_2}t} \|x(0)-x^*\| \tag{46}$$

<u>Remarks</u>: 1. The expressions $\frac{1}{\|x-x^*\|^\beta}\mathcal{V}(x)$ and $\frac{1}{\|x-x^*\|^\beta}\frac{\partial \mathcal{V}(x)}{\partial x} f(x)$

are well-defined for each $\underset{\sim}{x} \neq \underset{\sim}{x}^*$. Hence, when $\mathcal{V}(\underset{\sim}{x}) > 0$ for all $\underset{\sim}{x} \neq \underset{\sim}{x}^*$ and $\frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} f(\underset{\sim}{x}) > 0$ for all $x \neq x^*$ (these are the conditions of Lyapunov's Theorem [21]) then $\beta$, $\gamma_1$, $\gamma_2$, $\gamma_3$, and $\gamma_4$ in (44) and (45) exist if D is <u>bounded</u> and if

$$\lim_{\|\underset{\sim}{x}-\underset{\sim}{x}^*\|\to 0} \frac{1}{\|\underset{\sim}{x}-\underset{\sim}{x}^*\|^{\beta}} \mathcal{V}(\underset{\sim}{x}); \qquad \lim_{\|\underset{\sim}{x}-\underset{\sim}{x}^*\|\to 0} \frac{1}{\|\underset{\sim}{x}-\underset{\sim}{x}^*\|^{\beta}} \frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x}) \tag{47}$$

exist and are positive.

2. The proof of this theorem is straight-forward and need only be sketched. For any solution $\underset{\sim}{x}(t) \in D$ for all $t \geq 0$, the corresponding $\mathcal{V}(\underset{\sim}{x}(t))$ satisfies

$$-\frac{\gamma_4}{\gamma_1}\mathcal{V}(\underset{\sim}{x}(t) \leq \frac{d}{dt}\mathcal{V}(\underset{\sim}{x}(t)) \leq -\frac{\gamma_3}{\gamma_2}\mathcal{V}(\underset{\sim}{x}(t)) \tag{48}$$

because of (44) and (45); hence

$$\mathcal{V}(\underset{\sim}{x}(0))e^{-\frac{\gamma_4}{\gamma_1}t} \leq \mathcal{V}(\underset{\sim}{x}(t) \leq \mathcal{V}(\underset{\sim}{x}(0))e^{-\frac{\gamma_3}{\gamma_2}t} \tag{48b}$$

and (46) follows from this.

## IV. <u>Networks with Bounded Solutions</u>

We begin with two theorems which give conditions guaranteeing that the state equation (9a) has no finite escape-time solution. In the previous section, we analyzed the network of Fig. 3a where the diode was replaced by an arbitrary voltage-controlled resistor whose constitutive relation is $i_R = g_R(v_R)$. It was shown that when (42) is satisfied, there is no finite escape-time solution. This conclusion is rigorously extended in the following theorem (specifically, (50) is a generalization of (42)).

Theorem 1: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the state function (9a). Assume the capacitor-inductor function $h_p$ is a $C^1$-state function,[9] and there exists constants $k_D \geq 0$ and $\bar{\gamma} \geq \underline{\gamma} > 0$ such that for all $\|z_p'\| > k_D$, $\|z_p''\| > k_D$

$$\underline{\gamma}\|z_p'-z_p''\|^2 \leq (z_p'-z_p'')^T\left[h_p(z_p')-h_p(z_p'')\right] \leq \bar{\gamma}\|z_p'-z_p''\|^2 \tag{49}$$

Under these conditions, if there exists an arbitrary matrix $G_p \in \mathbb{R}^{n_p \times n_p}$, an arbitrary vector $\hat{y}_p \in \mathbb{R}^{n_p}$, constants $k_1 \geq 0$ and $k_2 \geq 0$ such that for all $u_S \in \mathbb{R}^{n_S}$, and for all $\|x_p\| \geq k_2$, we have

$$x_p^T\left[g_p(x_p,u_S)+G_p x_p+\hat{y}_p\right] \geq -k_1 \tag{50}$$

where $g_p(\cdot,\cdot)$ is the $n_p$-port function, then state equation (9a) has no finite escape-time solution. That is, for any continuous $u_S(t)$, for any initial time $t_0 \in \mathbb{R}^1$, each solution $z_p(t)$ of (9a) exists for all $t \geq t_0$.

Proof: We apply Corollary A and Theorem B-2. First (49) is the same as (A-19), hence from (A-21) of Corollary A we conclude that there exists a $C^2$-function $H_p$: $\mathbb{R}^{n_p} \rightarrow \mathbb{R}^1$ such that

$$\nabla H_p(z_p) = h_p(z_p) \qquad \forall\ z \in \mathbb{R}^n \tag{51a}$$

and for some $k > 0$, and $\gamma_1 > 0$

$$\gamma_1\|z_p\|^2 \leq H_p(z_p) \qquad \forall\ \|z_p\| > k \tag{51b}$$

Using the inequality of (51b), we see that $\lim_{\|z_p\|\rightarrow\infty} H_p(z_p) = +\infty$.

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[9] The condition that $h_p$ is a state function is equivalent to requiring that the capacitors and inductors be reciprocal. This is a weak condition and is satisfied by most capacitors and inductors of practical interest. This assumption is made throughout this paper.

Hence, in applying Theorem B-2, let $\mathcal{V}(\cdot) = H(\cdot)$, and equation (30) is satisfied. We have to show only (31). Using (50),

$$\frac{\partial H_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right) = \left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right)^T \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right) = \underset{\sim}{x}_p^T \underset{\sim}{g}_p(\underset{\sim}{x}_p,\underset{\sim}{u}_S)$$

$$\geq -\|\underset{\sim}{x}_p\|^2 \|\underset{\sim}{G}_p\| - \|\underset{\sim}{x}_p\| \cdot \|\hat{\underset{\sim}{y}}_p\| - k_3 \qquad \forall\ \|\underset{\sim}{x}_p\| > k_2 \tag{52}$$

where $\|\underset{\sim}{G}_p\|$ is the induced norm of the matrix $\underset{\sim}{G}_p$ and $\|\cdot\|$ is the Euclidean norm. Next, we make a series of modifications of (52). First, note that the first term on the right of (52) dominates for large $\|\underset{\sim}{x}_p\|$; i.e., for any $k_4 > \|\underset{\sim}{G}_p\|$ there exists $k_5 > 0$ such that

$$k_4\|\underset{\sim}{x}_p\|^2 \geq \|\underset{\sim}{x}_p\|^2\|\underset{\sim}{G}_p\| - \|\underset{\sim}{x}_p\| \cdot \|\tilde{\underset{\sim}{y}}_p\| - k_3 \qquad \forall\ \|\underset{\sim}{x}_p\| > k_5 \tag{53}$$

Then, since (A-20) of Corollary A holds, there exists $k_6 > 0$ and $k_7 > 0$ such that

$$k_6\|\underset{\sim}{z}_p\|^2 \geq \|\underset{\sim}{x}_p\|^2 = \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\|^2 \qquad \forall\ \|\underset{\sim}{z}_p\| > k_7 \tag{54}$$

Combining the last four equations, there exists $k_8 > 0$ such that

$$\frac{\partial H_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right) \geq -k_4\|\underset{\sim}{x}_p\|^2 \geq -k_4 k_6\|\underset{\sim}{z}_p\|^2$$

$$\geq -\frac{k_4 k_6}{\Upsilon_1} H_p(\underset{\sim}{z}_p) \qquad \forall\ \|\underset{\sim}{z}_p\| > k_8 \tag{55}$$

Define $\psi(u) \triangleq \frac{k_4 k_6}{\Upsilon_1} u$; $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and equations (29) and (31) are satisfied. ⌘

The condition (49) on the inductor-capacitor function $\underset{\sim}{h}_p$ is that

$h_p$ is "eventually" uniformly increasing. This will be true if the capacitors and inductors are "eventually" linear. The condition (50) on the resistive $n_p$-port function $g_p$ is satisfied if $g_p$ is eventually passive, for then (50) follows with $G_p = 0$ and $\hat{y}_p = 0$. When $k_1 = k_2 = 0$ and $\hat{y}_p = 0$, then (50) has the following interpretation: If $g_p(\cdot)$ is not "more active" than the matrix function $G_p \cdot$ (i.e., $x_p^T g_p(x_p, u_S) \geq x_p^T G_p x_p$) then there are no finite escape time solutions. This is an intuitive condition since, if (9a) is linear,

$$\dot{z}_p = -G_p \Gamma_p z_p - G_S u_S \tag{56a}$$

Then each solution [24]

$$z_p(t) = e^{-G_p \Gamma_p (t-t_0)} z_p(t_0) - \int_{t_0}^{t} e^{-G_p \Gamma_p (t-\sigma)} G_S u_S(\sigma) d\sigma \tag{56b}$$

exists for all $t \geq t_0$.

These interpretations of Theorem 1 are illustrated in

Example 1: Examine the network of Fig. 4. Voltage sources $E_1(t)$, $E_2(t)$ are continuous and bounded in time. We can write the state equation (9a) for this network, where

$$h_p(z_p) = \begin{pmatrix} e^{-(q_C)^2} \left[ (q_C)^3 - (q_C)^5 \right] + q_C \\ 5\phi_{L_1} - 2\phi_{L_2} \\ -2\phi_{L_1} + 3\phi_{L_2} \end{pmatrix} \tag{57a}$$

and

$$\underset{\sim}{g}_p(\underset{\sim}{x}_p) = \begin{pmatrix} \left[v_C - E_1(t)\right]\sin\left[v_C - E_1(t)\right] - \text{sgn}(v_C)\ln\left(1+|v_C|\right) + i_{L_1} \\ -v_C + e^{-\left(i_{L_1}+i_{L_2}\right)^2} \\ e^{-\left(i_{L_1}+i_{L_2}\right)^2} - i_{L_2} - E_2(t) \end{pmatrix} \quad (57b)$$

Now, (49) is true because $v_C \approx q_C$ for large $q_C$. Let $|E_1|$ and $|E_2|$ be the largest magnitude of bounded $E_1(t)$ and $E_2(t)$, respectively. Then (50) is satisfied, with $k_1 = k_2 = 0$, and

$$\underset{\sim}{G}_p = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \hat{\underset{\sim}{y}}_p = \begin{pmatrix} |E_1| \\ 1 \\ 1 + |E_2| \end{pmatrix} \quad (58)$$

Thus there are no finite escape-time solutions. Note that all the resistors are active; in fact resistor $R_2$ and $R_4$ have v-i curves lying solely in the second and fourth quadrants of the v-i plane. However, each resistor is "not more active" than a $-1\Omega$ resistor in the sense that $v_R i_R \geq -(v_R)^2$ for each resistor.

In the next theorem, we relax the condition (49) on the function $\underset{\sim}{h}_p$, and in turn place a stronger condition than (50) on $\underset{\sim}{g}_p$:

<u>Theorem 2</u>: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the state equation (9a). Assume the capacitor-inductor function $\underset{\sim}{h}_p$ is a $C^1$-state function, and there exists a $C^2$-function $H_p$: $\mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$. Assume $\underset{\sim}{h}_p$ and $H_p$ satisfy

$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty$$
$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty \quad (59)$$

Under these conditions, if there exist constants $k_1 \geq 0$ and $k_2 \geq 0$ such that for all $\|x_p\| > k_2$, for all $u_S \in \mathbb{R}^{n_S}$

$$x_p^T g_p(x_p, u_S) \geq -k_1 \tag{60}$$

Then (9a) has no finite escape-time solutions. That is, for any continuous $u_S(t)$ and for any initial time $t_0 \in \mathbb{R}^1$, each solution $z_p(t)$ of (9a) exists for all $t \geq t_0$.

Remark: If $h_p$ satisfies (49), then (59) follows from Corollary A. In fact, in this case Theorem 2 is a corollary of Theorem 1. Equation (59) also follows if $h_p$ is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ -- see Theorem A.

Proof: As in Theorem 1, we apply Theorem B-2 and let $\mathcal{V}(z_p) = H(z_p)$. Then, (30) follows from (59). To show (31),

$$\frac{\partial H_p(z_p)}{\partial z_p} g_p\left(h_p(z_p), u_S\right) = x_p^T g_p(x_p, u_S) \geq -k_1, \ \forall \ \|x_p\| > k_2, \ \forall \ u_S \in \mathbb{R}^{n_S} \tag{61}$$

Now, from the first equation of (59), there exists $k_3 > 0$ so that $[\|z_p\| > k_3] \Rightarrow [\|x_p\| > k_2]$. Hence, define $\psi(u) \triangleq k_1$ for all $u > 0$; $\psi$: $\mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and equations (29) and (31) are satisfied. ∎

In the remaining theorems of this paper, we assume $\mathcal{N}$ is autonomous. That is, $\mathcal{N}$ has only constant independent voltage and current sources, and is described by the state equation (9b). For results concerning nonautonomous networks, see [15]. Since $\mathcal{N}$ is time-invariant, we can assume without loss of generality that the network is initialized at time $t = 0$.

Theorem 3: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the state equation (9b). Assume the capacitor-inductor function $\underset{\sim}{h}_p$ is a $C^1$-state function, and there exists a $C^2$-function $H_p$: $\mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$. Assume $\underset{\sim}{h}_p$ and $H_p$ satisfy

$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty$$

$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty \tag{62}$$

Under these conditions:

1. If the $C^1$-function $\underset{\sim}{g}_p$ is eventually passive, then every solution $\underset{\sim}{z}_p(t)$ of (9b) is bounded.

2. If the $C^1$-function $\underset{\sim}{g}_p$ is eventually strictly passive, then every solution $\underset{\sim}{z}_p(t)$ of (9b) is eventually uniformly bounded, and $\mathcal{N}$ has at least one equilibrium point. In particular, if for some $k_0 \geq 0$, for all $\|\underset{\sim}{x}_p\| > k_0$,

$$\underset{\sim}{x}_p^T \underset{\sim}{g}_p(\underset{\sim}{x}_p) > 0 \tag{63}$$

then there exists a constant $\hat{k}_0 > 0$ such that

$$[\|\underset{\sim}{z}_p\| > \hat{k}_0] \Rightarrow [\|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| > k_0] \tag{64a}$$

and a constant $k_1 \in \mathbb{R}^1$ where

$$k_1 \triangleq \sup_{\|\underset{\sim}{z}_p\| \leq \hat{k}_0} H_p(\underset{\sim}{z}_p) \tag{64b}$$

and a <u>compact set</u> $Z_p \subseteq \mathbb{R}^{n_p}$, where

$$Z_p = \{\underset{\sim}{z}_p \in \mathbb{R}^{n_p}: \; H_p(\underset{\sim}{z}_p) \leq k_1\} \tag{65}$$

such that for each solution $\underset{\sim}{z}_p(t)$ of (9b), there exists $t_0 \geq 0$ such that

$$\underset{\sim}{z}_p(t) \in Z_p \qquad \forall\, t \geq t_0 \tag{66}$$

Furthermore, $\mathcal{N}$ has an equilibrium point $\hat{\underset{\sim}{z}}_p \in Z_p$.

Remark: The functions $\underset{\sim}{h}_p$ and $H_p$ satisfy (62) if either (i) there exist $k > 0$, $\overline{\gamma} \geq \underline{\gamma} > 0$ such that (14) is true, or if (ii) $\underset{\sim}{h}_p$ is a $C^1$-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ (Theorem A) and, in this latter case, compact $Z_p \subseteq \mathbb{R}^{n_p}$ in (65) is also convex.

Proof: We apply Theorem B-1. Pick $\mathcal{V}(\underset{\sim}{z}_p) = H(\underset{\sim}{z}_p)$, and (24) is satisfied by hypothesis. Now, to show (25) and (26), first note that from (62) we see that for any $k_0 \geq 0$ there exists $\hat{k}_0 \geq 0$ such that (64a) is satisfied. Now

$$\frac{\partial H(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p}\, \underset{\sim}{g}_p\!\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right) = \underset{\sim}{x}_p^T \underset{\sim}{g}_p(\underset{\sim}{x}_p) \tag{67}$$

If $\underset{\sim}{g}_p$ is eventually passive, there exists $k_0 \geq 0$, and hence a $\hat{k}_0 \geq 0$ such that the right side of (67) is non-negative for all $\|x_p\| > k_0$, for all $\|\underset{\sim}{z}_p\| > \hat{k}_0$. Similarly, if $\underset{\sim}{g}_p$ is eventually strictly passive, and (63) is true, then the right side of (67) is positive for all $\|\underset{\sim}{z}_p\| > \hat{k}_0$. ∎

Remark: The difference between the conclusions that solutions of (9b) are bounded, and that solutions of (9b) are eventually uniformly bounded is non-trivial: Examine the two networks of Fig. 5. For the network of Fig. 5a,

$$\underset{\sim}{g}_p = \begin{pmatrix} v_C + E \\ -i_L \end{pmatrix} \tag{68a}$$

The function $\underset{\sim}{g}_p$ is passive, and all solutions are bounded. However, the magnitude of each solution can be arbitrarily large. When two resistors are added as in Fig. 5b,

$$\underset{\sim}{g}_p = \begin{pmatrix} R_2 i_L + v_C + E \\ -i_L + \frac{1}{R_1}(v_C+E) \end{pmatrix} \tag{68b}$$

The function $\underset{\sim}{g}_p$ is eventually strictly passive, and all solutions are eventually uniformly bounded. In fact, as we shall see in Theorem 6, the network of Fig. 5b has a globally asymptotically stable equilibrium point.

Example 2: Let us return to the Wien Bridge Oscillator of Fig. 1 and equation (1). It was stated in the Introduction that with f(·) as shown in Fig. 1c, that the "saturation" characteristic of f(•) stabilizes the voltage and current waveforms of the network. Let us examine the precise condition under which this intuitively reasonable statement is valid.

Claim: If

$$\lim_{k\to\infty} \sup_{|v|>k} \left|\frac{f(v)}{v}\right| < 2 \tag{69}$$

then, all solutions are eventually uniformly bounded.

Remark: A sufficient condition for (69) is

$$\lim_{|v|\to\infty} \frac{f(v)}{v} = 0 \tag{70}$$

This is satisfied by the function f(·) in Fig. 1c. A much more arbitrary f(·) will also satisfy (70), such as

$$f(v) = \sin\left[v \ln(1+|v|)\right] + e^{-v^2} \tag{71}$$

Note that f(·) can be completely arbitrary for finite v.

Proof of Claim: We may write (1) in the form (9b), where

$$\underset{\sim}{h}_p\begin{pmatrix} q_{C_1} \\ q_{C_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{C_1R}\, q_{C_1} \\ \frac{1}{C_2R}\, q_{C_2} \end{pmatrix}$$

$$\underset{\sim}{g}_p\begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} = \begin{pmatrix} v_{C_1} + v_{C_2} - f(v_{C_2}) \\ v_{C_1} + 2v_{C_2} - f(v_{C_2}) \end{pmatrix} \tag{72}$$

The function $h_p$ is strongly uniformly increasing. We have only to show that $\underset{\sim}{g}_p$ is eventually strictly passive.

$$(v_{C_1}, v_{C_2})\underset{\sim}{g}_p\begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} = \left[v_{C_1} + v_{C_2} - \frac{f(v_{C_2})}{2}\right]^2 + \left[1 - \frac{1}{4}\left(\frac{f(v_{C_2})}{v_{C_2}}\right)^2\right](v_{C_2})^2 \tag{73}$$

Applying (70), there exists $\hat{k} > 0$ such that the second term of (73) is positive for $|v_{C_2}| > \hat{k}$. For $|v_{C_2}| \leq \hat{k}$, the second term is bounded, and the first term becomes arbitrarily large, positive as $|v_{C_1}| \to +\infty$. Hence, $\underset{\sim}{g}_p$ is eventually strictly passive. ∎

We next examine conditions placed upon the resistors of $\mathcal{N}$ such that $\underset{\sim}{g}_p$ has the appropriate properties of Theorem 3. First, we note that even if each resistor function $\underset{\sim}{g}_{R_\alpha}(\cdot)$ of $\mathcal{N}$ is eventually (strictly) passive, the composite resistor function $\underset{\sim}{g}_R(\cdot)$ may not be eventually (strictly) passive. This fact is illustrated by the two resistor v - i curves of Fig. 6; assume $\mathcal{N}$ is a 2-port made up of the two disconnected resistors of Fig. 6. Resistor $R_1$ is eventually strictly passive, while $R_2$ is strictly passive. Yet $\underset{\sim}{g}_R = \begin{pmatrix} g_{R_1} \\ g_{R_2} \end{pmatrix}$ is not eventually strictly passive. To show this, fix $v_{R_1} = 3/2$, then

$$\underset{\sim}{v}_R^T \underset{\sim}{g}_R(\underset{\sim}{v}_R) = \begin{cases} -9/4 + (v_{R_2})^2 & \forall\ |v_{R_2}| \leq 1 \\ -9/4 + \dfrac{1}{(v_{R_2})^2} & \forall\ |v_{R_2}| > 1 \end{cases}$$

$$< 0 \qquad \forall\ v_{R_2} \in R^1 \tag{74}$$

The reason that $\underset{\sim}{g}_R$ is not eventually strictly passive is because while $R_2$ is strictly passive, $|v_{R_2} i_{R_2}| \leq 1$ for all $v_{R_2}$. It is shown in [14] that if $\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} [\underset{\sim}{x}_{R_\alpha}]^T [\underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha})] = +\infty$ for each $\alpha = 1,2,\ldots,m_R$, then indeed $g_R$ is eventually strictly passive. However, with a condition of this form, it is no longer possible to prescribe an eventually passive $g_R$ that is not eventually strictly passive. Hence, in the following theorem, we prove only that $\underset{\sim}{g}_p$ is eventually strictly passive as in (ii) of Theorem 3.

Theorem 4: Assume the dynamic nonlinear network is described by the state equation (9b). Assume the capacitor-inductor $\underset{\sim}{h}_p$ is a $C^1$-state function, and there exists a $C^2$-function $H_p$: $\mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$. Assume $\underset{\sim}{h}_p$ and $H_p$ satisfy

$$\begin{aligned} &\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty \\ &\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty \end{aligned} \tag{75}$$

Assume further there is no loop and no cutset formed exclusively by capacitors and/or inductors. Then under these conditions, we have:

1. If $\mathcal{N}$ contains no independent sources, and each resistor function $\underset{\sim}{g}_{R_\alpha}$ is eventually strictly passive, satisfying

$$\lim_{\|\tilde{x}_{R_\alpha}\|\to\infty} (\tilde{x}_{R_\alpha})^T g_{R_\alpha}(\tilde{x}_{R_\alpha}) = +\infty \tag{76}$$

Then all voltage and current waveforms are eventually uniformly bounded, and $\mathcal{N}$ has at least one equilibrium point.

2. If $\mathcal{N}$ has constant independent sources such that there is no loop (resp., cutset) formed exclusively by capacitors and voltage sources (resp., inductors and current sources), and if each resistor function $g_{R_\alpha}$ is eventually strictly passive, satisfying

$$\lim_{\|\tilde{x}_{R_\alpha}\|\to\infty} \frac{1}{\|\tilde{x}_{R_\alpha}\|}(\tilde{x}_{R_\alpha})^T g_{R_\alpha}(\tilde{x}_{R_\alpha}) = +\infty \tag{77}$$

then all voltage and current waveforms are eventually uniformly bounded, and $\mathcal{N}$ has at least one equilibrium point.

Remarks: 1. In Theorem A and in Corollary A we show that (75) is true if either $\tilde{h}_p$ is a $C^1$-strictly increasing diffeomorphic state function, or if an equation of the form (14) is true. Similarly, when $\tilde{g}_{R_\alpha}$ has either of these properties, (76) and (77) are true.

2. By the conclusion that the voltage and current waveforms are eventually uniformly bounded, we mean the following: as in Theorem B-1 or Theorem 3, if $\begin{pmatrix} \tilde{v} \\ \tilde{i} \end{pmatrix} \in \mathbb{R}^{2(n_R+n_p+n_S)}$ denotes the voltage and current of every element of $\mathcal{N}$, there exists a compact set $X \subseteq \mathbb{R}^{2(n_R+n_p+n_S)}$ such that for each waveform $\begin{pmatrix} \tilde{v}(t) \\ \tilde{i}(t) \end{pmatrix}$ there exists $t_0 \geq 0$ so that $\begin{pmatrix} \tilde{v}(t) \\ \tilde{i}(t) \end{pmatrix} \in X$ for all $t \geq t_0$.

Proof: Applying Theorem 3, we have only to show in 1. and 2. above that $\tilde{g}_p$ is eventually strictly passive. This is proved in Theorems 8 and 9 of [14]. ∎

Example 3: Transistor Networks

A transistor may be modeled as a grounded two-port resistor using the Ebers-Moll equation [7]. Let $i_E$ and $v_E$ be the current and voltage respectively of the emitter-base junction, and let $i_C$ and $v_C$ be the current and voltage respectively of the collector-base junction. The resistive two-port is described by its constitutive relation:

$$\begin{pmatrix} i_E \\ i_C \end{pmatrix} = \underset{\sim}{g}_{tr}\begin{pmatrix} v_E \\ v_C \end{pmatrix} \triangleq \begin{bmatrix} 1 & -\alpha_R \\ -\alpha_F & 1 \end{bmatrix}\begin{pmatrix} I_{ES}(e^{v_E/v_T}-1) \\ I_{CS}(e^{v_C/v_T}-1) \end{pmatrix} \tag{78}$$

where the subscript "tr" denotes transistor. In (78), $I_{ES}$, $I_{CS}$, $\alpha_R$, $v_T$, and $\alpha_F$ are positive constants, and furthermore $\alpha_R < 1$, $\alpha_F < 1$, and $\alpha_R I_{CS} = \alpha_F I_{ES}$. Now, it can easily be shown that

$$\begin{pmatrix} v_E \\ v_C \end{pmatrix}^T \underset{\sim}{g}_{tr}\begin{pmatrix} v_E \\ v_C \end{pmatrix} = (1-\alpha_F)I_{ES}v_E(e^{v_E/v_T}-1) + (1-\alpha_R)I_{CS}v_C(e^{v_C/v_T}-1)$$

$$+ \alpha_R I_{CS}(v_E-v_C)\left(e^{v_E/v_T}-e^{v_C/v_T}\right) \tag{79}$$

and from this we can conclude that $\underset{\sim}{g}_{tr}$ is strictly passive and satisfies (76). However, (77) is not satisfied (to see this, in (79) set $v_E = v_C$ and let $v_E \to -\infty$). Hence, 2. of Theorem 4 is not directly applicable. However, we may still obtain a useful result when the network contains constant independent sources.

Proposition: Let $\mathcal{N}$ be a network containing capacitors, inductors, transistors, other resistors, and constant independent sources. Assume that the capacitors, inductors and resistors (other than transistors) satisfy the conditions of Theorem 4, 2; specifically, let the capacitor-inductor function $\underset{\sim}{h}_p$ be a state function (and hence there exists a $C^2$-function $H_p: \mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$) such that (75) is satisfied. Each resistor (other than transistors) is described by its constitutive relation $g_{R_\alpha}$ and let (77) be satisfied. Assume $\mathcal{N}$ is characterized by the state equation (9b). Under these conditions, if there is no loop and no cutset formed exclusively by any combination of capacitors, inductors, transistor emitter-base junctions, transistor collector-base junctions and sources, then all voltage and current waveforms of $\mathcal{N}$ are eventually uniformly bounded, and $\mathcal{N}$ has at least one equilibrium point.

Remarks: 1. A useful corollary of this proposition is that if all capacitors, inductors, and resistors (other than transistors) are linear and have positive capacitance, inductance, and resistance, then (75) and (77) follow; thus in this case if state equation (9b) exists and the interconnection condition above is satisfied, then all voltage and current waveforms of $\mathcal{N}$ are eventually uniformly bounded, and $\mathcal{N}$ has an equilibrium point.

2. In [8] a similar conclusion is reached when $\mathcal{N}$ has no external capacitors and inductors. Rather, capacitors exist in as elements of the transistor model. In the above proposition and the result in [8], the voltage and current sources may be time-varying so long as they are continuous and bounded functions of time.

Proof: This proof is a reiteration of material in [14]. Applying Theorem 8 of [14], using the i-shift Theorem and v-shift Theorem, respectively, each current source is placed in parallel with a resistor (other than a transistor) and each voltage source is placed in series with a resistor (other than a transistor). These resistors with sources attached may be viewed as composite resistors where each constitutive relation is $\hat{g}_{R_\alpha}$, $\alpha = 1,2,\ldots m_R$,

$$y_{R_\alpha} = \hat{g}_{R_\alpha}(x_{R_\alpha}) \triangleq g_{R_\alpha}(x_{R_\alpha} + b_\alpha) + c_\alpha \tag{80}$$

where $\underset{\sim}{b}_\alpha$ and $\underset{\sim}{c}_\alpha$ are vectors in $\mathbb{R}^{n_\alpha}$. It is easy to see that since (77) is satisfied for each $g_{R_\alpha}$, it is also satisfied by each $\hat{\underset{\sim}{g}}_{R_\alpha}$. We view $\mathcal{N}$ now as a network containing capacitors, inductors, transistors whose constitutive relations $\underset{\sim}{g}_{tr}$ satisfy (76), and resistors whose constitutive relations $\hat{\underset{\sim}{g}}_{R_\alpha}$ satisfy (76) and (77). The proposition follows from Theorem 4, 1. ⌑

As a final remark, note that the condition in Theorem 4 requiring that the state equation (9b) exists and that $\underset{\sim}{h}_p$ and $\underset{\sim}{g}_p$ be $C^1$-functions is a non-trivial condition; examine the network of Fig. 7a. The resistor is either a current-controlled resistor whose v-i curve is shown in Fig. 7b, or a voltage-controlled resistor whose v-i curve is shown in Fig. 7c. In the former case, $g_p$ does not exist, while in the latter case $g_p$ exists but is not continuous at $v_p = 0$. Both resistors are strictly passive, and applying the methods of Theorem 4 or Theorem 6 below, we might conclude that all voltage and current waveforms are eventually uniformly bounded. But, corresponding to $v_C(0) = 1$ for both networks,

$$\left.\begin{aligned} v_C(t) &= \sqrt{1-2t} \\ i_C(t) &= \frac{1}{\sqrt{1-2t}} \end{aligned}\right\} \quad t \in [0,2) \tag{81}$$

are admissible voltage and current waveforms. Thus, the network has a finite escape-time solution.

## V. Networks Containing a Globally Asymptotically Stable Equilibrium Point

Theorem 5: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the differential equation (9b). Assume the capacitor-inductor function

$h_p$ is a $C^1$-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Under these conditions,

1. If the $C^1$-function $g_p$ is strictly passive with respect to $x_p^* \in \mathbb{R}^{n_p}$, then $z_p^* = h_p^{-1}(x_p^*)$ is the globally asymptotically stable equilibrium point of (9b).

2. If the $C^1$-function $g_p$ is a strictly-increasing homeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, then there exists a unique $x_p^* \in \mathbb{R}^{n_p}$ such that $g_p(x_p^*) = 0$, and $z_p^* = h_p^{-1}(x_p^*)$ is the globally asymptotically stable equilibrium point of (9b).

Proof of 1: Since $h_p$ is bijective, there is a unique $z_p^* \in \mathbb{R}^{n_p}$ for every $x_p^* \in \mathbb{R}^{n_p}$ such that $x_p^* = h_p(z_p^*)$. The function

$$h_p(\cdot) - x_p^* \tag{82}$$

is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. It follows from Theorem A that there exists a $C^2$-function $H_p: \mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(z_p) \equiv h_p(z_p) - x_p^*$, and

$$\begin{aligned} &H_p(z_p^*) = 0 \\ &H_p(z_p) > 0 , \quad \forall\, z_p \neq z_p^* \\ &\lim_{\|z_p\| \to \infty} H_p(z_p) = +\infty \end{aligned} \tag{83}$$

We apply Lyapunov's Theorem where $\mathcal{V}(\cdot) = H_p(\cdot)$ is our Lyapunov function. To show that $z_p^*$ is the globally asymptotically stable equilibrium point, we will show that for any solution $z_p(t) \not\equiv z_p^*$, $\frac{d}{dt} H_p\left(z_p(t)\right) < 0$ for all $t \geq 0$. This is true if, and only if,

$$\frac{\partial H_p(z_p)}{\partial z_p}\, g_p\left(h_p(z_p)\right) > 0 \qquad \forall\, z_p \neq z_p^* \tag{84}$$

To show this,

$$\frac{\partial H_p(z_p)}{\partial z_p} g_p\left(h_p(z_p)\right) = \left[h_p(z_p)-x_p^*\right]^T g_p\left(h_p(z_p)\right) = (x_p-x_p^*)^T g_p(x_p) \quad (85)$$

and the right side of (85) is positive for all $x_p = h_p(z_p) \neq x_p^*$ since $g_p$ is strictly passive with respect to $x_p^*$.

<u>Proof of 2</u>: The function $g_p$ is a homeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, hence there exists a unique $x_p^* \in \mathbb{R}^{n_p}$ such that $g_p(x_p^*) = 0$. Since $g_p$ is strictly increasing, for all $x_p \neq x_p^*$,

$$(x_p-x_p^*)g_p(x_p) = (x_p-x_p^*)^T\left[g_p(x_p)-g_p(x_p^*)\right] > 0 \quad (86)$$

and thus $g_p$ is strictly passive with respect to $x_p^*$. The conclusion follows from 1. above. ∎

<u>Example 4</u>: Let us return to the Wien Bridge Oscillator of Fig. 1 and equation (1). It was stated in the Introduction that with f(·) as shown in Fig. 1c, when $C_1 = C_2$ and $A_v < 3$, an <u>ad hoc</u> "linear" analysis yields the conclusion that $\lim_{t\to\infty} v_{C_1}(t) = \lim_{t\to\infty} v_{C_2}(t) = 0$. Let us now examine the conditions under which this assertion is valid.

<u>Claim</u>: For any $C_1 > 0$ and $C_2 > 0$, if

$$f(0) = 0$$

$$\left|\frac{f(v)}{v}\right| < 2 \ , \quad \forall\, v \neq 0 \quad (87)$$

then $\begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the globally asymptotically stable equilibrium point of (1).

<u>Remark</u>: The condition (87) does not mean that f(·) must be passive or bounded as in Fig. 1c. For example, if

$$f(v) = v\left(\sin v\right) \quad (88a)$$

or

$$f(v) = \ln(1+|v|) \quad (88b)$$

then (87) is still satisfied.

Proof of Claim: The functions $\underset{\sim}{h}_p$ and $\underset{\sim}{g}_p$ for this network are given in (72). The function $\underset{\sim}{h}_p$ is a $C^1$-uniformly-increasing diffeomorphic state function mapping $\mathbb{R}^2$ onto $\mathbb{R}^2$. So, to apply Theorem 5, we have only to show that $\underset{\sim}{g}_p$ is strictly passive. Applying condition (87) to the right side of (73), we see that, indeed, $\underset{\sim}{g}_p$ is strictly passive. ∎

Using other analytic techniques we can also establish that the network may have oscillations;

Claim: Assume $C_1 = C_2 > 0$. If

$$f(0) = 0 \quad (89a)$$

$$\frac{df(0)}{dv} > 3 \quad (89b)$$

$$\lim_{k\to\infty} \sup_{|v|>k} \left|\frac{f(v)}{v}\right| < 2 \quad (89c)$$

then there is a non-constant periodic solution of (1).

Remark: The function f satisfying (89) need not be passive or bounded as in Fig. 1c. Indeed, (89) places conditions on f only at $v = 0$ and $v = \pm\infty$. For all other v, f may be arbitrary. For example, (89) remains valid with

$$f(v) = e^{-v^2}(-v^3+4v) + \frac{3}{2}\sin v\left[\ln\ (1+v^2)\right] \quad (90)$$

Proof of Claim: We apply the following special case of the Poincare-Bendixon Theorem [23]; the differential equation (1) has a non-constant periodic solution if (i) there is exactly one equilibrium point and it is unstable, and if (ii) all solutions are eventually uniformly bounded.

We have shown previously that when equation (89c) (this is also equation (69)) is satisfied, then all solutions are eventually uniformly bounded. Next, $\begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point of (1). It is the only equilibrium point since $[\dot{v}_{C_1} = \dot{v}_{C_2} = 0] \Rightarrow [\dot{v}_{C_1} - \dot{v}_{C_2} = 0] \Rightarrow [v_{C_2} = 0] \Rightarrow [v_{C_1} = 0]$. Finally, we use the linear methods mentioned in the introduction to show that the equilibrium point is unstable since $\frac{df(0)}{dv} > 3$. ∎

Theorem 6: Assume in the dynamic nonlinear network $\mathcal{N}$ that the capacitor-inductor function $\underset{\sim}{h}_p$ is a $C^1$-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Assume there is no loop and no cutset formed exclusively by capacitors and/or inductors, except possibly loops formed exclusively by capacitors and cutsets formed exclusively by inductors. Under these conditions,

1. If $\mathcal{N}$ contains no independent sources, and each internal resistor function $\underset{\sim}{g}_{R_\alpha}$ is strictly passive, then when $\mathcal{N}$ is described by the state equation (9b), it has a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^* = \underset{\sim}{h}_p^{-1}(\underset{\sim}{0})$.

2. If $\mathcal{N}$ has constant independent sources, and each internal resistor function $\underset{\sim}{g}_{R_\alpha}$ is a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, then (9b) describing $\mathcal{N}$ exists, and $\mathcal{N}$ has a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^* \in \mathbb{R}^{n_p}$.

Proof: This comes directly from Theorem 5 and [14; Theorems 9 and 11]. ∎

Example 5: In the hypothesis of Theorem 6 we allow loops of capacitors and cutsets of inductors. At first glance, this seems to cause a problem since, for example, if there is a loop of capacitors, then their voltages are linearly dependent, and no function $\underset{\sim}{g}_p$ in (9b) exists. However, Theorem 11 of [14] may be applied to these loops and cutsets. As an illustration of the methods involved, examine the network of Fig. 8a.

Here, it is assumed that each resistor, capacitor and inductor is uncoupled to any other element and has a constitutive relation which is a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^1$ onto $\mathbb{R}^1$. Thus, the conditions of Theorem 6 are satisfied, and the network voltage and current waveforms converge to a globally asymptotically stable equilibrium point. Observe that this conclusion is valid in spite of the fact that the three inductors form a cutset. In the following, we show how the results of [14] are applied so that the conclusion may be reached. Specifically, the network is transformed into an equivalent network having no such cutset of inductors.

By hypothesis, the inductors are flux-controlled as well as current-controlled; hence they are described by

$$\begin{pmatrix} i_{L_1} \\ i_{L_2} \\ i_{L_3} \end{pmatrix} = \underset{\sim}{f}_L \begin{pmatrix} \phi_{L_1} \\ \phi_{L_2} \\ \phi_{L_3} \end{pmatrix} \tag{91}$$

where $\underset{\sim}{f}_L = \underset{\sim}{h}_L^{-1}$ is a $C^1$-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^3$ onto $\mathbb{R}^3$. Applying Theorem 11 of [14], we replace one of the inductors, say $L_3$, with a short circuit, and the other two inductors are replaced by two inductors whose constitutive relation is

$$\begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} = \hat{\underset{\sim}{f}}_L \begin{pmatrix} \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} \triangleq \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \underset{\sim}{f}_L \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} \right) \tag{92}$$

where $\hat{f}_L$ is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^2$ onto $\mathbb{R}^2$. This transformed network has no cutset of inductors, and every voltage and current waveform of the other network elements is unaffected by the transformation. The inductor currents are also identical. Thus, in predicting the behavior of $\mathcal{N}$ in Fig. 8a, we need only study the behavior of the transformed network. This network is shown in Fig. 8b, where the constitutive relations are specified. We shall examine this network again in Section VI.

Let us return to the proof of <u>Theorem 5</u>. Recall that when $g_p$ is strictly passive with respect to $x^* \in \mathbb{R}^{n_p}$, then for any solution $z_p(t) \not\equiv z_p^*$ of (9b), we have $\frac{d}{dt}\mathcal{V}(z_p(t)) < 0$ for all $t \geq 0$, where $\mathcal{V}(z_p) = H_p(z_p)$ is the Lyapunov function. We note that the <u>strict</u> inequality may be relaxed [21]; namely, if $\frac{d}{dt}\mathcal{V}(z_p(t)) \leq 0$ for all $t \geq 0$, and $\frac{d}{dt}\mathcal{V}(z_p(t)) \equiv 0$ if, and only if $z_p(t) \equiv z_p^*$, then $z_p^*$ is the globally asymptotically stable equilibrium point of (9b). The difference between these two conditions on $\frac{d}{dt}\mathcal{V}(z_p(t))$ is rather subtle in application. To see this examine the networks of Fig. 9; in the network of Fig. 9a,

$$g_p = \begin{pmatrix} R_2 i_L + v_C \\ -i_L + \frac{v_C}{R_1} \end{pmatrix} \tag{93}$$

The function $g_p$ is strictly passive, the conditions of <u>Theorem 5</u> are satisfied, and $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the globally asymptotically stable equilibrium point. Now, suppose we replace resistor $R_2$ in Fig. 9a with a short circuit, forming the network of Fig. 9b. Here,

$$g_p = \begin{pmatrix} v_C \\ -i_L + \frac{v_C}{R_1} \end{pmatrix} \tag{94}$$

and $g_p$ is passive, but not strictly passive. Yet, $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is still the globally asymptotically stable equilibrium point of the network. One way to see this is to use the Lyapunov function $\mathcal{V}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \frac{C}{2}(v_C)^2 + \frac{L}{2}(i_L)^2$. Then, for any solution $\begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix}$ of the network,

$$\frac{d}{dt}\mathcal{V}\begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} = -\frac{1}{R_2}(v_C)^2 \tag{95a}$$

and

$$\left[\frac{d}{dt}\mathcal{V}\begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} \equiv 0\right] \Rightarrow [v_C(t) \equiv 0] \Rightarrow [i_C(t) \equiv i_R(t) \equiv 0] \Rightarrow [i_L(t) \equiv 0] \tag{95b}$$

There are a large number of networks such as that of Fig. 9b for which globally asymptotically stable equilibrium points may be shown using the methods above. There are other networks, of course, where this is not true. For example, examine the network of Fig. 9c. Here, there is no globally asymptotically stable equilibrium point. This is because, for any $\beta \in \mathbb{R}$,

$$\begin{pmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{pmatrix} = \begin{pmatrix} \beta \sin \omega t \\ -\beta \sin \omega t \end{pmatrix} \tag{96}$$

where $\omega = 1/\sqrt{LC}$, is a solution. Observe that the capacitor-inductor loop in Fig. 9b violates the hypothesis of Theorem 6. In Theorem 7 below, we use a different interconnection hypothesis which allows loops and cutsets such as that in Fig. 9b, but which does not allow those such as in Fig. 9c. The hypothesis is:

Inductor-Capacitor Loop-Cutset Hypothesis (L.C. Hypothesis)

Let the dynamic nonlinear network $\mathcal{N}$ contain capacitors, inductors, resistors and constant sources. The capacitors and inductors are described by $\underset{\sim}{h}_p$ in (4), where $\underset{\sim}{h}_p$ is a $C^1$-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Furthermore,

(i) Each loop (resp., each cutset) formed by an independent source exclusively with capacitors, inductors and other independent sources contains at least one capacitor, at least one inductor, and at least one current source (resp., voltage source).

(ii) Let $\mathcal{S}$ be any set of capacitors and inductors such that any capacitor or inductor in $\mathcal{S}$ forms a loop and/or cutset exclusively with other capacitors and inductors of $\mathcal{S}$. Let one of the following conditions be satisfied:

(a) There is a capacitor $C_j$ in $\mathcal{S}$ which is in a loop formed exclusively with elements of $\mathcal{S}$, but not in a cutset formed exclusively with elements of $\mathcal{S}$. This capacitor is not coupled[10] to any other capacitor of $\mathcal{S}$.

(b) There is an inductor $L_j$ in $\mathcal{S}$ which is in a cutset formed exclusively with elements of $\mathcal{S}$ but not in a loop formed exclusively with elements of $\mathcal{S}$. This inductor is not coupled to any other inductor of $\mathcal{S}$.

Remark: The L. C. Hypothesis is discussed in detail in [14]. It is used in Theorem 12 of [14] which in turn is used to prove:

Theorem 7: Assume the dynamic nonlinear network $\mathcal{N}$ satisfies the L. C. Hypothesis. Assume there is no loop formed exclusively by capacitors and no cutsets formed exclusively by the inductors. Under these conditions,

[10] That is, for any other capacitor $C_k$ in $\mathcal{S}$, $\frac{dv_{C_j}}{dq_{C_k}} \equiv \frac{dv_{C_k}}{dq_{C_j}} \equiv 0$.

1. If $\mathcal{N}$ contains no independent sources, if each internal resistor function $g_{R_\alpha}$ is strictly passive, if $\mathcal{N}$ is described by state equation (9b), and if each voltage and current waveform of $\mathcal{N}$ is a $C^1$-function of time,[11] then $\mathcal{N}$ has a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^* = \underset{\sim}{h}^{-1}(\underset{\sim}{0})$.

2. If $\mathcal{N}$ has constant independent sources, if $\underset{\sim}{h}_p$ is a $C^3$-function in $\mathbb{R}^{n_p}$ and if each resistor function $\underset{\sim}{g}_{R_\alpha}$ is a $C^3$-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, then the state equation (9b) describing $\mathcal{N}$ exists, and $\mathcal{N}$ has a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^* \in \mathbb{R}^{n_p}$.

Remark: We cannot allow loops of capacitors or cutsets of inductors as in Theorem 6. This is because if there exists, say, a cutset of inductors, and $\mathcal{N}$ is transformed as in Example 5 to eliminate the cutset, then the L. C. Hypothesis may no longer be satisfied.

Proof: First, the state equation (9b) describing $\mathcal{N}$ exists, and all voltage and current waveforms of $\mathcal{N}$ are $C^1$-functions of time. This is true by hypothesis in 1. above. In 2., since $\underset{\sim}{h}_p$ and each $g_{R_\alpha}$ are $C^3$-functions, the conclusion follows from a corollary of theorems in [2]. It suffices to prove 1., since 2. follows in a similar way.

Let $\mathcal{V}(\cdot) \triangleq H_p(\cdot)$, where $H_p$ is given in (84), and $\underset{\sim}{z}_p^* = \underset{\sim}{h}^{-1}(\underset{\sim}{0})$. As in Theorem 5, $\mathcal{V}(\cdot)$ satisfies the appropriate conditions of Lyapunov's Theorem. Then, for any solution $\underset{\sim}{z}_p(t)$ of (9b), using (85) (remember $\underset{\sim}{x}^* = \underset{\sim}{0}$),

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[11] The condition that each voltage and current waveform is a $C^1$-function of time is used in Theorem 12 of [14] to show that, for example, for any capacitor charge waveform $q_C(t)$ and capacitor current waveform $i_C(t)$, we have $i_C(t) \equiv \frac{d}{dt} q_C(t)$.

$$\frac{d}{dt}\mathcal{V}(\underset{\sim}{z}_p(t)) = -\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p(t)\right]^T \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p(t))\right) = -\underset{\sim}{x}_p^T(t)\underset{\sim}{g}_p\left(\underset{\sim}{x}_p(t)\right) = -\underset{\sim}{v}_R^T(t)\underset{\sim}{i}_R(t) \tag{97}$$

where the last equality comes from Tellegen's Theorem. Now, since each resistor is strictly passive, then the right side of (97) is positive at any time $t \geq 0$ unless $\underset{\sim}{v}_R(t) = \underset{\sim}{i}_R(t) = 0$. Now, from Theorem 12 of [14], $\underset{\sim}{v}_R(t) = \underset{\sim}{i}_R(t) = 0$ for all $t \geq 0$; that is, if, and only if, $\underset{\sim}{z}_p(t) = \underset{\sim}{z}_p^* = \underset{\sim}{h}_p^{-1}(\underset{\sim}{0})$ for all $t \geq 0$. ∎

As a final remark concerning the existence of a globally asymptotically stable equilibrium point of $\mathcal{N}$, we note that in these results we can extend the condition that the sources are constant to allow the sources to be asymptotically constant. That is, when $\mathcal{N}$ is described by state equation (9a) where $\lim_{t\to\infty} \underset{\sim}{u}(t) = \hat{\underset{\sim}{u}} \in \mathbb{R}^{n_s}$, and $\underset{\sim}{g}_p(\cdot,\hat{\underset{\sim}{u}})$ has the properties possessed by $\underset{\sim}{g}_p(\cdot)$ in the previous theorems, then the conclusion holds. This is proved in [15].

## VI. Exponential Decay of Transients to the Globally Asymptotically Stable Equilibrium Point

We return to Theorems 5 and 6 which give conditions such that $\mathcal{N}$ has a globally asymptotically stable equilibrium point. In this section, we show that under slightly stronger conditions the transients decay in an exponential way to the equilibrium point. We use Theorem B-3 to show this, but first we make the following observations: If $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, then in any compact convex set $D \subseteq \mathbb{R}^{n_p}$, $\underset{\sim}{h}_p$ is strongly uniformly increasing. This is because (see [19] or [14]) the eigenvalues of $\frac{\partial \underset{\sim}{h}_p}{\partial \underset{\sim}{z}_p}(\underset{\sim}{z}_p)$ are always real and positive, and they attain a maximum and

and minimum on any compact set. Furthermore, Theorem A-6 applies in $D \subseteq \mathbb{R}^1$, and there exists a $C^2$-function $H_p$: $D \to \mathbb{R}^1$ such that (20) and (21) are true in D. In a similar way, if $\underset{\sim}{g}_p$ is a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, then $\underset{\sim}{g}_p$ is strongly uniformly increasing on any compact convex set $D_0 \subseteq \mathbb{R}^{n_p}$. Also, if $\underset{\sim}{g}_p$ is a $C^1$-strictly passive function with respect to $\underset{\sim}{x}_p^* \in \mathbb{R}^{n_p}$, and $\frac{\partial \underset{\sim}{g}_p(\underset{\sim}{x}_p^*)}{\partial \underset{\sim}{x}_p}$ is positive-definite, then in any compact connected set $D_0 \subseteq \mathbb{R}^{n_p}$, $\underset{\sim}{x}_p^* \in D_0$, $\underset{\sim}{g}_p$ is "strongly uniformly passive" with respect to $\underset{\sim}{x}_p^*$ in $D_0$. That is, an equation of the form (14) is true where $\underset{\sim}{x}_p'' = \underset{\sim}{x}_p^*$ and $\underset{\sim}{x}_p'$ is arbitrary in $D_0$.

Theorem 8: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the state equation (9b). Assume the capacitor-inductor function $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Assume the $C^1$-function $\underset{\sim}{g}_p$ is strictly passive with respect to $\underset{\sim}{x}_p^*$, and $\frac{\partial \underset{\sim}{g}_p(\underset{\sim}{x}_p^*)}{\partial \underset{\sim}{x}_p}$ is positive-definite. Under these conditions, for each solution $\underset{\sim}{z}_p(t)$ of (9b), $\lim_{t\to\infty} \underset{\sim}{z}_p(t) = \underset{\sim}{z}_p^* = \underset{\sim}{h}_p^{-1}(\underset{\sim}{x}_p^*)$. Furthermore, let $D \subseteq \mathbb{R}^{n_p}$ be any convex, compact set such that $\underset{\sim}{z}_p(t) \in D$ for all $t \geq 0$. Then, there exists constants $\bar{\gamma}_h \geq \underline{\gamma}_h > 0$ and $\bar{\gamma}_g \geq \underline{\gamma}_g > 0$ such that for all $\underset{\sim}{z}_p'$, $\underset{\sim}{z}_p'' \in D$, we have the following basic inequalities:

$$\underline{\gamma}_h \|\underset{\sim}{z}_p'-\underset{\sim}{z}_p''\|^2 \leq (\underset{\sim}{z}_p'-\underset{\sim}{z}_p'')^T\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right] \leq \bar{\gamma}_h \|\underset{\sim}{z}_p'-\underset{\sim}{z}_p''\|^2$$

$$\underline{\gamma}_g \|\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{x}_p^*\|^2 \leq \left[\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{x}_p\right]^T \left[\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p')\right)\right] \leq \bar{\gamma}_g \|\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{x}_p^*\|^2 \qquad (98)$$

and, for each $t \geq 0$,

$$\left[\frac{\underline{\gamma}_h}{\bar{\gamma}_h}\right]^{1/2} e^{-t/\tau_{min}} \|\underset{\sim}{z}_p(0)-\underset{\sim}{z}_p^*\| \leq \|\underset{\sim}{z}_p(t)-\underset{\sim}{z}_p^*\| \leq \left[\frac{\bar{\gamma}_h}{\underline{\gamma}_h}\right]^{1/2} e^{-t/\tau_{max}} \|\underset{\sim}{z}_p(0)-\underset{\sim}{z}_p^*\| \tag{99a}$$

where

$$\tau_{min} \triangleq \frac{\underline{\gamma}_h}{(\bar{\gamma}_h)^2\bar{\gamma}_g} \quad \text{and} \quad \tau_{max} \triangleq \frac{\bar{\gamma}_h}{(\underline{\gamma}_h)^2\underline{\gamma}_g} \tag{99b}$$

Remarks: 1. If $\underset{\sim}{g}_p$ is a $C^1$-strictly-increasing diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, then, as in Theorem 5, there exists $\underset{\sim}{x}_p^* \in \mathbb{R}^{n_p}$ such such that $\underset{\sim}{g}_p$ is strictly passive with respect to $\underset{\sim}{x}_p^*$, and $\frac{\partial \underset{\sim}{g}_p(\underset{\sim}{x}_p^*)}{\partial \underset{\sim}{x}_p}$ is positive-definite. Thus, Theorem 8 is an extension of both 1. and 2. of Theorem 5.

2. We cannot extend Theorem 7 which uses the L. C. Hypothesis in a similar way. This is because $\underset{\sim}{g}_p$ is passive, not strictly passive in Theorem 7, and so no equation of the form (98) is possible for $\underset{\sim}{g}_p$.

3. Equation (99) describes the transient decay of the capacitor charge and inductor flux linkage. It is useful to have a similar expression for $\underset{\sim}{x}(t)$ which is the capacitor voltage and inductor current. Applying Theorem A-6 to (99), we have

$$\left[\frac{\underline{\gamma}_h}{\bar{\gamma}_h}\right]^{3/2} e^{-t/\tau_{min}} \|\underset{\sim}{x}_p(0)-\underset{\sim}{x}_p^*\| \leq \|\underset{\sim}{x}_p(t)-\underset{\sim}{x}_p^*\| \leq \left[\frac{\bar{\gamma}_h}{\underline{\gamma}_h}\right]^{3/2} e^{-t/\tau_{max}} \|\underset{\sim}{x}_p(0)-\underset{\sim}{x}_p^*\| \tag{100}$$

where $\tau_{max}$ and $\tau_{min}$ are given in (99b).

Proof: Equation (98) is valid in convex, compact $D \subseteq \mathbb{R}^{n_p}$ as discussed above. We apply Theorem A-6 to the function $\underset{\sim}{h}_p$; equation (20) holds, and for some $C^2$-function $H_p: \mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv h_p(\underset{\sim}{z}_p)$,

(21) is true. Let $H_p(\cdot) = \mathcal{V}(\cdot)$ in Theorem B-3. Then (41) is true with $\beta = 2$, $\gamma_1 = \frac{1}{2}\underline{\gamma}_h$, and $\gamma_2 = \frac{1}{2}\overline{\gamma}_h$. Hence, we only need to show (42). Using (98) and equation (20) of Theorem A-6, we obtain

$$\frac{\partial H_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right) = [\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{x}_p^*]^T \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right) \geq \underline{\gamma}_g \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{x}_p^*\|^2$$
$$\geq \underline{\gamma}_g(\underline{\gamma}_h)^2\|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\|^2 \tag{101a}$$

and

$$\frac{\partial H_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right) \leq \overline{\gamma}_p(\overline{\gamma}_h)^2\|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\|^2 \tag{101b}$$

Thus, (42) is true with $\beta = 2$, $\gamma_3 = \underline{\gamma}_g(\underline{\gamma}_h)^2$ and $\gamma_4 = \overline{\gamma}_g(\overline{\gamma}_h)^2$. Hence, from (43) we obtain (99). ∎

In the remainder of this section we discuss, present, and illustrate an algorithm for finding the transient decay time constant $\tau_{max}$ used in (99) and (100). Specifically, we will find $\overline{\gamma}_h \geq \underline{\gamma}_h > 0$, and $\underline{\gamma}_g > 0$ of (98). We will derive these constants without forming the state equation (9b), and without solving for the equilibrium point $\underset{\sim}{x}_p^* = \underset{\sim}{h}_p(\underset{\sim}{z}_p^*)$.

Preliminary Remarks: 1. Comparing Theorems 8 and 6, we conclude that (99) and (100) exist for networks satisfying either 1. or 2. of Theorem 6. That is, if each $\underset{\sim}{g}_{R_\alpha}$ is strictly passive and $\frac{\partial \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{0})}{\partial \underset{\sim}{x}_{R_\alpha}}$ is positive-definite, or if each $\underset{\sim}{g}_{R_\alpha}$ is a $C^1$-strictly-increasing diffeomorphism mapping $\mathbb{R}^{n}$ onto $\mathbb{R}^{n_{R_\alpha}}$, then we may apply Theorem 8 to derive (99) and (100). The algorithm below is directed towards networks containing the latter type of resistors. This is the more general case in that the equilibrium point $\underset{\sim}{x}_p^*$ is arbitrary, and $\mathcal{N}$ can contain independent sources. The algorithm is easily adapted to networks containing strictly passive resistors.

2. We will find $\bar{\gamma}_h$, $\underline{\gamma}_h$, and $\underline{\gamma}_g$ in (98), but we shall not derive $\bar{\gamma}_g$. The reasons why $\bar{\gamma}_g$ is not found will be given in Remark 2 immediately preceeding the algorithm. This means that we can obtain only the right half of the inequalities (99) and (100). These inequalities are useful in that they prescribe a "worse case time constant" $\tau_{max}$ for the network.

3. The following fact is proved in [14] and [19]: Let $D \subseteq \mathbb{R}^n$ be convex, and $\underset{\sim}{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ is $C^1$. Then $\underset{\sim}{f}$ is uniformly increasing on D, and in particular there exists a constant $\gamma > 0$ such that (13) is true in D if, and only if, the constant $\gamma$ also satisfies

$$0 < \gamma \leq \inf_{\underset{\sim}{x} \in D} \left[ \text{min. eigenvalue of } \frac{1}{2}\left( \frac{\partial \underset{\sim}{f}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} + \frac{\partial \underset{\sim}{f}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \right)^T \right] \tag{102}$$

There is a similar result for strongly uniformly increasing functions. We also have the following extension of the definition of a uniformly increasing function: Let $D \subseteq \mathbb{R}^n$ be convex and let the mapping $\underset{\sim}{x} \mapsto y$ be a scalar $C^1$-function from D into $\mathbb{R}^1$. We say that y is a <u>uniformly increasing function of $x_1$ uniformly in the remaining independent variables</u> (which are $x_2,\ldots,x_n$) if, and only if, there exists $\gamma > 0$ such that for every $\underset{\sim}{x}'$, $\underset{\sim}{x}'' \in D$, and corresponding $y'$, $y'' \in \mathbb{R}^1$, we have

$$(x_1'-x_1'')(y'-y'') \geq \gamma(x_1'-x_1'')^2 \tag{103a}$$

Furthermore, using the above result, this is true if, and only if, $\gamma$ also satisfies

$$0 < \gamma \leq \inf_{\underset{\sim}{x} \in D} \frac{\partial y}{\partial x_1} \tag{103b}$$

The following assumptions are essentially those stated in 2. of Theorem 6. They are made slightly stronger than in Theorem 6 so that $\bar{\gamma}_h$, $\underline{\gamma}_h$ and $\underline{\gamma}_g$ are the same for every compact $D \subseteq \mathbb{R}^{n_p}$ in (98).

Algorithm Assumptions: We assume that the dynamic nonlinear network $\mathcal{N}$ satisfies the hypothesis of 2. of Theorem 6. We assume that Theorem 11 of [14] has been applied if necessary so that $\mathcal{N}$ has no loops formed exclusively by capacitors, and no cutsets formed exclusively by inductors. In addition, we assume the following conditions on the elements characteristics:

1. The function $\tilde{h}_p$ is a strongly uniformly-increasing function in $\mathbb{R}^{n_p}$.

2. Each capacitor forms a loop exclusively with resistor branches of $\mathcal{N}$ and voltage sources. When resistor branch j is in such a loop, its current $i_{R_j}$ is a uniformly-increasing function of its voltage $v_{R_j}$ uniformly in all other resistor variables.

3. Each inductor forms a cutset exclusively with resistor branches of $\mathcal{N}$ and current sources. When resistor branch j is in such a cutset, its voltage $v_{R_j}$ is a uniformly-increasing function of its current $i_{R_j}$ uniformly in all other resistor variables.

Remark: In Algorithm Assumption 2, the only additional assumption beyond that of Theorem 6 is that $i_{R_j}$ is a uniformly-increasing function of $v_{R_j}$; indeed it follows from Theorems 2 and 8 of [14] and the hypothesis of Theorem 6 that such a loop always exists for each capacitor and that $i_{R_j}$ is a strictly-increasing function of $v_{R_j}$. A dual observation applies to Algorithm Assumption 3.

It is assumed that the capacitor-inductor function $\underset{\sim}{h}_p$ is given *a priori*. Thus (see 102)), we can define

$$\begin{aligned} \bar{\gamma}_h &\triangleq \sup_{\underset{\sim}{z}_p \in \mathbb{R}^{n_p}} \left[ \text{max. eigenvalue } \frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \right] \\ \underline{\gamma}_h &\triangleq \inf_{\underset{\sim}{z}_p \in \mathbb{R}^{n_p}} \left[ \text{min. eigenvalue } \frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p} \right] \end{aligned} \tag{104}$$

For example, if each capacitor and inductor is linear, strictly passive, and uncoupled, we have

$$\begin{aligned} \bar{\gamma}_h &= \max\left[ \max_{j=1,\ldots,n_C} \frac{1}{C_j}, \max_{j=1,\ldots,n_L} \frac{1}{L_j} \right] \\ \underline{\gamma}_h &= \min\left[ \min_{j=1,\ldots n_C} \frac{1}{C_j}, \min_{j=1,\ldots,n_L} \frac{1}{L_j} \right] \end{aligned} \tag{105}$$

On the other hand, the $n_p$-port function $\underset{\sim}{g}_p$ is not known *a priori*, and we want to find $\underline{\gamma}_g$ simply by using the internal resistor functions $\underset{\sim}{g}_{R_\alpha}$, $\alpha = 1,2,\ldots,m_R$.

## Analytical Methods Used to Derive $\underline{\gamma}_g$

For each capacitor (resp., inductor) let us form the loop (resp., cutset) as prescribed by Algorithm Assumption 2 (resp., 3). From KVL (resp., KCL), when $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ denotes the constant voltage and current sources, we obtain the equation (see [14; Theorem 2b])

$$\underset{\sim}{x}_p = \underset{\sim}{P}_0 \begin{pmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_R \end{pmatrix} + \underset{\sim}{P}_1 \underset{\sim}{u}_S \tag{106}$$

where the matrices $\underset{\sim}{P}_0 \in \mathbb{R}^{n_p \times 2n_R}$ and $\underset{\sim}{P}_1 \in \mathbb{R}^{n_p \times n_S}$ contain elements +1, -1 and 0, and every row of $\underset{\sim}{P}_0$ has a non-zero element. We partition the resistor branches into four mutually exclusive sets:

Set R0; resistor branch j is in Set R0 if, and only if, the columns of $\underset{\sim}{P}_0$ corresponding to its voltage $v_{R_j}$ and its current $i_{R_j}$ have zero elements.

Set R1; resistor branch j is in Set R1 if, and only if, the column of $\underset{\sim}{P}_0$ corresponding to its voltage $v_{R_j}$ has a non-zero element while the column corresponding to its current $i_{R_j}$ has all zero elements.

Set R2; resistor branch j is in Set R2 if, and only if, the column of $\underset{\sim}{P}_0$ corresponding to its voltage $v_{R_j}$ has all zero elements while the column corresponding to its current $i_{R_j}$ has a non-zero element.

Set R3; resistor branch j is in Set R3 if, and only if, the columns of $\underset{\sim}{P}_0$ corresponding to its voltage $v_{R_j}$ and current $i_{R_j}$ have non-zero elements.

Remark: We may define these four sets in the following equivalent way: Corresponding to the loops and cutsets represented by the linear equation (106), resistor branch j is in Set R0 if, and only if, it does not form a loop exclusively with capacitors and voltage sources and does not form a cutset exclusively with inductors and current sources. Resistor branch j is in Set R1 (resp., R2) if it is in such a loop (resp., cutset) but not in such a cutset (resp., loop). Resistor branch j is in Set R3 if, and only if, it forms a loop exclusively with capacitors and voltage sources, and it also forms a cutset exclusively with inductors and current sources.

Assume Set R1 contains $n_{R1}$ resistor branches. Let $\underset{\sim}{v}_{R1} \in \mathbb{R}^{n_{R1}}$ and $\underset{\sim}{i}_{R1} \in \mathbb{R}^{n_{R1}}$ be the resistor branch voltages and currents respectively. Define $\underset{\sim}{x}_{R1} \in \mathbb{R}^{n_{R1}}$ and $\underset{\sim}{y}_{R1} \in \mathbb{R}^{n_{R1}}$ by

$$\begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{y}_{R1} \end{pmatrix} \triangleq \begin{pmatrix} \underset{\sim}{v}_{R1} \\ \underset{\sim}{i}_{R1} \end{pmatrix} \tag{107}$$

For each branch j in Set R1, if it is part of an $n_\alpha$-port resistor, then by assumption, $x_{R1_j} = v_{R1_j}$ is one of the independent resistor port variables of the $n_\alpha$-port resistor. Denote the remaining $n_\alpha$-1 independent resistor port variables by $\hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}$. Define

$$\gamma_{R1_j} \triangleq \inf_{\substack{v_{R1_j} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}}} \frac{\partial i_{R1_j}}{\partial v_{R1_j}} \tag{108}$$

Because of <u>Algorithm Assumption 2</u>, we know $\gamma_R^j > 0$ for all $j = 1,\ldots,n_{R1}$.

Assume Set R2 contains $n_{R2}$ resistor branches. Let $\underset{\sim}{v}_{R2}$ $\mathbb{R}^{n_{R2}}$ and $\underset{\sim}{i}_{R2} \in \mathbb{R}^{n_{R2}}$ be the resistor branch voltages and currents respectively. Define $\underset{\sim}{x}_{R2} \in \mathbb{R}^{n_{R2}}$ and $\underset{\sim}{y}_{R2} \in \mathbb{R}^{n_{R2}}$ by

$$\begin{pmatrix} \underset{\sim}{x}_{R2} \\ \underset{\sim}{y}_{R2} \end{pmatrix} \triangleq \begin{pmatrix} \underset{\sim}{i}_{R2} \\ \underset{\sim}{v}_{R2} \end{pmatrix} \tag{109}$$

For each resistor branch j in Set R2, if it is part of an $n_\alpha$-port resistor, then by assumption, $x_{R2_j} = i_{R2_j}$ is one of the independent resistor variables. Let $\hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}$ denote the remaining $n_\alpha$-1 independent resistor port variables. Define

$$\gamma_{R2_j} \triangleq \inf_{\substack{i_{R2_j} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}}} \frac{\partial v_{R2_j}}{\partial i_{R2_j}} \tag{110}$$

Because of <u>Algorithm Assumption 3</u>, we know $\gamma_{R2_j} > 0$ for all $j = 1,..,n_{R2}$.

Assume Set R3 contains $n_{R3}$ resistor branches. Let $\underset{\sim}{v}_{R3} \in \mathbb{R}^{n_{R3}}$ and $\underset{\sim}{i}_{R3} \in \mathbb{R}^{n_{R3}}$ be the resistor branch voltages and currents respectively.

Define $\underset{\sim}{x}_{R3} \in \mathbb{R}^{2n_{R3}}$ and $\underset{\sim}{y}_{R3} \in \mathbb{R}^{2n_{R3}}$ by [12]

$$\underset{\sim}{x}_{R3} \triangleq \begin{pmatrix} \underset{\sim}{v}_{R3} \\ \underset{\sim}{i}_{R3} \end{pmatrix}$$

$$\underset{\sim}{y}_{R3} \triangleq \frac{1}{2}\begin{pmatrix} \underset{\sim}{i}_{R3} \\ \underset{\sim}{v}_{R3} \end{pmatrix} \tag{111}$$

For each resistor branch j in Set R3, if it is part of an $n_\alpha$-port resistor, then by assumption,

(i) $v_{R3_j}$ can be one of the $n_\alpha$ independent resistor port variables. Denote the remaining independent resistor port variables by $\hat{\underset{\sim}{x}}_{R(v)} \in \mathbb{R}^{n_\alpha - 1}$.

(ii) $i_{R3_j}$ can be one of the $n_\alpha$ independent resistor port variables. Denote the remaining independent resistor port variables by $\hat{\underset{\sim}{x}}_{R(i)} \in \mathbb{R}^{n_\alpha - 1}$.

Define

$$\gamma_{R3_j} \triangleq \frac{1}{2} \min \left[ \inf_{\substack{v_{R3_j} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_{R(v)} \in \mathbb{R}^{n_\alpha-1}}} \frac{\partial i_{R3_j}}{\partial v_{R3_j}}, \; \inf_{\substack{i_{R3_j} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_{R(i)} \in \mathbb{R}^{n_\alpha-1}}} \frac{\partial v_{R3_j}}{\partial i_{R3_j}} \right] \tag{112}$$

Because of <u>Algorithm Assumptions 2 and 3</u>, we know $\gamma_{R3_j} > 0$ for all

[12] Because resistor branch j in R4 is in <u>both</u> a loop and cutset represented in matrix $\underset{\sim}{P}_0$ in (106), it is necessary to view branch j as <u>both</u> a voltage-controlled branch and as a current-controlled branch. Thus, in (111) $\underset{\sim}{v}_{R3}$ and $\underset{\sim}{i}_{R3}$ are part of the independent variable $\underset{\sim}{x}_{R3}$ and also part of the dependent variable $\underset{\sim}{y}_{R3}$. The fraction $\frac{1}{2}$ appears so that $(\underset{\sim}{x}_{R3})^T(\underset{\sim}{y}_{R3}) \equiv (\underset{\sim}{v}_{R3})^T(\underset{\sim}{i}_{R3})$. In the same way, <u>both</u> $v_{R3_j}$ and $i_{R3_j}$ must be treated as independent resistor variables in (i) and (ii) which immediately follow equation (111).

$j = 1,..,n_{R3}$. Define $\underline{\gamma}_R > 0$[13]

$$\underline{\gamma}_R \triangleq \min \{\min_{j=1,..n_{R1}} \gamma_{R1_j}, \min_{j=1,..,n_{R2}} \gamma_{R2_j}, \min_{j=1,..n_{R3}} \gamma_{R3_j}\} \tag{113}$$

Claim: For

$$\underline{\gamma}_g \triangleq \underline{\gamma}_R \quad \frac{1}{\|\underset{\sim}{P}_0\|^2} \tag{114}$$

the right inequality of (98) is satisfied.

Proof of Claim: Let $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ denote the voltage source voltages and current source currents, and let $\underset{\sim}{w}_S \in \mathbb{R}^{n_S}$ denote the voltage source currents and current source voltages. It follows from applying Tellegen's Theorem (see Theorem 9 of [14]) that for every $\underset{\sim}{x}'_p, \underset{\sim}{x}''_p \in \mathbb{R}^{n_p}$, we have

$$\begin{aligned}(\underset{\sim}{x}'_p-\underset{\sim}{x}''_p)^T\left[\underset{\sim}{g}_p(\underset{\sim}{x}'_p)-\underset{\sim}{g}_p(\underset{\sim}{x}''_p)\right] &= (\underset{\sim}{v}'_R-\underset{\sim}{v}''_R)^T(\underset{\sim}{i}'_R-\underset{\sim}{i}''_R) + (\underset{\sim}{u}_S-\underset{\sim}{u}_S)^T(\underset{\sim}{w}_S-\underset{\sim}{w}_S) \\ &= (\underset{\sim}{v}'_R-\underset{\sim}{v}''_R)^T(\underset{\sim}{i}'_R-\underset{\sim}{i}''_R)\end{aligned} \tag{115}$$

Now, since every $\underset{\sim}{g}_{R_\alpha}$ is strictly increasing,

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[13] When the resistors are two-terminal elements, the expressions for $\gamma_{R1_j}$ in (108), $\gamma_{R2_j}$ in (110), and $\gamma_{R3_j}$ in (112) can be simplified considerably since there is no longer an independent variable for $\hat{\underset{\sim}{x}}_R$. Moreover, when the two-terminal resistors are linear and strictly passive, then (113) reduces to:

$$\underline{\gamma}_R = \min\{\min_{j=1,..,n_{R1}} \frac{1}{R_{R1_j}}, \min_{j=1,..,n_{R2}} R_{R2_j}, \frac{1}{2}\min_{j=1,..,n_{R3}} (\frac{1}{R_{R3_j}} + R_{R3_j})\}$$

where $R_{Rk_j}$ is the resistance of the jth resistor in set Rk, k = 1,2, and 3.

$$(\underset{\sim}{v}_R'-\underset{\sim}{v}_R'')^T(\underset{\sim}{i}_R'-\underset{\sim}{i}_R'') \geq (\underset{\sim}{x}_{R1}'-\underset{\sim}{x}_{R1}'')^T(\underset{\sim}{y}_{R1}'-\underset{\sim}{y}_{R1}'') + (\underset{\sim}{x}_{R2}'-\underset{\sim}{x}_{R2}'')^T(\underset{\sim}{y}_{R2}'-\underset{\sim}{y}_{R2}'') + (\underset{\sim}{x}_{R3}'-\underset{\sim}{x}_{R3}'')^T(\underset{\sim}{y}_{R3}'-\underset{\sim}{y}_{R3}'') \quad (116)$$

where the equality sign in (116) is attained if, and only if, Set R0 is an empty set. Now, as discussed in Preliminary Remark 3, since the constant $\gamma > 0$ in (13) can be the <u>same constant</u> $\gamma > 0$ in (103), it follows from the definition of $\underset{\sim}{\gamma}_R$ in (113) that

$$(\underset{\sim}{x}_{R1}'-\underset{\sim}{x}_{R1}'')^T(\underset{\sim}{y}_{R1}'-\underset{\sim}{y}_{R1}'') + (\underset{\sim}{x}_{R2}'-\underset{\sim}{x}_{R2}'')^T(\underset{\sim}{y}_{R2}'-\underset{\sim}{y}_{R2}'') + (\underset{\sim}{x}_{R3}'-\underset{\sim}{x}_{R3}'')^T(\underset{\sim}{y}_{R3}'-\underset{\sim}{y}_{R3}'')$$

$$\geq \underset{\sim}{\gamma}_R \left\| \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}' - \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}'' \right\|^2 \quad (117)$$

Finally, by deleting the all-zero columns of $\underset{\sim}{P}_0$ in (106) (these correspond to the resistor branches in Set R0) and reordering the remaining columns, we obtain the reduced matrix $\hat{\underset{\sim}{P}}_0 \in \mathbb{R}^{n_p \times (n_{R1}+n_{R2}+2n_{R3})}$ and the reduced equation

$$\underset{\sim}{x}_p = \hat{\underset{\sim}{P}}_0 \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix} + \underset{\sim}{P}_1 \underset{\sim}{u}_S \quad (118a)$$

and thus

$$\|\underset{\sim}{x}_p'-\underset{\sim}{x}_p''\| \leq \|\underset{\sim}{P}_0\| \left\| \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}' - \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}'' \right\| \quad (118b)$$

Furthermore, since the square of the induced norm $\|\underset{\sim}{P}_0\|^2$ = (max. eigenvalue of $\underset{\sim}{P}_0^T\underset{\sim}{P}_0$) = (max. eigenvalue of $\hat{\underset{\sim}{P}}_0^T\hat{\underset{\sim}{P}}_0$) = $\|\hat{\underset{\sim}{P}}_0\|^2$, we can combine (115), (116), (117), and (118) to obtain the inequality

$$(\underset{\sim}{x}_p'-\underset{\sim}{x}_p'')^T\left[\underset{\sim}{g}_p(\underset{\sim}{x}_p')-\underset{\sim}{g}_p(\underset{\sim}{x}_p'')\right] = (\underset{\sim}{v}_R'-\underset{\sim}{v}_R'')^T(\underset{\sim}{i}_R'-\underset{\sim}{i}_R'') \geq \underline{\gamma}_R \left\| \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}' - \begin{pmatrix} \underset{\sim}{x}_{R1} \\ \underset{\sim}{x}_{R2} \\ \underset{\sim}{x}_{R3} \end{pmatrix}'' \right\|^2$$

$$\geq \frac{\underline{\gamma}_R}{\|\underset{\sim}{P}_0\|^2} \|\underset{\sim}{x}_p'-\underset{\sim}{x}_p''\|^2 \qquad \blacksquare \qquad (119)$$

Remarks: 1. The classification of resistor branches into Sets R0, R1, R2, and R3 is not necessarily unique, and thus $\underline{\gamma}_g$ is not necessarily unique. For example, a capacitor may form two distinct loops exclusively with resistors and voltage sources as prescribed by Algorithm Assumption 2. This means both $\underline{\gamma}_R$, and $\underset{\sim}{P}_0$ may be different. At this point, we do not have an algorithm to find the optimal partitioning of the resistors.

2. We will now show why it is difficult to find $\overline{\gamma}_g$ in (98). While we may find $\overline{\gamma}_R$ similar to $\underline{\gamma}_R$ of (113), there is no easy way to derive function $\underset{\sim}{f}_N(\cdot)$ such that

$$\begin{pmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_R \end{pmatrix} = \underset{\sim}{f}_N(\underset{\sim}{x}_p) \qquad (120)$$

which is the inverse of (106). While such an equation may exist (see Theorem 4 of [14]), we cannot derive it simply by using KVL and KCL. Without such a function, we cannot write an equation similar to (118) (with the reversed inequality sign) and $\overline{\gamma}_g$ cannot be found.

3. As our induced matrix norm, we have chosen $\|\underset{\sim}{P}_0\| \triangleq$ [max. eigenvalue $\underset{\sim}{P}_0^T\underset{\sim}{P}_0]^{1/2}$. This is the best (that is, the smallest) expression for the induced norm when $\|\cdot\|$ is the Euclidean norm [24]. However, since all elements of $\underset{\sim}{P}_0$ are +1, -1 and 0, it is computationally easier to choose the equivalent but more conservative induced norm

$\|P_0\| \overset{\Delta}{=} [\text{number of non-zero elements in } P_0]^{1/2}$.

4. In the following algorithm, the procedure described above is used, although Sets R1, R2 and R3 are not overtly formed. Rather, we form $\underline{\gamma}_R$ and $P_0$ sequentially.

<u>Algorithm for Computing the Transient Decay Time Constant $\tau_{max}$</u>

<u>Step 0</u>: Order the $n_C$ capacitors, $n_L$ inductors ($n_C+n_L=n_p$) and $m_R$ resistors. Order the $n_R$ resistor branches. Form the capacitor-inductor function $h_p: \mathbb{R}^{n_p} \to \mathbb{R}^{n_p}$. Form the resistor functions $g_{R_\alpha}: \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$, $\alpha = 1,2,..,m_R$. Set $j = 1$.

<u>Step 1</u>: Set

$$\overline{\gamma}_h = \sup_{z_p \in \mathbb{R}^{n_p}} \left[ \text{max. eigenvalue of } \frac{\partial h_p(z_p)}{\partial z_p} \right]$$

$$\underline{\gamma}_h = \inf_{z_p \in \mathbb{R}^{n_p}} \left[ \text{min. eigenvalue of } \frac{\partial h_p(z_p)}{\partial z_p} \right]$$

<u>Comment</u>: In the remaining steps, we sequentially examine each capacitor and then each inductor, forming the matrix $P_0$ row by row, and sequentially solving for $\underline{\gamma}_R$.

<u>Step 2</u>: If $j = n_C + 1$, set $j = 1$, go to <u>Step 7</u>. Otherwise, find "loop $\mathcal{L}_j$" consisting of capacitor $C_j$, resistors and voltage sources as prescribed by <u>Algorithm Assumption 2</u>. Augment the $(j-1) \times 2n_R$ matrix $P_0$ with a row of zeroes; namely row $j$. Set $k = 1$.

<u>Step 3</u>: If $k = n_R + 1$, set $j = j + 1$ and go to <u>Step 2</u>.

Otherwise, if resistor branch $k$ is not in loop $\mathcal{L}_j$, set $k = k + 1$ and go to <u>Step 3</u>.

Otherwise, resistor branch $k$ is in loop $\mathcal{L}_j$.

Step 4: If resistor branch k is similarly directed with capacitor $C_j$ in loop $\mathcal{L}_j$, set the jkth element of $\underset{\sim}{P}_0$ (this is the element in the jth row and kth column of $\underset{\sim}{P}_0$) to +1.

Otherwise, set the jkth element of $\underset{\sim}{P}_0$ to -1.

Step 5: If the kth-column of $\underset{\sim}{P}_0$ has a non-zero entry in some row other than row j, set k = k + 1 and go to Step 3.

Otherwise, find resistor $R_\alpha$, $\alpha = 1,2,..,m_R$ such that resistor $R_\alpha$ contains resistor branch k. Invoking the Algorithm Assumption 2, $v_{R_k}$ is a possible independent port variable of the $n_\alpha$-port resistor $R_\alpha$. Let $\hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}$ be the remaining independent port variables. Set

$$\underline{\gamma} = \inf_{\substack{v_{R_k} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha - 1}}} \frac{\partial i_{R_k}}{\partial v_{R_k}}$$

Step 6: If j = 1, set k = k + 1, set

$$\underline{\gamma}_R = \underline{\gamma}$$

and go to Step 3.

Otherwise, set k = k + 1, set

$$\underline{\gamma}_R = \min(\underline{\gamma}, \underline{\gamma}_R)$$

and go to Step 3

Comment: Except for Step 11, the following steps dealing with inductors are dual of those dealing with capacitors.

Step 7: If $j = n_L + 1$, go to Step 13

Otherwise, find "cutset $\mathcal{C}_j$" consisting of inductor $L_j$, resistors, and current sources as prescribed by Algorithm Assumption 3. Augment the $(j-1+n_C) \times 2n_R$ matrix $\underset{\sim}{P}_0$ with a row of zeroes; namely row $(j+n_C)$. Set $k = 1$.

Step 8: If $k = n_R + 1$, set $j = j + 1$ and go to Step 7.

Otherwise, if resistor branch k is not in cutset $\mathcal{C}_j$, set $k = k + 1$ and go to Step 8.

Otherwise, resistor branch k is in cutset $\mathcal{C}_j$.

Step 9: If resistor branch k is similarly directed with inductor $L_j$ in $C_j$, set the $(n_C+j, n_R+k)$th element of $\underset{\sim}{P}_0$ to +1.

Otherwise, set the $(n_C+j, n_R+k)$th element of $\underset{\sim}{P}_0$ to -1.

Step 10: If the $(n_R+k)$th-column of $\underset{\sim}{P}_0$ has a non-zero entry in some row other than row $n_C + j$, set $k = k + 1$ and go to Step 8.

Otherwise, if the kth column of $\underset{\sim}{P}_0$ also has a non-zero element, go to Step 11.

Otherwise, the kth-column of $\underset{\sim}{P}_0$ has all zero elements, and the $(n_R+k)$th column of $\underset{\sim}{P}_0$ has all zero elements except in row j. Find resistor $R_\alpha$, $\alpha = 1,2,..,m_R$ such that resistor $R_\alpha$ contains resistor branch k. Invoking Algorithm Assumption 3, $i_{R_k}$ is a possible independent port variable of the $n_\alpha$-port resistor $R_\alpha$. Let $\hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha-1}$ be the remaining independent port variables. Set

$$\underline{\gamma} = \inf_{\substack{i_{R_k} \in \mathbb{R}^1 \\ \hat{\underset{\sim}{x}}_R \in \mathbb{R}^{n_\alpha-1}}} \frac{\partial v_{R_k}}{\partial i_{R_k}}$$

and go to Step 12.

Step 11: Find resistor $R_\alpha$, $\alpha = 1,2,..,m_R$ such that resistor $R_\alpha$ contains resistor branch k. Invoking both Algorithm Assumptions 2 and 3, both $v_{R_k}$ and $i_{R_k}$ can be possible independent port variables of the $n_\alpha$-port resistor. Let $\hat{x}_{R(v)} \in \mathbb{R}^{n_\alpha - 1}$ and $\hat{x}_{R(i)} \in \mathbb{R}^{n_\alpha - 1}$ be the respective remaining independent port variables in each case. Set

$$\gamma = \frac{1}{2} \min \left[ \inf_{\substack{v_{R_k} \in \mathbb{R}^1 \\ \hat{x}_{R(v)} \in \mathbb{R}^{n_\alpha - 1}}} \frac{\partial i_{R_k}}{\partial v_{R_k}}, \quad \inf_{\substack{i_{R_k} \in \mathbb{R}^1 \\ \hat{x}_{R(i)} \in \mathbb{R}^{n_\alpha - 1}}} \frac{\partial v_{R_k}}{\partial i_{R_k}} \right]$$

Step 12: If $j = 1$, $n_C = 0$, set $k = k + 1$, set

$$\underline{\gamma}_R = \gamma$$

and go to Step 7.

Otherwise, set $k = k + 1$, set

$$\underline{\gamma}_R = \min(\underline{\gamma}_R, \underline{\gamma})$$

and go to Step 7.

Step 13: Set

$$\bar{\lambda} = \text{max. eigenvalue } P_0^T P_0$$

$$\underline{\gamma}_g = \frac{1}{\bar{\lambda}} \underline{\gamma}_R$$

$$\tau_{max} = \frac{\bar{\gamma}_h}{(\underline{\gamma}_h)^2 \underline{\gamma}_g}$$

Stop

Example 6: Linear Time-Invariant Networks

The preceding algorithm is obviously applicable to networks containing linear time-invariant strictly passive elements. In the following examples we compare the estimated time constant $\tau_{max}$ derived using our algorithm with the actual maximum time constants of linear networks. Let us consider first the network shown in Fig. 10a. The network state equation is

$$\begin{pmatrix} \dot{v}_C \\ \dot{i}_L \end{pmatrix} = -\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{pmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}\right] \tag{121a}$$

The exact solution is given by:

$$\begin{aligned} \begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} &= v_C(0) \begin{pmatrix} -.35e^{-1.22t} + 1.35e^{-3.28t} \\ .485e^{-1.22t} + .485e^{-3.28t} \end{pmatrix} \\ &+ i_L(0) \begin{pmatrix} .971e^{-1.22t} - .971e^{-3.28t} \\ 1.35e^{-1.22t} - .35e^{-3.28t} \end{pmatrix} \\ &+ \begin{pmatrix} -.398(e^{-1.22t}-1) + .148(e^{-3.28t}-1) \\ -.553(e^{-1.22t}-1) + .053(e^{-3.28t}-1) \end{pmatrix} \end{aligned} \tag{121b}$$

Let us apply the algorithm. First, we obtain from (105)

$$\overline{\gamma}_h = 2; \quad \underline{\gamma}_h = 1 \tag{122}$$

Next, we find

$$\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \hat{P}_0 \begin{pmatrix} v_{R_3} \\ i_{R_1} \\ i_{R_2} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} v_{R_3} \\ i_{R_1} \\ i_{R_2} \end{pmatrix} \tag{123}$$

and compute

$$\overline{\lambda} \triangleq [\text{max. eigenvalue of } \hat{P}_0^T P_0] = \left[\text{max. eigenvalue of } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right] = 2 \tag{124}$$

and

$$\underline{\gamma}_R = \min[1\Omega,\ 1\Omega,\ 2\Omega] = 1 \tag{125}$$

Hence, we obtain from Step 13

$$\underline{\gamma}_g = 1/2 \quad ; \quad \tau_{max} = 4 \tag{126}$$

Now, since

$$\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} \tag{127}$$

is the globally asymptotically stable equilibrium point, it follows from (100) that for every solution $\left[v_C(t),\ i_L(t)\right]^T$,

$$\left\| \begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} - \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} \right\| \leq \sqrt{8} \left\| \begin{pmatrix} v_C(0) \\ i_L(0) \end{pmatrix} - \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} \right\| e^{-t/4} \tag{128}$$

Comparing $\tau_{max} = 4$ in (126) with the actual maximum time constant $\tau = 1/1.22 = .82$ in (121b), we see that our estimate of $\tau_{max}$ is within a factor of five of the actual time constant. This is an acceptable error.

The network of Fig. 10b is more "stiff" than that of Fig. 10a. The network state equation is given by

$$\begin{pmatrix} \dot{v}_{C_1} \\ \dot{v}_{C_2} \end{pmatrix} = - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \left[ \begin{bmatrix} 2 & -1 \\ -1 & 20 \end{bmatrix} \begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \tag{129}$$

Here, the two time constant are

$$\tau_1 = .256 \quad ; \quad \tau_2 = .025 \tag{130}$$

Using our algorithm, we obtain

$$\overline{\gamma}_h = \underline{\gamma}_h = 2 \quad ; \quad \underline{\gamma}_g = 1 \quad ; \quad \tau_{max} = 1/2 \tag{131}$$

Since

$$\begin{pmatrix} v_{C_1} \\ v_{C_2} \end{pmatrix} = \begin{pmatrix} 20/39 \\ 1/39 \end{pmatrix} \tag{132}$$

is the globally asymptotically stable equilibrium point, then

$$\left\| \begin{pmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{pmatrix} - \begin{pmatrix} 20/39 \\ 1/39 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} v_{C_1}(0) \\ v_{C_2}(0) \end{pmatrix} - \begin{pmatrix} 20/39 \\ 1/39 \end{pmatrix} \right\| e^{-2t} \tag{133}$$

The time constant $\tau_{max} = 1/2$ is within a factor of 2 of the actual maximum time constant in (130).

Example 7: The network of Fig 8b previously discussed in Example 5 satisfies our Algorithm Assumptions. We will compare the expression (99) with the computer-simulated network waveforms. In particular, the waveforms of this network are simulated using the CSMP [25].

The function $\underset{\sim}{h}_p$ is shown in Fig. 8b. We can find $\underset{\sim}{g}_p$,

$$\underset{\sim}{g}_p(\underset{\sim}{x}_p) = \underset{\sim}{g}_p\begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} v_C+(v_C)^3+1/3(v_C+i_{L_1}+i_{L_2}+1) \\ -v_C-1+2/3(v_C+i_{L_1}+i_{L_2}+1)^2 \\ -v_C-1+2/3(v_C+i_{L_1}+i_{L_2}+1)+2\left[\dfrac{i_{L_2}(1+i_{L_2})^2+1}{1+(i_{L_2})^2}\right] \end{pmatrix} \tag{134}$$

The globally asymptotically stable equilibrium point (this may be found by solving (134) or via computer analysis) is given by:

$$\begin{pmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{pmatrix} = \begin{pmatrix} -.1547 \\ 1.0205 \\ -.906 \end{pmatrix} \; ; \; \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} -.3094 \\ 2.098 \\ -1.754 \end{pmatrix} \tag{135}$$

Let us apply the algorithm; first,

$$\frac{dv_C}{dq_C} = 2 + \frac{(q_C)^2\left((q_C)^2+3\right)}{\left((q_C)^2+1\right)^2} \tag{136a}$$

Thus

$$2 \leq \frac{dv_C}{dq_C} \leq 3.125 \tag{136b}$$

The eigenvalues of

$$\begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix} \tag{137a}$$

are

$$\lambda = 2,3$$

and

$$\overline{\gamma}_h = 3.125 \quad ; \quad \underline{\gamma}_h = 2 \tag{138}$$

Next,

$$\begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \end{pmatrix} = \hat{P}_0 \begin{pmatrix} v_{R_2} \\ i_{R_1} \\ i_{R_3} \\ i_{R_4} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_{R_2} \\ i_{R_1} \\ i_{R_3} \\ i_{R_4} \end{pmatrix} \tag{139}$$

and

$$\overline{\lambda} \triangleq \text{max. eigenvalue of } [\hat{P}_0^T \hat{P}_0] = \max[1,0,2+\sqrt{2},\ 2-\sqrt{2}] = 2+\sqrt{2} \tag{140}$$

Finally

$$\underline{\gamma}_R = \min\left[\inf \frac{di_{R_2}}{dv_{R_2}},\ \inf \frac{dv_{R_1}}{di_{R_1}},\ \inf \frac{dv_{R_3}}{di_{R_3}},\ \inf \frac{dv_{R_4}}{di_{R_4}}\right]$$

$$= \min[1,2,1,2 - \frac{3\sqrt{3}}{4}] = 2 - \frac{3\sqrt{3}}{4} \tag{141}$$

Hence

$$\gamma_g = \frac{2-\frac{3\sqrt{3}}{4}}{2+\sqrt{2}} \cong .205 \quad ; \quad \tau_{max} \cong 3.8 \tag{142}$$

So for any solution $\begin{pmatrix} q_C(t) \\ \phi_{L_1}(t) \\ \phi_{L_2}(t) \end{pmatrix}$,

$$\left\| \begin{pmatrix} q_C(t) \\ \phi_{L_1}(t) \\ \phi_{L_2}(t) \end{pmatrix} - \begin{pmatrix} -.1547 \\ 1.0205 \\ -.906 \end{pmatrix} \right\| \leq 1.25 \left\| \begin{pmatrix} q_C(0) \\ \phi_{L_1}(0) \\ \phi_{L_2}(0) \end{pmatrix} - \begin{pmatrix} -.1547 \\ 1.0205 \\ -.906 \end{pmatrix} \right\| e^{-t/3.8} \tag{143}$$

The left and right sides of (143) were simulated using the CSMP and the waveforms displayed on the CRT. The results are shown in Fig. 11. The waveforms in Fig. 11a have as an initial state

$$\begin{pmatrix} q_C(0) \\ \phi_{L_1}(0) \\ \phi_{L_2}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{144}$$

which is close to the globally asymptotically stable equilibrium point given in (139). The upper waveform in Fig. 11a corresponds to the estimated waveforms which is the right side of (143). The lower waveform is the exact waveform given by the left side of (143). Next, we choose the initial state

$$\begin{pmatrix} q_C(0) \\ \phi_{L_1}(0) \\ \phi_{L_2}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{145}$$

which is not close to the equilibrium point. The results are shown in Fig. 11b, where the upper waveform corresponds to that estimated by the right side of (143) and the lower waveform corresponds to the exact waveform given by the left side of (143). The actual waveforms corresponding to the initial state (145) are shown in Fig. 11c. The upper waveform (on the vertical axis) is $q_C(t) - (-.1547)$, the middle waveform is $\phi_{L_2}(t) - (-.906)$ and the lower waveform $\phi_{L_1}(t) - (1.0205)$. In all cases we find that our estimates for transient decay are quite realistic upper bounds.

## VII. Conclusions:

A number of results concerning the qualitative behavior of nonlinear dynamic networks are presented. The hypotheses of these results are of two types: First, very general and practical conditions on the network state equation, and second, conditions upon the individual element constitutive relations and their interconnection. In the latter form, the hypotheses include (in general) the Fundamental Topological Assumption, namely there is no loop and no cutset formed exclusively by capacitors and/or inductors, and the L.C. Hypothesis. These conditions are simple, easy to verify, and therefore quite practical. For example, in [13] Varaiya and Liu develop a result similar in nature to 1. of Theorem 6 where it is required that for any set of network waveforms, $\left[\begin{pmatrix} v_R(t) \\ i_R(t) \end{pmatrix} \equiv 0\right] \Rightarrow [y_p(t) \equiv 0]$. This is precisely the case when either the Fundamental Topological Assumption or the L.C. Hypothesis is satisfied. In the same way, it would not be possible to develop the algorithm implementing Theorem 8 without an equation of the form (118a) which is derived using the Fundamental Topological Assumption [14; Theorem 2]. In [15] we apply these methods to nonautonomous networks. In particular we establish the existence of periodic network waveforms when sources are periodic, and we discuss the existence of a *unique*, steady-state waveform. A result similar to *Theorem 8* is developed where network waveforms converge *exponentially* to the unique, steady-state waveform, and, in this case, the algorithm presented here is directly applicable.

The results developed in this paper may be applied in a useful way to the study of the *structural sensitivity* of nonlinear dynamic networks. That is, we can answer the following question: Let $\mathcal{N}$ be a network. Network $\tilde{\mathcal{N}}$ is formed by altering slightly the constitutive relations of some of the elements of $\tilde{\mathcal{N}}$. Do the network waveforms of $\tilde{\mathcal{N}}$ behave in the same way as the waveforms of $\mathcal{N}$? Equivalently, let $\mathcal{N}$ be a *real* electrical dynamic nonlinear network and let $\tilde{\mathcal{N}}$ be its *mathematical model* used in computer simulation: Will the bahavior of $\mathcal{N}$ be the same as that predicted by the behavior of $\tilde{\mathcal{N}}$? In many cases we may apply the theorems presented here to answer these questions. For example, assume $\mathcal{N}$ is a *transistor network* satisfying the hypotheses of the *Proposition* of *Example 3*. We can form $\tilde{\mathcal{N}}$ by altering slightly the transistors, resistors, capacitor, inductors and sources of $\mathcal{N}$. But so long as each transistor function $\underset{\sim}{g}_{tr}$ still satisfies (76), each resistor function $g_{R_\alpha}$ still satisfies (77), and $\underset{\sim}{h}_p$ still satisfies (75), then the behavior of $\mathcal{N}$ is the same as the behavior of $\mathcal{N}$; namely, all waveforms are eventually uniformly bounded and there exists an equilibrium point.

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APPENDIX

Proof of Theorem A

Proof of A-1 [20]

Case 1: $n = 1$

Let $x' < x'' \in \mathbb{R}^1$. For any $\sigma \in (0,1)$, define $x_\sigma \triangleq (1-\sigma)x' + \sigma x''$.

Since f is strictly increasing,

$$\begin{aligned} F(x_\sigma) - F(x') &\triangleq \int_{x'}^{x_\sigma} f(\tau)d\tau < f(x_\sigma)(x_\sigma - x') \\ F(x'') - F(x_\sigma) &\triangleq \int_{x_\sigma}^{x''} f(\tau)d\tau > f(x_\sigma)(x'' - x_\sigma) \end{aligned} \tag{A-1}$$

Thus

$$\begin{aligned} F(x_\sigma) &< F(x') + f(x_\sigma)(x'' - x')\sigma \\ F(x_\sigma) &< F(x'') - f(x_\sigma)(x'' - x')(1-\sigma) \end{aligned} \tag{A-2}$$

Multiplying the first equation of (A-2) by $(1-\sigma)$, multiplying the second equation by $\sigma$, and adding, we obtain

$$(1-\sigma)F(x_\sigma) + \sigma F(x_\sigma) = F(x_\sigma) < (1-\sigma)F(x') + \sigma F(x'') \tag{A-3}$$

Case 2: $n > 1$

Let $\underset{\sim}{x}' \neq \underset{\sim}{x}'' \in \mathbb{R}^n$. Define $\hat{F} : \mathbb{R}^1 \to \mathbb{R}^1$, $\hat{F}(\sigma) \triangleq F\left((1-\sigma)\underset{\sim}{x}' + \sigma\underset{\sim}{x}''\right)$.

Now, $\hat{F}(0) = \hat{F}(\underset{\sim}{x}')$, $\hat{F}(1) = F(\underset{\sim}{x}'')$, so we have to show that

$$\hat{F}(\sigma) < (1-\sigma)\,\hat{F}(0) + \sigma\hat{F}(1) \qquad \forall\, \sigma \in (0,1) \tag{A-4}$$

To see this, note that

$$\frac{d^2\hat{F}(\sigma)}{d\sigma^2} = (\underset{\sim}{x}''-\underset{\sim}{x}')^T \left[\frac{\partial \underset{\sim}{f}\left((1-\sigma)\underset{\sim}{x}'+\sigma\underset{\sim}{x}''\right)}{\partial \underset{\sim}{x}}\right] (\underset{\sim}{x}''-\underset{\sim}{x}') \tag{A-5}$$

Since $\underset{\sim}{f}$ is a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^n \to \mathbb{R}^n$, the right side of (A-5) is positive [14; Theorem A]. Thus, $\frac{d\hat{F}(\sigma)}{d\sigma}$ is a strictly-increasing function on $\mathbb{R}^1$. It follows from the preceeding Case 1 that (A-4) is true. ∎

Proof of A-2: For any $\underset{\sim}{x} \neq \underset{\sim}{f}^{-1}(\underset{\sim}{0})$, there exists a unique $\underset{\sim}{e} \in \mathbb{R}^n$, $\|\underset{\sim}{e}\| = 1$ and a unique $r > 0$ so that $\underset{\sim}{x} = \underset{\sim}{f}^{-1}(0) + r\,\underset{\sim}{e}$. That is, every $\underset{\sim}{x} \neq \underset{\sim}{f}^{-1}(\underset{\sim}{0})$ may be uniquely represented by a vector emanating from $\underset{\sim}{f}^{-1}(\underset{\sim}{0})$. For each $\underset{\sim}{e} \in \mathbb{R}^n$, $\|\underset{\sim}{e}\| = 1$, we will show that $F(\underset{\sim}{x}) = F\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right)$ is a strictly-increasing function of $r > 0$. So, for $\underset{\sim}{e}$ fixed and $r > 0$, we have

$$\frac{dF(\underset{\sim}{x})}{dr} = \frac{dF\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right)}{dr} = \underset{\sim}{e}^T\,\underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right)$$

$$= \frac{1}{r}\left[\underset{\sim}{f}^{-1}(\underset{\sim}{0}) - \underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right]^T \left[\underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right) - \underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)\right]$$

$$= \frac{1}{r}\left[\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right) - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right]^T \left[\underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + r\,\underset{\sim}{e}\right) - \underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)\right] \quad \text{(A-6)}$$

Since $\underset{\sim}{f}$ is strictly-increasing, the right side of (A-6) is positive and hence F is a strictly-increasing function of $r > 0$. Since $F(0) = F\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0}) + 0\,\underset{\sim}{e}\right)$, it follows that $F(x) > 0$ for all $\underset{\sim}{x} \neq \underset{\sim}{f}^{-1}(\underset{\sim}{0})$. ∎

Proof of A-3: We first show that $\lim_{\|\underset{\sim}{x}\| \to \infty} F(\underset{\sim}{x}) = +\infty$. In particular, we will show that for any $N > 0$ there exists $M > 0$ such that if $\|\underset{\sim}{x} - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\| = r > M$, then $F(\underset{\sim}{x}) > N$.

We have already shown in the proof of Theorem A-2 that $F\left(r\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)$ is a strictly increasing function of r. That is, in equation (A-6) we have shown that $\frac{dF}{dr} > 0$. Next,

$$\frac{d^2F\left(r\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{dr^2} = \underset{\sim}{e}^T\,\frac{\partial \underset{\sim}{f}\left(r\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{\partial \underset{\sim}{x}}\,\underset{\sim}{e} \quad \text{(A-7)}$$

and since $\underset{\sim}{f}$ is a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$, the right side of (A-7) is positive for all $r > 0$, for all $\underset{\sim}{e} \in \mathbb{R}^n$, $\|\underset{\sim}{e}\| = 1$. Define

$$k_f \triangleq \inf_{\substack{r=1 \\ \underset{\sim}{e}\in\mathbb{R}^n,\ \|\underset{\sim}{e}\|=1}} \frac{dF\left(r\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{dr} \tag{A-8}$$

The constant $k_f$ is well-defined since $\frac{dF}{dr}$ is continuous, and the set $\{(r,\underset{\sim}{e}) : r=1,\ \|\underset{\sim}{e}\|=1\}$ is compact. Furthermore, because $\frac{dF}{dr} > 0$ and $\frac{d^2F}{dr^2} > 0$, it is easily shown that $k_f > 0$. Then, for any $\underset{\sim}{x} \in \mathbb{R}^n$ such that $\|\underset{\sim}{x} - f^{-1}(0)\| = r > 1$, we have

$$F(\underset{\sim}{x}) = F\left(r\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right) = \int_0^r \frac{dF\left(\rho\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho}\,d\rho$$

$$= \int_0^1 \frac{dF\left(\rho\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho}\,d\rho + \int_1^r \frac{dF\left(\rho\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho}\,d\rho$$

$$> \int_0^1 \frac{dF\left(\rho\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho}\,d\rho + \int_1^r \frac{dF\left(\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho}\,d\rho$$

$$\geq \int_0^1 \frac{dF\left(\rho\,\underset{\sim}{e} + \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)}{d\rho} + k_f(r-1) \tag{A-9}$$

Now, the first term on the right side of (A-9) is positive. Hence, for any $N > 0$ define $M \triangleq 1 + \frac{N}{k_f}$ . From (A-9) we conclude that if $\|\underset{\sim}{x}-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\| > M$, then $F(\underset{\sim}{x}) > N$.

By a simple extension of this conclusion, for any $\underset{\sim}{b} \in \mathbb{R}^n$, $\underset{\sim}{f}(\cdot) - \underset{\sim}{b}$ is also a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$, and $\nabla\left(F(\underset{\sim}{x}) - \underset{\sim}{x}^T\underset{\sim}{b}\right) \equiv \underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{b}$, thus

$$\lim_{\|\underset{\sim}{x}\| \to \infty} F(\underset{\sim}{x}) - \underset{\sim}{x}^T \underset{\sim}{b} = +\infty \qquad \forall\, \underset{\sim}{b} \in \mathbb{R}^n \tag{A-10}$$

Hence, the proof of this theorem is completed upon proving the following lemma (this lemma is stated but not proved in [19; pp. 110]):

Lemma A: Let H: $\mathbb{R}^n \to \mathbb{R}^1$ and let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^n$. Then

$$\left[\lim_{\|\underset{\sim}{x}\| \to \infty} H(\underset{\sim}{x}) - \underset{\sim}{x}^T \underset{\sim}{b} = +\infty \quad , \quad \forall\, \underset{\sim}{b} \in \mathbb{R}^n\right] \tag{A-11}$$

$$\Longrightarrow \left[\lim_{\|\underset{\sim}{x}\| \to \infty} H(\underset{\sim}{x})/\|\underset{\sim}{x}\| = +\infty\right] \tag{A-12}$$

Proof: We will use (A-11) to show that for every $\gamma > 0$ there exists $\beta > 0$ such that if $\|\underset{\sim}{x}\| > \beta$, then $H(\underset{\sim}{x})/\|\underset{\sim}{x}\| > \gamma$. That is,

$$[\|\underset{\sim}{x}\| > \beta] \Rightarrow [H(\underset{\sim}{x}) - \gamma\|\underset{\sim}{x}\| > 0] \tag{A-13}$$

Partition $\mathbb{R}^n$ into $2^n$ orthants. Orthant 1 is the set of $\underset{\sim}{x} = (x_1, .. x_n)^T \in \mathbb{R}^n$ such that $x_j \geq 0$ for each j = 1,..,n. Orthant 2 is the set of $\underset{\sim}{x} \in \mathbb{R}^n$ such that $x_1 < 0$, and $x_j \geq 0$ for all j = 2,3,..,n. More specifically, for any integer $k \in [1,2^n]$, let $a_n a_{n-1} \cdots a_2 a_1$ be the modulo 2 expression for the integer k - 1. Then, we say that $\underset{\sim}{x}$ is in orthant k (denoted by $\Theta_k$) if, and only if, for each j = 1,..,n,

$$\begin{aligned} x_j &\geq 0 \quad \text{if} \quad a_j = 0 \\ x_j &< 0 \quad \text{if} \quad a_j = 1 \end{aligned} \tag{A-14}$$

Now, suppose for each $k = 1,..,2^n$, for every $\gamma > 0$ there exists $\beta_k > 0$ such that

$$[\|\underset{\sim}{x}\| > \beta_k, \ \underset{\sim}{x} \in \Theta_k] \Rightarrow [H(\underset{\sim}{x}) - \gamma\|\underset{\sim}{x}\| > 0] \tag{A-15}$$

then, for $\beta \triangleq \max_{k=1,..,2^n} \beta_k$, equation (A-13) follows from (A-15). Thus, it suffices in this proof to show (A-15) for each orthant $\Theta_k$.

First, note that $\|\cdot\| : \underset{\sim}{x} \mapsto \sum_{j=1}^{n} |x_j|$ is a norm in $\mathbb{R}^n$, so by the equivalence of norms in $\mathbb{R}^n$ there exists $\alpha > 0$ such that $\|\underset{\sim}{x}\| \leq \alpha \sum_{j=1}^{n} |x_j|$ for all $\underset{\sim}{x} \in \mathbb{R}^n$. Next, for each orthant $\Theta_k$, define vector $\underset{\sim}{b} = (b_1,..b_n)^T \in \mathbb{R}^n$ in the following way: for any $\underset{\sim}{x} \in \Theta_k$, for each $j = 1,..,n$, define

$$\begin{aligned} b_j &= \gamma\alpha \quad \text{if} \quad x_j \geq 0 \\ b_j &= -\gamma\alpha \quad \text{if} \quad x_j < 0 \end{aligned} \tag{A-16}$$

Hence, by construction

$$\underset{\sim}{x}^T \underset{\sim}{b} = \gamma\alpha \sum_{j=1}^{n} |x_j| \geq \gamma \|\underset{\sim}{x}\| \qquad \forall\ \underset{\sim}{x} \in \Theta_k \tag{A-17}$$

We now apply the hypothesis (A-11), which we restate in the following manner: for any $\underset{\sim}{b} \in \mathbb{R}^n$ there exists $\beta_b > 0$ such that

$$[\|\underset{\sim}{x}\| > \beta_b] \Rightarrow [H(\underset{\sim}{x}) - \underset{\sim}{x}^T \underset{\sim}{b} > 0] \tag{A-18}$$

For the vector $\underset{\sim}{b}$ defined in (A-16), using (A-17) and the constant $\beta_b$ in (A-18), we obtain

$$H(\underset{\sim}{x}) - \gamma\|\underset{\sim}{x}\| \geq H(\underset{\sim}{x}) - \underset{\sim}{x}^T \underset{\sim}{b} > 0 \qquad \forall\ \|\underset{\sim}{x}\| > \beta_b,\ \underset{\sim}{x} \in \Theta_k \tag{A-19}$$

and for $\beta_k \triangleq \beta_b$, we have shown (A-15). Thus, Lemma A and Theorem A-3 are proved. ∎

Proof of A-4: First, note that for all $\underset{\sim}{x} \neq \underset{\sim}{0}$,

$$\frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}^T\left(\underset{\sim}{f}(\underset{\sim}{x})\right) \geq \frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}^T\left(\underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{f}(\underset{\sim}{0})\right) - \|\underset{\sim}{f}(\underset{\sim}{0})\| \tag{A-20}$$

Thus, it suffices to show that

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x}) = +\infty \tag{A-21}$$

assuming $\underset{\sim}{f}(\underset{\sim}{0}) = \underset{\sim}{0}$. Now, we have shown already that $\underset{\sim}{e}^T\underset{\sim}{f}\left(r\underset{\sim}{e}+\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right) = \underset{\sim}{e}^T\underset{\sim}{f}(r\underset{\sim}{e})$ is a strictly increasing function of $r > 0$. This means that $\underset{\sim}{e}^T\underset{\sim}{f}(r\underset{\sim}{e})$ is larger than $\frac{1}{r}\int_0^r \underset{\sim}{e}^T\underset{\sim}{f}(\rho\underset{\sim}{e})d\rho$ which is the average value of $\underset{\sim}{e}^T\underset{\sim}{f}(\rho\underset{\sim}{e})$ as $\rho$ varies from 0 to r. Then, for any $\underset{\sim}{x} \neq \underset{\sim}{0}$, we obtain the following inequality:

$$\frac{1}{\|\underset{\sim}{x}\|}\underset{\sim}{x}^T\underset{\sim}{f}(\underset{\sim}{x}) = \frac{1}{\|r\underset{\sim}{e}\|}(r\underset{\sim}{e})^T\underset{\sim}{f}(r\underset{\sim}{e}) = \underset{\sim}{e}^T\underset{\sim}{f}(r\underset{\sim}{e}) > \frac{1}{r}\int_0^r \underset{\sim}{e}^T\underset{\sim}{f}(\rho\underset{\sim}{e})d\rho = \frac{1}{r}\int_0^r \frac{dF(\rho\underset{\sim}{e})}{d\rho}\,d\rho$$

$$= \frac{1}{\|\underset{\sim}{x}\|}F(\underset{\sim}{x}) \tag{A-22}$$

Applying Theorem A-3, equation (A-21) follows from equation (A-22). ∎

Proof of A-5: The set K is non-empty since $\underset{\sim}{f}^{-1}(\underset{\sim}{0}) \in K$. It is closed since F is continuous. It is bounded because of (17). It remains to show that K is convex. For any $\sigma \in (0,1)$ and for each $\underset{\sim}{x}', \underset{\sim}{x}'' \in K$, it follows from the strict convexity of F that

$$F\left((1-\sigma)\underset{\sim}{x}'+\sigma\underset{\sim}{x}''\right) < (1-\sigma)F(\underset{\sim}{x}') + \sigma F(\underset{\sim}{x}'') \leq (1-\sigma)k + \sigma k = k \quad ∎ \tag{A-23}$$

Proof of A-6: Using the constants $\overline{\gamma} \geq \underline{\gamma} > 0$ of (14), for any $\underset{\sim}{x}' \neq \underset{\sim}{x}''$,

$$\|\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')\| = \frac{1}{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|}\|\underset{\sim}{x}'-\underset{\sim}{x}''\|\cdot\|\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')\|$$

$$\geq \frac{1}{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|}(\underset{\sim}{x}'-\underset{\sim}{x}'')^T\left(\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right) \geq \frac{1}{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|}\underline{\gamma}\|\underset{\sim}{x}' - \underset{\sim}{x}''\|^2 \tag{A-24}$$

and the right inequality of (20) follows directly. To show the left inequality, let us first observe that since the state function $\underset{\sim}{f}$ is strongly uniformly increasing, the inverse function $\underset{\sim}{f}^{-1}$ is also strongly uniformly increasing, and it can be shown that

$$\frac{1}{\underline{\gamma}} \|\underset{\sim}{y}' - \underset{\sim}{y}''\|^2 = (\underset{\sim}{y}'-\underset{\sim}{y}'')^T\left(\underset{\sim}{f}^{-1}(\underset{\sim}{y}')-\underset{\sim}{f}^{-1}(\underset{\sim}{y}'')\right) \geq \frac{1}{\overline{\gamma}} \|\underset{\sim}{y}' - \underset{\sim}{y}''\|^2 \qquad \text{(A-25)}$$

for each $\underset{\sim}{y}', \underset{\sim}{y}'' \in \mathbb{R}^n$. Equation (A-25) comes from the fact that the eigenvalues of the symmetric matrix $\frac{\partial \underset{\sim}{f}(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ are positive, real and lie between $\underline{\gamma}$ and $\overline{\gamma}$ [14]. Hence, the eigenvalues of $\frac{\partial \underset{\sim}{f}^{-1}(\underset{\sim}{y})}{\partial \underset{\sim}{y}}$ are positive, real and lie between $\frac{1}{\overline{\gamma}}$ and $\frac{1}{\underline{\gamma}}$.

Then, proceeding as in (A-15), we obtain:

$$\|\underset{\sim}{x}' - \underset{\sim}{x}''\| = \|\underset{\sim}{f}^{-1}(\underset{\sim}{y}') - \underset{\sim}{f}^{-1}(\underset{\sim}{y}'')\| \geq \frac{1}{\overline{\gamma}}\|\underset{\sim}{y}' - \underset{\sim}{y}''\| = \frac{1}{\overline{\gamma}}\|\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')\| \qquad \text{(A-28)}$$

which yields the second inequality of (20).

Next, we show (21) using the Mean Value Theorem [19]:

$$F(\underset{\sim}{x}') = \int_0^1 [\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})]^T\left\{\underset{\sim}{f}\left(\sigma\left[\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right]+\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)\right\}d\sigma$$

$$= \int_0^1 [x' -\underset{\sim}{f}^{-1}(\underset{\sim}{0})]^T\left\{\underset{\sim}{f}\left(\sigma\left[\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right]+\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right) -\left(\underset{\sim}{f}\ \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)\right\}d\sigma$$

$$= \int_0^1 \frac{1}{\sigma}\left[\sigma\left(\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)+\underset{\sim}{f}^{-1}(\underset{\sim}{0}) - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\right]^T\left\{\underset{\sim}{f}\left(\sigma\left[\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right]+\underset{\sim}{f}^{-1}(0)\right) - \underset{\sim}{f}\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right)\right\}d\sigma$$

$$\geq \int_0^1 \frac{1}{\sigma}\, \underline{\gamma}\|\sigma\left(\underset{\sim}{x}'-\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right) + \underset{\sim}{f}^{-1}(\underset{\sim}{0}) - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\|^2 d\sigma$$

$$= \int_0^1 \underline{\gamma}\, \| \underset{\sim}{x}' - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\|^2 \sigma d\sigma = \frac{1}{2}\, \underline{\gamma}\, \| \underset{\sim}{x}' - \underset{\sim}{f}^{-1}(\underset{\sim}{0})\|^2 \qquad \text{(A-27)}$$

This proves the right inequality of (21). The left inequality follows in the same way. ∎

Corollary A: Let $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-state function. Assume there exists constants $k > 0$, and $\bar{\gamma} \geq \underline{\gamma} > 0$ such that for all $\|\underset{\sim}{x}'\| > k$, $\|x''\| > k$,

$$\underline{\gamma}\|\underset{\sim}{x}' - \underset{\sim}{x}''\|^2 \leq (\underset{\sim}{x}'-\underset{\sim}{x}'')^T\left[\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right] \leq \bar{\gamma}\|\underset{\sim}{x}' - \underset{\sim}{x}''\|^2 \tag{A-28}$$

Then for all $\|\underset{\sim}{x}''\| > k$, $\|\underset{\sim}{x}'\| > k$, we have

$$\underline{\gamma}\|\underset{\sim}{x}' - \underset{\sim}{x}''\| \leq \|\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')\| \leq \bar{\gamma}\|\underset{\sim}{x}' - \underset{\sim}{x}''\| \tag{A-29}$$

Furthermore, there exists a $C^2$-function $F: \mathbb{R}^n \to \mathbb{R}^1$ and constants $k_1 > 0$, $\bar{\gamma}_1 \geq \underline{\gamma}_1 > 0$ such that

$$\begin{aligned} \nabla F(\underset{\sim}{x}) &= \underset{\sim}{f}(\underset{\sim}{x}) \quad , \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n \\ \underline{\gamma}_1\|\underset{\sim}{x}\|^2 \leq F(\underset{\sim}{x}) &\leq \bar{\gamma}_1\|\underset{\sim}{x}\|^2 \quad , \quad \forall \|\underset{\sim}{x}\| > k_1 \end{aligned} \tag{A-30}$$

Proof: This is proved essentially in the same manner as in the proof of Theorem A-6. Indeed, (A-29) follows from (A-28) as (20) follows from (14). In this case, $\underset{\sim}{f}^{-1}$ may not exist. However, $\underset{\sim}{f}$ is injective on the set $\{\underset{\sim}{x}: \|\underset{\sim}{x}\| > k\}$, and the proof is the same.

Let $\hat{F}: \mathbb{R}^n \to \mathbb{R}^1$ be any $C^2$-function such that $\nabla\hat{F}(\underset{\sim}{x}) \equiv \underset{\sim}{f}(\underset{\sim}{x})$. Define $\hat{k} \triangleq \inf_{\|\hat{\underset{\sim}{x}}\|=k} \hat{F}(\hat{\underset{\sim}{x}})$, and let $F(\underset{\sim}{x}) \triangleq \hat{F}(\underset{\sim}{x}) - \hat{k}$. By construction, $F(\hat{\underset{\sim}{x}}) \geq 0$ for all $\|\hat{\underset{\sim}{x}}\| = k$. For any $\underset{\sim}{x}$ such that $\|\underset{\sim}{x}\| > k$, let $\hat{\underset{\sim}{x}} = \frac{k}{\|\underset{\sim}{x}\|}\underset{\sim}{x}$. Then, applying the Mean Value Theorem, we obtain

$$\begin{aligned} F(\underset{\sim}{x}) \geq F(\underset{\sim}{x}) - F(\hat{\underset{\sim}{x}}) &= \int_0^1 [\underset{\sim}{x} - \hat{\underset{\sim}{x}}]^T [\underset{\sim}{f}\left(\sigma(\underset{\sim}{x}-\hat{\underset{\sim}{x}})+\hat{\underset{\sim}{x}}\right)]d\sigma \\ &\geq \frac{1}{2}\underline{\gamma}\|\underset{\sim}{x} - \hat{\underset{\sim}{x}}\|^2 - \|\underset{\sim}{x} - \hat{\underset{\sim}{x}}\| \cdot \|\underset{\sim}{f}(\hat{\underset{\sim}{x}})\| \end{aligned} \tag{A-31}$$

where the last inequality is derived as in (A-29). Define the constant

$k_0 \triangleq \sup_{\|\hat{x}\|=k} \|f(\hat{x})\|$, and noting that $\|x - \hat{x}\| = \|x\| - k$, we obtain

$$F(x) \geq \frac{1}{2} \underline{\gamma}\|x\|^2 - \|x\|[\underline{\gamma}k + k_0] - [\underline{\gamma} \frac{k^2}{2} + kk_0] \tag{A-32}$$

For any $\underline{\gamma}_1$, $0 < \underline{\gamma}_1 < \frac{1}{2} \underline{\gamma}$, there exists $k_0 > 0$ such that the left inequality of (A-21) follows from (A-32). The right inequality follows in a similar way. ⌘

## FIGURE CAPTIONS

Fig. 1. (a) The Wien-Bridge Oscillator (b) The Network Model (c) The Idealized Function $f(\cdot)$.

Fig. 2. The Dynamic Nonlinear Network $\mathcal{N}$.

Fig. 3. A Network whose Voltage and Current Waveforms Exhibit Finite Escape-Time Solutions in Negative Time.

Fig. 4. A Network Containing Highly Active Elements yet Having no Finite Escape-Time Solution.

Fig. 5. (a) A Network with Bounded Waveforms (b) A Network with Eventually Uniformly Bounded Waveforms.

Fig. 6. (a) The v-i Curve of Resistor $R_1$; the Function $g_{R_1}$ is Eventually strictly Passive (b) The v-i Curve of Resistor $R_2$; the Function $g_{R_2}$ is strictly Passive. The Composite Function $g_R = (g_{R_1}, g_{R_2})^T$ Is Not Eventually Strictly Passive.

Fig. 7. (a) A Network Containing Strictly Passive Elements when the Resistor has the v-i Curve of (b) or (c). Its state Equation does not Exist and the Network Exhibits Finite Escape-Time Phenomena.

Fig. 8. (a) A Network Containing a Cutset of Inductors. Each Element is Uncoupled and has a $C^1$-Strictly-Increasing Diffeomorphic Constitutive Relation (b) The Equivalent Network with the Cutset of Inductors Removed, and the Constitutive Relations of the Transformed Equivalent Network are Specified.

Fig. 9. (a) and (b); Networks with a Globally Asymptotically Stable Equilibrium Point (c) A Network which Oscillates.

Fig. 10. Two Linear Networks Used to Evaluate the Accuracy of the Algorithm.

Fig. 11. The Waveforms Defined by (143) for the Network of Fig. 8b.

(a) For Initial State (144), the Upper Waveform is the Right Side of (143) and the Lower Waveform is the Left Side.

(b) The Waveforms Corresponding to Initial State (145).

(c) The Waveforms $\phi_C(t) - (-.1547)$, $\phi_{L_2}(t) - (-.906)$, and $\phi_{L_1}(t) - 1.0205$.

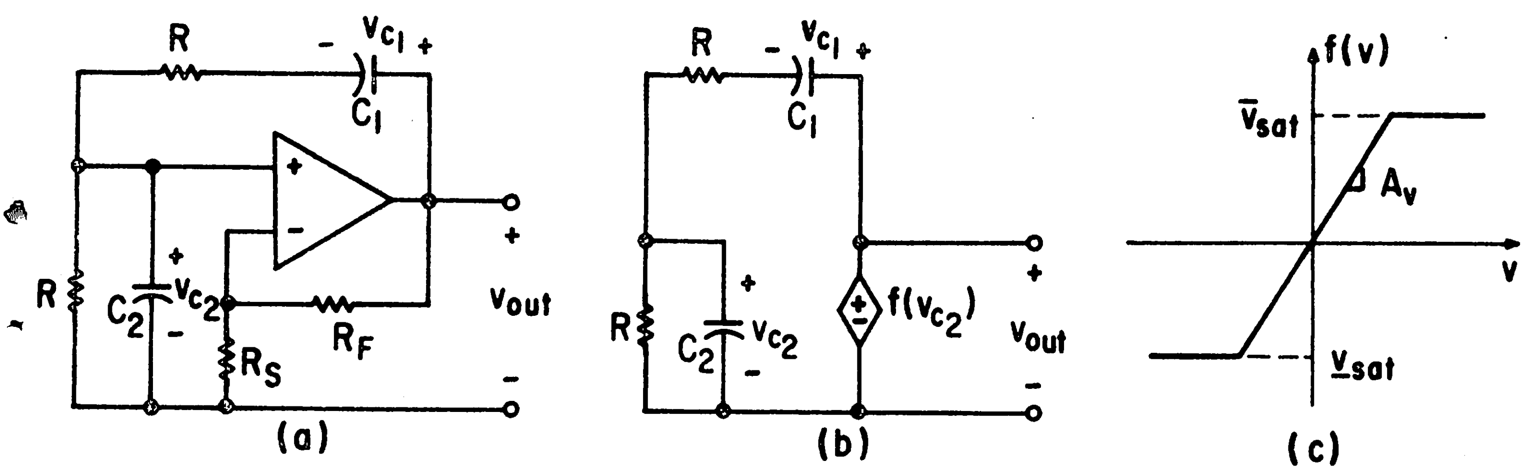



Fig. 1

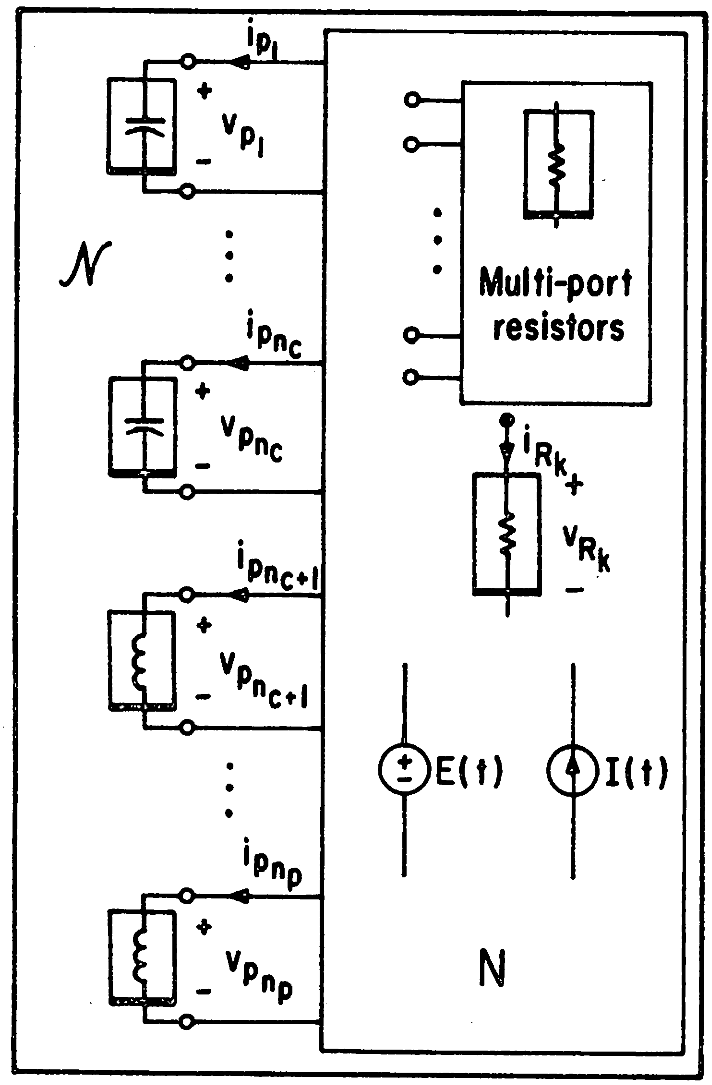



Fig. 2

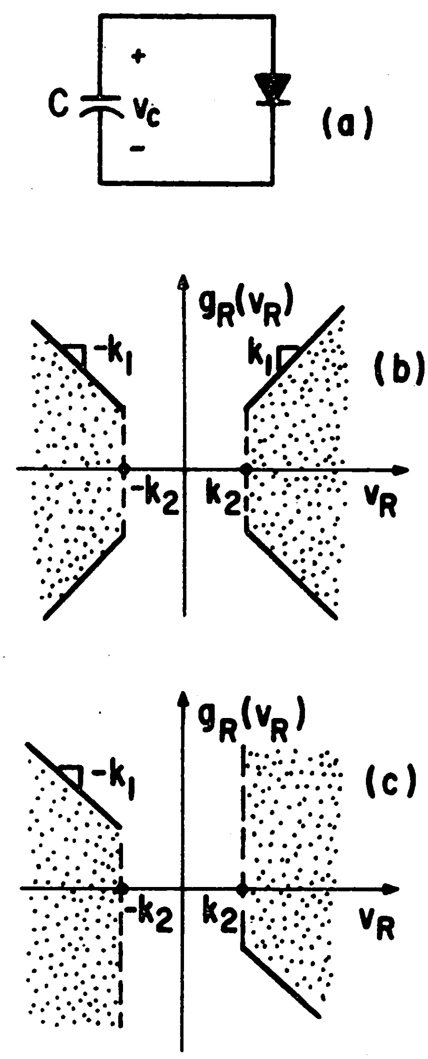
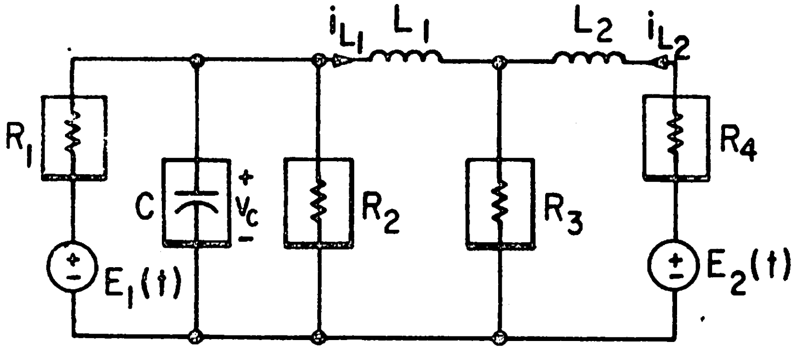



Fig. 3




$$v_c = e^{-(q_c)^2}\left[(q_c)^3 - (q_c)^5\right] + q_c$$

$$\begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \phi_{L_1} \\ \phi_{L_2} \end{pmatrix}$$

$$i_{R_1} = v_{R_1} \sin(v_{R_1})$$

$$i_{R_2} = -\mathrm{sgn}(v_{R_2}) \ln\left(1+|v_{R_2}|\right)$$

$$v_{R_3} = e^{-(i_{R_3})^2}$$

$$v_{R_4} = -i_{R_4}$$

Fig. 4

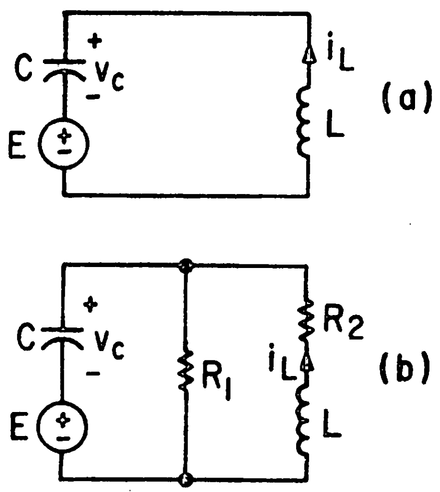



Fig.5

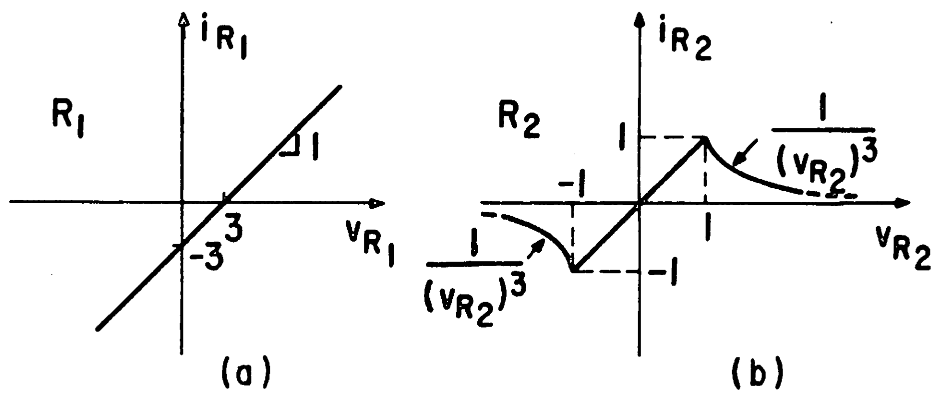



Fig.6

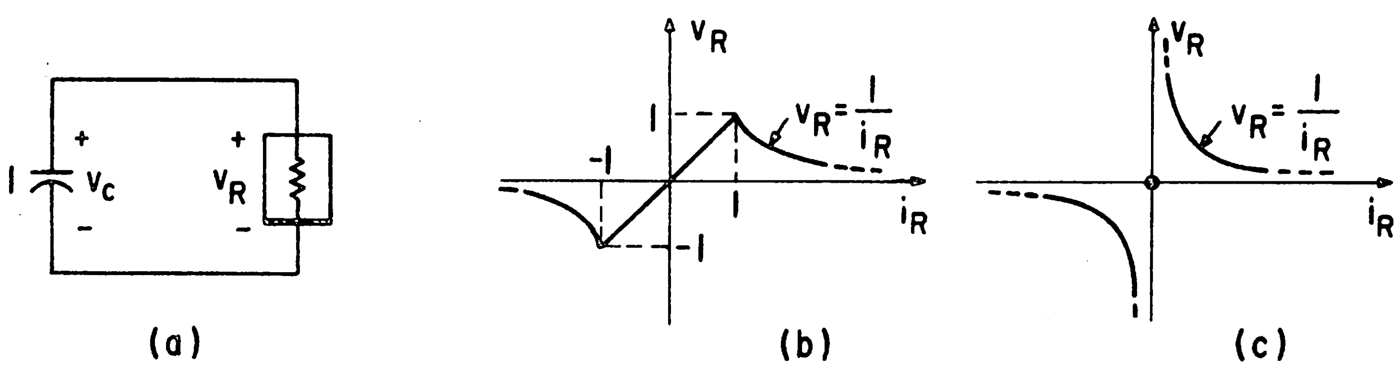
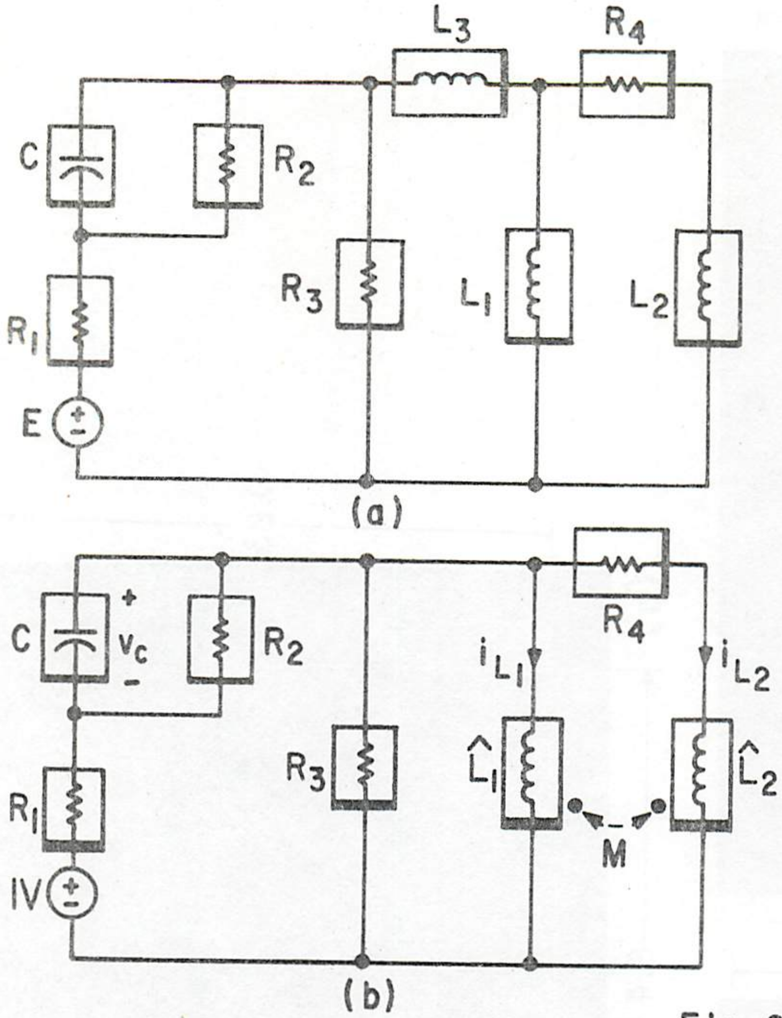



Fig. 7




$$v_c = q_c\left[2 + \frac{(q_c)^2}{1+(q_c)^2}\right]$$

$$\begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{pmatrix} \begin{pmatrix} \phi_{L_1} \\ \phi_{L_2} \end{pmatrix}$$

$$v_{R_1} = 2 i_{R_1}$$

$$i_{R_2} = v_{R_2} + (v_{R_2})^3$$

$$v_{R_3} = i_{R_3}$$

$$v_{R_4} = 2\left[\frac{i_{R_4}(1+i_{R_4})^2+1}{1+(i_{R_4})^2}\right]$$

Fig. 8

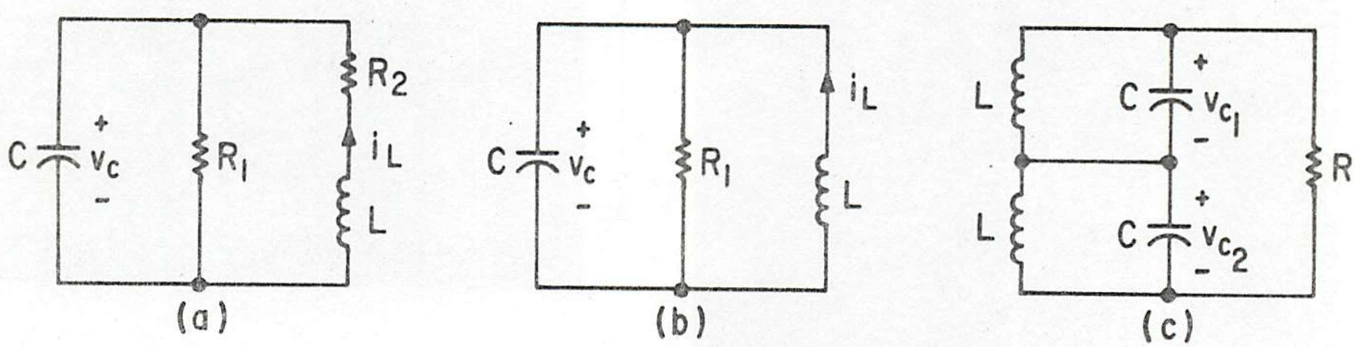



Fig. 9

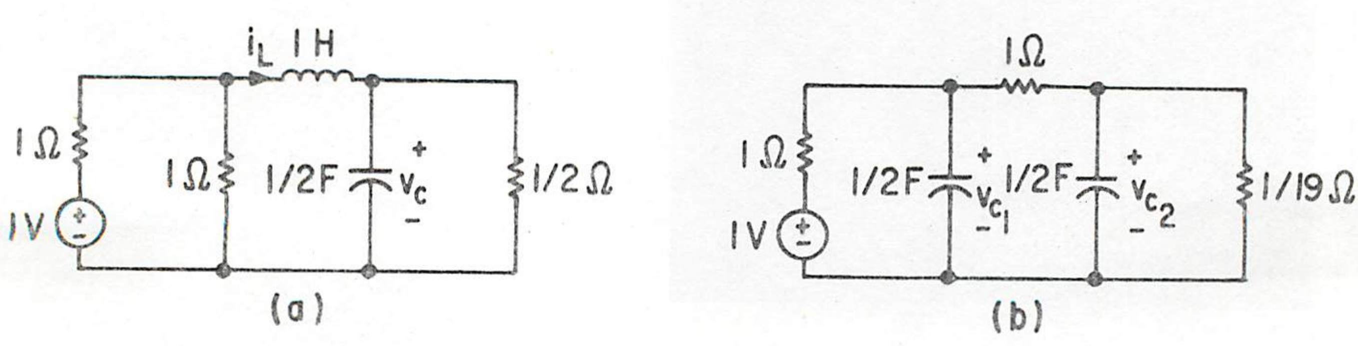



Fig. 10

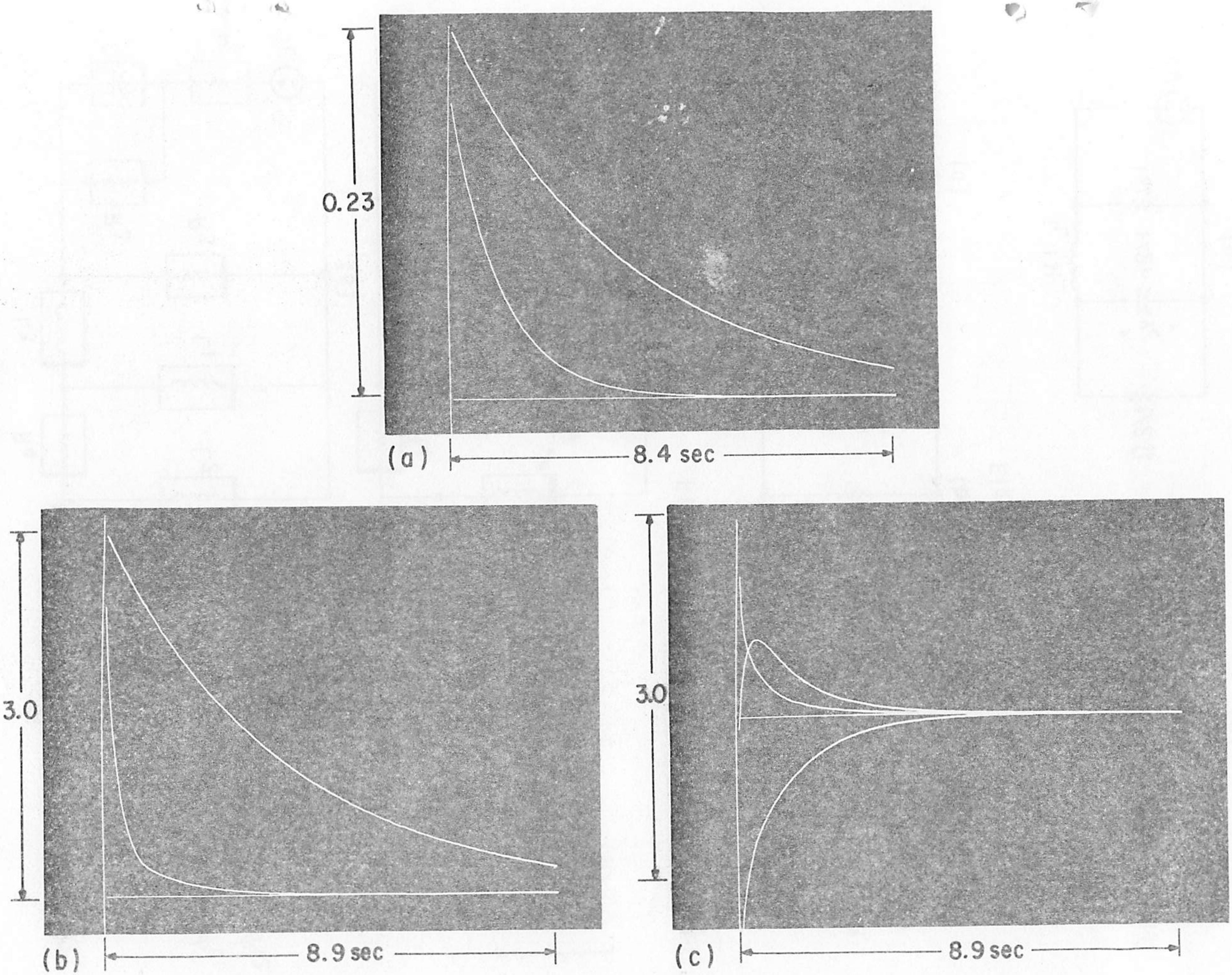



Fig. 11