A SECOND ORDER METHOD FOR UNCONSTRAINED OPTIMIZATION

by

H. Mukai and E. Polak

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Abstract: This paper presents a quadratically converging algorithm for unconstrained minimization. All the accumulation points that it constructs, satisfy second order necessary conditions of optimality. Thus, it avoids saddle and "inflection" points: a feature particularly useful in minimizing the modified Lagrangians in multiplier methods.

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Introduction

A shortcoming of Newton's method is that when applied to unconstrained minimization problems it may converge to a saddle or inflection point when started from a set of nonzero measure of initial points. This shortcoming in also inherited by the various "stabilized" or "globalized" versions of Newton's method, such as that due to Goldstein [1], Huang [3], Polak-Teodoru [9] or Polak [8]. This shortcoming becomes particularly important when any one of these methods is applied to the minimization of a modified lagrangian constructed in multiplier methods such as [2], [10]. The reason for this is that the saddle points of the modified lagrangian can be local maxima of the original minimization problem which one is trying to solve by minimizing, unconstrained, the modified lagrangian. Thus, there is a need, answered by the algorithm in this paper, for a quadratically converging, unconstrained minimization algorithm which "avoids" saddle and inflection points. The construction of this algorithm was facilitated by the existence of a constrained minimization algorithm, with similar properties, due to McCormick [6].

The algorithm in this paper uses either a part of McCormick's descent direction finding scheme or an alternate one, which may be preferable for large scale problems. It uses a substantially more efficient step size procedure and also a somewhat simpler scheme for ensuring global converge than the ones used by McCormick.

It is shown that, under certain conditions, (i) all the accumulation points constructed by the new algorithm satisfy second order necessary conditions of optimality and (ii) that it converges quadratically when it converges to a strong local minimum point. Computationally we have

found the method to perform quite well, particularly so, on problems with saddle and inflection points.

2. Preliminaries

We shall consider the optimization problem

$$\min\{f(x) \mid x \in \mathbb{R}^n\} \tag{1}$$

where f is a twice continuously differentiable function. To simplify notation, we shall denote by g the gradient and by H the Hessian of f, i.e.,

$$g(x) \triangleq \frac{\partial f(x)}{\partial x}^T, \qquad H(x) \triangleq \frac{\partial^2 f(x)}{\partial x^2} \tag{2}$$

Furthermore, $H(x) \geqq 0$, $H(x) > 0$ will be used to indicate that $H(x)$ is positive semidefinite or positive definite, respectively. The symbol $\|\cdot\|$ denotes the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product.

Definition 1: We shall say that $x \in \mathbb{R}^n$ is desirable if it satisfies both first and second order necessary conditions of optimality for (1) (see [5]), i.e., if $g(x) = 0$ and $H(x) \geqq 0$. We shall denote by Δ the set of desirable points. ¤

Let ϕ: $\mathbb{R}^n \to \mathbb{R}^1$ be defined by

$$\phi(x) = \min\{\tfrac{1}{2}\langle e,H(x)e\rangle \mid \langle g(x),e\rangle \le 0,\ \|e\| \le 1\}^\dagger \tag{3}$$

Noting that the following equality is valid,

$$\phi(x) = \min\{\tfrac{1}{2}\langle e,H(x)e\rangle \mid \|e\| \le 1\} \tag{4}$$

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† When an eigenvector subroutine is not available, $|e_i| \le 1$ for $i = 1,2,..,n$, may be substituted for $\|e\| \le 1$ in (3) to utilize a linear program subroutine without affecting the truth of the propositions and theorems to follow.

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We obtain (see B.3.20 in [7]) the following

Proposition 1: (a) The function $\phi(\cdot)$ is continuous and (b) a point $x \in \mathbb{R}^n$ is desirable if and only if $g(x) = 0$ and $\phi(x) = 0$. ¤

Consequently, we obtain the following alternative characterization of the set Δ:

$$\Delta = \{x \in \mathbb{R}^n \mid g(x) = 0,\ \phi(x) = 0\} \tag{5}$$

Since $g(\cdot)$ and $\phi(\cdot)$ are both continuous, it is clear that Δ is closed.

3. The Algorithm: Statement and Convergence

The algorithm uses three parameters α, β and ε_0 and requires an initial guess x_0. In the absence of experience with a problem, one could start with $\alpha = 0.5$, $\beta = 0.5$, and $\varepsilon_0 = \min\{10^{-20}, 10^{-3}|\det H(x_0)|\}$, assuming, of course, that $\det H(x_0) \neq 0$.

Algorithm

Data: $\alpha \in (0,1)$, $\beta \in (0,1)$, $0 < \varepsilon_0 << 1$, $x_0 \in \mathbb{R}^n$.

Step 0: Set $i = 0$.

Step 1: Compute $\phi(x_i)$ and an

$$e_i \in \{e \in \mathbb{R}^n \mid \langle g(x_i), e\rangle \le 0,\ \|e\| \le 1, \phi(x_i) = \tfrac{1}{2}\langle e, H(x_i)e\rangle\} \tag{6}$$

Comment: When $H(x_i) \geq 0$, $e_i = 0$, otherwise e_i is an eigenvector of $H(x_i)$ corresponding to the smallest eigenvalue.

Step 2: If $\phi(x_i) < 0$, go to step 6; else go to step 3.

Step 3: If $g(x_i) = 0$, stop; else go to step 4.

Step 4: If $|\det H(x_i)| < \varepsilon_0$, go to step 6; else go to step 5.

Step 5: Compute $h_i = -H(x_i)^{-1}g(x_i)$, set $\lambda_0 = 1$ and go to step 8.

Step 6: Compute $h_i = -g(x_i) + e_i$.

Step 7: If $\langle h_i, H(x_i)h_i\rangle \le 0$, set $\lambda_0 = 1$ and go to step 8; else set $\lambda_0 = \beta^{k_i}$ where $k_0 \ge 0$ is the smallest integer satisfying

$$\beta^{k_i} \le -\langle g(x_i),h_i\rangle / \langle h_i, H(x_i)h_i\rangle \tag{7}$$

Step 8: Compute the smallest nonnegative integer ℓ_i satisfying

$$f(x_i+\lambda_0\beta^{\ell_i}h_i) - f(x_i) \le \alpha[\lambda_0\beta^{\ell_i}\langle g(x_i),h_i\rangle + \frac{1}{2}(\lambda_0\beta^{\ell_i})^2\langle h_i, H(x_i)h_i\rangle] \tag{8}$$

Comment: When $h_i = -H(x_i)^{-1}g(x_i)$, as in step 5, the right hand side of (8) simplifies to $\alpha\beta^{\ell_i}\langle g(x_i),h_i\rangle[1 - \frac{1}{2}\beta^{\ell_i}]$.

Step 9: Set $x_{i+1} = x_i + \lambda_0\beta^{\ell_i}h_i$, set $i = i + 1$ and go to step 1. ⌑

To facilitate our convergence analysis, we introduce the following notation:

$$B(x,\varepsilon) = \{y \in \mathbb{R}^n \mid \|y - x\| \le \varepsilon\} \tag{9a}$$

$$\mathcal{E}(x) = \{e \in \mathbb{R}^n \mid \langle g(x),e\rangle \le 0,\ \|e\| \le 1, \phi(x) = \tfrac{1}{2}\langle e,H(x)e\rangle\}^{\dagger} \tag{9b}$$

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid \phi(x) = 0, |\det H(x)| \ge \varepsilon_0\} \tag{9c}$$

$$\mathcal{G}(x) = \{-g(x) + e \mid e \in \mathcal{E}(x)\} \tag{9d}$$

$$\mathcal{H}(x) = \begin{cases} \{-H(x)^{-1}g(x)\} & \text{if } x \in \mathcal{F} \\ \mathcal{G}(x) & \text{if } x \notin \mathcal{F} \end{cases} \tag{9e}$$

† $\mathcal{E}(x)$ may contain more than one point when $H(x)$ has repeated eigenvalues.

Let $\mathcal{N} = \{(x,h) \mid x \in \mathbb{R}^n, h \in \mathbb{R}^n, \langle g(x),h\rangle < 0\}$. We define $\lambda_0: \mathcal{N} \to \mathbb{R}^1$, $\theta: \mathcal{N} \times \mathbb{R}^1 \to \mathbb{R}^1$ and $\lambda: \mathcal{N} \to \mathbb{R}^1$ as follows:

$$\lambda_0(x,h) = \begin{cases} 1 \text{ if } \langle h,H(x)h\rangle \le 0 \\ \beta^k \text{ otherwise} \end{cases} \tag{9f}$$

where $k \ge 0$ is the smallest integer satisfying

$$\beta^k \le -\langle g(x),h\rangle / \langle h,H(x)h\rangle, \tag{9g}$$

$$\theta(x,h,\lambda) = \lambda\langle g(x),h\rangle + \frac{1}{2}\lambda^2\langle h,H(x)h\rangle \tag{9h}$$

$$\lambda(x,h) = \lambda_0(x,h)\beta^{\ell} \tag{9i}$$

where $\ell \ge 0$ is the smallest integer satisfying

$$f(x+\lambda_0(x,h)\beta^{\ell}h) - f(x) \le \alpha\theta(x,h,\lambda_0(x,h)\beta^{\ell}) \tag{9j}$$

<u>Lemma 1</u>: Let $x^* \in \mathbb{R}^n$. (i) If $g(x^*) \ne 0$, then there exist an $\varepsilon^* > 0$ and a $\delta^* > 0$ such that

$$\langle g(x),h\rangle \le -\delta^* < 0, \ \forall x \in B(x^*,\varepsilon^*), \ \forall h \in \mathcal{H}(x) \tag{10}$$

(ii) If $g(x^*) = 0$, and $H(x^*) \not\ge 0$, then there exist an $\varepsilon^* > 0$ and a $\delta^* > 0$ such that

$$\frac{1}{2}\langle h,H(x)h\rangle \le -\delta^* < 0, \ \forall x \in B(x^*,\varepsilon^*), \ \forall h \in \mathcal{H}(x) \tag{11}$$

<u>Proof</u>: Case (i) Suppose $x^* \in \mathbb{R}^n$ is such that $g(x^*) \ne 0$. Hence there exists an $\varepsilon_1 > 0$ such that $\|g(x)\|^2 \ge \frac{1}{2}\|g(x^*)\|^2 > 0$ for all $x \in B(x^*,\varepsilon_1)$. Therefore, for all $h \in \mathcal{G}(x)$

$$\langle g(x),h\rangle = -\|g(x)\|^2 + \langle g(x),e\rangle \leq -\|g(x)\|^2 \leq -\frac{1}{2}\|g(x^*)\|^2 \tag{12}$$

where $e = h + g(x) \in \mathcal{E}(x)$.

Now suppose that $x^* \notin \mathcal{F}$. Then, since $\mathcal{F}^c$ † is open, there exists an $\varepsilon^* \in (0,\varepsilon_1]$ such that $B(x^*,\varepsilon^*) \subset \mathcal{F}^c$ and hence $\mathcal{H}(x) = \mathcal{G}(x)$ for all $x \in B(x^*,\varepsilon^*)$. It now follows from (12) that (10) holds for this case.

Next, suppose that $x^* \in \mathcal{F}$. Then $H(x^*) > 0$ and there exists an $\varepsilon^* \in (0,\varepsilon_1]$ such that $\langle g(x),H(x)^{-1}g(x)\rangle \geq \frac{1}{2}\langle g(x^*),H(x^*)^{-1}g(x^*)\rangle > 0$ for all $x \in B(x^*,\varepsilon^*)$. Now, let $\delta^* = \frac{1}{2}\min\{\|g(x^*)\|^2, \langle g(x^*),H(x^*)^{-1}g(x^*)\rangle\}$. Then, for all $x \in B(x^*,\varepsilon^*)$, $h \in \mathcal{G}(x)$ or $h = -H(x)^{-1}g(x)$ and hence (10) holds for this case.

Case (ii): Suppose $x^* \in \mathbb{R}^n$ is such that $g(x^*) = 0$, $H(x^*) \geq 0$. In this case $\phi(x^*) < 0$ and hence $x^* \in \mathcal{F}^c$. Since $\mathcal{F}^c$ is open, there exists an $\varepsilon_2 > 0$ such that $B(x^*,\varepsilon_2) \subset \mathcal{F}^c$ and hence $\mathcal{H}(x) = \mathcal{G}(x)$ for all $x \in B(x^*,\varepsilon_2)$. Also, there exists an $M < \infty$ such that for all $x \in B(x^*,\varepsilon_2)$ and $e \in \mathcal{E}(x)$,

$$\left|\frac{1}{2}\langle g(x),H(x)g(x)\rangle - \langle g(x),H(x)e\rangle\right| \leq M\|g(x)\|. \tag{13}$$

Next, making use of (13), since $g(x^*) = 0$ and $\phi(x^*) < 0$, there exists an $\varepsilon^* \in (0,\varepsilon_2]$ such that for all $x \in B(x^*,\varepsilon^*)$, for all $e \in \mathcal{E}(x)$,

$$\frac{1}{2}\langle g(x),H(x)g(x)\rangle - \langle g(x),H(x)e\rangle + \frac{1}{2}\langle e,H(x)e\rangle \leq M\|g(x)\| + \phi(x) \leq \frac{1}{2}\phi(x^*) \tag{14}$$

Therefore, for all $x \in B(x^*,\varepsilon^*)$ and for all $h \in \mathcal{H}(x) = \mathcal{G}(x)$, $h = -g(x) + e$, with $e \in \mathcal{E}(x)$ and hence

$$\frac{1}{2}\langle h,H(x)h\rangle = \frac{1}{2}\langle g(x),H(x)g(x)\rangle - \langle g(x),H(x)e\rangle + \frac{1}{2}\langle e,H(x)e\rangle$$

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† We denote the complement of a set by a superscript c.

$$\leq \frac{1}{2}\,\phi(x^*) \triangleq -\delta^* < 0 \tag{15}$$

which conpletes our proof. ¤

<u>Lemma 2</u>: For any $x^* \in \Delta^c$ there exists a $\lambda^* \in (0,1]$ and an $\varepsilon^* > 0$ such that $\lambda_0(x,h) \geq \lambda^* > 0$ for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$.

<u>Proof</u>: Case (i): Suppose that $x^* \in \Delta^c$ is such that $g(x^*) \neq 0$. Then, by Lemma 1 (i), there exists an $\varepsilon^* > 0$ and a $\delta^* > 0$ such that

$$\langle g(x),h\rangle \leq -\delta^*,\ \forall x \in B(x^*,\varepsilon^*),\ \forall h \in \mathcal{H}(x^*) \tag{16}$$

Also, there exists an $M > 0$, such that

$$|\langle h,H(x)h\rangle| \leq M\ \forall x \in B(x^*,\varepsilon^*),\ \forall h \in \mathcal{H}(x) \tag{17}$$

Let $\lambda^* = \min\{\frac{\delta^*}{M}\beta,1\}$, where $\beta \in (0,1)$ is as in the Algorithm Data; then $\lambda^* \leq 1$. Now suppose that for some $x \in B(x^*,\varepsilon^*)$, and some $h \in \mathcal{H}(x)$, $\langle h,H(x)h\rangle > 0$. Then

$$-\langle g(x),h\rangle/\langle h,H(x)h\rangle \geq \delta^*/M\ , \tag{18}$$

hence from (9f,g) and (18), $\lambda_0(x,h) = \beta^k \geq \beta\delta^*/M \geq \lambda^*$. The case $\langle h,H(x)h\rangle \leq 0$ is trivial. Hence Lemma 2 holds for the case when $g(x^*) = 0$. Case (ii): Suppose that $x^* \in \Delta^c$ is such that $g(x^*) = 0$. Then we must have $H(x^*) \not\leq 0$ and then it follows from Lemma 1 (ii) that there exist an $\varepsilon^* > 0$ and a $\delta^* > 0$ such that (ii) holds for all $x \in B(x^*,\varepsilon^*)$. Consequently, $\lambda_0(x,h) = 1$ for all $x \in B(x^*,\varepsilon^*)$ for all $h \in \mathcal{H}(x)$. Setting $\lambda^* = 1$, we conclude our proof. ¤

<u>Lemma 3</u>: Suppose that $x \in \Delta^c$ and $h \in \mathcal{H}(x)$. The (i) for any $\lambda \in (0,\lambda_0(x,h)]$,

$$\theta(x,h,\lambda) \leq \frac{1}{2}\lambda\langle g(x),h\rangle \leq 0 \tag{19}$$

(ii) For any λ_1, λ_2 such that $0 < \lambda_1 < \lambda_2 \leq \lambda_0(x,h)$,

$$\theta(x,h,\lambda_2) < \theta(x,h,\lambda_1) \leq 0 \tag{20}$$

Proof: First, suppose that $x \in \Delta^c$ is such that $g(x) \neq 0$, and let $h \in \mathcal{H}(x)$. Then by Lemma 1 (i), $\langle g(x),h\rangle < 0$. If $\langle h,H(x)h\rangle \leq 0$, then, clearly, both (19) and (20) hold. If $\langle h,H(x)h\rangle > 0$ then $\lambda_0(x,h) \leq -\langle g(x),h\rangle/\langle h,H(x)h\rangle$ and hence for any $\lambda \in (0,\lambda_0(x,h)]$

$$\theta(x,h,\lambda) = \lambda\langle g(x),h\rangle\left(1 + \frac{\lambda}{2}\ \frac{\langle h,H(x)h\rangle}{\langle g(x),h\rangle}\right) \leq \lambda\langle g(x),h\rangle < 0 \tag{21}$$

Since in this case $\theta(x,h,\lambda)$ is monotonically decreasing in λ for $\lambda \in [0,\lambda_0(x,h)]$, (20) obviously holds.

Next, suppose that $x \in \Delta^c$ is such that $g(x) = 0$, and $H(x) \not\leq 0$. Since in this case $\phi(x) < 0$, $x \in \mathcal{F}^c$ and hence we have

$$\theta(x,h,\lambda) = \frac{1}{2}\lambda^2\langle h,H(x)h\rangle = \lambda^2\phi(x) < 0,\ \forall h \in \mathcal{H}(x),\ \forall\lambda > 0 \tag{22}$$

Consequently, (since $g(x) = 0$) (19) is satisfied, and so is (20), by inspection of (22). This completes our proof. ¤

Lemma 4: In any $x^* \in \Delta^c$ there exist a $\lambda^* \in 0$, a $\delta^* > 0$ and an $\varepsilon^* > 0$ such that $\theta(x,h,\lambda) \leq -\frac{1}{2}\delta^*\lambda^2$ for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$, for all $\lambda \in [0,\lambda^*]$.

Proof: Suppose that $x^* \in \Delta^c$. Then, by Lemma 2, there exist a $\lambda^* \in (0,1]$ and an $\varepsilon_1 > 0$ such that $\lambda_0(x,h) \geq \lambda^* > 0$, for all $x \in B(x^*,\varepsilon_1)$, for all $h \in \mathcal{H}(x)$. Now, suppose that $g(x^*) \neq 0$. Then by Lemma 1 (i), there exist an $\varepsilon^* \in (0,\varepsilon_1]$ and a $\delta^* > 0$ such that $\langle g(x),h\rangle \leq -\delta^*$ for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$. Hence, by Lemma 3 (see (19)),

$$\theta(x,h,\lambda) \leq \frac{1}{2}\lambda\langle g(x),h\rangle \leq -\frac{1}{2}\delta^*\lambda \leq -\frac{1}{2}\delta^*\lambda^2 \tag{23a}$$

Next suppose that $g(x^*) = 0$ and $H(x^*) \not\leq 0$. It then follows from Lemma 1 (ii) that there exist an $\varepsilon^* \in (0,\varepsilon_1]$ and a $\delta^* > 0$ such that $\frac{1}{2}\langle h,H(x)h\rangle \leq -\delta^* < 0$ for all $x \in B(x^*,\varepsilon^*)$ for all $h \in \mathcal{H}(x)$. Furthermore, we deduce from Lemma 1 (i) that for any $x \in \mathbb{R}^n$ such that $g(x) \neq 0$, $\langle g(x),h\rangle < 0$ for all $h \in \mathcal{H}(x)$. Consequently, we conclude that for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$, for all $\lambda \in [0,\lambda^*]$,

$$\theta(x,h,\lambda) = \lambda\langle g(x),h\rangle + \frac{1}{2}\lambda^2\langle h,H(x)h\rangle \leq \frac{1}{2}\lambda^2\langle h,H(x)h\rangle \leq -\frac{1}{2}\delta^*\lambda^2, \quad (23b)$$

which completes our proof. ⌑

Lemma 5: For any $x^* \in \Delta^c$, there exists a $\lambda^* > 0$ and an $\varepsilon^* > 0$ such that $\lambda(x,h) \geq \lambda^* > 0$ for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$.

Proof: Let $x^* \in \Delta^c$. Then it follows from Lemmas 2 and 4 that there exist $\lambda_1 \in (0,1]$, an $\varepsilon^* > 0$ and a $\delta^* > 0$ such that for all $x \in B(x^*,\varepsilon^*)$, for all $h \in \mathcal{H}(x)$, $\lambda_0(x,h) \geq \lambda_1$ and, in addition, for all $\lambda \in [0,\lambda_1]$, $\theta(x,h,\lambda) \leq -\frac{1}{2}\delta^*\lambda^2$. Clearly, there exists an $M \in (0,\infty)$ such that $\|h\| \leq M$ for all $h \in \mathcal{H}(x)$, $x \in B(x^*,\varepsilon^*)$. Now, since the function $\langle h,H(x)h\rangle$ of (x,h) is uniformly continuous on the compact set $B(x^*,\varepsilon^*) \times B(0,M)$, there exists a $\lambda_2 \in (0,\lambda_1]$ such that

$$|\langle h,[H(x+t\lambda h) - H(x)]h\rangle| \leq (1-\alpha)\delta^*/2, \quad (24)$$

for all $\lambda \in [0,\lambda_2]$, for all $t \in [0,1]$ and $(x,h) \in B(x^*,\lambda^*) \times B(0,M)$, where $\alpha \in (0,1)$ is as in the Algorithm Data. Hence, for any $x \in B(x^*,\varepsilon^*)$ and any $h \in \mathcal{H}(x)$, and any $\lambda \in [0,\lambda_2]$, we obtain

$$f(x+\lambda h) - f(x) - \alpha\theta(x,h,\lambda) = \lambda\langle g(x),h\rangle + \int_0^1 (1-t)\lambda^2\langle h,H(x+t\lambda h)h\rangle\,dt$$

$$-\alpha\theta(x,h,\lambda) = (1-\alpha)\theta(x,h,\lambda) + \int_0^1 (1-t)\lambda^2\langle h,[H(x+t\lambda h) - H(x)]h\rangle\,dt$$

$$\leq -\frac{1}{2}(1-\alpha)\delta^*\lambda^2 + \frac{1}{4}(1-\alpha)\delta^*\lambda^2 = -\frac{1}{4}(1-\alpha)\delta^*\lambda^2 \leq 0 \tag{25}$$

Let $\lambda^* = \beta^{\ell^*}$ where $\ell^* \geq 0$ is an integer such that $\beta^{\ell^*} \leq \lambda_2 < \beta^{\ell^*-1}$. Consequently, since (25) is satisfied for $\lambda = \lambda^*$, we must have that $\lambda(x,h) > \lambda^*$, (since $\lambda(x,h)$ is of the form $(\beta^k\beta^\ell)$ and $k+\ell$ is the smallest integer satisfying $f(x+\beta^{k+\ell}h) - f(x) - \alpha\theta(x,h,\beta^{k+\ell}) \leq 0$), which completes our proof. ¤

Theorem 1: Either the Algorithm stops at an $x_i \in \Delta$, or it constructs an infinite sequence, any accumulation point of which is in Δ.

Proof: The first part of the theorem is obvious. Next, given an $x_i \in \Delta^c$, it follows from Lemma 2 that k_i is finite and from Lemma 5 that ℓ_i is finite. Hence the Algorithm is well defined, i.e. it does not jam up. Now, for the sake of contradiction, suppose that $x^* \in \Delta^c$ is an accumulation point of $\{x_i\}_{i=0}^\infty$. Let $\{x_i\}_{i\in I}$ be a subsequence converging to x^*. Then by Lemmas 4 and 5, there exists an $i_0 \geq 0$, a $\delta^* > 0$ and a $\lambda^* \in (0,1]$ such that $\theta(x_i,h_i,\lambda^*) \leq -\frac{1}{2}\delta^*\lambda^{*2}$ and $\lambda(x_i,h_i) \geq \lambda^*$ for all $i \in I$ and $i \geq i_0$. Hence, making use of Lemmas 3, 4 and 5, we conclude that for all $i \in I$, $i \geq i_0$,

$$f(x_{i+1}) - f(x_i) \leq \alpha\theta(x_i,h_i,\lambda(x_i,h_i))$$

$$\leq \alpha\theta(x_i,h_i,\lambda^*) \leq -\frac{1}{2}\alpha\delta^*\lambda^{*2} < 0 \tag{26}$$

since $\{f(x_i)\}_{i=0}^\infty$ is a monotonically decreasing sequence and f is continuous, $f(x_i) \to f(x^*)$ as $i \to \infty$, but this is contradicted by (26) and hence we cannot have $x^* \in \Delta^c$. This completes our proof. ¤

4. Rate of Convergence

We recall that a point $x \in \mathbb{R}^n$ is said to be a strong local minimum point for (1) if $g(x) = 0$ and $H(x) > 0$. We shall now investigate the rate of convergence of the Algorithms when (1) has strong local minimum points.

Lemma 6: Suppose that x^* is a strong local minimum point for (1) and that $|\det H(x^*)| \geq \gamma\varepsilon_0$ for some $\gamma > 1$, with ε_0 as in the Data of the Algorithm. Then there exists an $\varepsilon^* > 0$ such that for any $x_i \in B(x^*,\varepsilon^*)$ the Algorithm constructs an x_{i+1} according to the formula

$$x_{i+1} = x_i - H(x_i)^{-1}g(x_i) \tag{27}$$

and $x_{i+1} \in B(x^*, \varepsilon^*)$.

Proof: Since by assumption $H(x^*) > 0$ and $|\det H(x^*)| \geq \gamma\varepsilon_0$, with $\gamma > 1$, there exists an $\varepsilon_1 > 0$ such that for all $x \in B(x^*,\varepsilon_1)$, $|\det H(x)| \geq \varepsilon_0$, and for some $0 < m \leq M$, $m\|y\|^2 \leq \langle y,H(x)^{-1}y\rangle \leq M\|y\|^2$ for all $y \in \mathbb{R}^n$. Hence, if $x_i \in B(x^*,\varepsilon_1)$, then $x_i \in \mathcal{F}$ and consequently the Algorithm sets $h_i = -H(x_i)^{-1}g(x_i)$ and $\lambda_0(x_i,h_i) = 1$. Now,

$$\begin{aligned} f(x_i+h_i) - f(x_i) - \alpha\theta(x_i,h_i,1) &= (1-\alpha)\theta(x_i,h_i,1) \\ &\quad + \int_0^1 (1-t)\langle h_i,[H(x_i+th_i) - H(x_i)]h_i\rangle\, dt \\ &\leq -\frac{1}{2}(1-\alpha)\,\langle g(x_i),H(x_i)^{-1}g(x_i)\rangle \\ &\quad + \frac{M^2}{2}\left(\sup_{t\in[0,1]} \|H(x_i+th_i) - H(x_i)\|\right)\|g(x_i)\|^2 \\ &\leq \frac{1}{2}\|g(x_i)\|^2\left(-m(1-\alpha)+M^2 \sup_{t\in[0,1]} \|H(x_i+th_i)-H(x_i)\|\right) \end{aligned} \tag{28}$$

Hence, since $H(\cdot)$ and $H(\cdot)^{-1}g(\cdot)$ are uniformly continuous on $B(x^*,\varepsilon_1)$, there exists an $\varepsilon_2 \in (0,\varepsilon_1]$ such that for any $x_i \in B(x^*,\varepsilon_1)$,

$$f(x_i+h_i) - f(x_i) - \alpha\theta(x_i,h_i,1) \leq 0 \tag{29}$$

so that $\lambda(x_i,h_i) = 1$. It now follows from the Kantorovich theorem [4] that there exists an $\varepsilon^* \in (0,\varepsilon_2]$ such that for all $x_i \in B(x^*,\varepsilon^*)$ and x_{i+1} constructed according to (27), $x_{i+1} \in B(x^*,\varepsilon^*)$, which completes our proof. ¤

Our final result is a direct consequence of Lemma 6 and the Kantorovich theorem, and hence we state it without proof.

Theorem 2: Suppose that there is a $\gamma > 1$ such that $|\det H(x^*)| > \gamma\varepsilon_0$ for all strong local minimum points of (1) in the level set $\{x|f(x) \leq f(x_0)\}$, and that the Algorithm has constructed an infinite sequence $\{x_i\}_{i=0}^{\infty}$. Then, either no accumulation point of $\{x_i\}_{i=0}^{\infty}$ is a strong local minimum point or $x_i \to x^*$ as $i \to \infty$, where x^* is a strong local minimum point, with

$$\frac{\|x_{i+1} - x^*\|}{\|x_i - x^*\|} \to 0 \text{ as } i \to \infty \tag{30}$$

Furthermore, if $H(\cdot)$ is Lipschitz continuous at x^*, then there exists an $M^* \in (0,\infty)$ and an i_0 such that

$$\|x_{i+1}-x^*\| \leq M^*\|x_i - x^*\|^2 \text{ for all } i \geq i_0. \tag{31}$$

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