THE LINGUISTIC APPROACH AND ITS APPLICATION TO DECISION ANALYSIS

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THE LINGUISTIC APPROACH AND ITS APPLICATION TO DECISION ANALYSIS†

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ABSTRACT

In a sharp departure from the conventional approaches to decision analysis, the linguistic approach abandons the use of numbers and relies instead on a systematic use of words to characterize the values of variables, the values of probabilities, the relations between variables, and the truth-values of assertions about them.

The linguistic approach is intended to be used in situations in which the system under analysis is too complex or too ill-defined to be amenable to quantitative characterization. It may be used, in particular, to define an objective function in linguistic terms as a function of the linguistic values of decision variables.

In cases in which the objective function is vector-valued, the linguistic approach provides a language for an approximate linguistic characterization of the trade-offs between its components. Such characterizations result in a fuzzy setof Pareto-optimal solutions, with the grade of membership of a solution representing the complement of the degree to which it is dominated by other solutions.

1. INTRODUCTION

The past two decades have witnessed many important theoretical

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advances in decision theory [1-18] as well as in such related fields as mathematical programming, statistical analysis, system simulation, game theory and optimal control. And yet, there are many observers who would agree that it is by no means easy to find concrete examples of successful applications of decision theory in practice. What, then, is the reason for the paucity of practical applications of a wide-ranging theory that had its inception more than three decades ago?

Although this as yet may not be a widely accepted view, our belief is that the limited applicability of decision theory to real-world problems is largely due to the fact that decision theory -- like most other mathematical theories of rational behavior -- fails to come to grips with the pervasive fuzziness and imprecision of human judgment, perception and modes of reasoning.[1] Thus, based as it is on the foundations of classical mathematics, decision theory aims at constructing a model of rational decision-making which is quantitative, rigorous and precise. Unfortunately, this may well be an unrealizable objective, for real-world decision processes are, for the most part, far too complex and much too ill-defined to be dealt with in this spirit. Indeed, to be able to cope with real-world problems, the mathematical theories of human cognition and rational behavior may have to undergo an extensive restructuring -- a restructuring which would entail an abandonment of the unrealistically high standards of precision which have become the norm in the literature and an acceptance of modes of logical inference which are approximate rather than exact.

The linguistic approach outlined in the present paper may be viewed as a step in this direction. In a sharp break with deeply entrenched traditions in science, the linguistic approach abandons the use of numbers and precise models of reasoning, and adopts instead a flexible system of verbal characterizations which apply to the values of variables, the relations between variables and the truth-values as well as the probabilities of assertions about them. The rationale for this seemingly retrograde step of employing words in place of numbers is that verbal characterizations are intrinsically approximate in nature and hence are better suited for the description of systems and processes which are as complex and as ill-defined as those which relate to human judgment and decision-making.

It should be stressed, however, that the linguistic approach

[1]In fact, far from being a negative characteristic of human thinking -- as it is usually perceived to be -- fuzziness may well be the key to the human ability to cope with problems (e.g., language translation, summarization of information, etc.) which are too complex for solution by machines that lack the capability to operate in a fuzzy environment.

is not the traditional non-mathematical way of dealing with humanistic systems. Rather, it represents a blend between the quantitative and the qualitative, relying on the use of words when numerical characterizations are not appropriate and using numbers to make the meaning of words more precise [19,20].

The central concept in the linguistic approach is that of a linguistic variable, that is, a variable whose values are words or structured combinations of words whose meaning is defined by a semantic rule [20]. For example, Age is a linguistic variable if its values are assumed to be young, not young, very young, not very young, more or less young, etc., rather than the numbers 0,1,2,..., 100. The meaning of a typical linguistic value, say not very young, is assumed to be a fuzzy subset of a universe of discourse, e.g., $U = [0,100]$, with the understanding that the meaning of not very young can be deduced from the meaning of young by the application of a semantic rule which is associated with the variable Age. In this sense, then, young is a primary term which plays a role akin to that of a unit of measurement. However, it is important to note that (a) the definition of young is purely subjective in nature; and (b) in contrast to the way in which the conventional units are used, the semantic rule involves nonlinear rather than linear operations on the meaning of the primary terms. These issues are discussed in greater detail in Section 2.

An important part of the linguistic approach relates to the treatment of truth as a linguistic variable with values such as true, very true, not very true, more or less true, etc. The use of such linguistic truth-values leads to what is called fuzzy logic [21] which provides a basis for approximate inference from possibly fuzzy premises whose validity may not be sharply defined. As an illustration, an approximate inference from (a) x is a small number, and (b) x and y are approximately equal, might be (c) y is more or less small. Similarly, an approximate inference from (a) (x is a small number) is very true, and (b) (x and y are approximately equal) is very true, might be (c) (y is more or less small) is true. In these assertions, small is assumed to be a specified fuzzy subset of the real line $R \triangleq (-\infty,\infty)$; approximately equal is a binary fuzzy relation in $R \times R$; and true and very true are fuzzy subsets of the unit interval $[0,1]$.[2] Because of limitations on space, we shall not discuss the applications of fuzzy logic to decision analysis in the present paper.

[2] A brief exposition of the basic properties of fuzzy sets is contained in the Appendix. A more detailed discussion of various aspects of the theory of fuzzy sets and its applications may be found in [22]. The most comprehensive treatise on the theory of fuzzy sets is the five-volume work of A. Kaufmann [23]. Some of the applications of the theory of fuzzy sets to decision analysis are discussed in [24-32].

Insofar as decision analysis is concerned, the linguistic approach serves, in the main, to provide a language for an approximate characterization of those components of a decision process which are either inherently fuzzy or are incapable of precise measurement. For example, if the probability of an outcome of a decision is not known precisely, it may be described in linguistic terms as likely or not very likely or very unlikely or more or less likely, and so forth. Or, if the degree to which an alternative α is preferred to an alternative β is not well-defined, it may be assigned a linguistic value such as strong or very strong or mild or very weak, etc. Similarly, a fuzzy relation between two variables x and y may be described in linguistic terms as "x is much larger than y" or "If x is small then y is large else x is approximately equal to y," etc.

As will be seen in Section 2, a linguistic characterization such as "x is small" may be viewed as a fuzzy restriction on the values of x. What is important to realize is that the assertion "x is small" conveys no information concerning the probability distribution of x; what it means, merely, is that "x is small" induces an elastic constraint on the values that may be assigned to x. Thus, if small is a fuzzy set in R whose membership function takes the value, say, 0.6 at $x = 8$, then the degree to which the constraint "x is small" is satisfied when the value 8 is assigned to x, is 0.6.

In what follows, we shall outline the main features of the linguistic approach and indicate some of its possible applications to decision analysis. It should be stressed that such applications are still in an exploratory stage and experience in the use of the linguistic approach may well suggest substantive changes in its implementation.[3]

2. LINGUISTIC VARIABLES AND FUZZY RESTRICTIONS

As stated in the Introduction, a linguistic variable is a variable whose values are words or sentences which serve as names of fuzzy subsets of a universe of discourse. In more specific terms, a linguistic variable is characterized by a quintuple (X,T(X),U,G,M) in which X is the name of the variable, e.g., Age; T(X) is the term-set of X, that is, the collection of its linguistic

[3] The linguistic approach has been applied to various problems in situation calculus by Yu. Klikov, G. Pospelov, D. Pospelov, V. Pushkin, D. Shapiro and others at the Computing Center of the Academy of Sciences, Moscow, under the direction of N.N. Moyseev. Other types of applications of the linguistic approach have recently been reported by P. King and E. Mamdani [33], R. Assilian [34], G. Retherford and G. Bloore [35], F. Wenstop [36], L. Pun [37], V. Dimitrov, W. Wechler and P. Barnev [38], and others.

values, e.g., T(X) = {young, not young, very young, not very young, ...}; U is a universe of discourse, e.g., in the case of Age, the set {0,1,2,3,...}; G is a syntactic rule which generates the terms in T(X); and M is a semantic rule which associates with each term, x, in T(X) its meaning, M(x), where M(x) denotes a fuzzy subset of U. Thus, the meaning, M(x), of a linguistic value, x, is defined by a compatibility -- or, equivalently, membership -- function $\mu_x: U \rightarrow [0,1]$ which associates with each u in U its compatibility with x. For example, the meaning of young might be defined in a particular context by the compatibility function

$$\mu_{\underline{young}}(u) = 1 \qquad \text{for} \quad 0 \le u \le 20 \tag{2.1}$$
$$= \frac{1}{1+\left(\frac{u-20}{10}\right)^2} \qquad \text{for} \quad u \ge 20$$

which may be viewed as the membership function of the fuzzy subset young of the universe of discourse U = $[0,\infty)$. Thus, the compatibility of the age 27 with young is approximately 0.66, while that of 30 is 0.5. The variable $u \in U$ is termed the base variable of X. The value of u at which $\mu_x(u) = 0.5$ is the cross-over point of x.

If X were a numerical variable, the assignment of a value, say a, to X would be expressed as

$$X = a\ . \tag{2.2}$$

In the case of linguistic variables, the counterpart of the assignment equation (2.2) is the proposition "X is x," where x is a linguistic value of X. From this point of view, x may be regarded as a fuzzy restriction on the values of the base variable u. This fuzzy restriction, which is denoted by $R_x(u)$ (or simply R(u)), is identical with the fuzzy subset M(x) which is the meaning of x. Thus, the proposition "X is x" translates into the relational assignment equation[4]

$$R(u) = x \tag{2.3}$$

which signifies that the proposition "X is x" may be interpreted as an elastic constraint on the values that may be assigned to u, with the membership function of x characterizing the compatibility, $\mu_x(u)$, of u with x.

As an illustration, consider the proposition "Edward is young." The translation of this proposition reads

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[4]As will be seen later, a relational assignment equation involves, more generally, the assignment of a fuzzy relation to a fuzzy restriction on the values of a base variable [39].

$$R(\text{Age}(\text{Edward})) = \underline{\text{young}} \tag{2.4}$$

where Age(Edward) is a numerical variable ranging over $[0,\infty)$, R(Age(Edward)) is a fuzzy restriction on its values, and young is a fuzzy subset of $[0,\infty)$ whose membership function is given by (2.1). To simplify the notation, a relational assignment equation such as (2.4) may be written as

$$\text{Age}(\text{Edward}) = \underline{\text{young}} \tag{2.5}$$

with the understanding that young is assigned not to the variable Age(Edward) but to the restriction on its values.

In this sense, each term, x, in the term-set of a linguistic variable X corresponds to a fuzzy restriction, R(u), on the values that may be assigned to the base variable u. A key idea behind the concept of a linguistic variable is that these fuzzy restrictions may be deduced from the fuzzy restrictions associated with the so-called primary terms in T(X). In effect, these fuzzy restrictions play the role of units which, upon calibration, make it possible to compute the meaning of the composite (that is non-primary) values of X from the knowledge of the meaning of the primary terms.

As an illustration of this technique, we shall consider an example in which $U = [0,\infty)$ and the term-set of X is of the form

$$T(X) = \{\underline{\text{small}}, \underline{\text{not}}\ \underline{\text{small}}, \underline{\text{very}}\ \underline{\text{small}}, \underline{\text{very}}\ (\underline{\text{not}}\ \underline{\text{small}}), \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{small}}, \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{small}}, \ldots\} \tag{2.6}$$

in which small is the primary term.

The terms in T(X) may be generated by a context-free grammar [40] $G = (V_T, V_N, S, P)$ in which the set of terminals, V_T, comprises (,), the primary term small and the linguistic modifiers very and not; the nonterminals are denoted by S, A and B and the production system is given by:

$$\begin{aligned} S &\rightarrow A \\ S &\rightarrow \underline{\text{not}}\ A \\ A &\rightarrow B \\ B &\rightarrow \underline{\text{very}}\ B \\ B &\rightarrow \underline{(S)} \\ B &\rightarrow \underline{\text{small}} \end{aligned} \tag{2.7}$$

Thus, a typical derivation yields

$$S \rightarrow \underline{\text{not}}\ A \rightarrow \underline{\text{not}}\ B \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ B \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{very}}\ B \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{small}} \tag{2.8}$$

In this sense, the syntactic rule associated with X may be viewed as the process of generating the elements of T(X) by a succession of substitutions involving the productions in G.

As for the semantic rule, we shall assume for simplicity that if μ_A is the membership function of A then the membership functions of not A and very A are given respectively by[5]

$$\mu_{\underline{\text{not}}\ A} = 1 - \mu_A \tag{2.9}$$

and

$$\mu_{\underline{\text{very}}\ A} = (\mu_A)^2 \ . \tag{2.10}$$

Thus, (2.10) signifies that the modifier very has the effect of squaring the membership function of its operand.

Suppose that the meaning of small is defined by the compatibility (membership) function

$$\mu_{\underline{\text{small}}}(u) = (1+(0.1u)^2)^{-1} \ , \quad u \geq 0 \ . \tag{2.11}$$

Then the meaning of very small is given by

$$\mu_{\underline{\text{very}}\ \underline{\text{small}}} = (1+(0.1u)^2)^{-2} \tag{2.12}$$

while the meanings of not very small and very (not small) are expressed respectively by

$$\mu_{\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{small}}} = 1 - (1+(0.1u)^2)^{-2} \tag{2.13}$$

and

$$\mu_{\underline{\text{very}}\ (\underline{\text{not}}\ \underline{\text{small}})} = (1 - (1+(0.1u)^2)^{-1})^2 \ . \tag{2.14}$$

In this way, we can readily compute the expression for the membership function of any term in T(X) from the knowledge of the membership function of the primary term small.

In effect, a linguistic variable X may be viewed as a microlanguage whose syntax and semantics are represented, respectively, by the syntactic and semantic rules associated with X. The sentences of this language are the linguistic values of X, with the meaning of each sentence represented as a fuzzy restriction on the values that may be assigned to the base variable, $u \in U$, of X.

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[5] A more detailed discussion of the effect of linguistic modifiers (hedges) may be found in [41], [42], and [43].

In the characterization of a decision process, we usually have to deal with a collection of interrelated linguistic variables. In this connection, it is helpful to have a set of rules for translating a proposition involving two or more linguistic variables into a set of relational assignment equations. The rules in question are as follows.[6]

Let X and Y be linguistic variables associated with possibly distinct universes of discourse U and V, and let P and Q be fuzzy subsets of U and V, respectively. Then, the conjunctive proposition p defined by

$$p \triangleq X \text{ is } P \text{ and } Y \text{ is } Q \tag{2.15}$$

translates into the relational assignment equation

$$R_p(u,v) = P \times Q \tag{2.16}$$

where R(u,v) is the restriction on the values that may be assigned to the ordered pair (u,v), $u \in U$, $v \in V$, and $P \times Q$ denotes the cartesian product of P and Q. Equivalently, (2.16) may be expressed as

$$R(u,v) = \bar{P} \cap \bar{Q} \tag{2.17}$$

where $\bar{P}$ and $\bar{Q}$ are the cylindrical extensions of P and Q, respectively, and $\bar{P} \cap \bar{Q}$ is their intersection. (See Appendix.)

Similarly, the disjunctive proposition

$$p \triangleq X \text{ is } P \text{ or } Y \text{ is } Q \tag{2.18}$$

translates into

$$R_p(u,v) = \bar{P} \cup \bar{Q} \tag{2.19}$$

where $\bar{P} \cup \bar{Q}$ is the union of the cylindrical extensions of P and Q.

The conditional proposition

$$p \triangleq \text{If } X \text{ is } P \text{ then } Y \text{ is } Q \tag{2.20}$$

translates into

$$R_p(u,v) = \bar{P}' \oplus \bar{Q} \tag{2.21}$$

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[6] Such rules will be referred to as semantic rules of Type II when it is necessary to distinguish them from the semantic rules which apply to individual variables (i.e., semantic rules of Type I).

where $\bar{P}'$ is the complement of $\bar{P}$ and $\oplus$ denotes the bounded sum. (See A36). More generally, the conditional proposition

$$p \triangleq \text{If } X \text{ is } P \text{ then } Y \text{ is } Q \text{ else } Y \text{ is } R \tag{2.22}$$

translates into

$$R_p(u,v) = (\bar{P}' \oplus \bar{Q}) \cap (\bar{P} \oplus R) \ . \tag{2.23}$$

Eq. (2.23) follows from (2.21) by the application of (2.15) and the fact that

$$p \triangleq X \text{ is not } P \tag{2.24}$$

translates into

$$R_p(u) = P' \tag{2.25}$$

where P' is the complement of P.

In cases where a linguistic truth-value, τ, such as <u>true</u>, <u>very true</u>, <u>more or less true</u>, etc. is associated with a proposition, as in

$$p \triangleq (X \text{ is } \underline{\text{small}}) \text{ is } \underline{\text{very true}} \tag{2.26}$$

the following rule of <u>truth-functional modification</u> may be used to translate p into a relational assignment equation:

$$p \triangleq (X \text{ is } A) \text{ is } \tau \tag{2.27}$$

translates into

$$R_p(u) = \mu_A^{-1} * \tau \tag{2.28}$$

where μ_A^{-1} is the inverse of μ_A and $*$ denotes the composition of the binary relation μ_A^{-1} with the unary fuzzy relation τ. (See A60.) It can readily be verified that the membership function of $R_p(u)$ is given by

$$\mu_{R_p}(u) = \mu_\tau(\mu_A(u)) \ , \quad u \in U \tag{2.29}$$

where μ_τ is the membership function of the linguistic truth-value τ and μ_A is that of A.

The basic translation rules stated above may be employed, in combination, to translate more complex propositions involving relations between two or more variables. As an illustration, consider the following proposition:

$$\pi \triangleq X \text{ is } \underline{\text{large}} \text{ and } Y \text{ is } \underline{\text{small}} \text{ or } X \text{ is } \underline{\text{not}}\ \underline{\text{large}} \text{ and } Y \text{ is } \underline{\text{very}}\ \underline{\text{small}} \tag{2.30}$$

which may be regarded as a linguistic characterization of the table shown below:

(2.31)

X	Y
large	small
not large	very small

For simplicity we shall assume that $U = V = \{0,1,2,4\}$ and that small and large are fuzzy sets defined by (see Appendix)

$$\underline{\text{small}} = 1/1 + 0.6/2 + 0.2/3 \ , \tag{2.32}$$

$$\underline{\text{large}} = 0.3/2 + 0.7/3 + 1/4 \ . \tag{2.33}$$

In this case, the application of (2.15) and (2.18) leads to the following expression for the restriction on (u,v) which is induced by the proposition in question:

$$R_p(u,v) = \underline{\text{large}} \times \underline{\text{small}} + \underline{\text{not}}\ \underline{\text{large}} \times \underline{\text{very}}\ \underline{\text{small}} \tag{2.34}$$

where $\times$ and $+$ represent the cartesian product and the union, respectively. Now, from (2.9) and (2.10) it follows that

$$\underline{\text{not}}\ \underline{\text{large}} = 1/1 + 0.7/2 + 0.3/3 \tag{2.35}$$

$$\underline{\text{very}}\ \underline{\text{small}} = 1/1 + 0.36/2 + 0.04/3 \tag{2.36}$$

and hence

$$\begin{aligned} R_p(u,v) = \ & 0.3/(2,1) + 0.7/(3,1) + 1/(4,1) \\ & + 0.3/(2,2) + 0.6/(3,2) + 0.6/(4,2) \\ & + 0.2/(2,3) + 0.2/(3,3) + 0.2/(1,3) \end{aligned} \tag{2.37}$$

in which a term such as 0.6/(3,2) signifies that the compatibility of the assignments $u = 3$ and $v = 2$ with p is 0.6.

As a further illustration, consider the proposition

$$q \triangleq \text{If } (X \text{ is } \underline{\text{large}} \text{ and } Y \text{ is } \underline{\text{small}} \text{ or } X \text{ is } \underline{\text{not}}\ \underline{\text{large}} \text{ and } Y \text{ is } \underline{\text{very}}\ \underline{\text{small}}) \text{ then } Z \text{ is } \underline{\text{very}}\ \underline{\text{small}} \tag{2.38}$$

in which the proposition in parentheses is that of the preceding example and the universe of discourse associated with Z is assumed to be the same as U.

In this case, using (2.20), we have

$$R_q(u,v,w) = \bar{R}'_p(u,v) \oplus \overline{\underline{\text{very}}\ \underline{\text{small}}} \tag{2.39}$$

where

$$\begin{aligned}\bar{R}_p(u,v) &= 0.3/\big((2,1,1)+(2,1,2)+(2,1,3)+(2,1,4)\big)+\cdots \\ &\quad + 0.2/\big((4,3,1)+(4,3,2)+(4,3,3)+(4,3,4)\big)\ ;\end{aligned} \tag{2.40}$$

$$\begin{aligned}\overline{\underline{\text{very}}\ \underline{\text{small}}} &= 1/\big((1,1,1)+(1,1,2)+(1,1,3)+(1,1,4)+ \\ &\quad +(1,2,1)+(1,2,2)+(1,2,3)+(1,2,4)+ \\ &\quad +(1,4,1)+(1,4,2)+(1,4,3)+(1,4,4)\big)+\cdots \\ &\quad + 0.04/\big((3,1,1)+(3,1,2)+(3,1,3)+(3,1,4)+ \\ &\quad +(3,4,1)+(3,4,2)+(3,4,3)+(3,4,4)\big)\ ;\end{aligned} \tag{2.41}$$

$\bar{R}'_p(u,v)$ is the complement of $\bar{R}_p(u,v)$ and $\oplus$ is defined by (A36).

To illustrate the rule of truth-functional modification, consider the proposition

$$p \triangleq (X \text{ is } \underline{\text{small}}) \text{ is } \underline{\text{very}}\ \underline{\text{true}} \tag{2.42}$$

where small is defined by (2.32) and

$$\underline{\text{true}} \triangleq 0.2/0.6 + 0.5/0.8 + 0.8/0.9 + 1/1\ . \tag{2.43}$$

In this case,

$$\underline{\text{very}}\ \underline{\text{true}} = 0.04/0.6 + 0.025/0.8 + 0.64/0.9 + 1/1 \tag{2.44}$$

and (2.29) yields

$$\begin{aligned}\mu_p(1) &= 1 \\ \mu_p(2) &= 0.2 \\ \mu_p(3) &= \mu_p(4) = 0\end{aligned} \tag{2.45}$$

which means that the compatibility of the assignment u = 2 with p is 0.2, while those of u = 3 and u = 4 are zero.

The above examples serve to illustrate one of the central features of the linguistic approach, namely, the mechanism for translating a proposition expressed in linguistic terms into a fuzzy restriction on the values which may be assigned to a set of base variables. Once the translation has been performed, the resulting fuzzy restrictions may be manipulated to yield the restrictions on whichever variables may be of interest. These restrictions, then, are translated into linguistic terms, yielding the final solution to the problem at hand.

In what follows, we shall illustrate this process by a few simple applications which are of relevance to decision analysis.

3. LINGUISTIC CHARACTERIZATION OF OBJECTIVE FUNCTIONS

In the literature of mathematical programming and decision analysis, it has become a universal practice to assume that the objective and utility functions are numerical functions of their arguments.

In most real-world problems, however, our perceptions of the consequences of a decision are not sufficiently precise or consistent to justify the assignment of numerical values to utilities or preferences. Thus, in most cases it would be more realistic to assume that the objective function is a linguistic function of the linguistic values of its arguments, and employ the techniques of the linguistic approach to assess the consequences of a particular choice of decision variables.

To be more specific, consider a simple case of a decision process in which the objective function $G(u_1,\dots,u_n)$ takes values in a space V while the decision variables $u_1,\dots,u_n$ take values in $U_1,\dots,U_n$, respectively. To simplify the discussion, we shall assume that $U_1 = U_2 = \cdots = U_n = U$.

The linguistic values of decision variables as well as those of the objective function are assumed to be of the form {<u>low</u>, <u>not low</u>, <u>very low</u>, <u>not very low</u>, ..., <u>medium</u>, <u>high</u>, <u>not high</u>, <u>very high</u>, <u>not very high</u>, <u>not low and not high</u>, <u>not very low and not very high</u>, ...}. It can readily be verified that these linguistic values can be generated by a context-free grammar whose production system is given below:

$$
\begin{array}{ll}
S \to A & C \to \underline{\text{very}}\ C \\
S \to S\ \underline{\text{and}}\ A & D \to \underline{\text{very}}\ D \\
A \to B & C \to \underline{\text{low}} \\
A \to \underline{\text{not}}\ B & D \to \underline{\text{high}} \\
B \to C & \\
B \to D & \\
B \to \underline{\text{medium}} &
\end{array}
\tag{3.1}
$$

in which S, A, B, C, D are non-terminals, S is the starting symbol, and <u>and</u>, <u>not</u>, <u>very</u>, <u>low</u>, <u>medium</u> and <u>high</u> are terminals, with <u>low</u>, <u>medium</u> and <u>high</u> playing the role of primary terms.

The simplicity of this grammar makes it possible to compute the meaning of various linguistic values by inspection. For example, the meaning of the value <u>not very low and not high</u> is given by

$$M(\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{low}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{high}}) = (\underline{\text{low}}^2)' \cap (\underline{\text{high}}') \tag{3.2}$$

where $\underline{\text{low}}^2$ is a fuzzy set whose membership function is the square of that of <u>low</u>, and ' and $\cap$ denote the complement and intersection, respectively.

It is important to note that the assumption that all of the decision variables and the objective function have the same term-set does not imply that the corresponding primary terms are also identical. Thus, <u>low</u>, for example, in the case of i-th decision variable need not have the same meaning as <u>low</u> for j-th decision variable ($j \neq i$) or G. To illustrate this point, suppose that $U = \{1,2,3,4\}$. Then <u>low</u> for u_1 might be defined as

$$\underline{\text{low}} = 1/1 + 0.8/2 + 0.2/3 \tag{3.3}$$

whereas <u>low</u> for u_2 may be

$$\underline{\text{low}} = 1/1 + 0.6/2 + 0.1/3\ . \tag{3.4}$$

Typically, a tabulation of the linguistic values of G as a function of the linguistic values of the decision variables would have a form such as shown below ($\underline{\text{low}}^2 \triangleq \underline{\text{very}}\ \underline{\text{low}}$, $\underline{\text{med}} \triangleq \underline{\text{medium}}$)

(3.5)

u_1	u_2	G
<u>low</u>	<u>low</u>	$\underline{\text{low}}^2$
<u>low</u>	<u>med</u>	<u>low</u>
<u>low</u>	$\underline{\text{low}}^2$	<u>not</u> <u>low</u>
--	--	--
<u>high</u>	<u>high</u>	$\underline{\text{high}}^2$

It should be noted that, in general, not all of the possible combinations of the linguistic values of decision variables will appear in the tableau of G.

The definition of G by a tableau of the form (3.5) induces a fuzzy restriction on the values that may be assigned to the decision variables and G. More specifically, let ℓ_{ij} denote the linguistic value of j-th decision variable in the i-th row of the tableau, and let v_i be the corresponding linguistic value of the objective function. Then the fuzzy restriction in question is expressed by

$$R_G(u_1,\ldots,u_n,v) = \ell_{11} \times \cdots \times \ell_{1n} \times v_1 + \cdots + \ell_{m1} \times \cdots \times \ell_{mn} \times v_m \tag{3.6}$$

where m is the number of rows in the tableau, and × and + denote the cartesian product and union, respectively. The fuzzy restriction $R_G(u_1,\ldots,u_n,v)$ on the values of $u_1,\ldots,u_n$ and v may be viewed as the meaning of the tableau of G in the same sense as the translation rules (2.15-2.29) express the meaning of various propositions as fuzzy restrictions on the values of the base variables.[7]

As a very simple illustration of (3.6), assume that the tableau of G is given by

u_1	u_2	G
low	low	$\underline{\text{low}}^2$
low	low'	low'

(3.7)

where low for u_1 and u_2 is defined by (3.3) and (3.4), respectively, and low for G has the same meaning as for u_2.

In this case, we have

$$\begin{aligned} R_G(u_1,u_2,v) &= (1/1+0.8/2+0.2/3) \times (1,1+0.6/2+0.1/3) \\ &\quad \times (1/1+0.36/2+0.01/3) \\ &\quad + (1/1+0.8/2+0.2/3) \times (0.4/2+0.9/3+1/4) \\ &\quad \times (0.4/2+0.9/3+1/4) \\ &= 1/(1,1,1)+0.6/(1,2,1)+0.36(1,1,2)+\cdots \\ &\quad + 0.9/(1,3,3)+\cdots+1/(1,4,4) \ . \end{aligned} \tag{3.8}$$

Note that the restriction defined by (3.8) is a ternary fuzzy relation in $U_1 \times U_2 \times V$.

An important aspect of the linguistic definition of G is that it provides a basis for an interpolation of G for values of the decision variables which are not in the tableau. Thus, since the meaning of the tableau is provided by the fuzzy (n+1)-ary relation $R_G(u_1,\ldots,u_n,v)$, we can assert that the result of substitution of

[7] A more detailed discussion of this point may be found in [44].

arbitrary linguistic values $\bar{\ell}_{i1},\ldots,\bar{\ell}_{in}$ for $u_1,\ldots,u_n$ is the composition of R_G with $\bar{\ell}_{i1},\ldots,\bar{\ell}_{in}$. This implies that the value of G corresponding to the prescribed values of $u_1,\ldots,u_n$ is given by

$$G(\bar{\ell}_{i1},\ldots,\bar{\ell}_{in}) = R_G(u_1,\ldots,u_n,v)*\bar{\ell}_{i1}*\cdots*\bar{\ell}_{in} \tag{3.9}$$

where $*$ denotes the operation of composition.[8]

As a very simple illustration of (3.9), assume that $R_G(u_1,u_2,v)$ is expressed by (for simplicity, $U_1 = U_2 = V = \{1,2\}$)

$$\begin{aligned} R_G(u_1,u_2,v) &= 0.8/(1,1,1) + 0.9/(1,2,1) + 0.3/(2,1,1) \\ &\quad + 0.7/(2,2,1) + 0.3/(1,1,2) + 0.2/(1,2,2) \\ &\quad + 0.6/(2,1,2) + 0.5/(2,2,2) \end{aligned} \tag{3.10}$$

and that

$$\bar{\ell}_1 = 0.3/1 + 0.5/2 \tag{3.11}$$

$$\bar{\ell}_2 = 0.9/1 + 0.2/2\ . \tag{3.12}$$

The ternary fuzzy relation (3.10) may be represented as two matrices

$$A \triangleq \begin{bmatrix} 0.8 & 0.9 \\ 0.3 & 0.7 \end{bmatrix} \qquad B \triangleq \begin{bmatrix} 0.3 & 0.2 \\ 0.6 & 0.5 \end{bmatrix} \tag{3.13}$$

Forming the max-min products of A and B with the row matrix [0.3 0.5] we obtain the matrix

$$C \triangleq \begin{bmatrix} 0.3 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \tag{3.14}$$

and forming the max-min product of this matrix with the row matrix [0.9 0.2] we arrive at

$$D = [0.3 \quad 0.5] \tag{3.15}$$

which implies that the interpolated value of G is

$$G(\bar{\ell}_1,\bar{\ell}_2) = 0.3/1 + 0.5/2\ . \tag{3.16}$$

---

[8]In the terminology of relational models of data, (3.9) may be viewed as an extension to fuzzy relations of the operation of disjunctive mapping [45].

To express this result in linguistic terms, it is necessary to approximate to the right-hand member of (3.16) by a linguistic value which belongs to the term-set of G. The issue of linguistic approximation is discussed in greater detail in [20] and [36].

In summary, if the objective function is defined in linguistic terms by a tableau of the form (3.5), the fuzzy restriction on the values of the decision variables which is induced by the definition is an (n+1)-ary fuzzy relation in $U_1 \times \cdots \times U_n \times V$ which is expressed by (3.6). By the use of this relation, the objective function may be interpolated for values of the decision variables which are not in the original tableau.

4. OPTIMIZATION UNDER MULTIPLE CRITERIA

The linguistic approach appears to be particularly well-adapted to the analysis of decision processes in which the objective function is vector- rather than scalar-valued.[9] The reason for this is that when more than one criterion of performance is involved, the trade-offs between the criteria are usually poorly defined. In such cases, then, linguistic characterizations of trade-offs or preference relations provide a more realistic conceptual framework for decision analysis than the conventional methods employing binary-valued preference relations.

A detailed exposition of the application of the linguistic approach to the optimization under multiple criteria will be presented in a separate paper. In what follows, we shall merely sketch very briefly the main ideas behind the method.

To simplify the notation, we shall assume that there are only two decision variables and two real-valued objective functions G_1 and G_2. The values of G_1 and G_2 at the points (u_1^1,u_2^1) and (u_1^2,u_2^2) are denoted by G_1^1 and G_2^2, respectively.

In the conventional formulation of the problem, the partial ordering defined by

$$(G_1^1,G_2^1) > (G_1^2,G_2^2) \Leftrightarrow G_1^1 \geq G_1^2 \text{ and } G_2^1 \geq G_2^2 \tag{4.1}$$

induces a pre-ordering in $U_1 \times U_2$ defined by

$$(u_1^1,u_2^1) > (u_1^2,u_2^2) \Leftrightarrow (G_1^1,G_2^1) > (G_1^2,G_2^2) . \tag{4.2}$$

[9] The literature on the optimization under multiple criteria is quite extensive. Of particular relevance to the discussion in this section are the references [46]-[48].

With each point (u_1^0,u_2^0) in $U_1 \times U_2$, we can associate the set of points $D(u_1^0,u_2^0)$ which dominate it; that is,

$$D(u_1^0,u_2^0) = \{(u_1,u_2) \mid (u_1,u_2) > (u_1^0,u_2^0)\} \quad . \qquad (4.3)$$

If C is a constraint set in $U_1 \times U_2$, then a point (u_1^0,u_2^0) in C is undominated if and only if the intersection of C with $D(u_1^0,u_2^0)$ is the singleton $\{(u_1^0,u_2^0)\}$. The set of all undominated points in C is the set of Pareto-optimal solutions to the optimization under the objective functions G_1 and G_2.

Generally, additional assumptions are made to induce a linear ordering in the set of undominated points in C or, at least, to disqualify some of the points in this set from contention as solutions to the optimization problem. The main shortcoming of these techniques is that the assumptions needed to induce a linear ordering tend to be rather arbitrary and hard to justify.

In the linguistic approach to this problem, the Pareto-optimal set is fuzzified and its size is "reduced" by making use of whatever information might be available regarding the trade-offs between G_1 and G_2. Since the trade-offs are usually poorly defined, they are allowed to be expressed in linguistic terms. Generally, the trade-offs are assumed to be defined indirectly via fuzzy preference relations [49], although in some cases it may be possible to define an overall objective function directly as a linguistic function of the linguistic values of G_1 and G_2.

As a simple illustration, a linguistic characterization of a fuzzy preference relation might have the following form.

Assume that the strength of preference is a linguistic variable whose values are <u>strong</u>, <u>very strong</u>, <u>not strong</u>, <u>not very strong</u>, <u>weak</u>, <u>not very strong and not very weak</u>, ... in which the primary terms <u>strong</u> and <u>weak</u> are fuzzy subsets of the unit interval. The meaning of such linguistic values may be computed in exactly the same way as the meaning of the linguistic values of X in Example 2.6.

Let ρ denote the degree to which (u_1^1,u_2^1) is preferred to (u_1^2,u_2^2). Then, a partial linguistic characterization of ρ may be expressed as:

If (G_1^1 is <u>much</u> <u>larger</u> than G_1^2 and (4.4)

G_2^1 is <u>approximately</u> <u>equal</u> to G_2^2 or

G_2^1 is <u>much</u> <u>larger</u> than G_2^2 and

G_1^1 is <u>approximately</u> <u>equal</u> to G_1^2)

then ρ is <u>strong</u> .

In this expression, the terms <u>much</u> <u>larger</u> and <u>approximately</u> <u>equal</u> play the role of linguistic values of a fuzzy binary relation in $V \times V$, while <u>strong</u> is a linguistic value of ρ. By the use of appropriate semantic rules, the expression in question can be translated into a fuzzy restriction on 5-tuples of the form $(u_1^1,u_2^1,u_1^2,u_2^2,v)$. Combined with similar fuzzy restrictions resulting from whatever other linguistic characterizations of ρ might be available, (4.4) yields a fuzzy preference relation ρ in $U_1 \times U_2 \times U_1 \times U_2 \times V$ which provides a basis for fuzzifying the Pareto-optimal set and thereby reducing the degree to which some of the points in this set may be regarded as contenders for inclusion in the set of optimal solutions.

More specifically, let $u^0 \triangleq (u_1^0,u_2^0)$ be a point in $U_1 \times U_2$. Furthermore, let $D(u^0)$ be the fuzzy set of points in $U_1 \times U_2$ which results from setting u^1 equal to u^0 in the fuzzy preference relation ρ. As in (4.3), $D(u^0)$ is the fuzzy set of points which dominate u^0.

It will be recalled that when $D(u^0)$ is a non-fuzzy set, the point u^0 is undominated and hence an element of the Pareto-optimal set if and only if the intersection of $D(u^0)$ with the constraint set C is the singleton $\{u^0\}$. More generally, if $D(u^0)$ is a fuzzy set then the degree to which u^0 belongs to the fuzzy Pareto-optimal set, P, may be related to the height[10] of the fuzzy set $D(u^0) \cap C - \{u^0\}$ by the relation

$$\mu_P(u^0) = 1 - \sup_{u^1} \left(D(u^0) \cap C - \{u^0\}\right) . \tag{4.5}$$

In this sense, then, the Pareto-optimal set is fuzzified, with each point u^0 assigned a grade of membership in the fuzzy Pareto-optimal set by (4.5).

The fuzzification of the Pareto-optimal set has the effect of reducing the degree of contention for optimality of those points

[10] The height of a fuzzy set is the supremum of its membership function over the universe of discourse.

which have a low grade of membership in the set. In general, the extent to which the size of the Pareto-optimal set is reduced in this fashion depends on the linguistic information provided by the trade-offs. Thus, if the fuzzy restrictions which are associated with the translations of the linguistic statements about the trade-offs are only mildly restrictive -- which is equivalent to saying that they convey little information about the trade-offs -- then the reduction in the size of the Pareto-optimal set will, in general, be slight. By the same token, the opposite will be the case if the restrictions in question are highly informative -- that is, have the effect of assigning low grades of membership to most of the points in $U_1 \times U_2 \times V$.

In sketching the application of the linguistic approach to optimization under multiple criteria, we have side-stepped several non-trivial problems. In the first place, the preference relation ρ which results from translation of linguistic propositions of the form (4.4) is a fuzzy set of Type 2 (i.e., has a fuzzy-set-valued membership function), which makes it more difficult to find the intersection of $D(u_0)$ with the constraint set as well as to compute the grade of membership of u^0 in the fuzzy set of Pareto-optimal solutions. Secondly, the preference relation represented by ρ may not be transitive (in the sense defined in [49]), in which case it may be necessary to construct the transitive closure of ρ. And finally, it may not be a simple matter to apply linguistic approximation to $\mu_P(u^0)$. Notwithstanding these difficulties, the linguistic approach sketched above or some variants of it may eventually provide a realistic way of dealing with practical problems involving decision-making under multiple criteria.

5. CONCLUDING REMARKS

In the foregoing discussion, we have attempted to outline some of the main ideas behind the linguistic approach and point to its possible applications in decision analysis. The specific problems discussed in Sections 3 and 4 are representative -- but not exhaustive -- of such applications. In particular, we have not touched upon the important subject of the manipulation of linguistic probabilities in problems of stochastic control nor upon the problem of multistage decision processes and inference from fuzzy data.

At this juncture, the linguistic approach to decision analysis is in its initial stages of development. Eventually, it may become a useful aid in decision-making relating to real-world problems.

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APPENDIX

FUZZY SETS - NOTATION, TERMINOLOGY AND BASIC PROPERTIES

The symbols U,V,W,..., with or without subscripts, are generally used to denote specific universes of discourse, which may be arbitrary collections of objects, concepts or mathematical constructs. For example, U may denote the set of all real numbers; the set of all residents in a city; the set of all sentences in a book; the set of all colors that can be perceived by the human eye, etc.

Conventionally, if A is a fuzzy subset of U whose elements are $u_1,\ldots,u_n$, then A is expressed as

$$A = \{u_1,\ldots,u_n\} . \tag{A1}$$

For our purposes, however, it is more convenient to express A as

$$A = u_1 + \cdots + u_n \tag{A2}$$

or

$$A = \sum_{i=1}^{n} u_i \tag{A3}$$

with the understanding that, for all i, j,

$$u_i + u_j = u_j + u_i \tag{A4}$$

and

$$u_i + u_i = u_i . \tag{A5}$$

As an extension of this notation, a finite <u>fuzzy</u> subset of U is expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A6}$$

or, equivalently, as

$$F = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{A7}$$

where the μ_i, $i=1,\ldots,n$, represent the <u>grades of membership</u> of the u_i in F. Unless stated to the contrary, the μ_i are assumed to lie in the interval [0,1], with 0 and 1 denoting <u>no</u> membership and <u>full</u> membership, respectively.

Consistent with the representation of a finite fuzzy set as a linear form in the u_i, an arbitrary fuzzy subset of U may be expressed in the form of an integral

$$F = \int_U \mu_F(u)/u \tag{A8}$$

in which $\mu_F\colon U \to [0,1]$ is the membership or, equivalently, the compatibility function of F; and the integral $\int_U$ denotes the union (defined by (A28)) of fuzzy singletons $\mu_F(u)/u$ over the universe of discourse U.

The points in U at which $\mu_F(u) > 0$ constitute the support of F. The points at which $\mu_F(u) = 0.5$ are the crossover points of F.

Example A9. Assume

$$U = a + b + c + d \;. \tag{A10}$$

Then, we may have

$$A = a + b + d \tag{A11}$$

and

$$F = 0.3a + 0.9b + d \tag{A12}$$

as nonfuzzy and fuzzy subsets of U, respectively.

If

$$U = 0 + 0.1 + 0.2 + \cdots + 1 \tag{A13}$$

then a fuzzy subset of U would be expressed as, say,

$$F = 0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1 \;. \tag{A14}$$

If $U = [0,1]$, then F might be expressed as

$$F = \int_0^1 \frac{1}{1+u^2} \;/\; u \tag{A15}$$

which means that F is a fuzzy subset of the unit interval [0,1] whose membership function is defined by

$$\mu_F(u) = \frac{1}{1+u^2} \;. \tag{A16}$$

In many cases, it is convenient to express the membership function of a fuzzy subset of the real line in terms of a standard function whose parameters may be adjusted to fit a specified membership function in an approximate fashion. Two such functions are defined below.

$$
\begin{aligned}
S(u;\alpha,\beta,\gamma) &= 0 && \text{for} \quad u \le \alpha && \text{(A17)}\\
&= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \text{for} \quad \alpha \le u \le \beta \\
&= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for} \quad \beta \le u \le \gamma \\
&= 1 && \text{for} \quad u \ge \gamma
\end{aligned}
$$

$$
\begin{aligned}
\pi(u;\beta,\gamma) &= S(u;\gamma-\beta,\gamma-\tfrac{\beta}{2},\gamma) && \text{for} \quad u \le \gamma && \text{(A18)}\\
&= 1 - S(u;\gamma,\gamma+\tfrac{\beta}{2},\gamma+\beta) && \text{for} \quad u \ge \gamma \,.
\end{aligned}
$$

In $S(u;\alpha,\beta,\gamma)$, the parameter β, $\beta = \frac{a+\gamma}{2}$, is the crossover point. In $\pi(u;\beta,\gamma)$, β is the bandwidth, that is the separation between the crossover points of π, while γ is the point at which π is unity.

In some cases, the assumption that μ_F is a mapping from U to [0,1] may be too restrictive, and it may be desirable to allow μ_F to take values in a lattice or, more particularly, in a Boolean algebra. For most purposes, however, it is sufficient to deal with the first two of the following hierarchy of fuzzy sets.

Definition A19. A fuzzy subset, F, of U is of type 1 if its membership function, μ_F, is a mapping from U to [0,1]; and F is of type n, $n = 2,3,\ldots$, if μ_F is a mapping from U to the set of fuzzy subsets of type n-1. For simplicity, it will always be understood that F is of type 1 if it is not specified to be of a higher type.

Example A20. Suppose that U is the set of all nonnegative integers and F is a fuzzy subset of U labeled small integers. Then F is of type 1 if the grade of membership of a generic element u in F is a number in the interval [0,1], e.g.,

$$\mu_{\underline{\text{small}}\ \underline{\text{integers}}}(u) = \left(1 + \left(\frac{u}{5}\right)^2\right)^{-1}, \quad u = 0,1,2,\ldots \,. \qquad \text{(A21)}$$

On the other hand, F is of type 2 if for each u in U, $\mu_F(u)$ is a fuzzy subset of [0,1] of type 1, e.g., for u = 10,

$$\mu_{\underline{\text{small}}\ \underline{\text{integers}}}(10) = \underline{\text{low}} \qquad \text{(A22)}$$

where low is a fuzzy subset of [0,1] whose membership function is defined by, say,

$$\mu_{\underline{\text{low}}}(v) = 1 - S(v;0,0.25,0.5)\ , \quad v \in [0,1] \qquad \text{(A23)}$$

which implies that

$$\underline{\text{low}} = \int_0^1 \bigl(1 - S(v;0,0.25,0.5)\bigr)/v \ . \tag{A24}$$

If F is a fuzzy subset of U, then its α-level-set, F_α, is a nonfuzzy subset of U defined by

$$F_\alpha = \{u \mid \mu_F(u) \geq \alpha\} \tag{A25}$$

for $0 < \alpha \leq 1$.

If U is a linear vector space, the F is convex if and only if for all $\lambda \in [0,1]$ and all u_1, u_2 in U,

$$\mu_F\bigl(\lambda u_1 + (1-\mu)u_2\bigr) \geq \min\bigl(\mu_F(u_1),\mu_F(u_2)\bigr) \ . \tag{A26}$$

In terms of the level-sets of F, F is convex if and only if the F_α are convex for all $\alpha \in (0,1]$.[11]

The relation of containment for fuzzy subsets F and G of U is defined by

$$F \subset G \Leftrightarrow \mu_F(u) \leq \mu_G(u) \ , \quad u \in U \ . \tag{A27}$$

Thus, F is a fuzzy subset of G if (A27) holds for all u in U.

Operations on Fuzzy Sets

If F and G are fuzzy subsets of U, their union, $F \cup G$, intersection, $F \cap G$, bounded-sum, $F \oplus G$, and bounded-difference, $F \ominus G$, are fuzzy subsets of U defined by

$$F \cup G \triangleq \int_U \mu_F(u) \vee \mu_G(u)/u \tag{A28}$$

$$F \cap G \triangleq \int_U \mu_F(u) \wedge \mu_G(u)/u \tag{A29}$$

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[11] This definition of convexity can readily be extended to fuzzy sets of type 2 by applying the extension principle (see (A75)) to (A26).

$$F \oplus G \triangleq \int_U 1 \wedge \left(\mu_F(u) + \mu_G(u)\right)/u \tag{A30}$$

$$F \ominus G \triangleq \int_U 0 \vee \left(\mu_F(u) - \mu_G(u)\right)/u \tag{A31}$$

where $\vee$ and $\wedge$ denote max and min, respectively. The <u>complement</u> of F is defined by

$$F' = \int_U \left(1 - \mu_F(u)\right)/u \tag{A32}$$

or, equivalently,

$$F' = U \ominus F\ . \tag{A33}$$

It can readily be shown that F and G satisfy the identities

$$(F \cap G)' = F' \cup G' \tag{A34}$$

$$(F \cup G)' = F' \cap G' \tag{A35}$$

$$(F \oplus G)' = F' \ominus G \tag{A36}$$

$$(F \ominus G)' = F' \oplus G \tag{A37}$$

and that F satisfies the resolution identity

$$F = \int_0^1 \alpha F_\alpha \tag{A38}$$

where F_α is the α-level-set of F; αF_α is a set whose membership function is $\mu_{\alpha F_\alpha} = \alpha \mu_{F_\alpha}$, and $\int_0^1$ denotes the union of the αF, with $\alpha \in (0,1]$.

Although it is traditional to use the symbol $\cup$ to denote the union of nonfuzzy sets, in the case of fuzzy sets it is advantageous to use the symbol + in place of $\cup$ where no confusion with the arithmetic sum can result. This convention is employed in the following example, which is intended to illustrate (A28), (A29), (A30), (A31) and (A32).

<u>Example A39</u>. For U defined by (A10) and F and G expressed by

$$F = 0.4a + 0.9b + d \tag{A40}$$

$$G = 0.6a + 0.5b \tag{A41}$$

we have

$$F + G = 0.6a + 0.9b + d \tag{A42}$$

$$F \cap G = 0.4a + 0.5b \tag{A43}$$

$$F \oplus G = a + b + d \tag{A44}$$

$$F \ominus G = 0.4b + d \tag{A45}$$

$$F' = 0.6a + 0.1b + c \tag{A46}$$

The linguistic connectives *and* (conjunction) and *or* (disjunction) are identified with $\cap$ and $+$, respectively. Thus,

$$F \ \underline{\text{and}}\ G \triangleq F \cap G \tag{A47}$$

and

$$F \ \underline{\text{or}}\ G \triangleq F + G\ . \tag{A48}$$

As defined by (A47) and (A48), *and* and *or* are implied to be *noninteractive* in the sense that there is no "trade-off" between *their operands*. When this is not the case, *and* and *or* are denoted by <*and*> and <*or*>, respectively, and are defined in a way that reflects the nature of the trade-off. For example, we may have

$$F \ \langle\underline{\text{and}}\rangle\ G \triangleq \int_U \mu_F(u)\mu_G(u)/u \tag{A49}$$

$$F \ \langle\underline{\text{or}}\rangle\ G \triangleq \int_U \big(\mu_F(u) + \mu_G(u) - \mu_F(u)\mu_G(u)\big)/u \tag{A50}$$

whose + denotes the arithmetic sum. In general, the interactive versions of *and* and *or* do not possess the simplifying properties of the connectives defined by (A47) and (A48), e.g., associativity, distributivity, etc.

If α is a real number, then F^α is defined by

$$F^\alpha \triangleq \int_V \big(\mu_F(n)\big)^\alpha/u\ . \tag{A51}$$

For example, for the fuzzy set defined by (A40), we have

$$F^2 = 0.16a + 0.81b + d \tag{A52}$$

and

$$F^{1/2} = 0.63a + 0.95b + d \ . \qquad \text{(A53)}$$

These operations may be used to approximate, very roughly, the effect of the linguistic modifiers very and more or less. Thus,

$$\underline{\text{very}}\ F \triangleq F^2 \qquad \text{(A54)}$$

and

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F \triangleq F^{1/2} \ . \qquad \text{(A55)}$$

If $F_1,\ldots,F_n$ are fuzzy subsets of $U_1,\ldots,U_n$, then the cartesian product of $F_1,\ldots,F_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$F_1 \times \cdots \times F_n = \int_{U_1\times\cdots\times U_n} \left(\mu_{F_1}(u_1) \wedge \cdots \wedge \mu_{F_n}(u_n)\right) / (u_1,\ldots,u_n) \ . \qquad \text{(A56)}$$

As an illustration, for the fuzzy sets defined by (A40) and (A41), we have

$$\begin{aligned} F \times G &= (0.4a + 0.9b + d) \times (0.6a + 0.5b) \qquad \text{(A57)} \\ &= 0.4/(a,a) + 0.4/(a,b) + 0.6/(b,a) \\ &\quad + 0.5/(b,b) + 0.6/(d,a) + 0.5/(d,b) \end{aligned}$$

which is a fuzzy subset of $(a+b+c+d) \times (a+b+c+d)$.

Fuzzy Relations

An n-ary fuzzy relation R in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$. The projection of R on $U_{i_1} \times \cdots \times U_{i_k}$, where $(i_1,\ldots,i_k)$ is a subsequence of $(1,\ldots,n)$, is a relation in $U_{i_1} \times \cdots \times U_{i_k}$ defined by

$$\begin{aligned} &\text{Proj } R \text{ on } U_{i_1} \times \cdots \times U_{i_k} \qquad \text{(A58)} \\ &\triangleq \int_{U_{i_1}\times\cdots\times U_{i_k}} \vee_{u_{j_1},\ldots,u_{j_\ell}} \mu_R(u_1,\ldots,u_n) / (u_1,\ldots,u_n) \end{aligned}$$

where $(j_1,\ldots,j_\ell)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if $n=6$ then $(1,3,6)$ is complementary to $(2,4,5)$), and $\vee_{u_{j_1},\ldots,u_{j_\ell}}$ denotes the supremum over $U_{j_1} \times \cdots \times U_{j_\ell}$.

If R is a fuzzy subset of $U_{i_1},\ldots,U_{i_k}$, then its <u>cylindrical extension</u> in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$\bar{R} = \int_{U_1\times\cdots\times U_n} \mu_R(U_{i_1},\ldots,u_{i_k})/(u_1,\ldots,u_n) \ . \tag{A59}$$

In terms of their cylindrical extensions, the <u>composition</u> of two binary relations R and S (in $U_1 \times U_2$ and $U_2 \times U_3$, respectively) is expressed by

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \tag{A60}$$

where $\bar{R}$ and $\bar{S}$ are the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Similarly, if R is a binary relation in $U_1 \times U_2$ and S is a unary relation in U_2, their composition is given by

$$R \circ S = \text{Proj } R \cap \bar{S} \text{ on } U_1 \ . \tag{A61}$$

<u>Example A62</u>. Let R be defined by the right-hand member of (A57) and

$$S = 0.4a + b + 0.8d \ . \tag{A63}$$

Then

$$\text{Proj } R \text{ on } U_1 (\triangleq a+b+c+d) = 0.4a + 0.6b + 0.6d \tag{A64}$$

and

$$R \circ S = 0.4a + 0.5b + 0.5d \ . \tag{A65}$$

The Extension Principle

Let f be a mapping from U to V. Thus,

$$v = f(u) \tag{A66}$$

where u and v are generic elements of U and V, respectively.

Let F be a fuzzy subset of U expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A67}$$

or, more generally,

$$F = \int_U \mu_F(u)/u \ . \tag{A68}$$

By the extension principle, the image of F under f is given by

$$f(F) = \mu_1 f(u_1) + \cdots + \mu_n f(u_n) \tag{A69}$$

or, more generally,

$$f(F) = \int_U \mu_F(u)/f(u) \ . \tag{A70}$$

Similarly, if f is a mapping from $U \times V$ to W, and F and G are fuzzy subsets of U and V, respectively, then

$$f(F,G) = \int_W \big(\mu_F(u) \wedge \mu_G(v)\big)/f(u,v) \ . \tag{A71}$$

<u>Example A72</u>. Assume that f is the operation of squaring. Then, for the set defined by (A14), we have

$$\begin{aligned} & f(0.3/0.5+0.6/0.7+0.8/0.9+1/1) \\ & = 0.3/0.25+0.6/0.49+0.8/0.81+1/1 \ . \end{aligned} \tag{A73}$$

Similarly, for the binary operation V ($\triangleq$ max), we have

$$\begin{aligned} & (0.9/0.1+0.2/0.5+1/1) \vee (0.3/0.2+0.8/0.6) \\ & = 0.3/0.2+0.2/0.5+0.8/1+0.8/0.6+0.2/0.6 \ . \end{aligned} \tag{A74}$$

It should be noted that the operation of squaring in (A73) is different from that of (A51) and (A52).