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Abstract

Fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning.

The fuzzy logic, FL, which is described in this paper has the following principal features. (a) The truth-values of FL are fuzzy subsets of the unit interval carrying labels such as _true_, _very true_, _not very true_, _false_, _more or less true_, etc.; (b) The truth-values of FL are _structured_ in the sense that they may be generated by a grammar and interpreted by a semantic rule; (c) FL is a _local_ logic in that, in FL, the truth-values as well as the connectives such as _and_, _or_, _if...then_ have a variable rather than fixed meaning; and (d) The rules of inference in FL are approximate rather than exact.

The central concept in FL is that of a _fuzzy restriction_, by which is meant a fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable. Thus, a fuzzy proposition such as "Nina is _young_" translates into a relational assignment equation of the form R(Age(Nina)) = _young_ in which Age(Nina) is a variable, R(Age(Nina)) is a fuzzy restriction on the values of Age(Nina), and _young_ is a fuzzy unary relation which is assigned as a value to R(Age(Nina)).

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In general, a composite fuzzy proposition translates into a system of relational assignment equations. In this paper, translation rules are developed for propositions of four basic types: Type I, of the general form "X is mF," where X is the name of an object or a variable, m is a linguistic modifier, e.g., *not*, *very*, *more or less*, *quite*, etc., and F is a fuzzy subset of a universe of discourse. Type II, of the general form, "X is F * Y is G" or "X is in relation R to Y," where * is a binary connective, e.g., *and*, *or*, *if*...*then*, etc., and R is a fuzzy relation, e.g., *much greater*. Type III, of the general form "QX are F," where Q is a fuzzy quantifier, e.g., *some*, *most*, *many*, *several*, etc., and F is a fuzzy subset of a universe of discourse. And, Type IV, of the general form, "X is F is τ," where τ is a linguistic truth-value such as *true*, *very true*, *more or less true*, etc. These rules may be used in combination to translate composite propositions whose constituents are instances of some of the four types in question, e.g., "'*Most tall* men are *stronger* than *most short* men' is *more or less true*," where the italicized words denote labels of fuzzy sets.

The translation rules for fuzzy propositions of Types I, II, III and IV induce corresponding truth valuation rules which serve to express the fuzzy truth-value of a fuzzy proposition in terms of the truth-values of its constitutents. In conjunction with linguistic approximation, these rules provide a basis for approximate inference from fuzzy premises, several forms of which are described and illustrated by examples.

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1. Introduction

Traditionally, logical systems have aimed at the construction of exact models of exact reasoning -- models in which there is no place for imprecision, vagueness or ambiguity.

In a sharp break with this deeply entrenched tradition, the model of reasoning embodied in fuzzy logic [1], [2], aims, instead, at an accomodation with the pervasive imprecision of human thinking and cognition. Clearly, we reason in approximate rather than precise terms when we have to decide on which route to take to a desired destination, where to find a space to park our car, or how to locate a lost object. Furthermore, we frequently use a mixture of precise and approximate reasoning in problem-solving situations, e.g., in looking for ways of proving a theorem, choosing a move in a game of chess, or trying to solve a puzzle. On the whole, however, it is evident that all but a small fraction of human reasoning is approximate in nature, and that such reasoning falls, in the main, outside of the domain of strict applicability of classical logic.

To provide an appropriate conceptual framework for approximate reasoning, fuzzy logic is based on the premise that human perceptions involve, for the most part, fuzzy sets, that is, classes of objects in which the

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transition from membership to non membership is gradual rather than abrupt.[1] It is such sets -- rather than sets in the traditional sense -- that correspond to the italicized words in the propositions "Nina is _very attractive_," "Mary is _extremely intelligent_," "_Most_ Swedes are _blond_," "It is _very true_ that John is _much taller_ than Betty," "_Many tall_ men are not _very agile_," "It is _quite likely_ that it will be a _warm_ day tomorrow," etc. We shall refer to such assertions as _fuzzy propositions_ in order to differentiate them from nonfuzzy propositions like "All men are mortal," "x is larger than y," Gisela has two sons," etc.

A distinctive feature of fuzzy logic is that the meaning of such terms as _beautiful_, _tall_, _small_, _approximately equal_, _very true_, etc. is assumed to be not merely subjective but also _local_ in the sense of having restricted validity in a specified domain of discourse. Thus, the definition of a _small_ number, for example, as a fuzzy subset of the real line may hold only for a designated set of propositions and is allowed to vary from one such set to another. The same applies, more importantly, to the definition of the linguistic truth-values _true_, _very true_, etc. as well as the connectives _and_, _or_ and _if_..._then_. It is in this sense that fuzzy logic may be viewed as a local logic in which the meaning of propositions, connectives and truth-values is, in general, of local rather than universal validity.

An important consequence of the local validity of meaning is that the inference processes in fuzzy logic are semantic rather than syntactic in nature. By this we mean that the consequence of a given set of premises depends in an essential way on the meaning attached to the fuzzy sets which appear in these premises. As a simple illustration, the consequence of the premises "X is a _small_ number," and "X and Y are _approximately equal_," depends

[1]Relevant aspects of the theory of fuzzy sets are discussed in references [3]-[60]. For convenience of the reader, a summarized exposition is presented in the Appendix. Alternative approaches to vagueness and inexact reasoning are discussed in [61]-[78].

on the meaning of small and approximately equal expressed as fuzzy subsets of the real line, R, and R^2, respectively. More specifically, the consequence in question may be expressed as "Y is H," where H is a fuzzy set which, as will be shown in Sec. 7, is given by the composition of the unary fuzzy relation small with the binary fuzzy relation approximately equal.

It is important to observe that fuzzy logic, in the sense used above, is a generic term which refers not to a unique logical system but to a collection of local logics in which the truth-values are fuzzy subsets of the truth-value set of an underlying multivalued logic. For example, if the underlying logic (i.e., base logic) is Lukasiewicz's L_{aleph_1} logic, then the truth-values of a fuzzy logic whose base logic is L_{aleph_1} would be fuzzy subsets of the unit interval.[2]

In this paper, our attention will be focussed on a particular fuzzy logic which, for convenience, will be referred to as FL [1]. The base logic for FL is L_{aleph_1} and its truth-value set is a countable collection of fuzzy subsets of the unit interval [0,1], carrying labels of the form true, very true, not very true, more or less true, not false and not true, etc.

The principal feature that distinguishes FL from classical logics as well as other types of fuzzy logics is that its truth-values are (a) linguistic and (b) structured in the sense that such truth-values may be generated by a grammar and interpreted by a semantic rule. Thus, as will be seen in Sec. 4, with true playing the role of a primary term, the non-primary truth-values in the truth-value set of FL may be generated by a context-free grammar and related to fuzzy subsets of [0,1] by an attributed grammar [1],[17], [110], [111].

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[2] In this sense, the conventional multivalued logics may be viewed as degenerate forms of fuzzy logics in which the fuzzy truth-values are singletons. Somes authors, e.g., [23], [42], [47], [56] employ the term fuzzy logic in a more restricted sense, interpreting a fuzzy logic as a multivalued logic with nonfuzzy truth-values. A succinct discussion of fuzzy logics and their relation to probability logics may be found in papers by B.R. Gaines [58], [59], [60].

The rationale for the use of linguistic truth-values in FL is the following. If p is a fuzzy proposition such as "Frances is very attractive," it would be inconsistent to attach a precise numerical truth-value to p, say 0.935, because the meaning of very attractive is not sharply defined. Thus, to be consistent, it would be logical to associate a fuzzy truth-value with p, that is, a fuzzy subset of [0,1] rather than a point in this interval. But, if we allowed any fuzzy subset of [0,1] to be a truth-value of FL, then the truth-value set of FL would be much too rich and much too difficult to manipulate. Thus, what suggests itself is the idea of allowing only a countable structured collection of fuzzy subsets of [0,1] to be used as the truth-values of FL. In this way, we trade a continuum of simple truth-values of L_{aleph_1} for a countable -- and actually, in most cases, a small -- collection of more complex truth-values of FL and gain a significant advantage in the process.

As will be seen in Sec. 6, the linguistic truth-values of FL do not form a closed system under the operations of conjunction, disjunction and implication. Thus, if the truth-values of p and q are, say, more or less true and not very true and not very false, then the truth-value of the conjunction "p and q" will not be, in general, a linguistic truth-value in the truth-value set of FL. Consequently, the use of linguistic truth-values in FL necessitates a linguistic approximation to fuzzy subsets of [0,1] by the linguistic truth-values of FL. The same applies, more generally, to the linguistic values for variables, relations and probabilities that might occur in fuzzy propositions, with the consequence that the inference processes in FL are, for the most part, approximate rather than exact. For example, as was stated earlier, the exact consequence of the premises "X is a small number," and "X and Y are approximately equal" is "Y is small $\circ$ approximately equal," where $\circ$ denotes the operation of composition. A linguistic approximation to the fuzzy set small ∘ approximately equal might be taken to be more or less small,[3] in which case the conclusion "Y is more or less small" becomes an approximate consequence of the premises in question.

In what follows, we shall begin our exposition of fuzzy logic with the introduction of the concept of a fuzzy restriction, by which is meant a fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable. In this capacity, a fuzzy restriction plays a basic role in FL which is somewhat similar to -- and yet distinct from -- that of a predicate in multivalued logic.

With the concept of a fuzzy restriction as a point of departure, the truth-value of a fuzzy proposition p may be defined as the degree of consistency of p with a reference proposition r. This, in turn, makes it possible to develop valuation rules for expressing the truth-value of a composite proposition in terms of the truth-values of its constitutents. However, in FL, unlike the traditional logics, these rules are derived from translation rules which relate the fuzzy restriction associated with a composite fuzzy proposition to those associated with its constituents.

Translation and valuation rules in FL are divided into four categories depending on the form of the fuzzy propositions to which they apply. Thus, rules of Type I apply to propositions of the general form "X is mF," where X is the name of an object or a variable, F is a fuzzy subset of a universe of discourse and m is a modifier such as not, very, more or less, quite, extremely, etc. Examples of propositions of this form are: "X is a very small number," and "Ruth is highly intelligent."

Rules of Type II apply to composite propositions of the form

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[3]As will be seen later, the effect of the modifier more or less on its operand may be characterized by a kernel function which represents the result of acting with more or less on a singleton.

"(X is F)*(Y is G)," or, more generally, "X is in relation R to Y," where R is a fuzzy relation and * is a binary connective such as _and_, _or_, _if···then···_, etc. (In FL, the conjunction and disjunction are allowed to be _interactive_ in the sense defined in Sec. 5.) Typical examples of such propositions are: "X is _small_ and Y is _very large_," "X is _much larger_ than Y," and "X and Y are _approximately equal_."

Rules of Type III apply to quantified fuzzy propositions of the form "QX are F," where Q is a fuzzy quantifier such as _most_, _many_, _several_, _few_, etc., as in "_Most_ Swedes are _tall_." As for rules of Type IV, they apply to qualified fuzzy propositions of the general form "X is F is τ," where τ is a linguistic truth-value. Examples of such propositions are: "Sally is _very attractive_ is _very true_," and "_Most_ Swedes are _tall_ is _more or less true_."

The basic rule of inference in fuzzy logic is the _compositional rule of inference_ which may be represented as

$$\begin{array}{l} X \text{ is } F \\ X \text{ is in relation } G \text{ to } Y \\ \hline Y \text{ is } LA(F \circ G) \end{array}$$

where F and G are, respectively, unary and binary fuzzy relations, $F \circ G$ is their composition and $LA(F \circ G)$ is a linguistic approximation to the unary fuzzy relation $F \circ G$. As was stated earlier, in consequence of the use of linguistic approximation, the inference processes in fuzzy logic are, for the most part, approximate rather than exact.

Although fuzzy logic represents a significant departure from the conventional approaches to the formalization of human reasoning, it constitutes -- so far at least -- an extension rather than a total abandonment of the currently held views on meaning, truth and inference [79]-[108]. It should be stressed that, at this juncture, fuzzy logic is still in its infancy.

Thus, our exposition of FL in the present paper should be viewed merely as a step toward the development of a logical system which may serve as a realistic model for human reasoning as well as a basis for a better understanding of the potentialities and limitations of machine intelligence.

2. The Concept of a Fuzzy Restriction

The concept of a fuzzy restriction [32] plays a central role in fuzzy logic, providing a basis for the characterization of the meaning as well as the truth-value of composite propositions. In what follows, we shall outline some of the basic properties of such restrictions with a view to making use of these properties in later sections for the definition of linguistic truth-values and the formulation of rules of approximate inference from fuzzy premises.

Let X be a variable which takes values in a universe of discourse $U = \{u\}$. Informally, a _fuzzy restriction_ is an elastic constraint on the values that may be assigned to X, expressed by a proposition of the form "X is F," where F is a fuzzy subset of U. For example, if X is a variable named _Temperature_ and F is a fuzzy subset of the real line labeled _high_, then the fuzzy proposition "_Temperature_ is _high_" may be interpreted as a fuzzy restriction on the values of _Temperature_.

If the fuzzy set _high_ is characterized by its membership function $\mu_{\underline{high}}: U \rightarrow [0,1]$, which associates with each temperature, u, its grade of membership, $\mu_{\underline{high}}(u)$, in the fuzzy set _high_, then $1 - \mu_{\underline{high}}(u)$ represents the degree to which the elastic constraint expressed by "_Temperature_ is _high_" must be stretched to accomodate the assignment of u to X. For example, if $\mu_{\underline{high}}(100°) = 0.9$, then we shall write

$$\underline{\text{Temperature}} = 100°\text{: } 0.9 \qquad (2.1)$$

to indicate that the assignment of 100° to Temperature is compatible to the degree 0.9 with the constraint "Temperature is high," or, equivalently, that the constraint in question must be stretched to the degree 0.1 to accomodate the assignment of 100° to Temperature.

In more general terms, a variable, X, which takes values in $U = \{u\}$ is a *fuzzy variable* if the restriction on the values that may be assigned to X is a fuzzy subset of U.[4] In relation to X, then, a fuzzy subset F of U is a *fuzzy restriction* if it serves as an elastic constraint on the values of X in the sense that the assignment equation for X has the form

$$X = u\text{: } \mu_F(u) \qquad (2.2)$$

where $\mu_F(u)$, the grade of membership of u in F, represents the *compatibility* of u with the fuzzy restriction F.

To express that F is a fuzzy restriction on the values of X, we write

$$R_X(u) = F \qquad (2.3)$$

where $R_X(u)$ denotes a fuzzy restriction on the elements of U which is associated with the variable X.[5] Thus, the assignment equation (2.2) may be said to imply -- or translate into -- the assignment equation (2.3). To distinguish (2.3) from (2.2), the latter will be referred to as a *relational assignment equation*.

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[4] In some contexts it is convenient to regard u as a variable ranging over U rather than as a particular element of U. In such cases, u will be referred to as a *base variable* for X.

[5] For convenience, $R_X(u)$ will usually be abbreviated to R_X or $R(u)$ or $R(X)$, with the understanding that $R(u)$ and $R(X)$ are labels of a fuzzy set rather than functions of u and X, respectively.

In general, a fuzzy proposition of the form "X is F" translates not into

$$R(X) = F \tag{2.4}$$

but into

$$R(A(X)) = F \tag{2.5}$$

where A is an implied attribute of X. For example, the proposition "Betty is <u>young</u>" translates into the relational assignment equation

$$R(\underline{Age}(Betty)) = \underline{young} \tag{2.6}$$

where <u>Age</u> is an attribute of Betty which is implied by <u>young</u>; <u>Age</u>(Betty) is a fuzzy variable; and <u>young</u> is a fuzzy subset of the real line defined by, say,

$$\mu_{\underline{young}}(u) = 1 - S(u;20,30,40) \tag{2.7}$$

where the S-function, $S(u;20,30,40)$, is expressed by (see A17)

$$\begin{aligned} S(u;20,30,40) &= 0 && \text{for } u \le 20 \\ &= 2\left(\frac{u-20}{20}\right)^2 && \text{for } 20 \le u \le 30 \\ &= 1 - 2\left(\frac{u-40}{20}\right)^2 && \text{for } 30 \le u \le 40 \\ &= 1 && \text{for } u \ge 40 \end{aligned} \tag{2.8}$$

In this definition of <u>young</u>, the age $u = 30$ is a <u>crossover point</u> in the sense that $\mu_{\underline{young}}(30) = 0.5$. For $u = 25$, we have $\mu_{\underline{young}}(25) = 0.875$, and hence "Betty is <u>young</u>" implies

$$\underline{Age}(Betty) = 25\text{: } 0.875 \tag{2.9}$$

In the foregoing discussion, we have restricted our attention to the case where X is a unary fuzzy variable with a base variable u ranging over a single universe of discourse U. In the more general case where X is an n-ary variable, $X = (X_1,\ldots,X_n)$, each of the n components of X is a fuzzy variable, X_i, $i = 1,\ldots,n$, whose base variable, u_i, ranges over a universe of discourse U_i. In this case, a fuzzy restriction on the values of X is an n-ary fuzzy relation, F, in the product space $U_1 \times \cdots \times U_n$, and the assignment equations (2.3) and (2.2) take the form

$$R_X(u_1,\ldots,u_n) = F \tag{2.10}$$

and

$$(X_1,\ldots,X_n) = (u_1,\ldots,u_n): \mu_F(u_1,\ldots,u_n) \tag{2.11}$$

respectively. As an illustration, if X_1 and X_2 are real numbers, then the proposition "X_1 is <u>much larger</u> than X_2" translates into the relational assignment equation

$$R(X_1,X_2) = \underline{\text{much larger}} \tag{2.12}$$

where <u>much larger</u> is a fuzzy relation in R^2 whose membership function may be defined as, say,

$$\begin{aligned}\mu_{\underline{\text{much larger}}}(u_1,u_2) &= 0 \quad \text{for} \quad u_1 \leq u_2 \\ &= \left(1 + \left(\frac{u_2-u_1}{10}\right)^{-2}\right)^{-1} .\end{aligned} \tag{2.13}$$

Correspondingly, for $u_1 = 2$ and $u_2 = 16$ we deduce

$$(X_1,X_2) = (2,16): 0.66 \tag{2.14}$$

An important concept that relates to n-ary fuzzy restrictions is that of <u>noninteraction</u>. Specifically, the components of an n-ary fuzzy variable are said to be <u>noninteractive</u> if and only if

$$R(X_1,\ldots,X_n) = R(X_1) \times \cdots \times R(X_n) \tag{2.15}$$

where $R(X_i)$ denotes the projection of $R(X_1,\ldots,X_n)$ on U_i and $\times$ denotes the cartesian product.[6] Equivalently, $X_1,\dots,X_n$ are noninteractive if and only if the n-ary assignment equation

$$(X_1,\dots,X_n) = (u_1,\dots,u_n): \mu_{R(X_1,\dots,X_n)}(u_1,\dots,u_n) \tag{2.16}$$

may be decomposed into n unary assignment equations

$$\begin{aligned} X_1 &= u_1: \mu_{R(X_1)}(u_1) \\ &\cdots\cdots\cdots \\ X_n &= u_n: \mu_{R(X_n)}(u_n) \end{aligned} \tag{2.17}$$

What is implied by (2.15) is that, if $X_1,\dots,X_n$ are noninteractive, then the assignment of values to any subset of the X_i has no effect on the fuzzy restrictions which apply to the remaining variables. For example, if X_1 and X_2 are noninteractive, then the assignment of a value, say u_1^0, to X_1 does not affect the fuzzy restriction on the values of X_2.[7] As we shall see in later sections, this property of noninteractive variables plays a basic role in the definition of logical connectives.

In the foregoing discussion of the concept of a fuzzy restriction, we have limited our attention to the translation of atomic fuzzy propositions of the form "X is F." In Sec. 4, we shall consider the more general problem

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[6] The membership function of the projection of $R(X_1,\dots,X_n)$ on U_i is defined by

$$\mu_{R(X_i)}(u_i) = \text{Sup}\ \mu_{R(X_1,\dots,X_n)}(u_1,\dots,u_n)$$

where the supremum is taken over $u_1,\dots,u_n$, excluding u_i. (See A58.)

If $F_1,\dots,F_n$ are fuzzy subsets of $U_1,\dots,U_n$, respectively, then the membership function of the cartesian product $F_1 \times \cdots \times F_n$ is given by

$$\mu_{F_1\times\cdots\times F_n}(u_1,\dots,u_n) = \mu_{F_1}(u_1) \wedge \cdots \wedge \mu_{F_n}(u_n)$$

where μ_{F_i} is the membership function of F_i and $\wedge$ stands for the infix form of min.

[7] A more detailed discussion of this aspect of noninteraction may be found in [2].

of translation of composite propositions which are formed from atomic propositions through the use of logical connectives such as and, or, if···then···, and fuzzy quantifiers such as most, many, few , etc. As a preliminary, in the following section we shall define the concept of a linguistic variable and apply it to the characterization of the truth-values of fuzzy logic.

3. Linguistic Variables and Truth-Values in Fuzzy Logic

As was pointed out in the Introduction, one of the important characteristics of fuzzy logic, FL, is that its truth-values are not points or sets but fuzzy subsets of the unit interval which are characterized by linguistic labels such as true, very true, not very true, etc.

To make the meaning of such truth-values more precise, we shall draw on the concept of a linguistic variable -- a concept which plays a basic role in approximate reasoning and which, as will be seen in the sequel, bears a close relation to the concept of a fuzzy restriction.

Essentially, a linguistic variable, X, is a nonfuzzy variable which ranges over a collection, $T(X)$, of structured fuzzy variables $X_1, X_2, X_3, \ldots$, with each fuzzy variable in $T(X)$ carrying a linguistic label, X_i, which characterizes the fuzzy restriction which is associated with X_i.

As an illustration, Age is a linguistic variable if its values are assumed to be the fuzzy variables labeled young, not young, very young, not very young, etc., rather than the numbers 0,1,2,3,... . The meaning of a linguistic value of Age, say very young, is identified with the fuzzy restriction which is associated with the fuzzy variable labeled very young. Thus, if the base variable for Age (i.e., numerical age) is assumed to range over the universe $U = \{0,1,\ldots,100\}$, then the linguistic values of Age may be interpreted as the labels of fuzzy subsets of U.

More generally, a linguistic variable is characterized by a quintuple $(X,T(X),U,G,M)$, where X is the name of the variable, e.g., Age; $T(X)$ is the term-set of X, that is, the collection of its linguistic values, e.g., $T(X)$ = {young, not young, very young, not very young,...}; U is a universe of discourse, e.g., in the case of Age, the set $\{0,1,2,\ldots,100\}$; G is a syntactic rule which generates the terms in $T(X)$; and M is a semantic rule which associates with each term, X_i, in $T(X)$ its meaning, $M(X_i)$, where $M(X_i)$ is a fuzzy subset of U which serves as a fuzzy restriction on the values of the fuzzy variable X_i.

A key idea behind the concept of a linguistic variable is that the fuzzy restriction associated with each X_i may be deduced from the fuzzy restrictions associated with the so-called primary terms in $T(X)$. In effect, these terms play the role of units which, upon calibration, make it possible to compute the meaning of the composite (i.e., non-primary) terms in $T(X)$ from the knowledge of the meaning of primary terms.

As an illustration, we shall consider an example in which $U = [0,\infty)$ and the term-set of X is of the form

$$T(X) = \{\text{small, not small, very small, very (not small),} \\ \text{not very small, very very small, } \ldots\} \qquad (3.1)$$

in which small is the primary term.

The terms in $T(X)$ may be generated by a context-free grammar $G = (V_T,V_N,S,P)$ in which the set of terminals, V_T, comprises (,), the primary term small and the linguistic modifiers very and not; the nonterminals are denoted by S, A and B, and the production system is given by:

$$\begin{aligned}
S &\to A \\
S &\to \underline{\text{not}}\ A \\
A &\to B \\
B &\to \underline{\text{very}}\ B \\
B &\to (S) \\
B &\to \underline{\text{small}}
\end{aligned} \tag{3.2}$$

Thus, a typical derivation yields

$$S \to \underline{\text{not}}\ A \to \underline{\text{not}}\ B \to \underline{\text{not}}\ \underline{\text{very}}\ B \to \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{very}}\ B \to \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{small}}\ . \tag{3.3}$$

In this sense, the syntactic rule associated with X may be viewed as the process of generating the elements of $T(X)$ by a succession of substitutions involving the productions in G.

As for the semantic rule, we shall assume for simplicity that if μ_A is the membership function of A then the membership functions of <u>not</u> A and <u>very</u> A are given respectively by

$$\mu_{\underline{\text{not}}\ A} = 1 - \mu_A \tag{3.4}$$

and

$$\mu_{\underline{\text{very}}\ A} = (\mu_A)^2\ . \tag{3.5}$$

Thus, (3.5) signifies that the modifier <u>very</u> has the effect of squaring the membership function of its operand.[8]

Suppose that the meaning of <u>small</u> is defined by the membership function

$$\mu_{\underline{\text{small}}}(u) = (1 + (0.1u)^2)^{-1}\ , \quad u \geq 0\ . \tag{3.6}$$

Then the meaning of <u>very</u> <u>small</u> is given by

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[8] A more detailed discussion of the effect of linguistic modifiers (hedges) may be found in [51], [52], [53], [54], [55] and [56].

$$\mu_{\underline{\text{very}}\ \underline{\text{small}}} = (1+(0.1u)^2)^{-2} \tag{3.7}$$

while the meanings of not very small and very (not small) are expressed respectively by

$$\mu_{\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{small}}} = 1 - (1+(0.1u)^2)^{-2} \tag{3.8}$$

and

$$\mu_{\underline{\text{very}}\ (\underline{\text{not}}\ \underline{\text{small}})} = (1-(1+(0.1u)^2)^{-1})^2 \ . \tag{3.9}$$

In this way, we can readily compute the expression for the membership function of any term in $T(X)$ from the knowledge of the membership function of the primary term small.

In summary, a linguistic variable X may be viewed, in effect, as a micro-language whose sentences are the linguistic values of X, with the meaning of each sentence represented as a fuzzy restriction on the values of a base variable, u, in a universe of discourse, U. The syntax and semantics of this language are, respectively, the syntactic and semantic rules associated with X.

In applying the concept of a linguistic variable to fuzzy logic, we assume that Truth is a linguistic variable with a term-set of the form[9]

T(Truth) = {true, false, not true, very true, not very true,
very (not true), not very true and not very false, ...}
(3.10)

in which the primary term is true.

In the case of FL, the universe of discourse, V, associated with Truth is assumed to be the unit interval $[0,1]$, and the logical operations

[9]More generally, the truth-values in T(Truth) could include, in addition to very, such linguistic modifiers (hedges) as quite, more or less, essentially, etc. As in the case of very, the meaning of these and other modifiers may be defined -- as a first approximation -- in terms of a set of standardized operations on the fuzzy sets which represent their operands.

on the linguistic truth-values are fuzzy extensions -- in the sense defined in Sec. 6 -- of the corresponding operations in Lukasiewcz's logic L_{aleph_1} [109]. Thus, L_{aleph_1} serves as a basic logic for FL, with the linguistic truth-values of FL being fuzzy subsets of the truth-value set of L_{aleph_1}.

So far, we have not addressed ourselves to a basic issue, namely, what is the significance of associating a numerical or linguistic truth-value with a fuzzy proposition? What does it mean, for example, to assert that "X is small is 0.8 true" or "Gail is highly intelligent is very true?"

Informally, we shall adopt the view that a truth-value, numerical or linguistic, represents the degree of consistency of p with a reference proposition r. Thus, in symbols[10]

$$v(p) \triangleq C(R(p),R(r)) \tag{3.11}$$

where v(p) denotes the truth-value of p; R(p) and R(r) represent, respectively, the restrictions associated with p and r; and C is a consistency function which maps ordered pairs of restrictions into points in [0,1] or fuzzy subsets of [0,1] and thereby defines the degree of consistency of p with r.

In general, r may be, like p, a fuzzy proposition. In the sequel, however, we shall take a more restricted point of view. Specifically, we shall assume that, if (a) p is a fuzzy proposition of the form

$$p \triangleq X \text{ is } F \tag{3.12}$$

which translates into

$$R(A(X)) = F \tag{3.13}$$

where A(X) is an implied attribute of X, and (b) v(p) is a numerical

[10]The symbol $\triangleq$ is used here and elsewhere in this paper to denote "is defined to be," or "denotes."

truth-value in [0,1], then the reference proposition r is a nonfuzzy proposition of the form

$$r \triangleq X \text{ is } u \tag{3.14}$$

where u is an element of U which represents a reference value of the variable $A(X)$.[11] Under these assumptions, then, the numerical truth-value of p is defined by

$$\begin{aligned} v(p) \triangleq t &= C(F,u) \\ &\triangleq \mu_F(u) \end{aligned} \tag{3.15}$$

where $\mu_F(u)$ is the grade of membership of u in F. In effect, (3.15) implies that the truth-value of p is equated, by definition, to the grade of membership of u in F, where u is a reference value of the variable $A(X)$. As an illustration, consider the proposition $p \triangleq$ Ilka is <u>tall</u>, where <u>tall</u> is defined by

$$\mu_{\underline{tall}}(u) = S(u;160;170;180)\ . \tag{3.16}$$

Then, if Ilka is, in fact, 172 cm tall and r is taken to be

$$r \triangleq \text{Ilka is 172 cm tall} \tag{3.17}$$

we have

$$\begin{aligned} v(\text{Ilka is } \underline{\text{tall}}) = t &= S(172;160;170;180) \\ &= 0.68 \end{aligned} \tag{3.18}$$

which thus represents the numerical truth-value of the fuzzy proposition $p \triangleq$ Ilka is <u>tall</u>.

We are now in a position to extend the notion of a numerical truth-value

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[11] What we rule out here is the possibility that the degree of consistency of two fuzzy propositions be a numerical truth-value. This case is more complex than that discussed in the present paper.

to fuzzy truth-values by interpreting a linguistic truth-value, τ, as the degree of consistency of p with a fuzzy reference proposition r. Thus, if r is of the form

$$r \triangleq X \text{ is } G \tag{3.19}$$

where G is a fuzzy subset of U, then a fuzzy truth-value, τ, may be associated formally with p by the expression

$$\tau = \mu_F(G) \tag{3.20}$$

where μ_F, as in (3.15), represents the membership function of F.[12]

To make (3.20) meaningful, it is necessary to extend the domain of definition of μ_F from U to $\mathcal{F}(U)$, where $\mathcal{F}(U)$ is the set of fuzzy subsets of U. This can be done by using the <u>extension principle</u> (A70), which is a basic rule for extending the definition of a function defined on a space U to $\mathcal{F}(U)$. Specifically, in application to (3.20), let G be represented symbolically in the "integral" form (see A8)

$$G = \int_U \mu_G(u)/u \tag{3.21}$$

where the integral sign denotes the union of fuzzy singletons $\mu_G(u)/u$, with $\mu_G(u)/u$ signifying that the compatibility of u with G (or, equivalently, the grade of membership of u in G) is $\mu_G(u)$. Then, on invoking the extension principle and treating μ_F as a function from U to $[0,1]$, we obtain

$$\mu_F(G) = \int_{[0,1]} \mu_G(u)/\mu_F(u) \tag{3.22}$$

[12] It should be noted that this interpretation of a fuzzy truth-value is contingent on the assumptions made in (3.15). Hence, a different set of assumptions concerning the consistency function C might lead to a different interpretation of τ.

which means that $\mu_F(G)$ is the union of fuzzy singletons $\mu_G(u)/\mu_F(u)$ in $[0,1]$.

When we have to make explicit that an expression, E, has to be evaluated by the use of the extension principle, we shall enclose E in angular brackets. With this understanding, then, a linguistic truth-value, τ, may be expressed as

$$\tau = \langle\mu_F(G)\rangle = \int_{[0,1]} \mu_G(u)/\mu_F(u) \; . \tag{3.23}$$

Adopting the interpretation of τ which is defined by (3.23), let $\mu_\tau\colon V \to [0,1]$ denote the membership function of τ. Then, the meaning of τ as a fuzzy subset of V may be expressed as

$$\tau = \int_0^1 \mu_\tau(v)/v \tag{3.24}$$

where $v \in V = [0,1]$ is the base variable for the fuzzy variable τ, and the integral sign, as in (3.21), denotes the union of fuzzy singletons $\mu_\tau(v)/v$, with $\mu_\tau(v)/v$ signifying that the compatibility of the numerical truth-value v with the linguistic truth-value τ is $\mu_\tau(v)$.

If the support of τ, that is, the set of points in V at which $\mu_\tau(v) \neq 0$, is a finite subset $\{v_1,\ldots,v_n\}$ of V, and μ_i is the compatibility of v_i with τ, $i = 1,\ldots,n$, then τ may be expressed as

$$\tau = \mu_1/v_1 + \cdots + \mu_n/v_n \tag{3.25}$$

or more simply as the linear form

$$\tau = \mu_1 v_1 + \cdots + \mu_n v_n \tag{3.26}$$

when no confusion between μ_i and v_i in a term of the form $\mu_i v_i$ can arise. It should be noted that in (3.25) and (3.26) the plus sign -- like the integral sign in (3.24) -- should be interpreted as the union rather than the arithmetic sum.

As an illustration of (3.24), if the membership function of <u>true</u> is assumed to be expressed as an S-function (see A17)

$$\mu_{\underline{true}}(v) = S(v;0.5,0.75,1) \tag{3.27}$$

then the meaning of <u>true</u> is the fuzzy subset of V expressed as

$$\underline{true} = \int_0^1 S(v;0.5,0.75,1)/v \ . \tag{3.28}$$

If V is assumed to be the finite set $\{0,0.1,0.2,\ldots,1\}$, then <u>true</u> may be defined as a fuzzy subset of V by, say,

$$\underline{true} = 0.3/0.6 + 0.5/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1 \ . \tag{3.29}$$

In this expression, a term such 0.7/0.8 signifies that the compatibility of the numerical truth-value 0.8 with the linguistic truth-value <u>true</u> is 0.7. It is important to note that the definition of <u>true</u> in (3.28) and (3.29) is entirely subjective as well as local in nature.

On occasion, we shall find it convenient to relate to a linguistic truth-value τ its <u>dual</u>, $D(\tau)$, which is defined by

$$\mu_{D(\tau)}(v) = \mu_\tau(1-v) \ , \quad v \in [0,1] \ . \tag{3.30}$$

or, equivalently,

$$D(\tau) = 1 - \tau \tag{3.31}$$

where for simplicity we have suppressed the angular brackets in the right-hand member of (3.31). Thus, if <u>true</u>, for example, is defined by (3.29), then

$$D(\underline{true}) = 0.3/0.4 + 0.5/0.3 + 0.7/0.2 + 0.9/0.1 + 1/0$$

and $D(\underline{true})$ will be assumed to be the meaning of false, i.e.,

$$\underline{false} \triangleq D(\underline{true}) \tag{3.32}$$

and conversely

$$\underline{true} = D(\underline{false}) \ . \tag{3.33}$$

As shown in [ ], the linguistic truth-values in T(Truth) can be generated by a context-free grammar whose production system is given by

$$\begin{array}{ll} S \rightarrow A & C \rightarrow D \\ S \rightarrow S \ \underline{or} \ A & C \rightarrow E \\ A \rightarrow B & D \rightarrow \underline{very} \ D \\ A \rightarrow A \ \underline{and} \ B & E \rightarrow \underline{very} \ E \\ B \rightarrow C & D \rightarrow \underline{true} \\ B \rightarrow \underline{not} \ C & E \rightarrow \underline{false} \\ C \rightarrow (S) & \end{array} \tag{3.34}$$

In this grammar, S, A, B, C, D, and E are nonterminals; and true, false, very, not, and, or, (,) are terminals. Thus, a typical derivation yields

$$\begin{aligned} S &\rightarrow A \rightarrow A \ \underline{and} \ B \rightarrow B \ \underline{and} \ B \rightarrow \underline{not} \ C \ \underline{and} \ B \rightarrow \underline{not} \ E \ \underline{and} \ B \\ &\rightarrow \underline{not} \ \underline{very} \ E \ \underline{and} \ B \rightarrow \underline{not} \ \underline{very} \ \underline{false} \ \underline{and} \ B \rightarrow \underline{not} \ \underline{very} \ \underline{false} \ \underline{and} \ \underline{not} \ C \\ &\rightarrow \underline{not} \ \underline{very} \ \underline{false} \ \underline{and} \ \underline{not} \ D \rightarrow \underline{not} \ \underline{very} \ \underline{false} \ \underline{and} \ \underline{not} \ \underline{very} \ D \\ &\rightarrow \underline{not} \ \underline{very} \ \underline{false} \ \underline{and} \ \underline{not} \ \underline{very} \ \underline{true} \end{aligned} \tag{3.35}$$

If the syntactic rule for generating the elements of T(Truth) is expressed as a context-free grammar, then the corresponding semantic rule may be conveniently expressed by a system of productions and relations in which each production in G is associated with a relation between the fuzzy subsets representing the meaning of the terminals and nonterminals.[13] For example, the production $A \rightarrow A$ and B induces the relation

$$A_L = A_R \cap B_R \tag{3.36}$$

where A_L, A_R, and B_R represent the meaning of A and B as fuzzy subsets of $[0,1]$ (the subscripts L and R serve to differentiate between the symbols on the left- and right-hand sides of a production), and $\cap$ denotes the intersection. Thus, in effect, (3.36) defines the meaning of the connective and.

Similarly, the production $B \rightarrow$ not C induces the relation

$$B_L = C_R' \tag{3.37}$$

where C_R' denotes the complement of the fuzzy set C_R (see A32), while $D \rightarrow$ very D induces

$$D_L = (D_R)^2 \tag{3.38}$$

which implies that the membership function of D_L is related to that of D_R by

$$\mu_{D_L} = (\mu_{D_R})^2 . \tag{3.39}$$

With this understanding, the dual system corresponding to (3.29) may be written as

$$\begin{array}{ll} S \rightarrow A & : S_L = A_R \\ S \rightarrow S \text{ or } A & : S_L = S_R \cup A_R \\ A \rightarrow B & : A_L = B_R \\ A \rightarrow A \text{ and } B & : A_L = A_R \cap B_R \end{array} \tag{3.40}$$

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[13] This technique is related to Knuth's method of synthesized attributes [1], [110].

$$B \rightarrow C \quad : B_L = C_R$$
$$B \rightarrow \underline{\text{not}}\ C \quad : B_L = C'_R$$
$$C \rightarrow S \quad : C_L = S_R$$
$$C \rightarrow D \quad : C_L = D_R$$
$$C \rightarrow E \quad : C_L = E_R$$
$$D \rightarrow \underline{\text{very}}\ D : D_L = (D_R)^2$$
$$E \rightarrow \underline{\text{very}}\ E : E_L = (E_R)^2$$
$$D \rightarrow \underline{\text{true}} \quad : D_L = \underline{\text{true}}$$
$$E \rightarrow \underline{\text{false}} \quad : E_L = \underline{\text{false}}$$

where $\cup$ denotes the union.

To employ this dual system to compute the meaning of a term, τ, generated by G, it is necessary, in principle, to construct a syntax tree for τ. Then, by advancing from the leaves of the tree to its root and successively computing the meaning of each node by the use of (3.40), we eventually arrive at the expression for the membership function of τ in terms of the membership function of the primary term true.

In practice, however, the linguistic values of Truth that one would commonly employ to characterize the truth-value of a fuzzy proposition, e.g., "Barbara is very intelligent," are likely to be sufficiently simple to make it possible to compute their meaning by inspection. For example,[14]

$$\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}} = (\underline{\text{true}}^2)' \tag{3.41}$$

$$\underline{\text{not}}\ \underline{\text{very}}(\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}}) = (((\underline{\text{true}}^2)')^2)' \tag{3.42}$$

$$\underline{\text{true}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}} = \underline{\text{true}} \cap (\underline{\text{true}}^2)' \tag{3.43}$$

$$\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}} = (\underline{\text{true}}^2)' \cap (\underline{\text{false}}^2)' \tag{3.44}$$

---

[14] It should be noted that in (3.41)-(3.44) true plays the role of a label of a fuzzy set in the left-hand member and that of the set itself in the right-hand member.

where ' denotes the complement and, in consequence of (3.32),

$$\underline{\text{false}} = 1 - \underline{\text{true}} \tag{3.45}$$

with (3.45) implying that the membership function of <u>false</u> is related to that of <u>true</u> by

$$\mu_{\underline{\text{false}}}(v) = \mu_{\underline{\text{true}}}(1-v) \ , \quad v \in [0,1] \ . \tag{3.46}$$

Note that <u>false</u> $\neq$ <u>not</u> <u>true</u>, since

$$\underline{\text{not}}\ \underline{\text{true}} = \underline{\text{true}}' \tag{3.47}$$

while <u>false</u> = 1 - <u>true</u>. The reason for defining the meaning of <u>false</u> by (3.45) rather than by equating <u>false</u> to <u>not</u> <u>true</u> will become clear in Sec. 6.

In the following two sections, we shall turn our attention to a problem that occupies a central place in fuzzy logic, namely, that of translating a fuzzy proposition into one or more relational assignment equations. Then, from the rules governing such translations, we shall be able to derive a set of valuation rules for computing the truth-values of composite fuzzy propositions.

4. <u>Translation Rules for Fuzzy Propositions - Types I and II</u>

As was stated in the Introduction, one of the basic problems in fuzzy logic is that of developing a set of rules for translating a given fuzzy proposition into a system of relational assignment equations.

In this section, we shall address ourselves to some of the simpler aspects of this problem, focusing our attention on what will be referred to as translation rules of Types I and II. In Sec. 5, we shall consider translation rules of Types III and IV, which apply to more complex propositions containing, respectively, quantifiers and truth-values. Implicit in all of these rules is Frege's principle [99], [112] that the meaning of a composite proposition is a function of the meanings of its constituents.

Translation Rules of Type I

Translation rules of this type apply to fuzzy propositions of the form $p \triangleq X$ is mF, where F is a fuzzy subset of $U = \{u\}$, m is a modifier such as not, very, more or less, slightly, somewhat, etc., and either X or $A(X)$ -- where A is an implied attribute of X -- is a fuzzy variable which takes values in U.

Translation rules of Type I may be subsumed under a general rule which, for convenience, will be referred to as the modifier rule. In essence, this rule asserts that the translation of a fuzzy proposition of the form $p \triangleq X$ is mF is expressed by

$$X \text{ is } mF \longrightarrow R(A(X)) = mF \tag{4.1}$$

where m is interpreted as an operator which transforms the fuzzy set F into the fuzzy set mF.

In particular, if $m \triangleq$ not, then the rule of negation asserts that the translation of $p \triangleq X$ is not F is expressed by

$$X \text{ is } \underline{\text{not}}\ F \longrightarrow X \text{ is } F' \longrightarrow R(A(X)) = F' \tag{4.2}$$

where F' is the complement of F, i.e.,

$$\mu_{F'}(u) = 1 - \mu_F(u)\ , \quad u \in U\ . \tag{4.3}$$

For example, if

$$\mu_{\underline{young}}(u) = 1 - S(u;20,30,40) \tag{4.4}$$

then $p \triangleq$ John is not young translates into

$$R(Age(John)) = \underline{young}' \tag{4.5}$$

where, in the notation of (3.21),

$$\underline{young}' = \int_0^\infty S(u;20,30,40)/u \;. \tag{4.6}$$

In general, m may be viewed as a restriction modifier which acts in a specified way on its operand. For example, the modifier very may be assumed to act -- to a first approximation -- as a concentrator which has the effect of squaring the membership function of its operand [51]. Correspondingly, the rule of concentration asserts that the translation of the fuzzy proposition $p = X$ is very F is expressed by

$$X \text{ is } \underline{very}\ F \rightarrow X \text{ is } F^2 \rightarrow R(A(X)) = F^2 \tag{4.7}$$

where

$$\underline{very}\ F = F^2 = \int_U (\mu_F(u))^2/u \tag{4.8}$$

and $A(X)$ is an implied attribute of X.

As an illustration, on applying (4.7), we find that "Sherry is very young" translates into

$$R(Age(Sherry)) = \underline{young}^2 \tag{4.9}$$

where

$$\underline{young}^2 = \int_0^\infty (1 - S(u;20,30,40))^2/u \;. \tag{4.10}$$

Similarly, on combining (4.2) with (4.7), we find that "Sherry is not very very young" translates into

$$R(\text{Age(Sherry)}) = (\underline{\text{young}}^4)' \tag{4.11}$$

where

$$(\underline{\text{young}}^4)' = \int_0^\infty (1 - (1 - S(u;20,30,40))^4)/u \ . \tag{4.12}$$

The effect of the modifier <u>more or less</u> is less susceptible to simple approximation than that of <u>very</u>. In some contexts, <u>more or less</u> acts as a dilator, playing a role inverse to that of <u>very</u>. Thus, to a first approximation, we may assume that, in such contexts, <u>more or less</u> may be defined by

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F = \sqrt{F} \tag{4.13}$$

where

$$\sqrt{F} = \int_U (\mu_F(u))^{1/2}/u \ .$$

Based on this definition of <u>more or less</u>, the <u>rule of dilation</u> asserts that

$$X \text{ is } \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F \rightarrow X \text{ is } \sqrt{F} \rightarrow R(A(X)) = \sqrt{F} \tag{4.14}$$

where $A(X)$ is an implied attribute of X. For example, "Doris is <u>more or less young</u>" translates into

$$R(\text{Age(Doris)}) = \sqrt{\underline{\text{young}}} = \int_0^\infty (1 - S(u;20,30,40))^{1/2}/u \tag{4.15}$$

while "Doris is <u>more or less</u> (<u>not very young</u>)" translates into

$$R(\text{Age(Doris)}) = ((\underline{\text{young}}^2)')^{1/2} \ . \tag{4.16}$$

In other contexts, <u>more or less</u> acts as a <u>fuzzifier</u> whose effect may be approximated by

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F = \int_U \mu_F(u)K(u) \tag{4.17}$$

where $K(u)$ is a specified fuzzy subset of U which depends on u as a parameter, $\mu_F(u)K(u)$ is a fuzzy set whose membership function is the product of $\mu_F(u)$ and the membership function of $K(u)$, and $\int_U$ denotes the union of the fuzzy sets $\mu_F(u)K(u)$, $u \in U$. When more or less is defined as a fuzzifier by (4.17), the fuzzy set $K(u)$ in the right-hand member of (4.17) is referred to as the kernel of the fuzzifier. Note that (4.17) implies that $K(u)$ may be interpreted as the result of acting with more or less on the singleton $\{u\}$ [51].

As an illustration, suppose that

$$U = 1+2+3+4 \tag{4.18}$$

and that a fuzzy subset of U labeled small is defined by

$$\underline{small} = 1/1 + 0.6/2 + 0.2/3 \ . \tag{4.19}$$

Furthermore, assume that the kernel of more or less is given by

$$\begin{aligned} K(1) &= 1/1 + 0.9/2 \\ K(2) &= 1/2 + 0.9/3 \\ K(3) &= 1/3 + 0.8/4 \end{aligned} \tag{4.20}$$

Then, on substituting (4.19) and (4.20) in (4.17), we obtain

$$\begin{aligned} \underline{more}\ \underline{or}\ \underline{less}\ \underline{small} &= K(1) + 0.6\ K(2) + 0.2\ K(3) \\ &= 1/1 + 0.9/2 + 0.6/2 + 0.54/3 \\ &\quad + 0.2/3 + 0.16/4 \\ &= 1/1 + 0.9/2 + 0.54/3 + 0.16/4 \end{aligned} \tag{4.21}$$

whereas, had we used (4.14), we would have

$$\underline{more}\ \underline{or}\ \underline{less}\ \underline{small} = 1/1 + 0.77/2 + 0.45/3 \ . \tag{4.22}$$

When _more or less_ is interpreted as a fuzzifier, the corresponding modifier rule will be referred to as the _rule of fuzzification_. In symbols, the statement of this rule reads:

$$X \text{ is } \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F \rightarrow R(A(X)) = \int_U \mu_F(u)K(u) \tag{4.23}$$

where $K(u)$ is the kernel of _more or less_ and $A(X)$ is an implied attribute of X. For example, the application of this rule to the proposition "X is _more or less small_," in which _small_ and _more or less_ are defined by (4.19) and (4.20), yields

$$\begin{aligned} R(X) &= \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{small}} \\ &= 1/1 + 0.9/2 + 0.54/3 + 0.16/4\ . \end{aligned} \tag{4.24}$$

By comparison, the application of the rule of dilation would yield

$$R(X) = 1/1 + 0.77/2 + 0.45/3\ . \tag{4.25}$$

In most practical applications, the difference between (4.24) and (4.25) would not be considered to be of significance.

Proceeding in a similar fashion, one can formulate, in principle, other concrete versions of the modifier rule for modifiers such as _slightly_, _quite_, _rather_, etc. In general, the definition of the effects of such modifiers presents many non-trivial problems which, at this stage of the development of the theory of fuzzy sets, are still largely unexplored [51]-[56].

Translation Rules of Type II

Translation rules of this type apply to composite fuzzy propositions which are generated from atomic fuzzy propositions of the form "X is F" through the use of various kinds of binary connectives such as the conjunction,

<u>and</u>, the disjunction, <u>or</u>, the conditional <u>if</u>...<u>then</u>..., etc.

More specifically, let $U = \{u\}$ and $V = \{v\}$ be two possibly different universes of discourse, and let F and G be fuzzy subsets of U and V, respectively.

Consider the atomic propositions "X is F" and "Y is G," and let q be their conjunction "X is F and Y is G." Then, the rule of <u>noninteractive conjunctive composition</u> or, for short, the <u>rule of conjunctive composition</u> asserts that the translation of q is expressed by

$$X \text{ is } F \text{ and } Y \text{ is } G \rightarrow (X,Y) \text{ is } F \times G \rightarrow R(A(X),B(Y)) = F \times G \qquad (4.26)$$

where $A(X)$ and $B(Y)$ are implied attributes of X and Y, respectively; $R(A(X),B(Y))$ is a fuzzy restriction on the values of the binary fuzzy variable $(A(X),B(Y))$; and $F \times G$ is the cartesian product of F and B. Thus, under this rule, the fuzzy proposition "Keith is <u>tall</u> and Adrienne is <u>young</u>" translates into

$$R(\text{Height(Keith)}),\text{Age(Adrienne)}) = \underline{\text{tall}} \times \underline{\text{young}} \qquad (4.27)$$

where <u>tall</u> and <u>young</u> are fuzzy subsets of the real line.

To clarify the reason for qualifying the term "conjunction" with the adjective "noninteractive," it is convenient to rewrite (4.26) in the equivalent form

$$X \text{ is } F \text{ and } Y \text{ is } G \rightarrow R(A(X),B(Y)) = \bar{F} \cap \bar{G} \qquad (4.28)$$

where $\bar{F}$ and $\bar{G}$ are the cylindrical extensions (see A59) of F and G, respectively, and $\bar{F} \cap \bar{G}$ is their intersection. In this form, the rule in question places in evidence the 1-1 correspondence between the noninteractive conjunction of fuzzy propositions, on the one hand, and the intersection of fuzzy cylindrical extensions, on the other.

The rationale for identifying "noninteraction" with set intersection is provided by the following lemma.[15]

Lemma. Let $M = \{\mu\}$, $N = \{\nu\}$, and let c be a mapping from $M \times N$ to the unit interval $[0,1]$. Then, under the following conditions on c:

(a) c is continuous in both arguments

(b) c is monotone non-decreasing in both arguments

(c) $c(\mu,0) = c(0,\nu) = 0$ for all μ, ν in $[0,1]$

(d) $c(\mu,\mu) = \mu$ for all μ in $[0,1]$

(e) For all μ in $[0,1]$, there do not exist $\alpha, \beta \in [0,1]$ such that $\alpha > \mu$, $\beta < \mu$ (or $\alpha < \mu$ and $\beta > \mu$) and $c(\alpha,\beta) = c(\mu,\mu)$

c must necessarily be of the form

$$c = \min(\mu,\nu) = \mu \wedge \nu \quad . \tag{4.29}$$

Note that condition (e) signifies that an increase in the first argument of c cannot be compensated by, or traded for, a decrease in the second argument of c, or vice-versa.

Proof. The proof is immediate. Let $\alpha > \mu$ and assume that $c(\alpha,\mu) > c(\mu,\mu) = \mu$. Now, $c(\alpha,0) = 0$ by (c) and hence from (a) it follows that there exists a β, $0 \leq \beta < \mu$, such that $c(\alpha,\beta) = \mu$. Since this contradicts (e), it follows that $c(\alpha,\mu) = \mu$ for $\alpha > \mu$ and hence that $c(\alpha,\mu) = \min(\alpha,\mu)$.

Q.E.D.

The main point of this lemma is that noncompensation implies and is implied by the form of dependence of c on μ and ν which is expressed by (4.29). Now, the intersection of $\bar{F}$ and $\bar{G}$ is defined by

[15] A thorough discussion of the rationale for the definitions of $\cap$ and $\cup$ for fuzzy sets may be found in [24].

$$\mu_{\bar{F} \cap \bar{G}}(u,v) = \mu_{\bar{F}}(u) \wedge \mu_{\bar{G}}(v) \tag{4.30}$$

and hence what we have called noninteractive conjunction -- or simply conjunction -- corresponds to noncompensation (in the sense of (e)) of the membership functions $\mu_{\bar{F}}$ and $\mu_{\bar{G}}$ which are associated with the operands of and.

To differentiate between noninteractive and interactive conjunction, the latter will be denoted by and*. With this understanding, the rule of interactive conjunction, in its general form, may be expressed as

$$X \text{ is } F \text{ and* } Y \text{ is } G \longrightarrow R(A(X),B(Y)) = F \otimes G \tag{4.31}$$

where $\otimes$ is a binary operation which maps F and G into a subset of $U \times V$ and thus provides a definition of and* in a particular context.

A simple example of an interactive conjunction is provided by the translation rule

$$X \text{ is } F \text{ and* } Y \text{ is } G \longrightarrow R(A(X),B(Y)) = FG \tag{4.32}$$

where

$$\mu_{FG} = \mu_F \mu_G \ . \tag{4.33}$$

Note that in this case, an increase in the grade of membership in F can be compensated for by a decrease in the grade of membership in G, and vice-versa.

It should be noted that while noninteractive conjunction is defined uniquely by (4.26), interactive conjunction is strongly application-dependent and has no universally valid definition. Thus, (4.33) constitutes but one of many ways in which interactive conjunction may be defined. In general, one would expect a definition of interactive conjunction to satisfy the conditions (a), (b), (c), and a weaker form of (d), namely, $c(\mu,\mu) \leq \mu$, but not (e).

The rules governing the translation of disjunctive propositions are dual of those of (4.26) and (4.31). Thus, the <u>rule of noninteractive disjunctive composition</u> -- or simply the <u>rule of disjunctive composition</u> asserts that

$$X \text{ is } F \text{ or } Y \text{ is } G \rightarrow R(A(X),B(Y)) = \bar{F} + \bar{G} \tag{4.34}$$

where $\bar{F}+\bar{G}$ denotes the union of the cylindrical extensions of F and G. Correspondingly, the <u>rule of interactive disjunction</u> reads

$$X \text{ is } F \text{ or* } Y \text{ is } G \rightarrow R(A(X),B(Y)) = F \otimes G \tag{4.35}$$

where $\otimes$ is an operation on F, G which defines or*, with the understanding that the conditions on or* are the same as on and*, except that 0 in (a) is replaced by 1.

Turning to conditional fuzzy propositions of the form "If X is F then Y is G," the translation rule for such propositions, which will be referred to as the <u>rule of conditional composition</u> may be expressed as[16]

$$\text{If } X \text{ is } F \text{ then } Y \text{ is } G \rightarrow R(A(X),B(Y)) = \bar{F}' \oplus \bar{G} \tag{4.36}$$

where $\oplus$ denotes the bounded sum[17] and $\bar{F}'$ is the complement of the cylindrical extension of F.

As an illustration, assume that <u>tall</u> and <u>young</u> are defined by

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[16] It is tacitly understood that the rule in question is noninteractive in nature. In the form defined by (4.36), it is consistent with the definition of implication in L_{aleph_1} logic. (See [1].)

[17] The bounded sum of F and G is defined by $\mu_{F\oplus G} = 1 \wedge (\mu_F + \mu_G)$, where + denotes the arithmetic sum. (See also A30.)

$$\underline{\text{tall}} = \int_U S(u;160,170,180)/u \tag{4.37}$$

$$\underline{\text{young}} = \int_V (1 - S(v;20,30,40))/v \tag{4.38}$$

where U and V may be taken to be the real line and the height is assumed to be measured in centimeters. Then, the fuzzy proposition "If Keith is <u>tall</u> then Adrienne is <u>young</u>" translates into

$$R(\text{Height(Keith)},\text{Age(Adrienne)}) = \overline{\underline{\text{tall}}'} \oplus \overline{\underline{\text{young}}} \tag{4.39}$$

or, more explicitly,

$$\begin{aligned} R(\text{Height(Keith)},\text{Age(Adrienne)}) &= \int_{U \times V} (1 \wedge (1 - \mu_{\underline{\text{tall}}}(u) + \mu_{\underline{\text{young}}}(v)))/(u,v) \\ &= \int_{U \times V} (1 \wedge (1 - S(u;160,170,180) \\ &\qquad + 1 - S(v;20,30,40)))/(u,v) \end{aligned} \tag{4.40}$$

If the conditional fuzzy proposition "If X is F then Y is G else Y is H" is interpreted as the conjunction of the propositions "If X is F then Y is G" and "If X is <u>not</u> F then Y is H," then by using in combination the rule of negation (4.2), the rule of conjuntive composition (4.26), and the rule of conditional composition (4.36), the translation of the proposition in question is found to be expressed by

$$\text{If } X \text{ is } F \text{ then } Y \text{ is } G \text{ else } Y \text{ is } H \longrightarrow R(A(X),B(Y)) = (F' \oplus \bar{G}) \cap (\bar{F} \oplus \bar{H}) \tag{4.41}$$

As a simple illustration, assume that $U = V = 1 + 2 + 3 + 4$,

$$F = \underline{\text{small}} = 1/1 + 0.6/2 + 0.2/3 \tag{4.42}$$

$$G = \underline{\text{large}} = 0.2/2 + 0.6/3 + 1/4 \tag{4.43}$$

and

$$H = \underline{\text{very}}\ \underline{\text{large}} = 0.02/2 + 0.36/3 + 1/4\ . \tag{4.44}$$

Then

$$F' = 0.4/2 + 0.8/3 + 1/4 \tag{4.45}$$

$$\begin{aligned}\bar{F}' = {} & 0.4/((2,1)+(2,2)+(2,3)+(2,4)) \\ & + 0.8/((3,1)+(3,2)+(3,3)+(3,4)) \\ & + 1/((4,1)+(4,2)+(4,3)+(4,4))\end{aligned} \tag{4.46}$$

$$\begin{aligned}\bar{G} = {} & 0.2/((1,2)+(2,2)+(3,2)+(4,2)) \\ & + 0.6/((1,3)+(2,3)+(3,3)+(4,3)) \\ & + 1/((1,4)+(2,4)+(3,4)+(4,4))\end{aligned} \tag{4.47}$$

$$\begin{aligned}F' \oplus G = {} & 0.2/(1,2) + 0.6/(1,3) + 1/(1,4) \\ & + 0.4/(2,1) + 0.6/(2,2) + 1/(2,3) + 1/(2,4) \\ & + 0.8/(3,1) + 1/(3,2) + 1/(3,3) + 1/(3,4) \\ & + 1/(4,1) + 1/(4,2) + 1/(4,3) + 1/(4,4)\end{aligned} \tag{4.48}$$

$$\begin{aligned}\bar{F} \oplus \bar{H} = {} & 1/(1,1) + 1/(1,2) + 1/(1,3) + 1/(1,4) \\ & + 0.6/(2,1) + 0.64/(2,2) + 0.96/(2,3) + 1/(2,4) \\ & + 0.2/(3,1) + 0.24/(3,2) + 0.56/(3,3) + 1/(3,4) \\ & + 0.04/(4,2) + 0.36/(4,3) + 1/(4,4)\end{aligned} \tag{4.49}$$

and hence the translation of "If X is small then Y is large else Y is very large" becomes

$$R(X,Y) = 0.2/(1,2) + 0.6/(1,3) + 1/(1,4) \tag{4.50}$$
$$+ 0.4/(2,1) + 0.6/(2,2) + 0.96/(2,3) + 1/(2,4)$$
$$+ 0.2/(3,1) + 0.24/(3,2) + 0.56/(3,3) + 1/(3,4)$$
$$+ 0.04/(4,2) + 0.36/(4,3) + 1/(4,4)$$

As in the case of the preceding example, translation rules may be used in combination to yield the meaning of composite fuzzy propositions which contain modifiers, conjunctions, disjunctions and implications. For example, if X, Y and Z are associated with the universes of discourse U, V and W, respectively, then using (4.7), (4.26) and (4.36) in combination, we find

$$X \text{ is } \underline{\text{very}}\ \underline{\text{small}} \text{ and (if } Y \text{ is } \underline{\text{small}} \text{ then } Z \text{ is } \underline{\text{very}}\ \underline{\text{large}}) \tag{4.51}$$
$$\rightarrow R(X,Y,Z) = \underline{\text{small}}^2 \times (\overline{\underline{\text{small}}}' \oplus \overline{\underline{\text{large}}^2})$$

where $\overline{\underline{\text{small}}}$ and $\overline{\underline{\text{large}}^2}$ are cylindrical extensions in $V \times W$ of small and large^2, respectively.

In addition to the rules discussed above, we shall regard as a rule of Type II the relational rule

$$X \text{ is in relation } F \text{ to } Y \rightarrow R(A(X),B(Y)) = F \tag{4.52}$$

or, equivalently,

$$X \text{ and } Y \text{ are } F \rightarrow R(A(X),B(Y)) = F \tag{4.53}$$

where F is a fuzzy relation in $U \times V$. For example, "Naomi is much taller than Maria" translates into

$$R(\text{Height(Naomi)},\text{Height(Maria)}) = \underline{\text{much}}\ \underline{\text{taller}} \tag{4.54}$$

where much taller is a fuzzy relation in R^2 defined by, say,

$$\underline{\text{much}}\ \underline{\text{taller}} = \int_{R^2} S(u-v;0,5,10)/(u,v) \quad . \tag{4.55}$$

Similarly, the fuzzy proposition "X and Y are approximately equal" translates into

$$R(X,Y) = \underline{\text{approximately}}\ \underline{\text{equal}}$$

where approximately equal is a fuzzy relation in R^2 defined by, say,

$$\underline{\text{approximately}}\ \underline{\text{equal}} = \int_{R^2} (1 + (\tfrac{u-v}{2})^2)^{-1}/(u,v) \ . \tag{4.56}$$

The rules and the examples given in the preceding discussion are intended merely to illustrate some of the basic ideas behind the characterization of the meaning of composite propositions by relational assignment equations. We proceed next to the somewhat more involved issues relating to the treatment of fuzzy quantification and truth-functional modification.

5. Translation Rules for Fuzzy Propositions - Types III and IV

As was stated earlier, translation rules of Type III apply to fuzzy propositions of the general form "QX are F," where Q is a fuzzy quantifier such as most, some, few, many, very many, not many, etc.; and F is a fuzzy subset of a universe of discourse $U = \{u\}$. Typical examples of propositions of this type are: "Most Swedes are tall," "Not many Italians are blond," "Some X's are large," etc.

Basically, what we are dealing with in cases of this type is not a single fuzzy proposition such as "X is F," but a fuzzy proposition concerning a collection of fuzzy or nonfuzzy propositions. More specifically, consider the proposition "Most Swedes are tall," and let $S_1,\ldots,S_N$ be a population of Swedes, with μ_i, $i = 1,\ldots,N$, representing the grade of membership of S_i in the fuzzy set tall.

Now, if F is a fuzzy subset of a finite universe of discourse $U = \{u_1, \ldots, u_n\}$, then the cardinality (or the power) of F is expressed by [22], [2]

$$|F| \triangleq \mu_1 + \cdots + \mu_N \tag{5.1}$$

where μ_i is the grade of membership of μ_i in F and $+$ is the arithmetic sum.[18] Using (5.1), the proportion of Swedes who are <u>tall</u> may be expressed as

$$r_{\underline{tall}} = \frac{\mu_1 + \cdots + \mu_N}{N} \tag{5.2}$$

and thus the proposition "<u>Most</u> Swedes are <u>tall</u>" translates into

$$R\left(\frac{\mu_1 + \cdots + \mu_N}{N}\right) = \underline{most} \tag{5.3}$$

where <u>most</u> is a fuzzy subset of the unit interval defined by, say,

$$\mu_{\underline{most}} = S(0.5, 0.75, 1) \ . \tag{5.4}$$

Stated in more general terms, the <u>rule of quantification</u> asserts that the translation of "QX are F" is given by

$$QX \text{ are } F \rightarrow R\left(\frac{\mu_1 + \cdots + \mu_N}{N}\right) = Q \tag{5.5}$$

or

$$QX \text{ are } F \rightarrow R(\mu_1 + \cdots + \mu_N) = Q \tag{5.6}$$

depending, respectively, on whether Q represents a fuzzy proportion (e.g., <u>most</u>) or a fuzzy number (e.g., <u>several</u>). Thus, in (5.5) Q is a fuzzy subset of the unit interval, while in (5.6) Q is a fuzzy subset of the

[18] In some instances it may be necessary to modify (5.1) by introducing a cutoff such that the μ_i below the cutoff are excluded from the right-hand member of (5.1).

integers $\{0,1,2,...\}$.

It is important to note that the relational assignment equations (5.5) and (5.6) define a fuzzy restriction not in U but in the N-cube $[0,1]^N$. It is this restriction, then, that constitutes the meaning of the fuzzy proposition "QX are F." As a simple illustration, let $N = 4$ and $\mu_1 = 0.8$, $\mu_2 = 0.6$, $\mu_3 = 1$ and $\mu_4 = 0.4$. Then $r_{\underline{tall}} = 0.7$ and, if most is defined by (5.4), $\mu_{\underline{most}}(0.7) = 0.65$. Thus, the grade of membership of the point $(0.8,0.6,1,0.4)$ in the fuzzy restriction associated with the proposition "Most Swedes are tall" is 0.65.

Another point that should be noted is that the quantifier some, in the sense used in classical logic, may be viewed as the complement of none, where none is a subset of $[0,1]$ (or $\{0,1,...,\}$) defined by

$$\begin{aligned} \mu_{\underline{none}}(u) &= 1 \quad \text{for} \quad u = 0 \\ &= 0 \quad \text{elsewhere} \end{aligned} \tag{5.7}$$

Thus,

$$\begin{aligned} \underline{some} &= \underline{none}' \\ &= \underline{not}\ \underline{none} \end{aligned} \tag{5.8}$$

The dual (see (3.30)) of none is all, with the membership function of all expressed by

$$\begin{aligned} \mu_{\underline{all}}(u) &= 1 \quad \text{for} \quad u = 1 \\ &= 0 \quad \text{elsewhere} \end{aligned} \tag{5.9}$$

Thus,

$$D(\underline{none}) = \underline{all} \tag{5.10}$$

and hence

$$\begin{aligned} \underline{some} &= \underline{not}\ \underline{none} \\ &= D(\underline{not}\ \underline{all}) \end{aligned} \tag{5.11}$$

In everyday discourse, however, <u>some</u> is usually used not in the nonfuzzy sense of (5.11), but in a fuzzy sense which may be approximated as

$$\underline{\text{some}} = \underline{\text{not}}\ \underline{\text{none}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{many}} \tag{5.12}$$

or, alternatively, as

$$\underline{\text{some}} = D(\underline{\text{most}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{all}}) \ . \tag{5.13}$$

Note that this interpretation of <u>some</u> as a fuzzy subset of $[0,1]$ differs substantially from the nonfuzzy definition expressed by (5.11).

When N is large, it is advantageous in many cases to use a limiting form of (5.5) as $N \to \infty$. Specifically, with reference to (5.1), let $\rho(u)du$ denote the proportion of Swedes whose height is in the interval $[u,u+du]$. Then, the proportion of Swedes who are <u>tall</u> is given by

$$r_{\underline{\text{tall}}} = \int_U \rho(u)\mu_{\underline{\text{tall}}}(u)du \tag{5.14}$$

where $\mu_{\underline{\text{tall}}}(u)$ denotes the grade of membership of a Swede whose height is u in the fuzzy subset of U labeled <u>tall</u>. This implies that

$$\underline{\text{Most}}\ \text{Swedes are}\ \underline{\text{tall}} \to R\left(\int_U \rho(u)\mu_{\underline{\text{tall}}}(u)du\right) = \underline{\text{most}} \tag{5.15}$$

and, more generally, that the translation of "QX are F" is given by

$$QX \text{ are } F \to R\left(\int_U \rho(u)\mu_F(u)du\right) = Q \tag{5.16}$$

where $\rho(u)du$ is the proportion of values of an implied attribute $A(X)$ which fall in the interval $[u,u+du]$.

As an illustration, suppose that <u>tall</u> and <u>most</u> are defined as fuzzy subsets of $U = [0,200]$ and $V = [0,1]$, respectively, by

$$\mu_{\underline{\text{tall}}} = S(160;170;180) \tag{5.17}$$

and

$$\mu_{\underline{most}} = S(0.5,0.75,1) \ . \tag{5.18}$$

Then, the compatibility of a distribution ρ with the restriction induced by the fuzzy proposition $p \triangleq$ Most Swedes are tall is given by

$$\mu_p(\rho) = S(\int_0^{200} \rho(u)S(u,160;170,180)du;0.5,0.75,1) \ . \tag{5.19}$$

Through this equation, the proposition in question defines a fuzzy set in the space of distributions $\{\rho\}$ in U, with the membership function of the set in question expressed by (5.19). This fuzzy set, then, may be viewed as a representation of the meaning of p.

Turning to translation rules of Type IV, let p be a fuzzy proposition and let p* be a fuzzy proposition which is derived from p by truth-functional modification, that is,

$$p^* \triangleq p \text{ is } \tau \tag{5.20}$$

where τ is a linguistic truth-value. As an illustration, if $p \triangleq$ Andrea is young, then p* might be

$$p^* \triangleq \text{Andrea is } \underline{\text{young}} \text{ is } \underline{\text{very}}\ \underline{\text{true}} \ . \tag{5.21}$$

Similarly, if p = X and Y are approximately equal, then p* might be

$$p^* \triangleq \text{X and Y are } \underline{\text{approximately}}\ \underline{\text{equal}} \text{ is } \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{true}} \ . \tag{5.22}$$

For concreteness, we shall focus our attention on fuzzy propositions of the form $p \triangleq X$ is F, where F is a fuzzy subset of $U = \{u\}$. Let t, $t \in [0,1]$, be a numerical truth-value of p. If we assume, as stated in Sec. 3, that t may be interpreted as the degree of consistency of the reference nonfuzzy proposition "X is u" with the fuzzy proposition "X is F," then

$$t = \mu_F(u) \tag{5.23}$$

and hence

$$u = \mu_F^{-1}(t) \tag{5.24}$$

where μ_F^{-1} is a function (or, more generally, a relation) which is inverse to μ_F. As an illustration, if

$$F = \underline{young} = \int_U (1 - S(u;20,30,40))/u \tag{5.25}$$

and $t = 0.5$, then

$$\begin{aligned} u &= \mu_F^{-1}(0.5) \\ &= 30 \text{ years} . \end{aligned} \tag{5.26}$$

Thus, "Andrea is <u>young</u> is 0.5 true" translates into "Andrea is 30 years old," and, more generally, "X is F is t" translates into

$$X \text{ is } \mu_F^{-1}(t) . \tag{5.27}$$

To extend (5.27) to linguistic truth-values, we may employ the extension principle in a manner similar to that of Sec. 3. Specifically, if g is a mapping from U to V and F is a fuzzy subset of U, then $g(F)$ is given by

$$\begin{aligned} g(F) &\triangleq \langle g(F) \rangle \\ &\triangleq \int_V \mu_F(u)/g(u) \end{aligned} \tag{5.28}$$

where the angular brackets signify that $\langle g(F) \rangle$ is to be evaluated by the use of the extension principle.[19] As a simple illustration, if

$$U = 0 + 0.1 + 0.2 + \cdots + 0.9 + 1 \tag{5.29}$$

and

$$F = 0.6/0.8 + 0.8/0.9 + 1/1 \tag{5.30}$$

then for

$$g(u) = 1 - u \tag{5.31}$$

we have

$$1 - (0.6/0.8 + 0.8/0.9 + 1/1) = 0.6/0.2 + 0.8/0.1 + 1/0 \tag{5.32}$$

while for

$$g(u) = u^2 \tag{5.33}$$

we obtain

$$(0.6/0.8 + 0.8/0.9 + 1/1)^2 = 0.6/0.64 + 0.8/0.81 + 1/1 \quad . \tag{5.34}$$

Equivalently, by regarding g as a binary relation from U to V and F as a unary fuzzy relation in U, $g(F)$ may be expressed as the composition of F and U, that is (see A60)

$$\langle g(F) \rangle = g \circ F \quad . \tag{5.35}$$

In particular, if the mapping $g: U \to V$ is 1-1, then (5.35) implies (through (5.28)) that

$$\mu_{g \circ F}(v) = \mu_F(u) \tag{5.36}$$

where $v = g(u)$ is the image of u.

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[19] The angular brackets may be suppressed whenever it is clear from the context that the evaluation is to be performed via the extension principle. If it is necessary to stipulate that the extension principle is <u>not</u> to be used, brackets of the form $\{\!\{ \; \}\!\}$ may be used for this purpose.

By applying these relations to (5.24), the rule of truth-functional modification may be expressed as the translation rule (of Type IV)

$$p^* = X \text{ is } F \text{ is } \tau \rightarrow q \triangleq X \text{ is } F^* \tag{5.37}$$

where F is a fuzzy subset of U, τ is a linguistic truth-value and F^* is a fuzzy subset of U which is related to F and τ by

$$F^* = \langle\mu_F^{-1}(\tau)\rangle = \mu_F^{-1} \circ \tau \tag{5.38}$$

where μ_F^{-1} is the inverse of the membership function of F and $\circ$ is the operation of composition. In particular, if μ_F is a 1-1 mapping from U to $[0,1]$, then

$$\mu_{F^*}(u) = \mu_\tau(\mu_F(u)) \tag{5.39}$$

where μ_τ is the membership function of τ.

On combining (5.37) with (5.38), the rule of truth-functional modification may be expressed as

$$X \text{ is } F \text{ is } \tau \rightarrow R(A(X)) = \mu_F^{-1} \circ \tau \tag{5.40}$$

where $A(X)$ is an implied attribute of X.

As a simple illustration, assume that $U = 1+2+3+4$ and consider the fuzzy proposition

$$p^* = X \text{ is } \underline{\text{small}} \text{ is } \underline{\text{very}}\ \underline{\text{true}} \tag{5.41}$$

where small is defined by

$$\underline{\text{small}} = 1/1 + 0.8/2 + 0.4/3 \tag{5.42}$$

and

$$\underline{\text{true}} = 0.2/0.6 + 0.5/0.8 + 0.8/0.9 + 1/1\ . \tag{5.43}$$

From (5.43) and (4.8), it follows that

$$\underline{very}\ \underline{true} = 0.04/0.6 + 0.25/0.8 + 0.64/0.9 + 1/1 \qquad (5.44)$$

and hence by (5.39) the translation of (5.41) is given by

$$R(X) = 1/1 + 0.25/2 \qquad (5.45)$$

which is approximately equivalent to

$$R(X) = \underline{very}\ \underline{very}\ \underline{small} \qquad (5.46)$$

if

$$\underline{very}\ \underline{very}\ \underline{small} = 1/1 + 0.4/2 + 0.03/3 \qquad (5.47)$$

is regarded as a linguistic approximation to the right-hand member of (5.45).

It is instructive to consider also a continuous version of this example. Assuming that $U = [0,\infty)$ and

$$\underline{small} = \int_0^\infty (1 + (\tfrac{u}{5})^2)^{-1}/u \qquad (5.48)$$

$$\underline{true} = \int_0^1 (1 + 16(1-v)^2)^{-1}/v \qquad (5.49)$$

and

$$\underline{very}\ \underline{true} = \int_0^1 (1 + 16(1-v)^2)^{-2}/v\ , \qquad (5.50)$$

we obtain from (5.39) and (5.40) the translation

$$X \text{ is } \underline{small} \text{ is } \underline{very}\ \underline{true} \qquad (5.51)$$

$$\rightarrow R(X) = \int_0^\infty (1 + 16(1 + (1 + (\tfrac{u}{5})^2)^{-1})^{-2})/u\ .$$

By way of comparison, $u = 4$ is compatible to the degree 0.6 with "X is small" and to the degree 0.2 with "X is small is very true."

An important conclusion that may be drawn from the rule of truth-functional modification is that the qualification of a fuzzy proposition p with a linguistic truth-value τ has the effect of transforming p^* into an unqualified fuzzy proposition q, with the fuzzy restriction associated with q related to that of p by (5.37). In this way, a qualified proposition such as "X is <u>small</u> is <u>very</u> <u>true</u>" may be approximated to by an unqualified proposition such as "X is <u>very</u> <u>very</u> <u>small</u>," and, more generally, $p \triangleq X \text{ is } F \text{ is } \tau$ may be replaced by $q \triangleq X \text{ is } F^*$.

It is important to recognize, however, that the rule of truth-functional modification rests in an essential way on the assumption that a numerical truth-value in a fuzzy proposition of the form $p^* = X \text{ is } F \text{ is } t$ serves as a measure of consistency of the nonfuzzy proposition $r \triangleq X \text{ is } u$ with the fuzzy proposition $p \triangleq X \text{ is } F$. If this assumption is not valid, it might still be possible to assert that a qualified fuzzy proposition of the form $p^* \triangleq X \text{ is } F \text{ is } \tau$ is equivalent to an unqualified fuzzy proposition of the form $q \triangleq X \text{ is } F^*$. However, the dependence of F^* on F and τ might not be correctly expressed by (5.38), since it is affected by the form of the reference proposition, r, as well as the criterion employed to define the consistency of p with r.

This concludes our discussion of translation rules of Types I, II, III and IV. As was stated earlier, these rules may be used in combination to yield translations of more complex composite fuzzy propositions, e.g., (If X is <u>large</u> is <u>true</u> and Y is <u>small</u> is <u>very</u> <u>true</u> then it is <u>more</u> <u>or</u> <u>less</u> <u>true</u> that <u>most</u> Z's are <u>small</u>) is <u>very</u> <u>true</u>. In general, the translations of such propositions assume the form of a system of relational assignment equations which, in graphical form, may be represented as a semantic network or a conceptual dependency graph [113]-[118].

6. Truth-Values of Composite Propositions

The translation rules stated in Secs. 4 and 5 provide a means of determining the restriction associated with a composite proposition from the knowledge of the restrictions associated with its constituents. In an analogous fashion, the truth valuation rules given in this section provide a means of computing the truth-value of a composite proposition from the knowledge of the truth-values of its constitutents.

As will be seen in the sequel, the rules for truth valuation may be inferred from the corresponding translations rules of Types I, II, III and IV. In what follows, we shall describe the basic idea behind this method and illustrate it by several examples.

Let p be a fuzzy proposition of the form "X is F" and let $t = v(p)$ be its numerical truth-value in $V = [0,1]$. We assume that F is a fuzzy subset of a universe of discourse $U = \{u\}$, and that $A(X)$, an implied attribute of X, is a fuzzy variable which takes values in U, with F representing a fuzzy restriction on the values of $A(X)$.

As was stated in Sec. 3, a proposition of the form "X is F is t true," e.g., "Paule is <u>tall</u> is 0.8 true" means that the grade of membership of Paule in the class of <u>tall</u> women is 0.8, or, equivalently, that

$$\mu_{\underline{tall}}(\text{Height}(\text{Paule})) = 0.8 \tag{6.1}$$

where $\mu_{\underline{tall}}$ is the membership function of the fuzzy subset <u>tall</u> of the real line.

Now, if the truth-value of the proposition "Paule is <u>tall</u>" is 0.8, then what is the truth-value of the proposition "Paule is <u>very</u> <u>tall</u>?" If we assume that the effect of the modifier <u>very</u> is defined by (4.8), then it follows from the concentration rule (4.7) that the grade of membership of Height(Paule) in <u>very</u> <u>tall</u> -- and hence the truth-value of the proposition

$p \triangleq$ Paule is very tall -- is given by

$$v(\text{Paule is } \underline{\text{very}}\ \underline{\text{tall}}) = 0.8^2 \tag{6.2}$$

and, more generally,

$$\begin{aligned} v(\text{Paule is } \underline{\text{very}}\ \underline{\text{tall}}) &= (v(\text{Paule is } \underline{\text{tall}}))^2 \\ &= t^2 \end{aligned} \tag{6.3}$$

where $v(p)$ stands for the truth-value of p. Thus, the rule for computing the numerical truth-value of a fuzzy proposition of the form "X is very F" from the knowledge of the numerical truth-value of the proposition "X is F," may be expressed as

$$X \text{ is } F \text{ is } t \text{ true } \Rightarrow X \text{ is } \underline{\text{very}}\ F \text{ is } t^2 \text{ true} \tag{6.4}$$

where t is the numerical truth-value of the fuzzy proposition "X is F."

Now, having this rule for numerical truth-values, we can readily extend it to linguistic truth-values by the application of the extension principle, as we have done in Secs. 3 and 5. Thus, for such values (6.4) becomes

$$X \text{ is } F \text{ is } \tau \Rightarrow X \text{ is } \underline{\text{very}}\ F \text{ is } \langle\tau^2\rangle \tag{6.5}$$

where the angular brackets indicate that the evaluation of $\langle\tau^2\rangle$ is to be performed by the use of the extension principle.

In more specific terms, this means that, if

$$\begin{aligned} v(X \text{ is } F) &= \tau \\ &= \int_0^1 \mu_\tau(u)/v\ , \quad v \in V\ , \end{aligned} \tag{6.6}$$

where μ_τ is the membership function of the linguistic truth-value τ, then

$$v(X \text{ is } \underline{\text{very}}\ F) = \langle \tau^2 \rangle \qquad (6.7)$$
$$= \int_0^1 \mu_\tau(v)/v^2 \quad .$$

As a simple illustration, suppose that $V = 0 + 0.1 + \cdots + 1$ and

$$v(\text{Paule is } \underline{\text{tall}}) = \underline{\text{very}}\ \underline{\text{true}} \qquad (6.8)$$

where

$$\underline{\text{true}} = 0.6/0.8 + 0.9/0.9 + 1/1 \qquad (6.9)$$

and

$$\underline{\text{very}}\ \underline{\text{true}} = \underline{\text{true}}^2 \qquad (6.10)$$
$$= 0.36/0.8 + 0.81/0.9 + 1/1 \ .$$

Then, by (6.7)

$$v(\text{Paule is } \underline{\text{very}}\ \underline{\text{tall}}) = \langle (\underline{\text{very}}\ \underline{\text{true}})^2 \rangle \qquad (6.11)$$
$$= \langle (0.36/0.8 + 0.81/0.9 + 1/1)^2 \rangle$$
$$= 0.36/0.64 + 0.81/0.81 + 1/1$$

and, if <u>true</u> is taken to be a rough linguistic approximation to the right-hand member of (6.11), i.e.,

$$\underline{\text{true}} = 0.6/0.8 + 0.9/0.9 + 1/1 \qquad (6.12)$$
$$= LA\ (0.36/0.8 + 0.81/0.9 + 1/1) \ ,$$

then we can infer from (6.11) that

$$v(\text{Paule is } \underline{\text{very}}\ \underline{\text{tall}}) \cong \underline{\text{true}} \ . \qquad (6.13)$$

More generally, let q be a fuzzy proposition of the form $q \triangleq X$ is mF where m is a modifier whose effect on F is described by the equation

$$\mu_{mF}(u) = g(\mu_F(u)) \ , \quad u \in U \qquad (6.14)$$

where g is a mapping from $[0,1]$ to $[0,1]$. Then, from the foregoing discussion it follows that

$$X \text{ is } F \text{ is } \tau \Rightarrow X \text{ is } mF \text{ is } \langle g(\tau)\rangle \tag{6.15}$$

where τ is the linguistic truth-value of $p \triangleq X$ is F, and

$$\langle g(\tau)\rangle = \int_0^1 \mu_\tau(v)/g(v) \tag{6.16}$$

where μ_τ is the membership function of τ. By analogy with (4.1), the rule expressed by (6.15) will be referred to as the <u>modifier</u> <u>rule</u> for truth valuation.

In particular, for the case where $m \triangleq$ <u>not</u>, (6.15) becomes

$$X \text{ is } F \text{ is } \tau \Rightarrow X \text{ is } \underline{\text{not}} \; F \text{ is } D(\tau) \tag{6.17}$$

where

$$D(\tau) = \langle 1-\tau\rangle \tag{6.18}$$

is the dual of τ (see (3.30)). For example, if $\tau =$ <u>true</u>, then

$$\begin{aligned} D(\underline{\text{true}}) &= \langle 1 - \underline{\text{true}}\rangle \\ &= \underline{\text{false}} \end{aligned}$$

and hence

$$X \text{ is } F \text{ is } \underline{\text{true}} \Rightarrow X \text{ is } \underline{\text{not}} \; F \text{ is } \underline{\text{false}} \tag{6.19}$$

where

$$\mu_{\underline{\text{false}}}(v) = \mu_{\underline{\text{true}}}(1-v) \;, \quad v \in V \;. \tag{6.20}$$

By analogy with (4.2), the rule expressed by (6.17) will be referred to as the <u>rule</u> <u>of</u> <u>negation</u> for truth valuation. It should be observed that the application of the rule of truth-functional modification to the left-hand member of the (6.17) -- and, more generally, (6.15) -- yields the same restriction as its application to the right-hand member.

Turning to rules of Type II, consider the composite proposition $p \triangleq X$ is F and Y is G, and assume that the numerical truth-values of the constituent propositions are

$$v(X \text{ is } F) = s \tag{6.21}$$

and

$$v(X \text{ is } G) = t \,. \tag{6.22}$$

Now, from the rule of conjunctive composition (4.26), it follows that p translates into

$$R(A(X),B(Y)) = F \times G \tag{6.23}$$

and consequently

$$\begin{aligned} v(X \text{ is } F \text{ and } Y \text{ is } G) &= \text{grade of membership of } (A(X),B(Y)) \text{ in } F \times G \\ &= \mu_F(A(X)) \wedge \mu_G(B(Y)) \end{aligned} \tag{6.24}$$

by the definition of $F \times G$ (A56).

On the other hand, we have (by (3.15))

$$v(X \text{ is } F) = \mu_F(A(X)) \tag{6.25}$$

$$v(Y \text{ is } G) = \mu_G(B(Y)) \tag{6.26}$$

and hence

$$\begin{aligned} v(X \text{ is } F \text{ and } Y \text{ is } G) &= v(X \text{ is } F) \wedge v(Y \text{ is } G) \\ &= s \wedge t \end{aligned} \tag{6.27}$$

or, equivalently,

$$(X \text{ is } F \text{ is } s \text{ true},\ Y \text{ is } G \text{ is } t \text{ true}) \Rightarrow (X \text{ is } F \text{ and } Y \text{ is } G) \text{ is } s \wedge t \text{ true} \tag{6.28}$$

As in the case of (6.15), we observe that

$$X \text{ is } F \text{ is } s \text{ true} \rightarrow X \text{ is } \mu_F^{-1}(s) \rightarrow A(X) = \mu_F^{-1}(s) \tag{6.29}$$

$$Y \text{ is } G \text{ is } t \text{ true} \rightarrow Y \text{ is } \mu_G^{-1}(t) \rightarrow B(Y) = \mu_G^{-1}(t) \tag{6.30}$$

and

$$(X \text{ is } F \text{ and } Y \text{ is } G) \text{ is } s \wedge t \text{ true} \rightarrow (A(X),B(Y)) \in \mu_{F\times G}^{-1}(s \wedge t) \ . \tag{6.31}$$

Thus, in this instance we obtain the inclusion relation

$$(\mu_F^{-1}(s),\mu_G^{-1}(t)) \in \mu_{F\times G}^{-1}(s \wedge t) \tag{6.32}$$

rather than equality, as in (6.15).[20]

To extend (6.28) to linguistic truth-values, we can invoke the extension principle, as we have done in the case of the modifier rule (4.1). In this way, we are led to the rule of conjunction for truth valuation, which asserts that

$$v(X \text{ is } F \text{ and } Y \text{ is } G) = \langle v(X \text{ is } F) \wedge v(Y \text{ is } G)\rangle \tag{6.33}$$

where the angular brackets signify that the evaluation is to be performed by the use of extension principle. Thus, if

$$v(X \text{ is } F) = \sigma \tag{6.34}$$

and

$$v(X \text{ is } G) = \tau \tag{6.35}$$

where σ and τ are linguistic truth-values with membership functions μ_σ and μ_τ, respectively, then (6.33) may be restated as

$$(X \text{ is } F \text{ is } \sigma, Y \text{ is } G \text{ is } \tau) \Rightarrow (X \text{ is } F \text{ and } Y \text{ is } G) \text{ is } \langle\sigma \wedge \tau\rangle \tag{6.36}$$

[20] This touches upon the issues of referential transparency and extensionality in fuzzy logic which are not as yet well understood.

where

$$<\sigma \wedge \tau> = \int_0^1 \mu_\sigma(u) \wedge \mu_\tau(v)/u \wedge v \ , \quad u, v \in [0,1] \ . \tag{6.37}$$

In a similar fashion, the rule of disjunction for truth evaluation is found to be expressed by

$$(X \text{ is } F \text{ is } \sigma, Y \text{ is } G \text{ is } \tau) \Rightarrow (X \text{ is } F \text{ or } Y \text{ is } G) \text{ is } <\sigma \vee \tau> \tag{6.38}$$

where

$$<\sigma \vee \tau> = \int_0^1 \mu_\sigma(u) \wedge \mu_\tau(v)/u \vee v \tag{6.39}$$

while the rule of implication reads

$$(X \text{ is } F \text{ is } \sigma, Y \text{ is } G \text{ is } \tau) \Rightarrow (\text{If } X \text{ is } F \text{ then } Y \text{ is } G) \text{ is } <<1-\sigma> \oplus \tau> \tag{6.40}$$

where $\oplus$ denotes the bounded sum (see A30) and[21]

$$<(1-\sigma) \oplus \tau> = \int_0^1 \mu_\sigma(u) \wedge \mu_\tau(v)/1 \wedge (1-u+v) \ . \tag{6.41}$$

As an illustration, assume that

$$\sigma \triangleq \underline{\text{true}} = 0.6/0.8 + 0.9/0.9 + 1/1 \tag{6.42}$$

$$\tau \triangleq \underline{\text{not}}\ \underline{\text{true}} = 1/(0+0.1+\cdots+0.7) + 0.4/0.8 + 0.1/0.9 \ . \tag{6.43}$$

Then

$$<\sigma \wedge \tau> = 1/(0+0.1+\cdots+0.7) + 0.4/0.8 + 0.1/0.9 \tag{6.44}$$

$$= \underline{\text{not}}\ \underline{\text{true}}$$

[21] As shown in [1], this expression for if...then... may be derived alternatively by applying the extension principle to the definition of implication in Lukasiewicz's L_{aleph_1} logic.

$$<\sigma \vee \tau> = \underline{\text{true}} \tag{6.45}$$

and

$$<<1-\sigma> \oplus \tau> = <\underline{\text{false}} \oplus \underline{\text{not}}\ \underline{\text{true}}> \tag{6.46}$$

$$= 1/(0+0.1+\cdots+0.7) + 0.9/0.8 + 0.6/0.9 \tag{6.47}$$

$$\cong \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{true}} \tag{6.48}$$

where the right-hand member of (6.48) is a linguistic approximation to the right-hand member of (6.47).

Proceeding in a similar fashion, we can develop valuation rules for composite propositions of more complex types than those considered in the previous discussion. We shall not pursue this subject further in the present paper.

7. Rules of Inference in Fuzzy Logic

Stated informally, the rules of inference in fuzzy logic constitute a collection of propositions -- some of which are precise and some are not -- which serve to provide a means of computing the fuzzy restriction associated with a variable $(X_1,\ldots,X_m)$ from the knowledge of the fuzzy restrictions associated with some other variables $Y_1,\ldots,Y_n$.

A typical example of an inference process in fuzzy logic is the following. Consider the fuzzy propositions

$$p \triangleq X \text{ is } \underline{\text{small}} \tag{7.1}$$

and

$$q \triangleq X \text{ and } Y \text{ are } \underline{\text{approximately}}\ \underline{\text{equal}} \tag{7.2}$$

where $U = V = 1+2+3+4$ and small and approximately equal are defined by

$$\underline{\text{small}} = 1/1 + 0.6/2 + 0.2/3$$

and

$$\underline{\text{approximately}}\ \underline{\text{equal}} = 1/((1,1)+(2,2)+(3,3)+(4,4)) + 0.5/((1,2)+(2,1)+(2,3)+(3,2) + (3,4)+(4,3)) \ . \tag{7.4}$$

By using (2.4) and (4.53), the translations of these propositions are found to be

$$R(X) = \underline{\text{small}} \tag{7.5}$$

and

$$R(X,Y) = \underline{\text{approximately}}\ \underline{\text{equal}} \ . \tag{7.6}$$

Now, let us replace p by its cylindrical extension, $\bar{p}$, which reads

$$\bar{p} = X \text{ is } \underline{\text{small}} \text{ and } Y \text{ is } \underline{\text{unrestricted}} \tag{7.7}$$

and form the conjunctive composition of $\bar{p}$ and q, i.e.,

$$\bar{p} \text{ and } q = (X \text{ is } \underline{\text{small}}\ \underline{\text{and}}\ Y \text{ is } \underline{\text{unrestricted}})\ \underline{\text{and}}\ (X \text{ and } Y \text{ are } \underline{\text{approximately}}\ \underline{\text{equal}}) \tag{7.8}$$

which by (4.28) translates into

$$\bar{p} \text{ and } q \triangleq R^*(X,Y) = (\underline{\text{small}} \times U) \cap (\underline{\text{approximately}}\ \underline{\text{equal}}) \tag{7.9}$$

implying that the membership function of the restriction defined by (7.9) is given by

$$\mu_{R^*}(u,v) = \mu_{\underline{\text{small}}}(u) \wedge \mu_{\underline{\text{approximately}}\ \underline{\text{equal}}}(u,v) \ . \tag{7.10}$$

From the restriction $R^*(X,Y)$ defined by (7.10), we can infer the fuzzy restriction associated with Y by projecting $R^*(X,Y)$ on the universe of discourse associated with X, that is

$$R(Y) = \text{Proj } R^*(X,Y) \text{ on } U \tag{7.11}$$

which, by the definition of projection (see (A58), (A60)) is equivalent to

$$\begin{aligned} R(Y) &= R(X) \circ R(X,Y) \\ &= \underline{\text{small}} \circ \underline{\text{approximately}}\ \underline{\text{equal}} \end{aligned} \tag{7.12}$$

where the right-hand member denotes the composition of the unary fuzzy relation small with the binary fuzzy relation approximately equal. Expressed in terms of membership functions of $R(Y)$, small and approximately equal, (7.12) reads

$$\mu_{R(Y)}(v) = \vee_u(\mu_{\underline{\text{small}}}(u) \wedge \mu_{\underline{\text{approximately}}\ \underline{\text{equal}}}(u,v)) \tag{7.13}$$

where $\vee_u$ denotes the supremum over $u \in U$.

To compute $\mu_{R(Y)}$ from (7.13), it is convenient to represent the right-hand member of (7.13) as the max-min product[22] of the relation matrices of small and approximately equal. In this way, we obtain

$$[1 \quad 0.6 \quad 0.2 \quad 0] \circ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} = [1 \quad 0.6 \quad 0.5 \quad 0.2] \tag{7.14}$$

which implies that

$$R(Y) = 1/1 + 0.6/2 + 0.5/3 + 0.2/4 \quad . \tag{7.15}$$

To approximate to the right-hand member of (7.15) by a linguistic value of Y, we note that if more or less is defined as a fuzzifier (see (4.17)) with

[22] In this product, the operations of $+$ and product are replaced by $\vee$ and $\wedge$, respectively.

$$K(1) = 1/1 + 0.7/2 \tag{7.16}$$
$$K(2) = 1/2 + 0.7/3$$
$$K(3) = 1/3 + 0.7/4$$
$$K(4) = 1/4$$

then <u>more</u> <u>or</u> <u>less</u> <u>small</u> becomes

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{small}} = 1/1 + 0.7/2 + 0.42/3 + 0.14/4 \tag{7.17}$$

which is a reasonably close approximation to (7.15) in the sense that

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{small}} = LA(1/1 + 0.6/2 + 0.5/3 + 0.2/4)\ . \tag{7.18}$$

In this way, then, from the fuzzy propositions $p \triangleq X$ is <u>small</u> and $q \triangleq X$ and Y are <u>approximately</u> <u>equal</u> we can infer exactly the fuzzy proposition

$$Y \text{ is } 1/1 + 0.6/2 + 0.5/3 + 0.2/4 \tag{7.19}$$

and approximately

$$Y \text{ is } \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{small}}\ . \tag{7.20}$$

The essential features of the procedure which we have employed in the above example may be summarized as follows.

Let p and q be fuzzy propositions of the form

$$p \triangleq X \text{ is } F \tag{7.21}$$

$$q \triangleq X \text{ is in relation } G \text{ to } Y \tag{7.22}$$

where F is a fuzzy subset of U and G is a fuzzy relation in $U \times V$. Then, from p and q we can infer exactly

$$r \triangleq Y \text{ is } F \circ G \tag{7.23}$$

and approximately

$$r \triangleq Y \text{ is } LA(F \circ G) \tag{7.24}$$

where $\circ$ is the operation of composition and LA stands for "linguistic approximation."[23] We shall refer to this rule as the <u>compositional rule of inference</u> [7], [1], [2]. It should be noted that this rule is an instance of a <u>semantic</u> rule in the sense that r depends on the meaning of F and G through the composition $F \circ G$.

A special but important case of the compositional rule of inference results when G is a function from U to V, with q having the form

$$q \triangleq Y \text{ is } g(X) \ . \tag{7.25}$$

In this case, the composition of F and G yields

$$F \circ G = \langle g(F) \rangle \tag{7.26}$$

where the angular brackets signify that $\langle g(F) \rangle$ is to be evaluated by the use of the extension principle. Thus, the rule of inference which applies to this case may be expressed as

$$\begin{array}{l} p \triangleq X \text{ is } F \\ \underline{q \triangleq Y \text{ is } g(X)} \\ r \triangleq Y \text{ is } \langle g(F) \rangle \end{array} \tag{7.27}$$

and we shall refer to it as the <u>transformational rule of inference</u>.[24]

As a simple illustration of (7.27), suppose that $U = V = 0 + 1 + 2 + 3 + \cdots$,

$$F \triangleq \underline{\text{small}} \triangleq 1/0 + 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5 \tag{7.28}$$

---

[23] Exposition of a least squares approach to linguistic approximation may be found in [53].

[24] The transformational rule of inference is closely related to the rule for computing the membership function of a set induced by a mapping [3].

and g is the operation of squaring. Then,

$$<\underline{\text{small}}^2> = 1/0 + 1/1 + 0.8/4 + 0.6/9 + 0.4/16 + 0.2/25 \tag{7.29}$$

and, we have

$$\begin{array}{l} p \triangleq X \text{ is } \underline{\text{small}} \\ q \triangleq Y \text{ is } X^2 \\ \hline r \triangleq Y \text{ is } 1/0+1/1+0.8/4+0.6/9+0.4/16+0.2/25 \end{array}$$

Another important special case of (7.23) is the <u>rule of compositional modus ponens</u>. Specifically, for the case where q is of the form

$$q \triangleq \text{If } X \text{ is } F \text{ then } Y \text{ is } G \tag{7.30}$$

the translation rule of conditional composition (4.36) asserts that

$$\text{If } X \text{ is } G \text{ then } Y \text{ is } H \rightarrow (A(X),B(Y)) = \bar{G}' \oplus \bar{H} \tag{7.31}$$

where $\bar{G}'$ is the cylindrical extension of the complement of G, $\bar{H}$ is the cylindrical extension of H, $\oplus$ is the bounded sum, and $A(X)$ and $B(Y)$ are the implied attributes of X and Y, respectively.

On applying (7.31) to the case where q is of the form (7.30), we obtain the <u>rule of compositional modus ponens</u>, which reads

$$\begin{array}{l} p \triangleq X \text{ is } F \\ q \triangleq \text{If } X \text{ is } G \text{ then } Y \text{ is } H \\ \hline r \triangleq Y \text{ is } F \circ (\bar{G}' \oplus \bar{H}) \end{array} \tag{7.32}$$

or, as a linguistic approximation,

$$r \triangleq Y \text{ is } LA(F \circ (\bar{G}' \oplus \bar{H})) \quad . \tag{7.33}$$

As a simple example which does not involve linguistic values, assume that $U = V = 1+2+3+4$ and

$$F = 0.2/2 + 0.6/3 + 1/4 \quad , \tag{7.34}$$

$$G = 0.6/2 + 1/3 + 0.5/4 \quad , \tag{7.35}$$

$$H = 1/2 + 0.6/3 + 0.2/4 \quad . \tag{7.36}$$

Then

$$\bar{G}' \oplus \bar{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 1 & 0.6 \\ 0 & 1 & 0.6 & 0.6 \\ 0.5 & 1 & 1 & 0.7 \end{bmatrix} \tag{7.37}$$

and

$$F \circ (\bar{G}' \oplus \bar{H}) = [0 \quad 0.2 \quad 0.6 \quad 1] \circ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 1 & 0.6 \\ 0 & 1 & 0.6 & 0.6 \\ 0.5 & 1 & 1 & 0.7 \end{bmatrix} \tag{7.38}$$

$$= [0.5 \quad 1 \quad 1 \quad 0.7]$$

from which we can infer that

$$Y \text{ is } 0.5/1 + 1/2 + 1/3 + 0.7/4 \ . \tag{7.39}$$

As should be expected, the compositional rule of <u>modus ponens</u> reduces to the conventional rule of <u>modus ponens</u> when F is nonfuzzy and $F = G$. Thus, under these assumptions it can readily be verified that

$$F \circ (\bar{F}' \oplus G) = G \ . \tag{7.40}$$

When F is fuzzy, however, (7.40) does not hold true, except as an approximation. The explanation for this phenomenon [32] is that the implicit part of q, namely, "If X is <u>not</u> F then Y is unrestricted" overlaps the explicit part, "If X is F then Y is H" resulting in an "interference" term which vanishes when F is nonfuzzy.

Underlying the rules of inference which we have formulated in the foregoing discussion is a basic principle -- to which we shall refer as the <u>projection principle</u> -- which asserts that if $R(X_1,\ldots,X_n)$ is a fuzzy restriction associated with an n-ary fuzzy variable $(X_1,\ldots,X_n)$ which takes values in $U_1\times\cdots\times U_n$, then the restriction on $(X_{i_1},\ldots,X_{i_k})$, where $(i_1,\ldots,i_k)$ is a subsequence of the index sequence $(1,2,\ldots,n)$, is given by the projection of $R(X_1,\ldots,X_n)$ on $U_{j_1}\times\cdots\times U_{j_m}$, where $(j_1,\ldots,j_m)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if $n = 5$ and $(i_1,i_2) = (1,3)$, then $(j_1,j_2,j_3) = (2,4,5)$). Thus,

$$R(X_{i_1},\ldots,X_{i_k}) = \text{Proj } R(X_1,\ldots,X_n) \text{ on } U_{j_1}\times\cdots\times U_{j_m} \tag{7.41}$$

which implies that

$$\mu_{R(X_{i_1},\ldots,X_{i_k})}(u_{i_1},\ldots,u_{i_k}) = \bigvee_{(u_{j_1},\ldots,u_{j_m})}\mu_{R(X_1,\ldots,X_n)}(u_1,\ldots,u_n)\ . \tag{7.42}$$

The rationale for the projection principle is that, by virtue of (7.42), the projection of $R(X_1,\ldots,X_n)$ on $U_{j_1}\times\cdots\times U_{j_m}$ yields the <u>maximal</u> (i.e., largest) restriction which is consistent with $R(X_1,\ldots,X_n)$. Thus, by employing the projection principle, we are, in effect, finding the largest restriction on the variables of interest which is consistent with the restrictions on the variables which enter into the premises.

We shall conclude our discussion of inference rules in fuzzy logic with an example of semantic inference from a quantified fuzzy proposition. Specifically, let us consider the fuzzy proposition

$$p \triangleq \underline{\text{Most}} \text{ Swedes are } \underline{\text{tall}} \tag{7.43}$$

which by (5.3) translates into

$$R\left(\frac{\mu_1 + \cdots + \mu_N}{N}\right) = \underline{most} \tag{7.44}$$

where μ_i, $i = 1,\ldots,N$, is the grade of membership of S_i in the fuzzy set tall.

Now, suppose that we wish to find the answer to the question "How many Swedes are very tall?" To this end, we note that if μ_i is the grade of membership of S_i in tall, then the grade of membership of S_i in very tall is μ_i^2. Consequently, the numerical proportion of Swedes who are very tall is given by

$$r_{\underline{very\ tall}} = \frac{\mu_1^2 + \cdots + \mu_N^2}{N} . \tag{7.45}$$

The relational assignment equation (7.44) defines a fuzzy set D in $[0,1]^N$ whose membership function is expressed by

$$\mu_D(\mu_1,\ldots,\mu_n) = \mu_{\underline{most}}\left(\frac{\mu_1 + \cdots + \mu_N}{N}\right) . \tag{7.46}$$

On the other hand, (7.45) defines a mapping from $[0,1]^N$ to $[0,1]$ which induces a fuzzy set $P_{\underline{very\ tall}}$ in $[0,1]$, with P standing for Proportion.

By the transformational rule of inference (7.27), the membership function of $P_{\underline{very\ tall}}$ may be expressed as

$$\mu_P(r_{\underline{very\ tall}}) = \max_{\mu_1,\ldots,\mu_N} \mu_{\underline{most}}\left(\frac{\mu_1 + \cdots + \mu_N}{N}\right) \tag{7.47}$$

with the relation (7.45), i.e.,

$$r_{\underline{very\ tall}} = \frac{\mu_1^2 + \cdots + \mu_N^2}{N} \tag{7.48}$$

playing the role of a constraint. Thus, the determination of $P_{\underline{very\ tall}}$

reduces to the solution of a nonlinear program expressed by (7.47) and (7.45).

It is apparent by inspection that the maximizing values of $\mu_1,\ldots,\mu_N$ are given by

$$\mu_1 = \cdots = \mu_N = \sqrt{r_{\underline{very}\ \underline{tall}}} \tag{7.49}$$

and hence that

$$\mu_p(r_{\underline{very}\ \underline{tall}}) = \mu_{\underline{most}}(\sqrt{r_{\underline{very}\ \underline{tall}}}) \tag{7.50}$$

which is equivalent to

$$P_{\underline{very}\ \underline{tall}} = <\underline{most}^2> \tag{7.51}$$

where the angular brackets indicate that $<\underline{most}^2>$ is to be evaluated by the use of the extension principle.[25]

To summarize, from

$$p \triangleq \underline{Most}\ \text{Swedes are}\ \underline{tall} \tag{7.52}$$

we can infer that

$$q \triangleq <\underline{Most}>^2\ \text{Swedes are}\ \underline{very}\ \underline{tall} \tag{7.53}$$

where

$$<\underline{Most}^2> = \int_0^1 \mu_{\underline{most}}(v)/v^2 \ . \tag{7.54}$$

Thus, if Most is defined by, say,

$$\mu_{\underline{most}}(v) = S(v;0.5,0.75,1)\ , \quad v \in [0,1] \tag{7.55}$$

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[25] For numerical values of $r_{\underline{very}\ \underline{tall}}$ and most it can readily be shown that $\underline{most}^2 \leq r_{\underline{very}\ \underline{tall}} \leq \underline{most}$. Extending these inequalities to fuzzy sets leads to the expression $r_{\underline{very}\ \underline{tall}} = (\geq \circ <\underline{most}^2>) \cap (\leq \circ \underline{most})$ where $\geq \circ (\underline{most}^2)$ denotes the composition of the nonfuzzy binary relation $\geq$ with the unary fuzzy relation $<\underline{most}^2>$. Since $\underline{most} \subset <\underline{most}^2>$, this result is consistent with (7.51).

where the S-function is expressed by (A17), then

$$\mu_{\underline{most}^2}(v) = S(\sqrt{v};0.5,0.75,1) \;. \tag{7.56}$$

In a similar fashion, from the premise "Most Swedes are tall," we can obtain answers to such questions as "How many Swedes are very very tall?", "How many Swedes are not very tall?" and, more generally, "How many Swedes are m tall?" where m is a modifier. As is typical of inference processes in fuzzy logic, the answers to such questions are fuzzy restrictions rather than points in or subsets of U. In this lies one of the basic differences between inference in fuzzy logic, which is inherently approximate in nature, and the traditional deductive processes in mathematics and its applications.

8. Concluding Remarks

Our exposition of fuzzy logic in the present paper has touched upon only a few of the many basic issues which arise in relation to this -- as yet largely unexplored -- conceptual model of human reasoning and perception.

Clearly, the problems, the aims and the concerns of fuzzy logic are substantially different from those which animate the traditional logical systems. Thus, axiomatization, decidability, completeness, consistency, proof procedures and other issues which occupy the center of the stage in such systems are, at best, of peripheral importance in fuzzy logic. In part, these differences stem from the use of linguistic variables in fuzzy logic but, more fundamentally, they reflect the fact that, in fuzzy logic, the conception of truth is local rather than universal and fuzzy rather than precise.

Appendix

Fuzzy Sets -- Notation, Terminology and Basic Properties

The symbols $U, V, W, \ldots,$ with or without subscripts, are generally used to denote specific universes of discourse, which may be arbitrary collections of objects, concepts or mathematical constructs. For example, U may denote the set of all real numbers; the set of all residents in a city; the set of all sentences in a book; the set of all colors that can be perceived by the human eye, etc.

Conventionally, if A is a fuzzy subset of U whose elements are $u_1, \ldots, u_n$, then A is expressed as

$$A = \{u_1, \ldots, u_n\} . \tag{A1}$$

For our purposes, however, it is more convenient to express A as

$$A = u_1 + \cdots + u_n \tag{A2}$$

or

$$A = \sum_{i=1}^{n} u_i \tag{A3}$$

with the understanding that, for all i, j,

$$u_i + u_j = u_j + u_i \tag{A4}$$

and

$$u_i + u_i = u_i . \tag{A5}$$

As an extension of this notation, a finite <u>fuzzy</u> subset of U is expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A6}$$

or, equivalently, as

$$F = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{A7}$$

where the μ_i, $i = 1,\ldots,n$, represent the <u>grades of membership</u> of the u_i in F. Unless stated to the contrary, the μ_i are assumed to lie in the interval $[0,1]$, with 0 and 1 denoting <u>no</u> membership and <u>full</u> membership, respectively.

Consistent with the representation of a finite fuzzy set as a linear form in the u_i, an arbitrary fuzzy subset of U may be expressed in the form of an integral

$$F = \int_U \mu_F(u)/u \qquad \text{(A8)}$$

in which $\mu_F\colon U \to [0,1]$ is the <u>membership</u> or, equivalently, the <u>compatibility function</u> of F; and the integral $\int_U$ denotes the union (defined by (A28)) of <u>fuzzy singletons</u> $\mu_F(u)/u$ over the universe of discourse U.

The points in U at which $\mu_F(u) > 0$ constitute the <u>support</u> of F. The points at which $\mu_F(u) = 0.5$ are the <u>crossover</u> points of F.

<u>Example A9</u>. Assume

$$U = a + b + c + d \ . \qquad \text{(A10)}$$

Then, we may have

$$A = a + b + d \qquad \text{(A11)}$$

and

$$F = 0.3a + 0.9b + d \qquad \text{(A12)}$$

as nonfuzzy and fuzzy subsets of U, respectively.

If

$$U = 0 + 0.1 + 0.2 + \cdots + 1 \qquad \text{(A13)}$$

then a fuzzy subset of U would be expressed as, say,

$$F = 0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1 \ . \qquad \text{(A14)}$$

If $U = [0,1]$, then F might be expressed as

$$F = \int_0^1 \frac{1}{1+u^2}/u \qquad \text{(A15)}$$

which means that F is a fuzzy subset of the unit interval $[0,1]$ whose membership function is defined by

$$\mu_F(u) = \frac{1}{1+u^2} \,. \qquad \text{(A16)}$$

In many cases, it is convenient to express the membership function of a fuzzy subset of the real line in terms of a standard function whose parameters may be adjusted to fit a specified membership function in an approximate fashion. Two such functions are defined below.

$$\begin{aligned} S(u;\alpha,\beta,\gamma) &= 0 && \text{for } u \leq \alpha && \text{(A17)} \\ &= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \text{for } \alpha \leq u \leq \beta \\ &= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for } \beta \leq u \leq \gamma \\ &= 1 && \text{for } u \geq \gamma \end{aligned}$$

$$\begin{aligned} \pi(u;\beta,\gamma) &= S(u;\gamma-\beta,\gamma-\tfrac{\beta}{2},\gamma) && \text{for } u \leq \gamma && \text{(A18)} \\ &= 1 - S(u;\gamma,\gamma+\tfrac{\beta}{2},\gamma+\beta) && \text{for } u \geq \gamma \,. \end{aligned}$$

In $S(u;\alpha,\beta,\gamma)$, the parameter β, $\beta = \frac{a+\gamma}{2}$, is the crossover point. In $\pi(u;\beta,\gamma)$, β is the bandwidth, that is the separation between the crossover points of π, while γ is the point at which π is unity.

In some cases, the assumption that μ_F is a mapping from U to $[0,1]$ may be too restrictive, and it may be desirable to allow μ_F to take values in a lattice or, more particularly, in a Boolean algebra. For most purposes, however, it is sufficient to deal with the first two of the following hierarchy of fuzzy sets.

Definition A19. A fuzzy subset, F, of U is of type 1 if its membership function, μ_F, is a mapping from U to [0,1]; and F is of type n, n = 2,3,..., if μ_F is a mapping from U to the set of fuzzy subsets of type n-1. For simplicity, it will always be understood that F is of type 1 if it is not specified to be of a higher type.

Example A20. Suppose that U is the set of all nonnegative integers and F is a fuzzy subset of U labeled small integers. Then F is of type 1 if the grade of membership of a generic element u in F is a number in the interval [0,1], e.g.,

$$\mu_{\underline{\text{small}}\ \underline{\text{integers}}}(u) = \left(1 + \left(\frac{u}{5}\right)^2\right)^{-1}, \quad u = 0,1,2,\ldots \qquad \text{(A21)}$$

On the other hand, F is of type 2 if for each u in U, $\mu_F(u)$ is a fuzzy subset of [0,1] of type 1, e.g., for u = 10,

$$\mu_{\underline{\text{small}}\ \underline{\text{integers}}}(10) = \underline{\text{low}} \qquad \text{(A22)}$$

where low is a fuzzy subset of [0,1] whose membership function is defined by, say,

$$\mu_{\underline{\text{low}}}(v) = 1 - S(v;0,0.25,0.5)\ , \quad v \in [0,1] \qquad \text{(A23)}$$

which implies that

$$\underline{\text{low}} = \int_0^1 \left(1 - S(v;0,0.25,0.5)\right)/v\ . \qquad \text{(A24)}$$

If F is a fuzzy subset of U, then its α-level-set, F_α, is a nonfuzzy subset of U defined by

$$F_\alpha = \{u \mid \mu_F(u) \geq \alpha\} \tag{A25}$$

for $0 < \alpha \leq 1$.

If U is a linear vector space, the F is <u>convex</u> if and only if for all $\lambda \in [0,1]$ and all u_1, u_2 in U,

$$\mu_F(\lambda u_1 + (1-\mu)u_2) \geq \min(\mu_F(u_1), \mu_F(u_2)) \ . \tag{A26}$$

In terms of the level-sets of F, F is convex if and only if the F_α are convex for all $\alpha \in (0,1]$.[26]

The relation of containment for fuzzy subsets F and G of U is defined by

$$F \subset G \Leftrightarrow \mu_F(u) \leq \mu_G(u) \ , \quad u \in U \ . \tag{A27}$$

Thus, F is a fuzzy subset of G if (A27) holds for all u in U.

<u>Operations on Fuzzy Sets</u>

If F and G are fuzzy subsets of U, their <u>union</u>, $F \cup G$, <u>intersection</u>, $F \cap G$, <u>bounded-sum</u>, $F \oplus G$, and <u>bounded-difference</u>, $F \ominus G$, are fuzzy subsets of U defined by

$$F \cup G \triangleq \int_U \mu_F(u) \vee \mu_G(u)/u \tag{A28}$$

$$F \cap G \triangleq \int_U \mu_F(u) \wedge \mu_G(u)/u \tag{A29}$$

$$F \oplus G \triangleq \int_U 1 \wedge (\mu_F(u) + \mu_G(u))/u \tag{A30}$$

$$F \ominus G \triangleq \int_U 0 \vee (\mu_F(u) - \mu_G(u))/u \tag{A31}$$

[26]This definition of convexity can readily be extended to fuzzy sets of type 2 by applying the extension principle (see (A70)) to (A26).

where $\vee$ and $\wedge$ denote max and min, respectively. The <u>complement</u> of F is defined by

$$F' = \int_U (1 - \mu_F(u))/u \tag{A32}$$

or, equivalently,

$$F' = U \ominus F . \tag{A33}$$

It can readily be shown that F and G satisfy the identities

$$(F \cap G)' = F' \cup G' \tag{A34}$$

$$(F \cup G)' = F' \cap G' \tag{A35}$$

$$(F \oplus G)' = F' \ominus G \tag{A36}$$

$$(F \ominus G)' = F' \oplus G \tag{A37}$$

and that F satisfies the resolution identity

$$F = \int_0^1 \alpha F_\alpha \tag{A38}$$

where F_α is the α-level-set of F; αF_α is a set whose membership function is $\mu_{\alpha F_\alpha} = \alpha \mu_{F_\alpha}$, and $\int_0^1$ denotes the union of the αF, with $\alpha \in (0,1]$.

Although it is traditional to use the symbol $\cup$ to denote the union of nonfuzzy sets, in the case of fuzzy sets it is advantageous to use the symbol $+$ in place of $\cup$ where no confusion with the arithmetic sum can result. This convention is employed in the following example, which is intended to illustrate (A28), (A29), (A30), (A31) and (A32).

Example A39. For U defined by (A10) and F and G expressed by

$$F = 0.4a + 0.9b + d \tag{A40}$$

$$G = 0.6a + 0.5b \tag{A41}$$

we have

$$F + G = 0.6a + 0.9b + d \tag{A42}$$

$$F \cap G = 0.4a + 0.5b \tag{A43}$$

$$F \oplus G = a + b + d \tag{A44}$$

$$F \ominus G = 0.4b + d \tag{A45}$$

$$F' = 0.6a + 0.1b + c \tag{A46}$$

The linguistic connectives and (conjunction) and or (disjunction) are identified with $\cap$ and $+$, respectively. Thus,

$$F \text{ and } G \triangleq F \cap G \tag{A47}$$

and

$$F \text{ or } G \triangleq F + G\ . \tag{A48}$$

As defined by (A47) and (A48), and and or are implied to be noninteractive in the sense that there is no "trade-off" between their operands. When this is not the case, and and or are denoted by and* and or* respectively, and are defined in a way that reflects the nature of the trade-off. For example, we may have

$$F \text{ and* } G \triangleq \int_U \mu_F(u)\mu_G(u)/u \tag{A49}$$

$$F \text{ or* } G \triangleq \int_U (\mu_F(u) + \mu_G(u) - \mu_F(u)\mu_G(u))/u \tag{A50}$$

whose $+$ denotes the arithmetic sum. In general, the interactive versions of and and or do not possess the simplifying properties of the connectives defined by (A47) and (A48), e.g., associativity, distributivity, etc.

If α is a real number, then F^{α} is defined by

$$F^{\alpha} \triangleq \int_U (\mu_F(n))^{\alpha}/u \ . \tag{A51}$$

For example, for the fuzzy set defined by (A40), we have

$$F^2 = 0.16a + 0.81b + d \tag{A52}$$

and

$$F^{1/2} = 0.63a + 0.95b + d \ . \tag{A53}$$

These operations may be used to approximate, very roughly, the effect of the linguistic modifiers <u>very</u> and <u>more</u> <u>or</u> <u>less</u>. Thus,

$$\underline{\text{very}}\ F \triangleq F^2 \tag{A54}$$

and

$$\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ F \triangleq F^{1/2} \ . \tag{A55}$$

If $F_1,\ldots,F_n$ are fuzzy subsets of $U_1,\ldots,U_n$, then the <u>cartesian product</u> of $F_1,\ldots,F_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$F_1 \times \cdots \times F_n = \int_{U_1\times\cdots\times U_n} (\mu_{F_1}(u_1) \wedge \cdots \wedge \mu_{F_n}(u_n))/(u_1,\ldots,u_n) \ . \tag{A56}$$

As an illustration, for the fuzzy sets defined by (A40) and (A41), we have

$$\begin{aligned} F \times G &= (0.4a + 0.9b + d) \times (0.6a + 0.5b) \\ &= 0.4/(a,a) + 0.4/(a,b) + 0.6/(b,a) \\ &\quad + 0.5/(b,b) + 0.6/(d,a) + 0.5/(d,b) \end{aligned} \tag{A57}$$

which is a fuzzy subset of $(a+b+c+d) \times (a+b+c+d)$.

Fuzzy Relations

An n-ary fuzzy relation R in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$. The projection of R on $U_{i_1} \times \cdots \times U_{i_k}$, where $(i_1,\ldots,i_k)$ is a subsequence of $(1,\ldots,n)$, is a relation in $U_{i_1} \times \cdots \times U_{i_k}$ defined by

$$\text{Proj } R \text{ on } U_{i_1} \times \cdots \times U_{i_k} \triangleq \int_{U_{i_1} \times \cdots \times U_{i_k}} \vee_{u_{j_1},\ldots,u_{j_\ell}} \mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n) \quad \text{(A58)}$$

where $(j_1,\ldots,j_\ell)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if $n = 6$ then $(1,3,6)$ is complementary to $(2,4,5)$), and $\vee_{u_{j_1},\ldots,u_{j_\ell}}$ denotes the supremum over $U_{j_1} \times \cdots \times U_{j_\ell}$.

If R is a fuzzy subset of $U_{i_1},\ldots,U_{i_k}$, then its cylindrical extension in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$\bar{R} = \int_{U_1 \times \cdots \times U_n} \mu_R(U_{i_1},\ldots,U_{i_k})/(u_1,\ldots,u_n) \ . \quad \text{(A59)}$$

In terms of their cylindrical extensions, the composition of two binary relations R and S (in $U_1 \times U_2$ and $U_2 \times U_3$, respectively) is expressed by

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \quad \text{(A60)}$$

where $\bar{R}$ and $\bar{S}$ are the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Similarly, if R is a binary relation in $U_1 \times U_2$ and S is a unary relation in U_2, their composition is given by

$$R \circ S = \text{Proj } R \cap \bar{S} \text{ on } U_1 \ . \quad \text{(A61)}$$

Example A62. Let R be defined by the right-hand member of (A57) and

$$S = 0.4a + b + 0.8d \ . \tag{A63}$$

Then

$$\text{Proj } R \text{ on } U_1 \ (\triangleq a+b+c+d) = 0.4a + 0.6b + 0.6d \tag{A64}$$

and

$$R \circ S = 0.4a + 0.5b + 0.5d \ . \tag{A65}$$

The Extension Principle

Let g be a mapping from U to V. Thus,

$$v = g(u) \tag{A66}$$

where u and v are generic elements of U and V, respectively.

Let F be a fuzzy subset of U expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A67}$$

or, more generally,

$$F = \int_U \mu_F(u)/u \ . \tag{A68}$$

By the extension principle, the image of F under g is given by

$$g(F) = \mu_1 g(u_1) + \cdots + \mu_n g(u_n) \tag{A69}$$

or, more generally,

$$g(F) = \int_U \mu_F(u)/g(u) \ . \tag{A70}$$

Similarly, if g is a mapping from $U \times V$ to W, and F and G are fuzzy subsets of U and V, respectively, then

$$g(F,G) = \int_W (\mu_F(u) \wedge \mu_G(v))/g(u,v) \ . \tag{A71}$$

<u>Example A72</u>. Assume that g is the operation of squaring. Then, for the set defined by (A14), we have

$$g(0.3/0.5+0.6/0.7+0.8/0.9+1/1) = 0.3/0.25 + 0.6/0.49 + 0.8/0.81 + 1/1 \,. \tag{A73}$$

Similarly, for the binary operation $\vee$ ($\triangleq$ max), we have

$$(0.9/0.1+0.2/0.5+1/1) \vee (0.3/0.2+0.8/0.6) = 0.3/0.2 + 0.2/0.5 + 0.8/1 + 0.8/0.6 + 0.2/0.6 \,. \tag{A74}$$

It should be noted that the operation of squaring in (A73) is different from that of (A51) and (A52).

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