COMPLETE STABILITY OF AUTONOMOUS NONLINEAR NETWORKS

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Memorandum No. ERL-M596

21 October 1976

ELECTRONICS RESEARCH LABORATORY

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ABSTRACT

Several sufficient conditions are presented which guarantee that an autonomous nonlinear reciprocal network is completely stable in the sense that all trajectories of the network tend to an equilibrium state and hence no oscillation or other exotic mode of spurious behavior is possible. Stability criteria are derived with the help of the concept of the generalized inverse of a matrix for both complete and non-complete networks. The results on non-complete networks depend crucially on the introduction of a pseudo-potential function called pseudo-hybrid content and on the imposition of a local solvability condition. Most of the hypotheses are algorithmic in the sense that either explicit bounds are provided for computation purposes, or equivalent topological tests are given for checking the non-quantitative conditions.

Most results presented are applicable to networks containing multi-port and multi-terminal elements which are represented by coupled two-terminal elements. Examples are given which demonstrate that some of our results on complete stability are the best possible that can be obtained for the class of networks under consideration.

†Research supported in part by the National Science Foundation under Grant No. ENG75-22282 and the Naval Electronics System Command Contract N00039-75-C-D034.

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I. INTRODUCTION

This paper is concerned with the problem of complete stability for autonomous nonlinear reciprocal networks. Given a dynamic network $\mathcal{N}$ described by an autonomous system of differential equations $\dot{z} = f(z)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$, a point $z^* \in \mathbb{R}^n$ is called an equilibrium state of the network if $f(z^*) = 0$. Practical networks containing locally active elements often have more than one equilibrium states. By complete stability [1,2], we mean the property that any trajectory $z(t)$, $t \in [0, \infty)$ of the network eventually settles down to one of the equilibrium states, i.e., $\lim_{t\to\infty} z(t) = z^*$ for some z^* which depends on the initial state z_0. Obviously a network will never oscillate or display other exotic modes of dynamic operations--such as almost periodic spurious oscillations--if it is completely stable.

Complete stability is one of the most important considerations in the design of dynamic nonlinear networks. It is well known that practical networks can suddenly burst into undesirable oscillations even though it is not expected to do so in the original design. A clear understanding of the mechanisms which "provoke" instability and oscillation is therefore essential in any serious analysis and design of nonlinear electronic circuits.

This paper is essentially an extension of the classic results due to Brayton and Moser [1]. In section II we shall make use of the concept of the generalized inverse of a matrix to derive a sufficient condition which guarantees complete stability for a class of networks more general than that discussed in [1]. The use of the generalized matrix inverse not only shows that a topological condition ($B_{\mathcal{L}_2 \mathcal{J}_2}$ be of maximal rank) required by Brayton and Moser is unnecessary, but it also shows that the associated stability bound is the best possible that can be obtained. In particular, while Brayton and Moser shows that their complete stability bound for a complete iterated ladder network is the best possible as the number of ladder sections "n" tends to infinity, we are able to demonstrate the same result using a non-complete finite network containing only eight elements (Fig. 2). The entire section II is devoted to the so called complete n-ports. Given a network $\mathcal{N}$, an associated n-port N is created if we extract all energy-storage elements, i.e., inductors and capacitors, and consider them as loads connected across external ports. A capacitor gives rise to a voltage port and an inductor gives rise to a current port. The n-port N is said to be topologically complete if given any branch in $\mathcal{N}$, either the branch voltage or the branch current can be determined by the port variables (i.e., voltages across the voltage ports and currents through the current ports) directly from KCL and KVL without invoking any element constitutive relations. For complete n-ports, a mixed potential function called hybrid content can be defined explicitly in terms of the fundamental loop matrix and the element characteristics. We then apply Liapounov's direct method [2] to a modified form of the hybrid content to ensure the complete stability of the network. In section III, we extend the result to n-ports which are not necessarily complete. Our results in this section depend crucially on the introduction of the concepts of local solvability and pseudo-potential functions [3]. The local solvability condition is essentially an application of the local "implicit function theorem" which guarantees that all trajectories are uniquely defined for all time $t \geq t_o$. This condition is weaker than that usually invoked for guaranteeing the existence of a global state equations. Consequently, our state equations need not be defined globally. Nevertheless, our condition guarantees that the state equation exists in an open neighborhood of each point in $\mathbb{R}^n$ and that the trajectories can be continued indefinitely in forward time and can be interpreted therefore as a smooth "flow" on a "differentiable manifold" [4]. The concept of a pseudo-potential function allows a non-complete n-port to be expressed as the pseudo-gradient of a pseudo-hybrid content [3] to be defined explicitly in terms of topological matrices and the elements' constitutive relations. This pseudo-hybrid content allows us to formulate the state equation in a form analogous to that obtained for a complete network. Using several identities[1] derived in [3], we were able to derive a complete stability criteria for non-complete networks.

Most of the results in this paper are stated first for networks containing uncoupled two-terminal elements for simplicity. After it is obvious that the method of proof remains applicable in the more general case, they are then extended to allow couplings among various elements.

In this paper, a two-terminal resistor is characterized by either $i = \hat{i}(v)$, $-\infty < v < \infty$, or $v = \hat{v}(i)$, $-\infty < i < \infty$, where v and i are the branch voltage and current of the resistor, respectively, and $\hat{v}$ and $\hat{i}$ are continuous functions. In case the resistor is characterized by $i = \hat{i}(v)$ (resp., $v = \hat{v}(i)$), it is said to be voltage controlled (v.c.) (resp.,

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These identities depend on a rather remarkable topological property for resistive nonlinear networks which permit the differentiation operation to commute with the composition operation in the sense that

$$\left.\frac{\partial \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{y}=\underset{\sim}{h}(\underset{\sim}{x})} = \frac{\partial \underset{\sim}{f}\left(\underset{\sim}{x},\underset{\sim}{h}(\underset{\sim}{x})\right)}{\partial \underset{\sim}{x}}$$

It is easily seen that this commutative property is not valid for arbitrary functions. Its validity here rests on the additional constraints imposed by KVL and KCL.

current controled (c.c.)).

Let R be a c.c. resistor, define the quantity $G(i) \triangleq \int_0^i \hat{v}(x)dx$ as the content [5] of R and the quantity $G^*(i^*) \triangleq \int_0^{i^*} \hat{v}(-x)dx$ as the conjugate content of R, where $i^* \triangleq -i$. Notice that $dG/di = dG^*/di^* = \hat{v}(i)$. Similarly, let R be a v.c. resistor, then the co-content [5] and the conjugate co-content of R are defined by $\hat{G}(v) \triangleq \int_0^v \hat{i}(y)dy$ and $\hat{G}^*(v^*) \triangleq \int_0^{v^*} \hat{i}(-y)dy$, with $v^* \triangleq -v$, respectively. Again, $d\hat{G}/dv = d\hat{G}^*/dv^* = \hat{i}(v)$.

Without loss of generality, multi-terminal or multi-port resistors will be treated as coupled two-terminal resistors. This will allow our representing these elements in the form of a graph made up of two-terminal branches and hence standard results from network topology remain applicable. We assume that each multi-terminal or multi-port resistor is either voltage-controlled or current-controlled. Independent sources are considered as two-terminal resistors. In particular, a voltage source is considered as a c.c. resistor having a well-defined content function and a current source is considered as a v.c. resistor having a well-defined co-content function.

ive m-port is said to be reciprocal if the line integral $\int_0^{\underset{\sim}{z}_R} \underset{\sim}{h}(\underset{\sim}{z}'_R)\cdot d\underset{\sim}{z}'_R$ exists. It is well known that such integral exists if, and only if, the Jacobian matrix $\partial \underset{\sim}{h}(\underset{\sim}{z}_R)/\partial \underset{\sim}{z}_R$ is symmetric. In the special case where R is v.c., the above integral is called the co-content of the multi-terminal resistor or multi-port R and is denoted by $\hat{G}(\underset{\sim}{v}_R)$. Similarly, $G(\underset{\sim}{i}_R) \triangleq \int_0^{\underset{\sim}{i}_R} \underset{\sim}{h}(\underset{\sim}{i}'_R)\cdot d\underset{\sim}{i}'_R$ is called the content of R when the integral exists and R is c.c..

The conjugate content and co-content of R is defined in the same way as that for ordinary resistors.

For convenience, the symbols G, $\hat{G}$, G^*, $\hat{G}^*$, H, $\hat{\mathcal{G}}$ and $\mathcal{H}$ are all reserved for scalar functions in this paper. Vectors are denoted by lower case bold-face letters while matrices are denoted by capital bold-face letters. We use $\|\cdot\|$ to denote the norm of either a vector or a matrix while we use $|S|$ to denote the cardinality of a set S. In general, any convenient norm can be chosen. Finally we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote, respectively, the range space and the null space of a matrix A.

II. COMPLETE STABILITY OF COMPLETE NETWORKS

In this section we shall present a fairly general sufficient condition which ensures complete stability for complete networks [1,3]. The extension to the more general noncomplete networks will be given in the next section.

2.1 The n-port Formulation.

Let $\mathcal{N}$ be a network containing capacitors, inductors, two-terminal resistors and independent sources. For simplicity, let us first assume that the resistors are uncoupled. Let $\mathcal{J}_1$ be a subtree made up of "composite" branches each of which consists of a capacitor and all v.c. resistors (possibly none) connected in parallel with it and let $\mathcal{L}_2$ be a subcotree made up of composite branches each of which consists of an inductor and all c.c. resistors (possibly none) connected in series with it. The composite branches are shown in Fig. 1. If we extract all elements in $\mathcal{J}_1$ and $\mathcal{L}_2$ and consider them as loads connected across an n-port N, then we say N is complete if there is a subtree $\mathcal{J}_2$ made up of c.c. resistors such that $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ form a tree of $\mathcal{N}$, and if all remaining elements are v.c. resistors forming closed loops <u>exclusively</u> with branches in $\mathcal{J}_1$. If we denote these v.c. resistors by the subcotree $\mathcal{L}_1$, then $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is the cotree associated with $\mathcal{J}$. It follows from the completeness of the n-port N that the fundamental loops associated with branches in $\mathcal{L}_1$ contain branches from $\mathcal{J}_1$ only, i.e., $\underset{\sim}{v}_{\mathcal{L}_1} + \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{0}$, where $\underset{\sim}{v}_{\mathcal{L}_1}$ and $\underset{\sim}{v}_{\mathcal{J}_1}$ denote the branch voltage of the elements in $\mathcal{L}_1$ and $\mathcal{J}_1$, respectively, and $\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}$ denotes an appropriate submatrix of the following <u>fundamental loop matrix</u> $\underset{\sim}{B}$:

$$\underset{\sim}{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{J}_1 & \mathcal{J}_2 \end{array} & \\ \left[\begin{array}{cc|cc} \underset{\sim}{1}_{\mathcal{L}_1\mathcal{L}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{J}_2} \\ \underset{\sim}{0}_{\mathcal{L}_2\mathcal{L}_1} & \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \end{array}\right] & \begin{array}{c} \mathcal{L}_1 \\ \mathcal{L}_2 \end{array} \end{array}$$

where the upper right-hand corner submatrix is always a <u>zero matrix</u>. If we let $\underset{\sim}{i}_{\mathcal{L}_j}$, $\underset{\sim}{v}_{\mathcal{L}_j}$, $\underset{\sim}{i}_{\mathcal{J}_j}$, and $\underset{\sim}{v}_{\mathcal{J}_j}$ denote the current and voltage vectors for elements in $\mathcal{L}_j$ and $\mathcal{J}_j$, respectively, then we can write:

$$\text{KCL: } -\underset{\sim}{C}\frac{d}{dt}\underset{\sim}{v}_{\mathcal{J}_1} = -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{i}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right) + \underset{\sim}{i}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) - \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{i}_{\mathcal{L}_2} \quad (1)$$

$$\text{KVL: } -\underset{\sim}{L}\frac{d}{dt}\underset{\sim}{i}_{\mathcal{L}_2} = \underset{\sim}{v}_{\mathcal{L}_2}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{v}_{\mathcal{J}_2} \circ \left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2}\right) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1} \quad (2)$$

where $\underset{\sim}{C} = \underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ and $\underset{\sim}{L} \triangleq \underset{\sim}{L}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)$ denote the incremental capacitance and inductance matrix, respectively. Letting $\underset{\sim}{i}^*_{\mathcal{L}_2} \triangleq -\underset{\sim}{i}_{\mathcal{L}_2}$, we define the <u>hybrid content</u> $H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ of the complete n-port by

$$\boxed{\begin{aligned} H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right) \triangleq\ & \hat{G}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right) + G^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) + \hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) \\ & + G^*_{\mathcal{J}_2} \circ \left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}^*_{\mathcal{L}_2}\right) + \underset{\sim}{i}^{*T}_{\mathcal{L}_2}\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1} \end{aligned}} \quad (3)$$

where $\hat{G}_{\mathcal{L}_1}\left(\underset{\sim}{v}_{\mathcal{L}_1}\right) \triangleq \sum_{j\in\mathcal{L}_1} \hat{G}_j(v_j)$ (resp., $\hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) \triangleq \sum_{j\in\mathcal{J}_1} \hat{G}_j(v_j)$) denotes the sum of the co-contents of all v.c. resistors in $\mathcal{L}_1$ (resp., $\mathcal{J}_1$), and $G^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \triangleq \sum_{j\in\mathcal{L}_2} G^*_j(i^*_j)$ (resp., $G^*_{\mathcal{J}_2}\left(\underset{\sim}{i}^*_{\mathcal{J}_2}\right) \triangleq \sum_{j\in\mathcal{J}_2} G^*_j(i^*_j)$) denotes the sum of the conjugate contents of all c.c. resistors in $\mathcal{L}_2$ (resp., $\mathcal{J}_2$). The symbol "$\circ$" denotes the composition operation; for example, $\hat{G}_{\mathcal{L}_1}\left(\underset{\sim}{v}_{\mathcal{L}_1}\right) = \hat{G}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right)$. Observe that the first four terms in (3) are potential functions associated with the resistors, whereas the last term does not involve any constitutive relations.

Now consider the general case where the resistors are coupled to each other. The hybrid content $H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ is well-defined so long as the couplings are reciprocal; i.e., each internal resistive m-port is reciprocal. For example, assume the resistors in $\mathcal{L}_1 \cup \mathcal{J}_1$ are coupled to each other. Instead of summing separately the co-contents of the individual resistors in $\mathcal{L}_1$ and $\mathcal{J}_1$, respectively, the term $\hat{G}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}\right) + \hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ in the definition of $H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ in (3) will be replaced by a single co-content function $\hat{G}_{\mathcal{L}_1\cup\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) = \sum_j \hat{G}_j\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{L}_1}\right)\Big|_{\underset{\sim}{v}_{\mathcal{L}_1} = -\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}}$ where each $\hat{G}_j$ is the co-content of an m-port whose branches are in $\mathcal{L}_1 \cup \mathcal{J}_1$. The grouping of the content and co-content functions depends on the actual coupling among the resistors.

In contrast to the content and co-content functions in H, the term $\underset{\sim}{i}^{*T}_{\mathcal{L}_2}\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}$ is independent of the branch characteristics and is purely topological. This term has an important physical meaning: Partitioning the network $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ where $\mathcal{N}_1$ contains all branches in $\mathcal{L}_1 \cup \mathcal{J}_1$ and $\mathcal{N}_2$ contains all branches in $\mathcal{L}_2 \cup \mathcal{J}_2$, the term $\underset{\sim}{i}^{*T}_{\mathcal{L}_2}\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}$ is then equal to the instantaneous power delivered from $\mathcal{N}_1$ to $\mathcal{N}_2$ [6,7]. See Appendix A-1 for a rigorous proof.

Using the preceding explicit definition for the hybrid content $H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right)$, the state equations of any complete network can be expressed as follows [3]:

$$\frac{d}{dt}\begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}^*_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} -\underset{\sim}{C}^{-1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{L}^{-1}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right) \\ \dfrac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}} H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right) \end{bmatrix} \tag{4}$$

Observe that the "complete" resistive n-port N is described by the gradient of the hybrid content H.[2] The matrices $\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ and $\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ in (4) are assumed throughout this paper to be symmetric and positive definite.

Ideal transformers located in $\mathcal{L}_1$ or $\mathcal{J}_2$ of a complete network may also be included in this formulation [6,7]. Since the characteristic of an ideal transformer introduces an algebraic relation among the network variables $\underset{\sim}{v}_{\mathcal{J}_1}$ or $\underset{\sim}{i}^*_{\mathcal{L}_2}$, each transformer reduces the number of network variables by one. Furthermore, since transformers are nonenergic [8], no extra terms will be introduced into the definition of H. A discussion on the inclusion of ideal transformers on a complete network is given in Appendix A-2.

Remark. Given a network, since the capacitive and the inductive branches are fixed, it is relatively easy to check when it is non-complete. For a complete n-port, KVL and KCL yield

[2] Comparing (4) with the Brayton-Moser state equation, we see that our hybrid content is identical to the mixed potential of Brayton and Moser [1].

$$\underset{\sim}{v}_{\mathcal{L}_1} + \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{i}_{\mathcal{J}_2} - \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2} = \underset{\sim}{0}.$$

Therefore, any branch which forms a loop with branches in $\mathcal{J}_1$ (the capacitive branches) must be v.c. and must be assigned to $\mathcal{L}_1$. Hence, our first step is to check each branch which forms a loop with $\mathcal{J}_1$. If there is at least one such branch which is not v.c., the n-port is non-complete. Suppose now that all such branches are v.c., and have been assigned to $\mathcal{L}_1$. We then check the remaining branches other than those in $\mathcal{L}_2$. These are resistors which should be assigned to $\mathcal{J}_2$. Each of them must form a cut set with $\mathcal{L}_2$ and hence must be c.c.. Again, the n-port is non-complete if there is at least one such branch which is not c.c..

2.2 Criteria for Completely Stable Networks

Let us now derive some complete stability criteria for the network described by (4) with the help of the following well-known theorem:

Theorem 1. Complete Stability Criteria [2]

Consider the system

$$\dot{\underset{\sim}{z}} = \underset{\sim}{f}(\underset{\sim}{z})$$

where $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ is continuous. This system is completely stable if there exists a scalar function $V(\underset{\sim}{z}): \mathbb{R}^n \to \mathbb{R}^1$ having continuous first partial derivatives and satisfying the following properties:

(i) the trajectory derivative $\dot{V}(t) < 0$ for any initial state except at the equilibrium states.

(ii) all solutions are bounded.

Applying Theorem 1, we are now ready to derive the following sufficient condition for complete stability.

Theorem 2. Let $\mathcal{N}$ be a complete network containing two-terminal uncoupled resistors described by (3) and (4). Then $\mathcal{N}$ is completely stable if the following conditions are satisfied:

(i) All elements in $\mathcal{J}_2$ are linear and positive resistors, i.e., $\underset{\sim}{v}_{\mathcal{J}_2} = \underset{\sim}{R}_{\mathcal{J}_2} \underset{\sim}{i}_{\mathcal{J}_2}$ where $\underset{\sim}{R}_{\mathcal{J}_2}$ is a constant, diagonal and positive definite matrix.

(ii) All elements in series with the inductors in $\mathcal{L}_2$ are constant voltage sources[3], i.e., $\underset{\sim}{v}_{\mathcal{L}_2} = \underset{\sim}{E}_{\mathcal{L}_2}$ (see Fig. 1). Furthermore, $\underset{\sim}{E}_{\mathcal{L}_2} \in \mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\right)$ where $\mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\right)$ denotes the range space of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$.

(iii) $\mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right) \subset \mathcal{R}\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}\right)$.

(iv) Let $\underset{\sim}{R} \triangleq \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \underset{\sim}{R}_{\mathcal{J}_2} \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}$ and $\underset{\sim}{R}^I$ = the generalized inverse of $\underset{\sim}{R}$, as defined in Appendix A-3, then

$$\|\underset{\sim}{K}\|^2 \triangleq \left\| \underset{\sim}{L}^{\frac{1}{2}} \underset{\sim}{R}^I \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{C}^{-\frac{1}{2}} \right\|^2 < 1 - \delta \text{ for some } \delta > 0.$$

where $\|\underset{\sim}{K}\|$ denotes any convenient norm of the matrix $\underset{\sim}{K}$.

(v) All solutions of (4) are bounded.

Proof. See Appendix A-3.

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[3] Condition (ii) requires that any element in series with an inductor must be either a short circuit or an independent voltage source. Moreover, there must exist a vector $\underset{\sim}{v}_{\mathcal{J}_2}$ such that $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2} = \underset{\sim}{E}_{\mathcal{L}_2}$. Observe that a short circuit or an independent voltage source is a legitimate element for $\mathcal{L}_2$ because each can be considered as a special case of a current-controlled resistor.

Remarks:

1. Theorem 2 is an extension of the result in [1] in that instead of requiring the rows of $B_{\mathcal{L}_2\mathcal{J}_2}$ to be linearly independent, we only require that $\mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$. In the special case where the rows of $B_{\mathcal{L}_2\mathcal{J}_2}$ are linearly independent, then condition (iii) and the condition $E_{\mathcal{L}_2} \in \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$ are always satisfied. In this case $R^I = R^{-1}$ and we obtain the result in [1].

2. In Theorem 2, as well as in several subsequent theorems, we require that all solutions of the network be bounded. This hypothesis is satisfied by most networks of practical interest and can be ensured by rather mild conditions, see, for example [9-10].

3. The condition $E_{\mathcal{L}_2} \in \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$ is also rather weak. In fact, networks which do not obey this condition can usually be transformed into equivalent networks which do. For example, suppose "e_s" is a constant voltage source connected in series with a resistor R_k, and suppose our topological algorithm for partitioning $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ requires that R_k be assigned to $\mathcal{J}_2$ and that the voltage source e_s be assigned to $\mathcal{L}_2$. Applying the v-shift theorem, the voltage source e_s can be shifted in series with the remaining branches of the fundamental cut set associated with R_k, thereby creating a source vector $e_{\mathcal{L}_2}$. Since the n-port is complete, $i_{\mathcal{J}_2} = B^T_{\mathcal{L}_2\mathcal{J}_2} i_{\mathcal{L}_2}$, the remaining branches in this cut set consists of branches in $\mathcal{L}_2$ only[4]. Since $i_{R_k} = \left(B^T_{\mathcal{L}_2\mathcal{J}_2}\right)_k i_{\mathcal{L}_2}$ where $\left(B^T_{\mathcal{L}_2\mathcal{J}_2}\right)_k$ is the k th row of $B^T_{\mathcal{L}_2\mathcal{J}_2}$, it follows that $e_{\mathcal{L}_2} = \left[\left(B^T_{\mathcal{L}_2\mathcal{J}_2}\right)_k\right]^T e_s \in \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$.

[4]A specific example illustrating this transformation property is given in Example 1 after Lemma 1.

We have demonstrated therefore that voltage sources in series with resistors in $\mathcal{J}_2$ are actually allowed in so far as conditions (i) and (ii) are concerned.

4. It is important to ensure that either the rows of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$ are linearly independent, or, if they are not, that $\mathcal{R}(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2})$. The following lemma provides a simple topological algorithm for checking either one of these two conditions.

Lemma 1. Let $\mathcal{N}$ be a connected network which has been partitioned into $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{J}_1$, and $\mathcal{J}_2$ in accordance with the preceding rules.

(i) Let $\mathcal{N}'$ be the sub-network obtained by shrinking (short-circuiting) all branches except those belonging to $\mathcal{J}_2$. Let b' and n' be the number of branches and nodes in $\mathcal{N}'$, respectively, so that $\mathcal{N}'$ has b' - n' + 1 independent loops. Then the rows of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$ are linearly independent if, and only if, b' - n' +1 = $|\mathcal{L}_2|$, where $|\mathcal{L}_2|$ denotes the original number of branches in $\mathcal{L}_2$. Equivalently, let $\mathcal{N}''$ be the sub-network obtained by shrinking all branches in $\mathcal{J}_1 \cup \mathcal{L}_1$. Then the rows of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$ are linearly independent if, and only if, $\mathcal{N}''$ contains no loop formed exclusively of branches belonging to $\mathcal{L}_2$.

(ii) $\mathcal{R}(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2})$ if, and only if, upon open-circuiting the branches in $\mathcal{L}_1$ and $\mathcal{J}_2$, the current $\underset{\sim}{i}_{\mathcal{J}_1} = \underset{\sim}{0}$ identically, i.e., branches in $\mathcal{J}_1$ are not contained in any loop in the reduced network.

Proof. (i) It can be shown by a straightforward though somewhat tedious procedure that the rows of the submatrix $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$ span all loops of the reduced graph $\mathcal{N}'$ in the sense that each loop in $\mathcal{N}'$ is a linear combination of rows of $B_{\mathcal{L}_2\mathcal{T}_2}$. It follows from basic graph theory that $B_{\mathcal{L}_2\mathcal{T}_2}$ must contain <u>at least</u> $b' - n' + 1$ linearly independent rows. Now suppose all rows of $B_{\mathcal{L}_2\mathcal{T}_2}$ are linearly independent, then since each row of $B_{\mathcal{L}_2\mathcal{T}_2}$ designates one loop in $\mathcal{N}'$, there are exactly $|\mathcal{L}_2|$ linearly independent loops in $\mathcal{N}'$ and hence $|\mathcal{L}_2| = b' - n' + 1$. Conversely, suppose $b' - n' + 1 = |\mathcal{L}_2|$. Then all rows of $B_{\mathcal{L}_2\mathcal{T}_2}$ must be linearly independent.

Now, let us prove the equivalent statement. Let b'' and n'' be the number of branches and nodes in $\mathcal{N}''$, respectively, so that $\mathcal{N}''$ has $b'' - n'' + 1 = |\mathcal{L}_2|$ linearly independent loops. Suppose $\mathcal{N}''$ contains no loop formed exclusively of $\mathcal{L}_2$ branches. Then, shrinking each $\mathcal{L}_2$ branch will reduce b'' as well as n'' by 1, and hence $b' = b'' - |\mathcal{L}_2|$ and $n' = n'' - |\mathcal{L}_2|$. Consequently, $b' - n' + 1 = (b''-|\mathcal{L}_2|) - (n''-|\mathcal{L}_2|) + 1 = b'' - n'' + 1 = |\mathcal{L}_2|$. Since $\mathcal{N}''$ reduces to $\mathcal{N}'$ upon shrinking all branches in $\mathcal{L}_2$, it follows from the first part of this lemma that the rows of $B_{\mathcal{L}_2\mathcal{T}_2}$ are linearly independent. Now, conversely, suppose the rows of $B_{\mathcal{L}_2\mathcal{T}_2}$ are linearly independent. We claim that $\mathcal{N}''$ contains no loop formed exclusively of $\mathcal{L}_2$ branches. Suppose not. Then, shrinking each $\mathcal{L}_2$ branch will reduce b'' but not necessarily n'' by 1. In particular, let $b_\alpha, b_\beta, \ldots, b_\rho$ denote $\mathcal{L}_2$ branches which formed a loop exclusively by themselves. Let us first shrink all branches in this loop except b_α and b_β. In this step, b'' and n'' both decrease by 1 for <u>each</u> short-circuited branch. However, since the remaining two branches b_α and b_β now formed a loop and hence shared a common pair of nodes, it follows that if we shrink also these two branches, then b'' decreases by 2 but n'' decreases by only 1. Hence, we have $b' = b'' - |\mathcal{L}_2|$ and $n' > n'' - |\mathcal{L}_2|$. Consequently, $b' - n' + 1 < (b''-|\mathcal{L}_2|) - (n''-|\mathcal{L}_2|) + 1 = b'' - n'' + 1 = |\mathcal{L}_2|$. But then the first part of the <u>Lemma</u> would imply that the rows of $B_{\mathcal{L}_2\mathcal{T}_2}$ are linearly dependent; and we obtain again a contradiction.

(ii) Write KCL as follows:

$$\begin{bmatrix} -B^T_{\mathcal{L}_1\mathcal{J}_1} & -B^T_{\mathcal{L}_2\mathcal{J}_1} & 1_{\mathcal{J}_1\mathcal{J}_1} & 0 \\ 0_{\mathcal{L}_1\mathcal{J}_2} & -B^T_{\mathcal{L}_2\mathcal{J}_2} & 0 & 1_{\mathcal{J}_2\mathcal{J}_2} \end{bmatrix} \begin{bmatrix} i_{\mathcal{L}_1} \\ i_{\mathcal{L}_2} \\ i_{\mathcal{J}_1} \\ i_{\mathcal{J}_2} \end{bmatrix} = 0. \qquad (5)$$

Now to prove <u>sufficiency</u>, suppose $\mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$. This implies that $\mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_1}) \supset \mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_2})$. We wish to prove that $i_{\mathcal{J}_1} = 0$ whenever branches in $\mathcal{L}_1$ and $\mathcal{J}_2$ are open-circuited. Since $i_{\mathcal{J}_2} = 0$, the second equation in (5) implies that $-B^T_{\mathcal{L}_2\mathcal{J}_2}\, i_{\mathcal{L}_2} = 0$. This means that $-B^T_{\mathcal{L}_2\mathcal{J}_1}\, i_{\mathcal{L}_2} = 0$. Thus the first equation in (5) becomes $-B^T_{\mathcal{L}_1\mathcal{J}_1}\, i_{\mathcal{L}_1} + i_{\mathcal{J}_1} = 0$. But $i_{\mathcal{L}_1} = 0$ and hence $i_{\mathcal{J}_1} = 0$ identically. It remains to prove <u>necessity</u>.

Suppose $i_{\mathcal{J}_1} = 0$ identically after we have open-circuited all branches in $\mathcal{L}_1$ and $\mathcal{J}_2$ but $\mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_1}) \not\supset \mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_2})$. This implies that there exists $\bar{i}_{\mathcal{L}_2} \neq 0$ such that

$$B^T_{\mathcal{L}_2\mathcal{J}_2}\, \bar{i}_{\mathcal{L}_2} = 0 \quad \text{but} \quad B^T_{\mathcal{L}_2\mathcal{J}_1}\, \bar{i}_{\mathcal{L}_2} \neq 0$$

Substituting these relations into the first equation in (5), we obtain $B^T_{\mathcal{L}_2\mathcal{J}_1}\, \bar{i}_{\mathcal{L}_2} = i_{\mathcal{J}_1} \neq 0$, a contradiction. Hence $\mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_1}) \supset \mathcal{N}(B^T_{\mathcal{L}_2\mathcal{J}_2})$ and hence $\mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$.

□

5. It can be easily shown that if the matrix R is non-singular, then condition (v) can be replaced by the growth condition

$$\left\| B_{\mathcal{L}_2\mathcal{J}_1} v_{\mathcal{J}_1} \right\| + \hat{G}_{\mathcal{J}_1} v_{\mathcal{J}_1} + \hat{G}_{\mathcal{L}_1} \circ \left(-B_{\mathcal{L}_1\mathcal{J}_1} v_{\mathcal{J}_1}\right) \to \infty, \text{ as } \left\| v_{\mathcal{J}_1} \right\| \to \infty$$

given in [1]. This condition is sufficient to guarantee that

$$H^*(x) \to \infty \quad \text{as } \|x\| \triangleq \left\| \left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2} \right) \right\| \to \infty.$$

In the case where R is singular, however, this property cannot be guaranteed by the preceding growth condition. Nevertheless, from a practical point of view, condition (v) is preferable because it is usually satisfied for most networks containing eventually-passive elements [10].

Example 1. Consider the circuit shown in Fig. 2(a). Assume that the capacitor and the three inductors are linear with $C > 0$ and $L_j > 0$, $j = 1, 2, 3$. Assume also that resistors R_5 and R_6 are linear and positive. Since the voltage source E is not in series with inductors, let us apply the v-shift theorem to the independent source E and obtain the equivalent circuit shown in Fig. 2(b). Pick $\mathcal{J}_1 = \{C \| R_4\}$, $\mathcal{J}_2 = \{R_5, R_6\}$, $\mathcal{L}_1 = \phi$ (the empty set), and $\mathcal{L}_2 = \{L_1\text{-E}, L_2, L_3\text{-E}\}$. Labelling the branches in the order $\mathcal{L}_2$, $\mathcal{J}_1$ and $\mathcal{J}_2$, we obtain the following fundamental loop matrix:

$$B = \left[\begin{array}{ccc|c|cc} 1 & & & 0 & 1 & 1 \\ & 1 & & -1 & 0 & 1 \\ & & 1 & 1 & 1 & 0 \end{array}\right]$$
$$\qquad\quad \underbrace{\qquad\quad}_{\mathcal{L}_2} \quad \underbrace{\quad}_{\mathcal{J}_1} \quad \underbrace{\qquad}_{\mathcal{J}_2}$$

Therefore,

$$B_{\mathcal{L}_2\mathcal{J}_2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B_{\mathcal{L}_2\mathcal{J}_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

As an illustration of the application of Lemma 1(i), we observe that there is a loop made up exclusively of branches belonging to $\mathcal{L}_2$ in the reduced network $\mathcal{N}'$ and hence we can conclude that the rows of $B_{\mathcal{L}_2\mathcal{J}_2}$ are not linearly independent. This conclusion is easily verified from the above matrix. Observe that even though the rows of $B_{\mathcal{L}_2\mathcal{J}_2}$ are not linearly independent, we have nonetheless $\mathcal{R}(B_{\mathcal{L}\mathcal{J}}) \subset \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2})$. This conclusion follows immediately from Lemma 1(ii). By hypotheses, $R_{\mathcal{J}_2} = \mathrm{diag}(R_5, R_6)$ is a positive-definite and diagonal matrix. Moreover,

$$v'_{\mathcal{L}_2} = E_{\mathcal{L}_2} = E[1\ 0\ 1]^T \in \mathcal{R}(B_{\mathcal{L}_2\mathcal{J}_2}).$$

where the "prime" denotes voltage across the voltage source. Hence conditions (i), (ii) and (iii) of Theorem 2 are satisfied.

Consider next

$$R \triangleq B_{\mathcal{L}_2\mathcal{J}_2} R_{\mathcal{J}_2} B^T_{\mathcal{L}_2\mathcal{J}_2} = \begin{bmatrix} R_5 + R_6 & R_6 & R_5 \\ R_6 & R_6 & 0 \\ R_5 & 0 & R_5 \end{bmatrix}$$

We can compute R^I once R_5 and R_6 are given. Then, for a fixed value of C, an upper bound for the L_i's can be found by requiring $\|K\|^2 < 1$ to ensure complete stability. As a numerical example, let $R_5 = R_6 \triangleq R = 1\ \mathrm{M}\Omega$; $L_i = L$, $i = 1, 2, 3$ and $E = 0$. Furthermore, let $\hat{i}_4(\cdot)$ be defined as in Fig. 2(c), i.e.,

$$\hat{i}_4(v_4) = \begin{cases} -Gv_4 \quad , & |v_4| < 1 \\ Gv_4 - 2G, & v_4 \geq 1 \\ Gv_4 + 2G, & v_4 \leq -1 \end{cases}$$

where we assume that the value of G is such that $RG = 1-\varepsilon$, $0 < \varepsilon < 1$. Using the definition for $\underset{\sim}{R}^I$ in Appendix A-3, we obtain

$$\underset{\sim}{R}^I = \frac{1}{9\times 10^6}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 5 & -4 \\ 1 & -4 & 5 \end{bmatrix}, \text{ and } \quad \underset{\sim}{K} = \frac{L^{\frac{1}{2}}}{10^6}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} C^{-\frac{1}{2}} .$$

Letting $\|\underset{\sim}{K}\|^2 < 1$,[5] we obtain $L < 10^{12}C$.

To see how conservative this bound is, let us derive the condition which allows an oscillation within the range of the negative resistance. Hence, let us suppose the circuit oscillates with an amplitude of v_4 less than 1. In this case $i_4 = -Gv_4$ is linear. The characteristic polynomial of the linear circuit is given by

$$p(s) = Ls[L^2Cs^3 + (4RLC - L^2G)s^2 + (3R^2C + L - 4RGL)s + (3R - 3R^2G)]$$

The zero $s = 0$ corresponds to dc current flowing around the loop of inductors. We now find conditions on L such that p(s) has a pair of imaginary zeros. Applying the Routh Criterion and using our assumption $RG = 1-\varepsilon$, we found that when $L = RC/G$, $p(s) = 0$ has a pair of imaginary roots

$s = \pm j\sqrt{\dfrac{3\varepsilon R}{4RLC-L^2G}}$. Hence, this network is <u>not</u> completely stable when

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[5]Here we define the norm of the vector $\underset{\sim}{K} = [k_1 \; k_2 \; k_3]^T$ by $\|\underset{\sim}{K}\| = \max_i |k_i|$. To obtain the sharpest estimate, it is desirable to choose this L^∞-norm whenever $\underset{\sim}{K}$ is a vector because it gives the smallest value of all L^P-norms.

$L \geq RC/G = 10^{12}(1-\varepsilon)C$. Since $\varepsilon > 0$ can be chosen arbitrarily small in magnitude, this bound can be made arbitrarily close to the upper bound $L = 10^{12}C$ for complete stability. Hence the bound derived earlier for this example is the best possible that can be obtained.[6]

Interchanging the roles of capacitors and inductors, we can easily state the dual version of Theorem 2:

<u>Theorem 2'</u> Let $\mathcal{N}$ be a complete network containing two-terminal uncoupled resistors described by (3) and (4). Then $\mathcal{N}$ is <u>completely stable</u> if the following conditions are satisfied:

(i) All elements in $\mathcal{L}_1$ are linear and positive resistors, i.e., $\underset{\sim}{i}_{\mathcal{L}_1} = \underset{\sim}{G}_{\mathcal{L}_1} \underset{\sim}{v}_{\mathcal{L}_1}$ where $\underset{\sim}{G}_{\mathcal{L}_1}$ is constant, diagonal and positive definite matrix.

(ii) All elements in parallel with the capacitors in $\mathcal{J}_1$ are constant current sources, i.e., $\underset{\sim}{i}'_{\mathcal{J}_1} = \underset{\sim}{I}_{\mathcal{J}_1}$ (see Fig. 1). Furthermore, $\underset{\sim}{I}_{\mathcal{J}_1} \in \mathcal{R}(\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1})$.

(iii) $\mathcal{R}(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}) \subset \mathcal{R}(\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1})$.

(iv) Let $\underset{\sim}{G} \triangleq \underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1} \underset{\sim}{G}_{\mathcal{L}_1} \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}$ and $\underset{\sim}{G}^I \triangleq$ the generalized inverse of $\underset{\sim}{G}$, then

$$\|\underset{\sim}{S}\|^2 = \left\| \underset{\sim}{C}^{\frac{1}{2}} \underset{\sim}{G}^I \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{L}^{-\frac{1}{2}} \right\|^2 < 1 - \delta \text{ for some } \delta > 0,$$

where $\|\underset{\sim}{S}\|$ denotes any convenient <u>norm</u> of the matrix $\underset{\sim}{S}$.

(v) All solutions of (4) are bounded.

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[6] Brayton and Moser have demonstrated that their bound approaches the best that can be obtained in the <u>limiting</u> case of an <u>infinite</u> network [1]. However, their results can not be applied to this example because the matrix $\underset{\sim}{R}$ is <u>singular</u> which in turn is due to the fact that the rows of the matrix $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2}$ are not linearly independent.

Let us now generalize Theorem 2 to allow coupling:

Theorem 3. Let $\mathcal{N}$ be a complete network described by (3) and (4), where the resistors in $\mathcal{J}_2$, or the resistors in $\mathcal{L}_1 \cup \mathcal{J}_1$, may be coupled to each other within each set, so long as the coupling remain reciprocal. Then $\mathcal{N}$ is completely stable under the same conditions as in Theorem 2 provided all resistors in $\mathcal{J}_2$ are linear and described by $\underset{\sim}{v}_{\mathcal{J}_2} = \underset{\sim}{R}_{\mathcal{J}_2} \underset{\sim}{i}_{\mathcal{J}_2}$, where $\underset{\sim}{R}_{\mathcal{J}_2}$ is symmetric and positive definite.

Proof. The proof follows similarly, *mutatis mutaudis*, from that given for Theorem 2. ⌑

Remark: A dual *generalized* version of Theorem 2' can obviously be stated.

III. COMPLETE STABILITY OF NON-COMPLETE NETWORKS.

In general, the capacitor voltages and inductor currents do not form a complete set of variables for most networks, i.e., the n-port N obtained by extracting all capacitors and inductors as external ports is not complete. The network equations will then take the following form:

$$\frac{d}{dt} \underset{\sim}{x} = \underset{\sim}{f}_1 (\underset{\sim}{x}, \underset{\sim}{y}) \tag{6a}$$

$$\underset{\sim}{f}_2(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{0}, \tag{6b}$$

with $[\underset{\sim}{x}(t_o), \underset{\sim}{y}(t_o)] = [\underset{\sim}{x}_o, \underset{\sim}{y}_o]$, where $\underset{\sim}{f}_1: \mathbb{R}^{n_x + n_y} \to \mathbb{R}^{n_x}$ and $\underset{\sim}{f}_2: \mathbb{R}^{n_x + n_y} \to \mathbb{R}^{n_y}$, $n_x + n_y = n$. By a *trajectory* through $(\underset{\sim}{x}_o, \underset{\sim}{y}_o)$ of the above system we mean a function $[\underset{\sim}{x}(t), \underset{\sim}{y}(t)]$, $t \geq t_o$ which satisfies Eq. (6) and that $[\underset{\sim}{x}(t_o), \underset{\sim}{y}(t_o)] = [\underset{\sim}{x}_o, \underset{\sim}{y}_o]$. Similarly, by an *equilibrium state* we mean a point $[\underset{\sim}{x}, \underset{\sim}{y}] \in \mathbb{R}^n$ on the trajectory such that $\underset{\sim}{f}_1 (\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{0}$.

Eq. (6) defines a differential-algebraic system. It is important to notice that the "initial state" $[\underset{\sim}{x}_o, \underset{\sim}{y}_o] \in \mathbb{R}^n$ is generally not arbitrary. A vector $[\underset{\sim}{x}_o, \underset{\sim}{y}_o] \in \mathbb{R}^n$ which satisfies KCL, KVL and all the branch characteristics is henceforth called a <u>feasible state</u>. Obviously, any valid initial state must be feasible. The following special case of Eq. (6) is of particular importance.

<u>Definition 1</u>. A system described by (6) is said to be <u>locally solvable</u> if, given <u>any</u> feasible state $[\underset{\sim}{x}, \underset{\sim}{y}] \in \mathbb{R}^n$, (6b) is solvable for $\underset{\sim}{y}$ in terms of $\underset{\sim}{x}$ in a neighborhood $B = B_x \times B_y$ of $[\underset{\sim}{x}, \underset{\sim}{y}]$. That is, there exists a continuously differentiable function $\underset{\sim}{s}$: $B_x \subset \mathbb{R}^{n_x} \to B_y \subset \mathbb{R}^{n_y}$ such that $\underset{\sim}{y} = \underset{\sim}{s}(\underset{\sim}{x})$ for all $[\underset{\sim}{x}, \underset{\sim}{y}] \in B$.

Locally solvable systems are defined by "implicit" differential equations with initial states restricted by a set of algebraic equations in (6b) [4]. For locally solvable networks, the state equation can be written in the form

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}_1\left(\underset{\sim}{x}, \underset{\sim}{s}(\underset{\sim}{x})\right)$$

over a neighborhood $B_x \subset \mathbb{R}^{n_x}$ about each point $\underset{\sim}{x} \in \mathbb{R}^{n_x}$ where local solvability holds. Moreover, the locus of $\underset{\sim}{y}(t)$ about the corresponding neighborhood $B_y \subset \mathbb{R}^{n_y}$ is given by

$$\frac{d\underset{\sim}{y}}{dt} = \frac{\partial \underset{\sim}{s}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}_1\left(\underset{\sim}{x}, \underset{\sim}{s}(\underset{\sim}{x})\right)$$

Hence for locally solvable systems, the trajectory $(\underset{\sim}{x}(t), \underset{\sim}{y}(t))$ in $\mathbb{R}^{n_x + n_y}$ is uniquely defined through each point $(\underset{\sim}{x}, \underset{\sim}{y}) \in \mathbb{R}^{n_x + n_y}$. If we define a scalar function $V(\underset{\sim}{x}, \underset{\sim}{y})$: $\mathbb{R}^{n_x + n_y} \to \mathbb{R}^1$, then the <u>trajectory derivative</u> defined by

$$\dot{V}(t) = \frac{\partial V(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{x}} \dot{\underset{\sim}{x}} + \frac{\partial V(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{y}} \dot{\underset{\sim}{y}} = \left[\frac{\partial V(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{x}} + \frac{\partial V(\underset{\sim}{x},\underset{\sim}{y})}{\partial \underset{\sim}{y}} \cdot \frac{\partial \underset{\sim}{s}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \right] \underset{\sim}{f}_1\left(\underset{\sim}{x}\ ,\ \underset{\sim}{s}(\underset{\sim}{x})\right)$$

is also well defined <u>for all time</u> $t \geq t_o$.[7] Hence the same proof for the <u>Complete Stability Criteria</u> in <u>Theorem 1</u> can be used to prove the following result:

<u>Theorem 1'</u>. <u>Generalized Complete Stability Criteria</u>

The locally solvable implicit differential-algebraic equation (6) is <u>completely stable</u> in the sense that all trajectories tend to an equilibrium state if there exists a scalar function $V(\underset{\sim}{x},\underset{\sim}{y})$: $\mathbb{R}^{n_x + n_y} \to \mathbb{R}^1$ having continuous first partial derivatives and satisfying the following properties:

(i) the trajectory derivative $\dot{V}(t) < 0$ for any feasible initial state except at the equilibrium states.

(ii) all solutions are bounded.

<u>Remark</u>.

Condition (ii) of <u>Theorem 1'</u> can be replaced by the following sufficient condition:

$$V(\underset{\sim}{x},\ \underset{\sim}{y}) \to \infty \qquad \text{as } \|(\underset{\sim}{x},\ \underset{\sim}{y})\| \to \infty$$

3.1 <u>Complete Stability of RC or RL Networks.</u>

Before we deal with the most general case, let us consider networks which contain only one kind of energy storage elements, i.e., either capacitors or inductors. Let $\mathcal{N}$ be a reciprocal network containing

[7]The function $\underset{\sim}{s}(\underset{\sim}{x})$ may of course have to be updated from time to time by one which is valid over the appropriate neighborhood of points along the trajectory. This is because $\underset{\sim}{s}(\underset{\sim}{x})$ is only a <u>local</u> coordinate system [4].

capacitors, v.c. resistors and constant current sources. (This implies that any constant voltage source must be connected via a plier-type entry in series with some v.c. resistor and considered as part of a composite v.c. resistor). If the capacitors do not form loops, then one can always pick a tree $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ where $\mathcal{J}_1$ consists of capacitors only. Let $\mathcal{L}$ be the cotree corresponding to $\mathcal{J}$. If we label the tree branches before the links, we obtain the following <u>fundamental cut set equations</u>:

$$\begin{bmatrix} \underset{\sim}{1}_{\mathcal{J}_1\mathcal{J}_1} & \underset{\sim}{0}_{\mathcal{J}_1\mathcal{J}_2} & \underset{\sim}{Q}_{\mathcal{J}_1\mathcal{L}} \\ \underset{\sim}{0}_{\mathcal{J}_2\mathcal{J}_1} & \underset{\sim}{1}_{\mathcal{J}_1\mathcal{J}_1} & \underset{\sim}{Q}_{\mathcal{J}_2\mathcal{L}} \end{bmatrix} \begin{bmatrix} -\underset{\sim}{i}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{J}_2} \\ \underset{\sim}{i}_{\mathcal{L}} \end{bmatrix} = \underset{\sim}{0}$$

where $\underset{\sim}{i}_{\mathcal{J}_1}$ is the current vector flowing <u>out</u> of the positive terminals of the capacitors. Now, let us extract the branches in $\mathcal{J}$ as external ports and define the <u>pseudo-co-content</u> $\hat{\mathcal{G}}$ of the n-port N as [3]

$$\hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) \triangleq \hat{G}\left(\underset{\sim}{v}_{\mathcal{L}}, \underset{\sim}{v}_{\mathcal{J}_2}\right) \Big|_{\underset{\sim}{v}_{\mathcal{L}} = \bar{\underset{\sim}{Q}}^T \underset{\sim}{v}_{\mathcal{J}}}$$

$$= \hat{G}\left(\bar{\underset{\sim}{Q}}^T \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{v}_{\mathcal{J}_2} \end{bmatrix}, \underset{\sim}{v}_{\mathcal{J}_2}\right)$$

where $\underset{\sim}{v}_{\mathcal{J}_1} \in \mathbb{R}^{n_1}$ and $\underset{\sim}{v}_{\mathcal{J}_2} \in \mathbb{R}^{n_2}$, and $\bar{\underset{\sim}{Q}}^T \triangleq \left[\underset{\sim}{Q}^T_{\mathcal{J}_1\mathcal{L}}, \underset{\sim}{Q}^T_{\mathcal{J}_2\mathcal{L}}\right]$. Notice that the resistors in $\mathcal{L} \cup \mathcal{J}_2$ may be coupled to each other. It follows from the above definition of $\hat{\mathcal{G}}$ that

$$\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \underset{\sim}{Q}_{\mathcal{J}_1\mathcal{L}}\, \underset{\sim}{i}_{\mathcal{L}} = \underset{\sim}{i}_{\mathcal{J}_1}$$

$$\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \underset{\sim}{i}_{\mathcal{J}_2} + \underset{\sim}{Q}_{\mathcal{J}_2\mathcal{L}} \underset{\sim}{i}_{\mathcal{L}} = \underset{\sim}{0}.$$

Since $\mathcal{J}_1$ consists of capacitors only, $\underset{\sim}{i}_{\mathcal{J}_1} = -\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)\left(d\underset{\sim}{v}_{\mathcal{J}_1}/dt\right)$; hence:

$$\frac{d}{dt} \underset{\sim}{v}_{\mathcal{J}_1} = -\underset{\sim}{C}^{-1}(\underset{\sim}{v}_{\mathcal{J}_1}) \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right), \tag{7}$$

$$\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \underset{\sim}{0}. \tag{8}$$

As usual, $\underset{\sim}{C}^{-1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ is assumed to be symmetric and positive definite.

For the most general case, a trajectory may not exist corresponding to an arbitrary feasible initial point $\left[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right]_0 \in \mathbb{R}^{n_1+n_2}$. Moreover, even if a trajectory through $\left[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right]_0$ exists, it may not be unique. If the system is locally solvable, however, a unique trajectory always exists for any feasible initial point $\left[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right]_0$.

<u>Lemma 2</u>. The n-port N is locally solvable if the matrix

$$\underset{\sim}{M} \triangleq \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \underset{\sim}{i}_{\mathcal{J}_2}\left(\underset{\sim}{v}_{\mathcal{J}_2}\right) + \underset{\sim}{Q}_{\mathcal{J}_2\mathcal{L}} \left[\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{L}}} \underset{\sim}{i}_{\mathcal{L}}(\underset{\sim}{v}_{\mathcal{L}})\right] \underset{\sim}{Q}^T_{\mathcal{J}_2\mathcal{L}}$$

is nonsingular for all $\left[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right] \in \mathbb{R}^{n_1+n_2}$.

<u>Proof</u>. It follows from the <u>implicit function theorem</u> that the implicit algebraic equation (8) is locally solvable if $\partial^2 \hat{\mathcal{G}} / \partial \underset{\sim}{v}^2_{\mathcal{J}_2}$ is nonsingular. By direct calculation, we obtain $\underset{\sim}{M} = \partial^2 \hat{\mathcal{G}}/\partial \underset{\sim}{v}^2_{\mathcal{J}_2}$.

∎

Let us now consider the complete stability of capacitive n-ports.

Theorem 4. Consider the non-complete nonlinear RC network $\mathcal{N}$ described by (7) and (8), where $\underset{\sim}{C}(\underset{\sim}{v}_{\mathcal{J}_1})$ is symmetric and positive definite. Then $\mathcal{N}$ is completely stable if the following conditions are satisfied:

(i) Equation (8) is locally solvable for $\underset{\sim}{v}_{\mathcal{J}_2}$ as a function of $\underset{\sim}{v}_{\mathcal{J}_1}$.

(ii) all solutions are bounded.

Proof. Since N is locally solvable, it is described by a set of implicit differential equations. Choosing $\hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2})$ as the scalar function in Theorem 1', it suffices to show that $\dot{\hat{\mathcal{G}}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}) < 0$ for all $[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}] \in \mathbb{R}^{n_1+n_2}$, except at the equilibrium points at which it vanishes. Now, applying (7) and (8) and recalling that $\underset{\sim}{C}(\underset{\sim}{v}_{\mathcal{J}_1})$ is positive-definite, we obtain

$$\dot{\hat{\mathcal{G}}} = [\dot{\underset{\sim}{v}}^T_{\mathcal{J}_1}, \dot{\underset{\sim}{v}}^T_{\mathcal{J}_2}] \begin{bmatrix} \dfrac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}) \\[2ex] \dfrac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}) \end{bmatrix}$$

$$= \dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2})$$

$$= -\left[\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2})\right]^T \underset{\sim}{C}^{-1}(\underset{\sim}{v}_{\mathcal{J}_1}) \left[\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2})\right] \leq 0.$$

Notice that the equality holds only when $\partial \hat{\mathcal{G}}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}) / \partial \underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{0}$, i.e., at equilibrium points. $\square$

Let us now consider a Corollary whose hypotheses can be easily verified:

Corollary 1. A non-complete nonlinear RC network $\mathcal{N}$ is completely stable if the following conditions are satisfied:

(i) All non-monotonic resistors are connected in parallel with the capacitors and are possibly coupled among themselves only.

(ii) All other resistors are strictly increasing.

(iii) All solutions are bounded.

Proof. Consider each parallel combination of a capacitor and a non-monotonic resistor as a composite branch and extract it across an external port. Let the remaining n-port N be described by

$$\underset{\sim}{i}_{\mathcal{J}_1} = \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}'\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right)$$

$$\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}'\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \underset{\sim}{0}$$

as was defined in (7) and (8). Observe that we have added a "prime" to $\hat{\mathcal{G}}$ in order to distinguish it from the overall resistive n-port which include the non-monotonic resistors. Hence $\hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \hat{\mathcal{G}}'\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) + \hat{G}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ where $\hat{G}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ is the co-content of the non-monotonic resistors across the capacitors. Equations (7) and (8) now assume the form

$$\underset{\sim}{C} \frac{d}{dt} \underset{\sim}{v}_{\mathcal{J}_1} = -\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right)$$

$$\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \hat{\mathcal{G}}'\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}\right) = \underset{\sim}{0}$$

where the second equation involves only strictly-increasing resistors. Hence the matrix $\underset{\sim}{M}$ in Lemma 2 is nonsingular and the network is locally solvable. It follows from Theorem 4 that the network is completely stable. ◻

Remark.

The concept of local solvability is introduced to ensure the existence of unique trajectories for all times. If this condition is not satisfied, a trajectory may not be defined beyond some finite time. To see this consider the simple circuit shown in Fig. 3(a). Let $C = 1\ F$, $R_2 = -1\ \Omega$ and let R_3 be defined by $i_3 = v_3^{1/3}$. Choosing $\{C, R_2\}$ as $\mathcal{J}_1$, we obtain the equations:

$$\dot{v}_1 = v_2,$$

$$v_2^3 - v_2 + v_1 = 0.$$

Observe that this circuit is not locally solvable at $v_2 = \pm 1/\sqrt{3}$. Indeed, the condition of Lemma 2 is violated at these two points. To investigate what happens when a trajectory reaches these points, we plot the second equation $v_1 = v_2 - v_2^3$ in Fig. 3(b). Observe that since $\dot{v}_1 > 0$ in the upper half plane and $\dot{v}_1 < 0$ in the lower half plane, the trajectory in the vicinity of points A and B must converge toward these points in finite time t^*. Consequently an impasse occurs whenever a trajectory arrives at either A or B and the solution no longer exists for $t > t^*$. To overcome this dilemma, one could either modify the circuit model by introducing a parasitic inductor in series with the resistor R_3, thereby increasing the order of the state equation [11], or one could postulate a jump hypothesis [12] and obtain a discontinuous oscillation. In either case, the circuit oscillates and is therefore not completely stable.

A dual of Theorem 4 can be easily formulated when inductors are the only energy storage elements. In particular, let $\mathcal{N}$ be a network containing inductors, c.c. resistors and constant voltage sources. Assume the inductors do not form cut sets among themselves. Let $\mathcal{L}_1$ be the set of inductor brnaches and let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a cotree. Denote the corresponding tree by $\mathcal{J}$. Then the pseudo-content $\mathcal{G}\left(\underset{\sim}{i}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{L}_2}\right)$ of the n-port N obtained by extracting all cotree branches as external current ports is defined by [3]

$$\mathcal{G}\left(\underset{\sim}{i}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{L}_2}\right) = G\left(\underset{\sim}{i}_{\mathcal{J}}, \underset{\sim}{i}_{\mathcal{L}_2}\right)\Bigg|_{\underset{\sim}{i}_{\mathcal{J}}=\bar{\underset{\sim}{B}}^T\underset{\sim}{i}_{\mathcal{L}}} = G\left(\bar{\underset{\sim}{B}}^T\begin{bmatrix}\underset{\sim}{i}_{\mathcal{L}_1}\\ \underset{\sim}{i}_{\mathcal{L}_2}\end{bmatrix}, \underset{\sim}{i}_{\mathcal{L}_2}\right)$$

where $\underset{\sim}{B} = [\bar{\underset{\sim}{B}}, \underset{\sim}{1}]$ is the fundamental loop matrix of $\mathcal{N}$. We will state the dual theorem without proof:

<u>Theorem 5</u>. The non-complete nonlinear RL network $\mathcal{N}$ described above is <u>completely stable</u> if the following conditions are satisfied:

(i) The incremental inductance matrix $\underset{\sim}{L}\left(\underset{\sim}{i}_{\mathcal{L}_2}\right)$ is symmetric and positive definite.

(ii) The network is <u>locally solvable</u> for $\underset{\sim}{i}_{\mathcal{L}_2}$ as a function of $\underset{\sim}{i}_{\mathcal{L}_1}$.

(iii) All solutions are bounded.

Remark. In Theorem 4 and 5 we assume there are no loops of capacitors and no cut sets of inductors, respectively. In case these conditions are not satisfied, techniques are available for eliminating any such loops or cut sets. See [9] for details.

3.2 Complete Stability of RLC Networks.

Let us now consider networks which contain both capacitors and inductors. For simplicity, assume first that the resistors are uncoupled, and that there are neither capacitor loops nor inductor cut sets. Let us first assign all capacitive branches (i.e. composite branches of capacitors with v.c. resistors in parallel) to a subtree $\mathcal{J}_1$ and all inductive branches (i.e. composite branches of inductors with c.c. resistors in series) to a subcotree $\mathcal{L}_2$. To complete the tree, we add as many v.c. resistors as tree branches, forming another subtree $\mathcal{J}_2$. The remaining subset of v.c. resistors which cannot be included in the tree (because they formed loops with branches in $\mathcal{J}_1$ and $\mathcal{J}_2$) must be assigned as elements of the cotree $\mathcal{L}$ and will be denoted by $\mathcal{L}_1$. Let us next fill up the tree with c.c. resistors and denote them by $\mathcal{J}_3$ so that $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$. Whatever branches that have not yet been assigned are necessarily c.c. resitors which we denote by $\mathcal{L}_3$. Clearly $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. To summarize, $\mathcal{J}_1$ = {capacitive composite tree branches}, $\mathcal{J}_2$ = {v.c. tree branches}, $\mathcal{J}_3$ = {c.c. tree branches}, $\mathcal{L}_1$ = {v.c. cotree branches}. $\mathcal{L}_2$ = {inductive composite co-tree branches} and $\mathcal{L}_3$ = {c.c. cotree branches}. The fundamental loop matrix B is then given by:

$$
\begin{array}{c}
 \\ \underset{\sim}{B} = \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{J}_1 & \mathcal{J}_2 & \mathcal{J}_3 \end{array} \\
\left[\begin{array}{ccc|ccc}
\underset{\sim}{1}_{\mathcal{L}_1\mathcal{L}_1} & & & \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} & \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_2} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{J}_3} \\
 & \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\
 & & \underset{\sim}{1}_{\mathcal{L}_3\mathcal{L}_3} & \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_1} & \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_2} & \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3}
\end{array}\right]
\end{array}
\begin{array}{c} \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \end{array}
$$

Upon defining the <u>pseudo-hybrid</u> <u>content</u> $\mathcal{H}$ by [3]

$$
\boxed{\begin{aligned}
\mathcal{H}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}, \underset{\sim}{v}_{\mathcal{J}_2}, \underset{\sim}{i}^*_{\mathcal{L}_3}\right) &\triangleq \hat{G}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} - \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2}\right) + G^*_{\mathcal{L}_3}\left(\underset{\sim}{i}^*_{\mathcal{L}_3}\right) \\
&+ \hat{G}_{\mathcal{J}_2}\left(\underset{\sim}{v}_{\mathcal{J}_2}\right) + G^*_{\mathcal{J}_3} \circ \left(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_3} \underset{\sim}{i}^*_{\mathcal{L}_2} + \underset{\sim}{B}^T_{\mathcal{L}_3\mathcal{J}_3} \underset{\sim}{i}^*_{\mathcal{J}_3}\right) + \hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) + G^*_{\mathcal{L}_2}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \\
&+ \underset{\sim}{i}^{*T}_{\mathcal{L}_2} \circ \left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2}\right) + \underset{\sim}{i}^{*T}_{\mathcal{L}_3} \circ \left(\underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} + \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2}\right)
\end{aligned}}
$$

we obtain the following system of differential-algebraic equations for the network $\mathcal{N}$:

$$
\begin{aligned}
\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) \frac{d}{dt} \underset{\sim}{v}_{\mathcal{J}_1} &= -\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \mathcal{H} \\
\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right) \frac{d}{dt} \underset{\sim}{i}^*_{\mathcal{L}_2} &= \frac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}} \mathcal{H} \\
\frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \mathcal{H} = \underset{\sim}{0} \quad &\text{and} \quad \frac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_3}} \mathcal{H} = \underset{\sim}{0}
\end{aligned} \tag{9}
$$

As will be shown latter, the resistors in $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{L}_1$ and those in $\mathcal{J}_3$ respectively, are allowed to be coupled to each other. In this case, the partial sum $\hat{G}_{\mathcal{J}_1} + \hat{G}_{\mathcal{J}_2} + \hat{G}_{\mathcal{L}_1}$ in $\mathcal{H}$ will be replaced by

$$
\hat{G}_{\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{L}_1} \triangleq \hat{G}\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{v}_{\mathcal{J}_2}, \underset{\sim}{v}_{\mathcal{L}_1}\right)\Bigg|_{\underset{\sim}{v}_{\mathcal{L}_1} = -\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} - \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2}}
$$

Theorem 6. Let $\mathcal{N}$ be a non-complete nonlinear RLC network containing uncoupled two-terminal resistors described by (9), where the incremental capacitance matrix $\underset{\sim}{C}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right)$ and the incremental inductance matrix $\underset{\sim}{L}\left(\underset{\sim}{i}^*_{\mathcal{L}_2}\right)$ are assumed to be symmetric (not necessarily diagonal) and positive definite. Then $\mathcal{N}$ is completely stable if the following conditions are satisfied:

(i) There exists a tree $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ as defined above, where $\mathcal{J}_3$ consists of linear resistors which may be coupled to each other; i.e., let $\underset{\sim}{v}_{\mathcal{J}_3} = \underset{\sim}{R}_{\mathcal{J}_3} \underset{\sim}{i}_{\mathcal{J}_3}$, where $R_{\mathcal{J}_3}$ is a symmetric and positive definite constant matrix.

(ii) All elements in $\mathcal{L}_3$ and all elements in series with the inductors in $\mathcal{L}_2$ are constant voltage sources, i.e. $\underset{\sim}{v}'_{\mathcal{L}_2} = \underset{\sim}{E}_{\mathcal{L}_2}$ and $\underset{\sim}{v}_{\mathcal{L}_3} = \underset{\sim}{E}_{\mathcal{L}_3}$. Furthermore,

$$\begin{bmatrix} \underset{\sim}{E}_{\mathcal{L}_2} \\ \underset{\sim}{E}_{\mathcal{L}_3} \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix}\right)$$

(iii) $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} = \underset{\sim}{0}$ and $\underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_2} = \underset{\sim}{0}$.

(iv)

$$\mathcal{R}\left(\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_1} \end{bmatrix}\right) \subset \mathcal{R}\left(\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix}\right).$$

(v) Let

$$\underset{\sim}{R} \triangleq \begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix} \underset{\sim}{R}_{\mathcal{J}_3} \begin{bmatrix} \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_3} & \underset{\sim}{B}^T_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix},$$

and let

$\underset{\sim}{R}^I \triangleq$ the generalized inverse of $\underset{\sim}{R}$ as defined in Appendix A-3. Partition $\underset{\sim}{R}^I$

as follow:

$$\underset{\sim}{R}^{I} = \begin{bmatrix} \underset{\sim}{R}^{I}_{11} & \underset{\sim}{R}^{I}_{12} \\ \underset{\sim}{R}^{I}_{21} & \underset{\sim}{R}^{I}_{22} \end{bmatrix}$$

where $\underset{\sim}{R}^{I}_{11}$ is of dimension $|\mathcal{L}_2| \times |\mathcal{L}_2|$, then

$$\|\underset{\sim}{K}\|^2 \triangleq \left\| \underset{\sim}{L}^{\frac{1}{2}} \underset{\sim}{R}^{I}_{11} \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \underset{\sim}{C}^{-\frac{1}{2}} \right\|^2 < 1 - \delta, \quad \text{for some } \delta > 0.$$

(vi) The system is locally solvable and all solutions are bounded.

Proof. See Appendix A-4.

Example 2. Consider the circuit shown in Fig. 4(a), where R_3, R_4, and R_5 are v.c. resistors. Since R_1 is linear, shift E by v-shift theorem as in Fig. 4(b). Let $\mathcal{J}_1 = \{C\text{-}R_5\}$, $\mathcal{J}_2 = \{R_3\}$, $\mathcal{J}_3 = \{R_1, R_2\}$[8] so that $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ is a tree. The associated cotree $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ is partitioned as follow: $\mathcal{L}_1 = \{R_4\}$, $\mathcal{L}_2 = \{L\text{-}E_1\}$ and $\mathcal{L}_3 = \{E_2\}$. The fundamental loop matrix $\underset{\sim}{B}$ is given by:

$$\underset{\sim}{B} = \begin{array}{c} \begin{array}{cccccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{J}_1 & \mathcal{J}_2 & \overbrace{\quad\quad}^{\mathcal{J}_3} \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & & & -1 & 1 & 0 & 0 \\ & 1 & & 0 & 0 & -1 & 1 \\ & & 1 & -1 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \end{array} \end{array}$$

It is easily verified that conditions (i) to (v) in Theorem 6 are satisfied.

[8] Elements in $\{\cdot\}$ are written in the order of their branch numbers, thus R_1 is numbered prior to R_2.

Hence, the circuit is completely stable if the system is locally solvable and if all solutions are bounded. Notice that in this case $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} = \underset{\sim}{0}$ and condition (v) is automatically satisfied by default.

To illustrate the significance of condition (v), consider the next example.

<u>Example 3</u>. Consider the network shown in Fig. 4(c), where R_3, R_4, and R_5 are non-monotonic v.c. resistors. Following the tree selection algorithm described earlier, we choose $\mathcal{J}_1 = \{C_1, C_2-R_5\}$, $\mathcal{J}_2 = \{R_3\}$ and $\mathcal{J}_3 = \{R_1, R_2\}$ so that $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ formed a tree. Partition the associated cotree $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_3$ with $\mathcal{L}_1 = \{R_4\}$, $\mathcal{L}_2 = \{L_1-E, L_2\}$ and $\mathcal{L}_3 = \phi$. The fundamental loop matrix $\underset{\sim}{B}$ is given by

$$\underset{\sim}{B} = \begin{array}{c} \begin{array}{cccccccc} \mathcal{L}_1 & \multicolumn{2}{c}{\overbrace{\quad\quad}^{\mathcal{L}_2}} & \multicolumn{2}{c}{\overbrace{\quad\quad}^{\mathcal{J}_1}} & \mathcal{J}_2 & \multicolumn{2}{c}{\overbrace{\quad\quad}^{\mathcal{J}_3}} \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \}\mathcal{L}_1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\mathcal{L}_2 \end{array}$$

Observe that conditions (i), (ii), and (iii) of <u>Theorem 6</u> are satisfied by inspection. Similarly, condition (iv) is also satisfied upon application of the following <u>Lemma 3</u>. To check condition (v) the following calculations are needed:

$$\underset{\sim}{R} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R_1 & \\ & R_2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} R_1+R_2 & R_1 \\ R_1 & R_1 \end{bmatrix}$$

$$\underset{\sim}{R}^{I}_{11} = \underset{\sim}{R}^{-1} = \frac{1}{R_1 R_2} \begin{bmatrix} R_1 & -R_1 \\ -R_1 & R_1+R_2 \end{bmatrix},$$

$$\underset{\sim}{K} = \frac{1}{R_1R_2}\begin{bmatrix} -R_1C_1^{-1/2}L_1^{1/2} & 0 \\ (R_1+R_2)L_2^{1/2}C_1^{-1/2} & -R_2L_2^{1/2}C_2^{-1/2} \end{bmatrix}$$

By requiring each element in K to be less than 1/2 in magnitude so that $\|\underset{\sim}{K}\| < 1$, we obtain the following upper bounds for the two inductors L_1 and L_2 in order for condition (v) to hold:

$$L_1 < \frac{1}{4} R_2^2C_1$$

$$L_2 < \min\left\{\frac{1}{4} R_1^2C_2, \frac{1}{4}\left(\frac{R_1R_2}{R_1+R_2}\right)^2 C_1\right\}$$

It follows from Theorem 6 that if the above parameter relations are satisfied, then the network of Fig. 4(c) is completely stable provided condition (vi) is also satisfied. Observe that condition (vi) is satisfied by most networks of practical interest and can be checked using the results in [3]. It is condition (v), however, which is of main practical importance because it furnishes a quantitative upper bound on the values of the linear inductors in terms of the values of the linear capacitors and resistors. In case the resistors in $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{L}_1$, or those in $\mathcal{J}_3$, are coupled to each other, we obtain the following direct extension of Theorem 6.

Theorem 7. Let $\mathcal{N}$ be a non-complete nonlinear RLC network as described above. The linear resistors in $\mathcal{J}_3$ may be coupled to each other provided $\underset{\sim}{R}_{\mathcal{J}_3}$ is symmetric. The nonlinear resistors in $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{L}_1$ may be coupled to each other so long the coupling is reciprocal. Then $\mathcal{N}$ is completely stable if all conditions in Theorem 6 are satisfied.

Remarks.

1. Dual versions of Theorems 6 and 7 can be obtained by following the same

procedure used in driving Theorem 2' upon interchanging the roles of capacitors and inductors.

2. Conditions on the topology of the network $\mathcal{N}$ similar to those in Lemma 1 which ensure that (iv) is true can be obtained in a similar fashion:

Lemma 3. Let $\mathcal{N}$ be a connected network which has been partitioned into $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ in accordance with the preceding rules.

(i) Let $\mathcal{N}'$ be the sub-network obtained by shrinking (short-circuiting) all branches except those belonging to $\mathcal{J}_3$. Let b' and n' be the number of branches and nodes in $\mathcal{N}'$; respectively, so that $\mathcal{N}'$ has b' - n' + 1 independent loops. Then the rows of $\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix}$ are linearly independent if, and only if, $b' - n' + 1 = |\mathcal{L}_2| + |\mathcal{L}_3|$, where $|\mathcal{L}_2|$ and $|\mathcal{L}_3|$ denote the original number of branches in $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively. Equivalently, let $\mathcal{N}''$ be the subnetwork obtained by shrinking all branches in

$\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{L}_1$. Then the rows of $\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix}$ are linearly independent if, and only if, $\mathcal{N}''$ contains no loop formed exclusively of branches belonging to $\mathcal{L}_2 \cup \mathcal{L}_3$.

(ii) $$\mathcal{R}\left(\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_1} \end{bmatrix}\right) \subset \mathcal{R}\left(\begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_3} \end{bmatrix}\right)$$ if, and only if, upon open-circuiting all branches in $\mathcal{L}_1$ and $\mathcal{J}_3$, the current $\begin{bmatrix} \underset{\sim}{i}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{J}_2} \end{bmatrix} = \underset{\sim}{0}$

identically, i.e., branches in $\mathcal{J}_1$ and $\mathcal{J}_2$ are not contained in any loop in the reduced network.

Proof. The proof is quite similar to that of Lemma 1 and is therefore omitted.

IV. CONCLUDING REMARKS

A remark concerning the relationship between this paper and a recent paper by Chua and Green is in order: While Theorems 6 and 7 of [10] also deal with complete stability, our results in this paper are much more general in the sense that the networks considered in [10] are essentially restricted to two-element kind RC or RL networks having global state equations. In this paper we deal with RLC networks and their state equations need only exist locally through each point in $\mathbb{R}^n$.

It is important to observe that the local solvability hypothesis guarantees a unique trajectory through each point $(\underset{\sim}{x}, \underset{\sim}{y}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ but not necessarily through each point $\underset{\sim}{x} \in \mathbb{R}^{n_x}$. In fact, for a locally solvable network which can not be described by a global state equation, each point $\underset{\sim}{x} \in \mathbb{R}^{n_x}$ may correspond to several points $\{\underset{\sim}{y}_a, \underset{\sim}{y}_b, \cdots, \underset{\sim}{y}_m\} \subset \mathbb{R}^{n_y}$, each of which satisfying the implicit algebraic equation. From the computer-simulation point of view [13], this situation is equivalent to the existence of multiple dc solutions when the capacitors are replaced by dc voltage sources and the inductors are replaced by dc current sources. In this case, the point $\underset{\sim}{y}_k$ which the internal equation solution algorithm--usually a modified Newton-Raphson method--converged to will be selected by the computer. The local solvability hypothesis will then guarantee that the numerical integration process can proceed without ever reaching an "impasse point" of the sort exemplified in Fig. 4. In other words, the local solvability hypothesis is the weakest requirement that one needs to ensure that a given network may be meaningfully simulated on a computer regardless of the initial condition.

Each of the complete stability results presented in the preceding sections requires several conditions to be satisfied. Most of these conditions are topological in nature and are directly verifiable. The condition involving the norm of a matrix, however, is quantitative in nature and has to be calculated for each specific network. This quantitative condition is the one which gives rise to an upper bound on the value of the inductor parameters (resp., capacitor parameters) as a function of the capacitor (resp., inductor) and resistor parameters, and are therefore extremely useful.

Notwithstanding the complexities of the hypotheses of the theorems some of them are in fact the best possible that can be obtained for the class of networks under consideration. One should recognize that complete stability is a very strong qualitative property not possessed by many practical networks. Consequently any theorem which guarantees complete stability must necessarily impose rather severe conditions. The subtle problem here is to ensure that the conditions are no more severe than are necessary. For otherwise, a theorem on complete stability may turn out to be just a theorem on global asymptotic stability where the severity of the hypothesis forces the network to have only one equilibrium state [10]. When we talk about complete stability in this paper, however, we are primarily concerned with the more interesting cases where the network can possess multiple equilibrium states.

Finally we remark that our results in this paper is restricted exclusively to networks containing reciprocal elements. Generalization of these results to non-reciprocal dynamic networks having more than one equilibrium states remain an outstanding open problem.

ACKNOWLEDGEMENT

The authors wo ld like to thank Professor A. San-Giovanni Vincentelli and Mr. L. K. Chen, W. Kim, and S. Suwannukul for their criticisms and suggestions on an earlier version of this paper.

A-1. Physical Interpretation of $\underset{\sim}{i}^{*T}_{\mathcal{L}_2}\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{v}_{\mathcal{J}_1}$

Consider the network $\mathcal{N}$ shown in Fig. 5(a), where $\mathcal{N}_1 \triangleq \mathcal{L}_1 \cup \mathcal{J}_1$ and $\mathcal{N}_2 \triangleq \mathcal{L}_2 \cup \mathcal{J}_2$. Partition the set of nodes of $\mathcal{N}$ into three subsets S_1, S_2 and S_3, where S_1 is the set of all nodes in $\mathcal{N}_1$ which are not also nodes of $\mathcal{N}_2$, S_2 is the set of all nodes in $\mathcal{N}_2$ which are not also nodes of $\mathcal{N}_1$, and S_3 is the set of all nodes which are common to both $\mathcal{N}_1$ and $\mathcal{N}_2$. The reduced incidence matrix $\underset{\sim}{A}$ of $\mathcal{N}$ is of the form

$$\underset{\sim}{A} = \begin{array}{c} \begin{array}{cccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{J}_1 & \mathcal{J}_2 \end{array} \\ \begin{bmatrix} \underset{\sim}{A}_{\mathcal{L}_1}(S_1) & \underset{\sim}{0} & \underset{\sim}{A}_{\mathcal{J}_1}(S_1) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}_{\mathcal{L}_2}(S_2) & \underset{\sim}{0} & \underset{\sim}{A}_{\mathcal{J}_2}(S_2) \\ \underbrace{\underset{\sim}{A}_{\mathcal{L}_1}(S_3)}_{\underset{\sim}{A}_{\mathcal{L}_1}} & \underbrace{\underset{\sim}{A}_{\mathcal{L}_2}(S_3)}_{\underset{\sim}{A}_{\mathcal{L}_2}} & \underbrace{\underset{\sim}{A}_{\mathcal{J}_1}(S_3)}_{\underset{\sim}{A}_{\mathcal{J}_1}} & \underbrace{\underset{\sim}{A}_{\mathcal{J}_2}(S_3)}_{\underset{\sim}{A}_{\mathcal{J}_2}} \end{bmatrix} \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \end{array} \triangleq \left[\underset{\sim}{A}_{\mathcal{L}_1}\ \underset{\sim}{A}_{\mathcal{L}_2}\ \underset{\sim}{A}_{\mathcal{J}_1}\ \underset{\sim}{A}_{\mathcal{J}_2}\right] \tag{A-1}$$

where $\underset{\sim}{A}_{\mathcal{J}_1}(S_1)$ is the part of $\underset{\sim}{A}_{\mathcal{J}_1}$ which is connected to the nodes in S_1, etc. Notice that since $\mathcal{N}$ can be torn into two separate parts by removing all nodes in S_3, we have $\underset{\sim}{A}_{\mathcal{L}_2}(S_1) = \underset{\sim}{0}$, $\underset{\sim}{A}_{\mathcal{J}_2}(S_1) = \underset{\sim}{0}$, $\underset{\sim}{A}_{\mathcal{L}_1}(S_2) = \underset{\sim}{0}$ and $\underset{\sim}{A}_{\mathcal{J}_1}(S_2) = \underset{\sim}{0}$.

Denoting the node voltages of $\mathcal{N}$ with respect to an arbitrary datum node in either S_1 or S_2 by $\underset{\sim}{v}_n$ and the part of $\underset{\sim}{v}_n$ associated with the nodes in S_3 by $\underset{\sim}{v}_n(S_3)$, the instantaneous power W delivered from $\mathcal{N}_2$ to $\mathcal{N}_1$ is given by

$$W = \underset{\sim}{i}^T(S_3)\ \underset{\sim}{v}_n(S_3) \tag{A-2}$$

where $\underset{\sim}{i}(S_3)$ is the "net-current" vector flowing from $\mathcal{N}_2$ to $\mathcal{N}_1$ through the corresponding nodes in S_3. This current vector can be visualized as follow. Let us split each node in S_3 into two "half-nodes" connected by a short circuit as shown in Fig. 5 (b). The net-current vector $\underset{\sim}{i}(S_3)$ is defined as the currents flowing from the "primed" nodes to the "double primed" nodes. We now make the following two observations:

<u>Observation 1</u>.

The net-current vector $\underset{\sim}{i}(S_3)$ is given explicitly by

$$\underset{\sim}{i}(S_3) = \underset{\sim}{A}_{\mathcal{J}_1}(S_3)\ \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1}\ \underset{\sim}{i}_{\mathcal{L}_2} \tag{A-3}$$

where $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}$ is defined in Sec. 2.1 ($\underset{\sim}{A}$ and $\underset{\sim}{B}$ are written with respect to the same branch labellings).

<u>Proof</u>. Let us consider first the matrix $\underset{\sim}{T} \triangleq \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)$. Consider a typical row, say the ℓ-th row of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}$, written $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_\ell$. This row designates a path consisting of branches in $\mathcal{J}_1$ which is part of the fundamental loop associated with branch ℓ in $\mathcal{L}_2$. Call this path p_ℓ. Traversing p_ℓ in the direction of branch ℓ, we can classify the nodes encountered <u>which are also in S_3</u> into three categories: (i) the <u>starting node</u> $n_s \in S_3$ at which we enter $\mathcal{N}_1$ from $\mathcal{N}_2$. (ii) the <u>final node</u> $n_f \in S_3$ at which we leave $\mathcal{N}_1$ and enter $\mathcal{N}_2$ again. (iii) an <u>intermediate node</u> $n_m \in S_3$ which is neither the starting node nor the final node but is only being passed through. (traversing through this node would bring us back into $\mathcal{N}_1$). Observe that while n_s and n_f are unique, there may be more than one inter-

mediate nodes.

To clarify the general statements to be made in the following proof, consider the network shown in Fig. 5(c) along with its partitioned network in Fig. 5(d) relative to the choice of $\mathcal{J}_1 = \{C_1, C_2, C_3\}$, $\mathcal{J}_2 = \{R_2\}$, $\mathcal{L}_1 = \{R_1\}$, and $\mathcal{L}_2 = \{L_1, L_2\}$. Observe that $\mathcal{N}_1 = \{R_1, C_1, C_2, C_3\}$, $\mathcal{N}_2 = \{L_1, L_2, R_2\}$, and hence $S_1 = \{n_5\}$, $S_2 = \{n_4\}$, and $S_3 = \{n_1, n_2, n_3\}$ where n_k denotes node ⓚ. The two paths ℓ_1 and ℓ_2 corresponding to the two fundamental loops associated with links L_1 and L_2 are given respectively by $\ell_1 = \{L_1, n_1, C_1, n_5, C_2, n_2, C_3, n_3, R_2, n_4\}$ and $\ell_2 = \{L_2, n_2, C_3, n_3, R_2, n_4\}$. The portions of these paths which correspond to $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}$ are given respectively by $p_1\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right) = \{n_1, C_1, n_5, C_2, n_2, C_3, n_3\}$ and $p_2\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right) = \{n_2, C_3, n_3\}$. Hence the starting node of p_1 is n_1, its final node is n_3, while $n_2 \in S_3$ is an intermediate node.

Before we discuss the meaning of each row of T, let us also consider the matrix $\underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)$. Each column of $\underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)$ corresponds to one node in S_3 and the nonzero components in that column (either 1 or -1) represent the branches in $\mathcal{J}_1$ which are incident with the node. The sign convention is the usual one: + 1 for branches incident <u>from</u> (leaving) the node and - 1 for branches incident <u>to</u> (entering) the node.

Let us now consider the path associated with $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_\ell$, i.e., p_ℓ. Let $b_s \in \mathcal{J}_1$ be the branch in p_ℓ connected to the starting node n_s. Then we have $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell, b_s} = \pm 1$ and $\left(\underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)\right)_{b_s, n_s} = \pm 1$, respectively, where $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell, b_s}$ denotes the (ℓ, b_s) component of the matrix $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}$, etc.. In this case, $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell, b_s} \cdot \left(\underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)\right)_{b_s, n_s} = 1$. Similarly, let b_f be the branch in p_ℓ connected to the final node n_f. We then have $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell, b_f} = \pm 1$

and $\left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b_f,n_f} = \mp 1$ respectively. In this case, $\left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b_f} \cdot \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b_f,n_f} = -1$. Finally, let $b'_m \in \mathcal{J}_1$ and $b''_m \in \mathcal{J}_1$ be the branches in p_ℓ such that we pass b'_m, n_m, b''_m in that order in the traversal of the path, where $n_m \in S_3$ belongs to category (iii) defined above. In this case, we have:

$$\left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b'_m} = \pm 1 \text{ and } \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b'_m,n_m} = \mp 1, \text{respectively, and}$$

$$\left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b''_m} = \pm 1 \text{ and } \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b''_m,n_m} = \pm 1, \text{respectively.}$$

Here $\left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b'_m} \cdot \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b'_m,n_m} + \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b''_m} \cdot \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b''_m,n_m} = 0.$

Similarly, all product terms of the form $\left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \cdot \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b,n}$, $n \in S_3$ other than those mentioned above are equal to zero. Hence the elements in the ℓ-th row of $\mathbf{T} = \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_\ell \mathbf{A}^T_{\mathcal{J}_1}(S_3)$ are given by

$$T_{\ell,n_s} = \sum_{b\in\mathcal{J}_1} \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \cdot \left(\mathbf{A}_{\mathcal{J}_1}(S_3)\right)_{b,n_s}$$

$$= \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b_s} \cdot \left(\mathbf{A}_{\mathcal{J}_1}(S_3)\right)_{b_s,n_s} = 1,$$

$$T_{\ell,n_f} = \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b_f} \cdot \left(\mathbf{A}_{\mathcal{J}_1}(S_3)\right)_{b_f,n_f} = -1,$$

$$T_{\ell,n_m} = \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b'_m} \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b'_m,n_m} + \left(\mathbf{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b''_m} \left(\mathbf{A}^T_{\mathcal{J}_1}(S_3)\right)_{b''_m,n_m} = 0,$$

and $T_{\ell,n} = 0$ for all $n \in p_\ell$, $n \neq n_s, n_f, n_m$.

Thus, each row of $\mathbf{T}$ has only two nonzero elements $+1$ and -1.

The + 1 corresponds to the starting node n_s and the - 1 corresponds to the final node n_f. For example, referring to our earlier network shown in Fig. 5(d), we have

$$\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} = \begin{array}{cc} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \\ \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} L_1 \\ L_2 \end{array} \end{array}, \quad \underset{\sim}{A}_{\mathcal{J}_1}(S_3) = \begin{array}{cc} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} & \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \end{array}, \text{ and}$$

$$\underset{\sim}{T} \triangleq \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \underset{\sim}{A}^T_{\mathcal{J}_1}(S_3) = \begin{array}{cc} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} & \begin{array}{l} \leftarrow \text{corresponds to } p_1 \\ \leftarrow \text{corresponds to } p_2 \end{array} \end{array}$$

Referring to Figs. 5(b) and (d), we observe that each element of $\underset{\sim}{T}^T \underset{\sim}{i}_{\mathcal{L}_2}$ corresponding to each node $n_k \in S_3$ can be interpreted as the current flowing in the short-circuit branch from right to left. Hence, in general, $\underset{\sim}{i}(S_3) = \underset{\sim}{T}^T \underset{\sim}{i}_{\mathcal{L}_2} = \underset{\sim}{A}_{\mathcal{J}_1}(S_3)\, \underset{\sim}{B}^T_{\mathcal{L}_2 \mathcal{J}_1}\, \underset{\sim}{i}_{\mathcal{L}_2}$ is equal to the net-current vector flowing from $\mathcal{N}_2$ into $\mathcal{N}_1$ through the common nodes in S_3. This concludes our proof of observation 1.

Let us now proceed to prove that $w = \underset{\sim}{i}^T_{\mathcal{L}_2}\, \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1}\, \underset{\sim}{v}_{\mathcal{J}_1}$. Observe that since $\underset{\sim}{A}_{\mathcal{J}_1}(S_2) = \underset{\sim}{0}$, we can write

$$\underset{\sim}{i}^T_{\mathcal{L}_2}\, \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1}\, \underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{i}^T_{\mathcal{L}_2}\, \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \left(\underset{\sim}{A}^T_{\mathcal{J}_1}\, \underset{\sim}{v}_n \right)$$

$$= \underset{\sim}{i}^T_{\mathcal{L}_2}\, \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \left[\underset{\sim}{A}^T_{\mathcal{J}_1}(S_1) \underset{\sim}{v}_n(S_1) + \underset{\sim}{A}^T_{\mathcal{J}_2}(S_3) \underset{\sim}{v}_n(S_3) \right] \tag{A-4}$$

where $\underset{\sim}{v}_n(S_1)$ and $\underset{\sim}{v}_n(S_3)$ denotes the subvectors of $\underset{\sim}{v}_n$ corresponding to nodes in S_1 and S_3, respectively. We now pause to prove the following:

Observation 2.

$$\underset{\sim}{M} \triangleq \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{A}^T_{\mathcal{J}_1}(S_1) = \underset{\sim}{0} \tag{A-5}$$

Proof. Consider the (ℓ, n_k) element

$$(\underset{\sim}{M})_{\ell,n_k} = \sum_{b\in\mathcal{J}_1} \left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \left(\underset{\sim}{A}^T_{\mathcal{J}_1}(S_1)\right)_{b,n_k}$$

$$= \sum_{b\in\mathcal{J}_1} \left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \left(\underset{\sim}{A}_{\mathcal{J}_1}(S_1)\right)_{n_k,b}$$

where $\ell \in \mathcal{L}_2$, $n_k \in S_1$. Using the same interpretation for elements of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)$ given in our earlier proof of Observation 1, we note that the product $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \left(\underset{\sim}{A}_{\mathcal{J}_1}(S_1)\right)_{n_k,b}$ is nonzero only if branch b is both in the path p_ℓ and incident with node n_k. Since $n_k \in S_1$, n_k is an interior node of p_ℓ. Therefore, there must be another branch $b' \in p_\ell$ which is also incident with n_k. Furthermore, $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b'} \left(\underset{\sim}{A}_{\mathcal{J}_1}(S_1)\right)_{n_k,b'}$ is of opposite sign as $\left(\underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1}\right)_{\ell,b} \cdot \left(\underset{\sim}{A}_{\mathcal{J}_1}(S_1)\right)_{n_k,b}$. This proves $\underset{\sim}{M} = \underset{\sim}{0}$.

Finally, substituting (A-5) into (A-4) and making use of (A-3), we obtain

$$\underset{\sim}{i}^T_{\mathcal{L}_2} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} = \underset{\sim}{i}^T_{\mathcal{L}_2} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{A}^T_{\mathcal{J}_1}(S_3)\underset{\sim}{v}_n(S_3) = \underset{\sim}{i}^T(S_3)\underset{\sim}{v}_n(S_3) = W. \tag{A-6}$$

Noting now that $\underset{\sim}{i}^*_{\mathcal{L}_2} = -\underset{\sim}{i}_{\mathcal{L}_2}$, we conclude that the topological term $\underset{\sim}{i}^*_{\mathcal{L}_2} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1}$, in the hybrid content $H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{J}_2}\right)$ can be interpreted as the instantaneous power delivered from $\mathcal{N}_1$ into $\mathcal{N}_2$ through the common nodes in S_3. □

A-2. Complete Network with Ideal Transformers

Suppose $\mathcal{N}$ is a complete network and has been partitioned into $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$, and $\mathcal{T}_2$ as described in Sec. II. Let N_T be an ideal 2-port transformer as shown in Fig. 6(a) and let this transformer be represented as two coupled 2-terminal elements as shown in Fig. 6(b) with the coupling relationship given by

$$v_1 = kv_2, \quad i_2 = - ki_1 \tag{A-7}$$

where k is the transformer turns-ratio. Notice that an ideal transformer is neither v.c. nor c.c. However, since $i_1v_1 + i_2v_2 = 0$, it is nonenergic [8]. Now suppose each of the two windings of the transformer is added across an arbitrary pair of nodes belonging to branches in $\mathcal{T}_1$. Then the two corresponding branches must be added to the original graph, each of which forms a loop exclusively with branches in $\mathcal{T}_1$. In other words, the augmented network remains complete with the number of branches in $\mathcal{L}_2$ increased by two. Now observe that the equivalent transformer representation in Fig. 6(b) can be replaced by two independent current sources as shown in Fig. 6(c) with the additional constraint $v_1 = k\, v_2$ between the terminal voltages v_1 and v_2. Since the co-content of each independent current source is simply equal to $\int_0^{v_k} i_k dv_k = i_k v_k$, the total co-content of the two current sources add up to zero in view of the nonenergic property of the transformer. The same argument can be used to show that when there are more than one transformer, or when the transformers have more than two ports, the total co-content of each transformer is zero. In other words, so long as all ports of the ideal n-port transformers can be augmented with the branches in $\mathcal{L}_1$, the overall network remains complete and the hybrid content $H\left(v_{\mathcal{J}_1}, i^*_{\mathcal{L}_2}\right)$ defined by (3) in Sec. II remains invariant. A dual property of course also holds when all ports can be augmented with the branches in $\mathcal{J}_2$.

The invariance of the hybrid content makes use of only the current relation of the ideal transformers. The voltage relation has not been used so far and must therefore be considered as another independent equation. Now since, by construction, all elements in $\mathcal{L}_1$ must necessarily form loops with elements in $\mathcal{J}_1$, each voltage relation introduces a linear constraint among the voltage state variables $v_{\mathcal{J}_1} = v_C$. This constraint is analogous to the presence of a loop of capacitors and can therefore be used to eliminate one of the "n" state variables [9]. The presence of an ideal 2-port transformer with both windings in $\mathcal{L}_1$ therefore leads to a reduced order state equation. The interesting question to pose at this point is whether this can still be expressed in terms of the gradient of a new potential function, and if so, whether any qualitative property of the original network is preserved. The answer turns out to be yes in both cases. To derive this property, let us assume for simplicity that only capacitors (no inductors) are present and that there is only one ideal 2-port transformer to be augmented in $\mathcal{L}_1$. The same property can be proved to hold in the general case but the notations becomes rather unwieldy.

Let n be the number of tree (capacitor) voltages of $\mathcal{N}$, with the tree voltages $v' = [v_1, v_2, \cdots, v_n] \in \mathbb{R}^n$. The presence of the transformer will eliminate one tree branch voltage, say v_n. To be specific, assume the branch associated with v_n forms a loop with the second port of the transformer so that the voltage relationship $v_1 = kv_2$ leads to a linear constraint $v_1 = \underset{\sim}{a}^T \underset{\sim}{v} = k\left[\underset{\sim}{b}^T, 1\right]\begin{bmatrix}\underset{\sim}{v}\\ v_n\end{bmatrix}$, where $\underset{\sim}{v} = [v_1, v_2, \cdots, v_{n-1}]$ is the reduced set of independent variables alluded to earlier, and where $\underset{\sim}{a}$, $\underset{\sim}{b}$ are column vectors consisting of 1's and 0's. Now, let us replace the branches of the transformer by two independent current sources as shown in Fig. 6(c) along with the constraint $v_1 = kv_2$. The co-content function $P(v_1, v_2, \cdots, v_n)$ of the augmented network is then given by

$$P(v_1, v_2, \cdots, v_n) = i[\underset{\sim}{a}^T\underset{\sim}{v}] + (-ki)\left[\underset{\sim}{b}^T\underset{\sim}{v} + v_n\right] + \hat{G}(v') \tag{A-8}$$

where $\hat{G}(\underset{\sim}{v}')$ is the co-content function of $\mathcal{N}$ with the transformer removed. The state equation of $\mathcal{N}$ is given by

$$\underset{\sim}{C}'\ \dot{\underset{\sim}{v}}' = -\frac{\partial P}{\partial \underset{\sim}{v}'} \tag{A-9}$$

where $\underset{\sim}{C}'$ is an n×n, symmetric, and positive definite matrix with the additional constraint $\underset{\sim}{a}^T\underset{\sim}{v} = k[\underset{\sim}{b}^T\underset{\sim}{v} + v_n]$. Let us next eliminate v_n and obtain a new co-content function from (A-8). Now, since $v_n = \frac{1}{k}(\underset{\sim}{a}^T - k\underset{\sim}{b}^T)\underset{\sim}{v}$, we can write $\underset{\sim}{v}' = \begin{bmatrix}\underset{\sim}{1}_{n-1}\\ \hline \frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\end{bmatrix}\underset{\sim}{v}$ and (A-9) becomes

$$\underset{\sim}{C}'\begin{bmatrix}\underset{\sim}{1}_{n-1}\\ \hline \frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\end{bmatrix}\dot{\underset{\sim}{v}} = -\frac{\partial P}{\partial \underset{\sim}{v}'}\ . \tag{A-10}$$

Pre-multiplying both sides of (A-10) by $[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}]$, we obtain

$$[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}]\underset{\sim}{C}'\begin{bmatrix}\underset{\sim}{1}_{n-1}\\ \frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\end{bmatrix}\dot{\underset{\sim}{v}} = -\left[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right]\frac{\partial P}{\partial \underset{\sim}{v}'}\ .$$

We now prove that $\left[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right]\frac{\partial P}{\partial \underset{\sim}{v}'}$ is equal to $\frac{\partial \tilde{G}(\underset{\sim}{v})}{\partial \underset{\sim}{v}}$ where

$$\tilde{G} \stackrel{\Delta}{=} \hat{G}(v')\Big|_{v_n = \left[\frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\right]\underset{\sim}{v}.} \tag{A-11}$$

Observe that

$$\left[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right]\frac{\partial P}{\partial \underset{\sim}{v}'} = \left[\underset{\sim}{1}_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right]\begin{bmatrix} i\underset{\sim}{a} - ki\underset{\sim}{b} + \frac{\partial \hat{G}}{\partial \underset{\sim}{v}} \\ \\ -ki + \frac{\partial \hat{G}}{\partial v_n} \end{bmatrix}$$

$$= \frac{\partial \hat{G}}{\partial \underset{\sim}{v}} + \left(\frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right)\frac{\partial \hat{G}}{\partial v_n} = \frac{\partial \hat{G}}{\partial \underset{\sim}{v}} + \frac{\partial \hat{G}}{\partial v_n}\cdot\frac{\partial v_n}{\partial \underset{\sim}{v}}$$

$$= \frac{\partial \hat{G}(\underset{\sim}{v}, v_n)\Big|_{v_n = \left[\frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\right]\underset{\sim}{v}}}{\partial \underset{\sim}{v}} = \frac{\partial \tilde{G}(\underset{\sim}{v})}{\partial \underset{\sim}{v}}$$

Therefore the new state equation of $\mathcal{N}$ is given by

$$\underset{\sim}{C}\,\dot{\underset{\sim}{v}} = -\frac{\partial \tilde{G}(\underset{\sim}{v})}{\partial \underset{\sim}{v}}, \tag{A-13}$$

where $\tilde{\underset{\sim}{G}}(\underset{\sim}{v})$ is the new co-content function and

$\underset{\sim}{C} \stackrel{\Delta}{=} \left[1_{n-1}, \frac{1}{k}\underset{\sim}{a} - \underset{\sim}{b}\right]\underset{\sim}{C}'\begin{bmatrix}\underset{\sim}{1}_{n-1} \\ \\ \frac{1}{k}\underset{\sim}{a}^T - \underset{\sim}{b}^T\end{bmatrix}$ is an $(n-1) \times (n-1)$, symmetric, and

positive definite matrix. Hence we have proved that the reduced-order state equation can be expressed in terms of the <u>gradient</u> of a potential function $\tilde{\underset{\sim}{G}}(v)$, and that the new capacitance matrix $\underset{\sim}{C}$ remains a positive-definite matrix so long as the original matrix $\underset{\sim}{C}'$ is positive definite. It follows that the various <u>qualitative properties</u> described in [9-10] are preserved in the augmented network.

A-3. Proof of Theorem 2.

Let us define $\underset{\sim}{R}^I$ first. Since $\underset{\sim}{R} = \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_2} \underset{\sim}{R}_{\mathcal{J}_2} \underset{\sim}{B}^T_{\mathcal{L}_2 \mathcal{J}_2}$ is symmetric and at least positive semidefinite, there exists an orthogonal matrix $\underset{\sim}{S}$ such that

$$\underset{\sim}{R} = \underset{\sim}{S} \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_m & & \\ & 0 & & & 0 & \\ & & & & & \ddots \;0 \end{bmatrix} \underset{\sim}{S}^T = \begin{bmatrix} \lambda_1^{\frac{1}{2}} \underset{\sim}{S}^1, \cdots, \lambda_m^{\frac{1}{2}} \underset{\sim}{S}^m \end{bmatrix} \begin{bmatrix} \lambda_1^{\frac{1}{2}} \underset{\sim}{S}^{1^T} \\ \vdots \\ \lambda_m^{\frac{1}{2}} \underset{\sim}{S}^{m^T} \end{bmatrix} = \underset{\sim}{F}\, \underset{\sim}{F}^T ,$$

where $\lambda_k > 0$ are the eigenvalues of $\underset{\sim}{R}$, and $\underset{\sim}{S}^k$ is the k <u>th</u> column of $\underset{\sim}{S}$. If $\underset{\sim}{R}$ is nonsingular, m = n, otherwise m < n. The generalized inverse $\underset{\sim}{R}^I$ of $\underset{\sim}{R}$ is then defined as:

$$\underset{\sim}{R}^I = \underset{\sim}{F} \begin{bmatrix} \lambda_1^{-2} & & 0 \\ & \ddots & \\ 0 & & \lambda_m^{-2} \end{bmatrix} \underset{\sim}{F}^T = \underset{\sim}{F}(\underset{\sim}{F}^T \underset{\sim}{F})^{-2} \underset{\sim}{F}^T$$

Notice that in this case $\underset{\sim}{R}^I$ is symmetric and positive semidefinite. Furthermore, for any $\underset{\sim}{x} \in \mathcal{R}(\underset{\sim}{R})$, $\underset{\sim}{R}\, \underset{\sim}{R}^I\, \underset{\sim}{x} = \underset{\sim}{R}^I\, \underset{\sim}{R}\, \underset{\sim}{x} = \underset{\sim}{x}$, as it should.

We are now ready to prove the theorem. Under conditions (i) and (ii), the hybrid content H of $\mathcal{N}$ is given by:

$$H\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right) = G_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1 \mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1}\right) + \underset{\sim}{i}^{*T}_{\mathcal{L}_2} \underset{\sim}{E}_{\mathcal{L}_2} + \hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) - \frac{1}{2} \underset{\sim}{i}^{*T}_{\mathcal{L}_2} \underset{\sim}{R}\, \underset{\sim}{i}^*_{\mathcal{L}_2}$$

$$+ \underset{\sim}{i}^{*T}_{\mathcal{L}_2} \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} . \qquad \text{(A-14)}$$

where $\underset{\sim}{R} \triangleq \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_2} \underset{\sim}{R}_{\mathcal{J}_2} \underset{\sim}{B}^T_{\mathcal{L}_2 \mathcal{J}_2}$.

Define H^* as follows:

$$H^*\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^*_{\mathcal{L}_2}\right) \triangleq H + \left(\frac{\partial H}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}}\right)^T \underset{\sim}{R}^I \left(\frac{\partial H}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}}\right) \tag{A-15}$$

In order to prove complete stability, we only need to show that $\frac{d}{dt} H^* \leq 0$ along any trajectory of (4), with the equality holding only at equilibrium states. Using (A-15) and (4), we obtain

$$\frac{d}{dt} H^* = \dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} H + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \frac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}} H + \frac{d}{dt}\left(\frac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}} H\right)^T \underset{\sim}{R}^I \left(\frac{\partial}{\partial \underset{\sim}{i}^*_{\mathcal{L}_2}} H\right)$$

$$= -\dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \underset{\sim}{C}\dot{\underset{\sim}{v}}_{\mathcal{J}_1} + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} - \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\underset{\sim}{R}^I\underset{\sim}{R}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2}$$

$$+ \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\underset{\sim}{R}^I \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \dot{\underset{\sim}{v}}_{\mathcal{J}_1} - \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{R}\underset{\sim}{R}^I \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} + \dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \underset{\sim}{B}^T_{\mathcal{L}_2 \mathcal{J}_1} \underset{\sim}{R}^I \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} \tag{A-16}$$

It follows from (2) and hypotheses (ii) and (iii) that $\underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} \in \mathcal{R}(\underset{\sim}{R})$.

Hence $\dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\underset{\sim}{R}^I\underset{\sim}{R}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} = \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} = \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{R}\underset{\sim}{R}^I\underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2}$ and we obtain

$$-\frac{d}{dt} H^* = \dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \underset{\sim}{C}\dot{\underset{\sim}{v}}_{\mathcal{J}_1} + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} - 2\dot{\underset{\sim}{v}}^T_{\mathcal{J}_1} \underset{\sim}{B}^T_{\mathcal{L}_2 \mathcal{J}_1} \underset{\sim}{R}^I \underset{\sim}{L}\dot{\underset{\sim}{i}}^*_{\mathcal{L}_2}$$

$$\geq \left\| \underset{\sim}{C}^{\frac{1}{2}} \dot{\underset{\sim}{v}}_{\mathcal{J}_1} - \underset{\sim}{K}\,\underset{\sim}{L}^{\frac{1}{2}} \dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} \right\|^2 + \left(1 - \|\underset{\sim}{K}\|^2\right) \left\| \underset{\sim}{L}^{\frac{1}{2}} \dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} \right\|^2$$

$$\geq 0 \qquad \text{for all } \left[\dot{\underset{\sim}{v}}_{\mathcal{J}_1}, \dot{\underset{\sim}{i}}^*_{\mathcal{L}_2}\right] \tag{A-17}$$

and

$$\frac{d}{dt} H^* = 0 \qquad \text{only when } \dot{\underset{\sim}{v}}_{\mathcal{J}_1} = \underset{\sim}{0} \text{ and } \dot{\underset{\sim}{i}}^*_{\mathcal{L}_2} = \underset{\sim}{0}. \tag{A-18}$$

It follows from (A-17), (A-18) and condition (v) that the network is completely stable.

A-4. Proof of Theorem 6.

The proof is quite similar to that of Theorem 2. Constructing $\underset{\sim}{R}^{I}$ in the same way as in Appendix A-3, we obtain the following expression for $\mathcal{H}$:

$$\mathcal{H} = \hat{G}_{\mathcal{J}_1}\left(\underset{\sim}{v}_{\mathcal{J}_1}\right) + \hat{G}_{\mathcal{J}_2}\left(\underset{\sim}{v}_{\mathcal{J}_2}\right) + \hat{G}_{\mathcal{L}_1} \circ \left(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} - \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_2} \underset{\sim}{v}_{\mathcal{J}_2}\right)$$
$$+ \underset{\sim}{i}^{*T}_{\mathcal{L}_2} \underset{\sim}{E}_{\mathcal{L}_2} + \underset{\sim}{i}^{*T}_{\mathcal{L}_3} \underset{\sim}{E}_{\mathcal{L}_3} - \frac{1}{2}\left[\underset{\sim}{i}^{*T}_{\mathcal{L}_2}, \underset{\sim}{i}^{*T}_{\mathcal{L}_3}\right] \underset{\sim}{R} \begin{bmatrix} \underset{\sim}{i}^{*}_{\mathcal{L}_2} \\ \underset{\sim}{i}^{*}_{\mathcal{L}_3} \end{bmatrix}$$
$$+ \left[\underset{\sim}{i}^{*T}_{\mathcal{L}_2}, \underset{\sim}{i}^{*T}_{\mathcal{L}_3}\right] \begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_3\mathcal{J}_1} \end{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \tag{A-19}$$

Define $\mathcal{H}^*$ as follow:

$$\mathcal{H}^*\left(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}^{*}_{\mathcal{L}_2}, \underset{\sim}{v}_{\mathcal{J}_2}, \underset{\sim}{i}^{*}_{\mathcal{L}_3}\right) \triangleq \mathcal{H} + \left[\frac{\partial \mathcal{H}^T}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \quad \frac{\partial \mathcal{H}^T}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}}\right] \underset{\sim}{R}^{I} \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \\ \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}} \end{bmatrix} \tag{A-20}$$

We only need to prove that

$\dot{\mathcal{H}}^* \leq 0$ along any trajectory, with equality holding only at equilibrium states. First, notice that

$$\frac{\partial}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \mathcal{H} = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_3} \underset{\sim}{v}_{\mathcal{J}_3} + \underset{\sim}{E}_{\mathcal{L}_2} \tag{A-21}$$

$$\frac{\partial}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}} \mathcal{H} = \underset{\sim}{B}_{\mathcal{L}_3 \mathcal{J}_1} \underset{\sim}{v}_{\mathcal{J}_1} + \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_3} \underset{\sim}{v}_{\mathcal{J}_3} + \underset{\sim}{E}_{\mathcal{L}_3} \tag{A-22}$$

It follows from (ii) and (iv) that $\begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \\ \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}} \end{bmatrix} \in \mathcal{R}(\underset{\sim}{R})$.

Differentiating next $\mathcal{H}^{*}$ with respect to time, we obtain

$$\dot{\mathcal{H}}^{*} = \dot{\mathcal{H}} + \frac{d}{dt} \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}}^{T} & \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}}^{T} \end{bmatrix} \underset{\sim}{R}^{I} \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \\ \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}} \end{bmatrix} \tag{A-23}$$

The function $\dot{\mathcal{H}}$ can be obtained from (9):

$$\dot{\mathcal{H}} = \dot{\underset{\sim}{v}}^{T}_{\mathcal{J}_1} \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_1}} \mathcal{H} + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \frac{\partial}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}} \mathcal{H} + \dot{\underset{\sim}{v}}^{T}_{\mathcal{J}_2} \frac{\partial}{\partial \underset{\sim}{v}_{\mathcal{J}_2}} \mathcal{H} + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_3} \frac{\partial}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}} \mathcal{H}$$

$$= - \dot{\underset{\sim}{v}}^{T}_{\mathcal{J}_1} \underset{\sim}{C} \dot{\underset{\sim}{v}}_{\mathcal{J}_1} + \dot{\underset{\sim}{i}}^{*T}_{\mathcal{L}_2} \underset{\sim}{L} \dot{\underset{\sim}{i}}^{*}_{\mathcal{L}_2} \tag{A-24}$$

The second term of $\dot{\mathcal{H}}^{*}$ is equal to:

$$2 \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_2}}^{T} & \dfrac{\partial \mathcal{H}}{\partial \underset{\sim}{i}^{*}_{\mathcal{L}_3}}^{T} \end{bmatrix} \underset{\sim}{R}^{I} \left\{ \begin{bmatrix} \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_3 \mathcal{J}_1} \end{bmatrix} \dot{\underset{\sim}{v}}_{\mathcal{J}_1} - \underset{\sim}{R} \begin{bmatrix} \dot{\underset{\sim}{i}}^{*}_{\mathcal{L}_2} \\ \dot{\underset{\sim}{i}}^{*}_{\mathcal{L}_3} \end{bmatrix} \right\}. \tag{A-25}$$

Now, since $\partial \mathcal{H} / \partial \underset{\sim}{i}^{*}_{\mathcal{L}_3} \equiv \underset{\sim}{0}$ and $\partial \mathcal{H} / \partial \underset{\sim}{i}^{*}_{\mathcal{L}_2} = \underset{\sim}{L}\, \underset{\sim}{i}^{*}_{\mathcal{L}_2}$, we have

$$\begin{bmatrix} \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_2}} & \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_3}} \end{bmatrix} \mathbf{R}^I \begin{bmatrix} \mathbf{B}_{\mathcal{L}_2\mathcal{J}_1} \\ \\ \mathbf{B}_{\mathcal{L}_3\mathcal{J}_1} \end{bmatrix} \dot{\mathbf{v}}_{\mathcal{J}_1} = \dot{\mathbf{i}}^*_{\mathcal{L}_2} \mathbf{L} \mathbf{R}^T_{11} \mathbf{B}_{\mathcal{L}_2\mathcal{J}_1} \dot{\mathbf{v}}_{\mathcal{J}_1} \tag{A-26}$$

Making use of (A-22), we obtain

$$\begin{bmatrix} \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_2}} & \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_3}} \end{bmatrix} \mathbf{R}^I \mathbf{R} \begin{bmatrix} \dot{\mathbf{i}}^*_{\mathcal{L}_2} \\ \\ \dot{\mathbf{i}}^*_{\mathcal{L}_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_2}} & \dfrac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_3}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{i}}^*_{\mathcal{L}_2} \\ \\ \dot{\mathbf{i}}^*_{\mathcal{L}_3} \end{bmatrix}$$

$$= \frac{\partial \mathcal{H}^T}{\partial \mathbf{i}^*_{\mathcal{L}_2}} \dot{\mathbf{i}}^*_{\mathcal{L}_2} \quad (\text{since } \partial \mathcal{H}/\partial \mathbf{i}^*_{\mathcal{L}_3} \equiv \mathbf{0} \ \underline{\text{along the trajectory}})$$

$$= \dot{\mathbf{i}}^{*T}_{\mathcal{L}_2} \mathbf{L} \dot{\mathbf{i}}^*_{\mathcal{L}_2} \ . \tag{A-27}$$

Finally, following the same procedure as in the derivation of (A-17) and (A-18), we obtain

$$\dot{\mathcal{H}}^* = -\dot{\mathbf{v}}_{\mathcal{J}_1} \mathbf{C} \dot{\mathbf{v}}_{\mathcal{J}_1} - \dot{\mathbf{i}}^*_{\mathcal{L}_2} \mathbf{L} \dot{\mathbf{i}}^*_{\mathcal{L}_2} + 2\dot{\mathbf{i}}^*_{\mathcal{L}_2} \mathbf{L} \mathbf{R}^I_{11} \mathbf{B}_{\mathcal{L}_2\mathcal{J}_1} \dot{\mathbf{v}}_{\mathcal{J}_1} \ .$$

$$\leq 0 \quad \text{along any trajectory}$$

$$= 0 \quad \text{only if } \dot{\mathbf{v}}_{\mathcal{J}_1} = 0 \text{ and } \dot{\mathbf{i}}^*_{\mathcal{L}_2} = 0 \tag{A-28}$$

It follows from (A-31) and condition (vi) that the network is completely stable.

□

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FIGURE CAPTIONS

Fig. 1 (a) A typical composite branch in $\mathcal{J}_1$. The resistor R_t is the parallel combination of all v.c. resistors connected across the capacitor. Note the current into R_t is denoted by i_t'.

(b) A typical composite branch in $\mathcal{L}_2$. The resistor R_ℓ is the series combination of all c.c. resistors connected in series with the inductor. Note the voltage across R_ℓ is denoted by v_ℓ'.

Fig. 2 (a) The circuit for Example 1.

(b) Equivalent circuit obtained by applying the v-shift theorem.

(c) The $v_4 - i_4$ curve for v.c. resistor R_4.

Fig. 3 (a) A simple RC circuit.

(b) Graphical illustration of impasse points A and B and the discontinuous oscillation resulting from invoking the jump postulate.

Fig. 4 (a) The circuit for Example 2.

(b) The same circuit of (a) with the voltage source E shifted in series with the remaining branches in the cut set.

(c) The circuit for Example 3.

Fig. 5 The partition of network $\mathcal{N}$ into $\mathcal{N}_1 \cup \mathcal{N}_2$, where $\mathcal{N}_1 = \mathcal{L}_1 \cup \mathcal{J}_1$ and $\mathcal{N}_2 = \mathcal{L}_2 \cup \mathcal{J}_2$.

Fig. 6 An ideal 2-port transformer and its various equivalent representations.

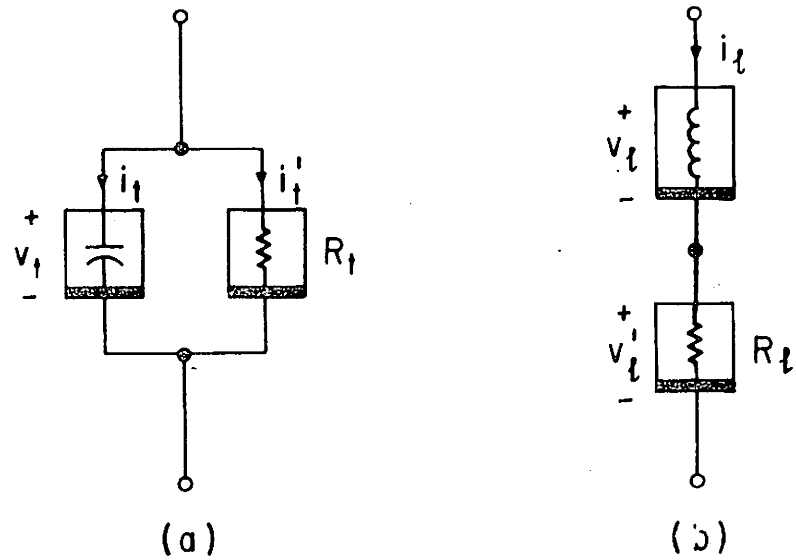


Fig. 1

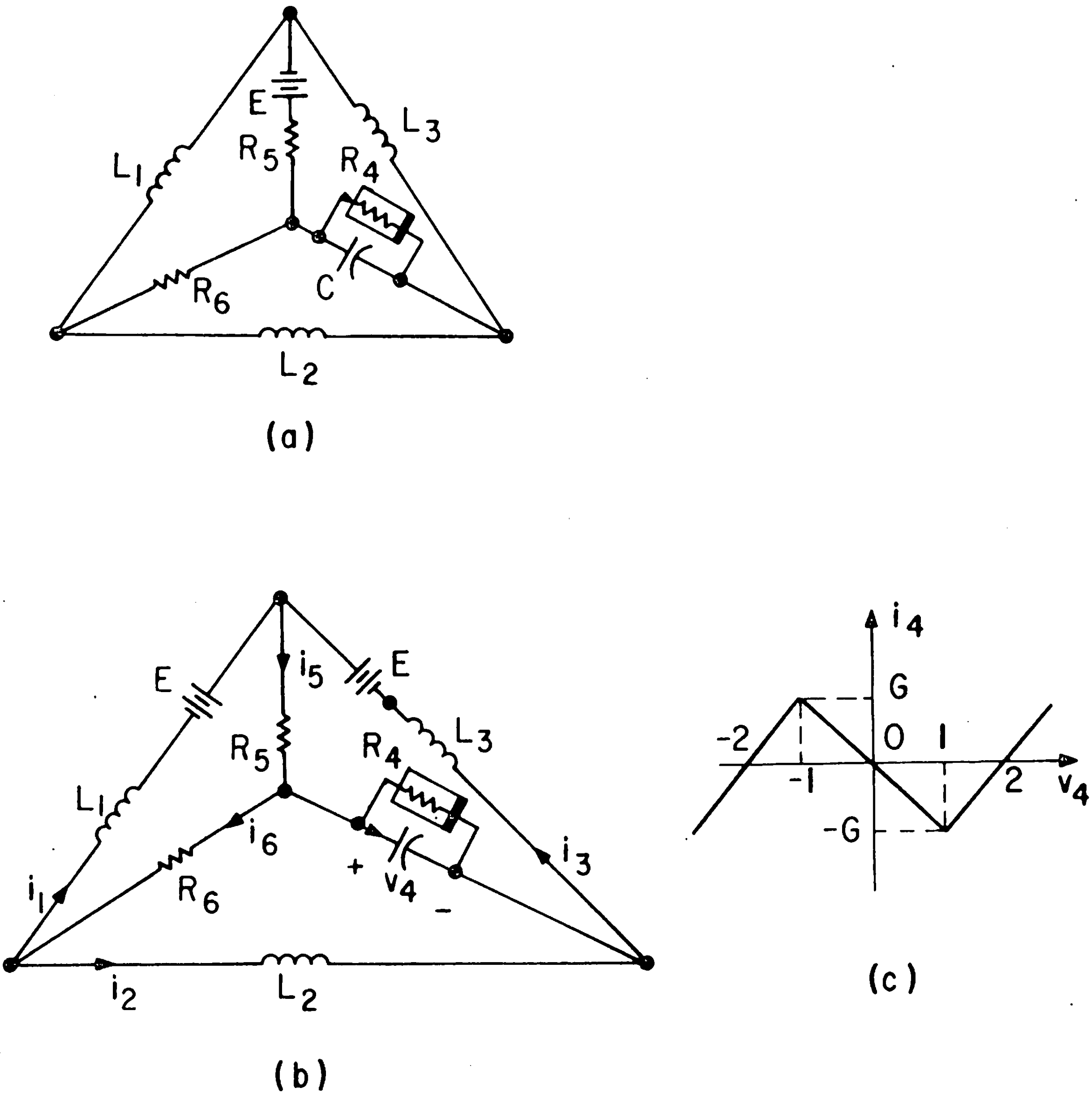
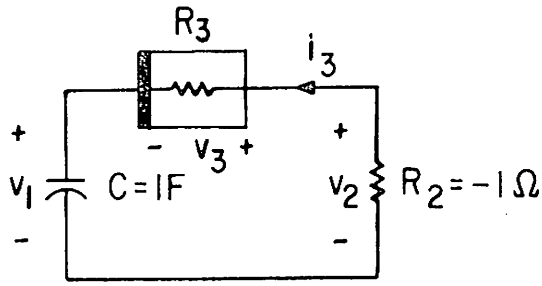
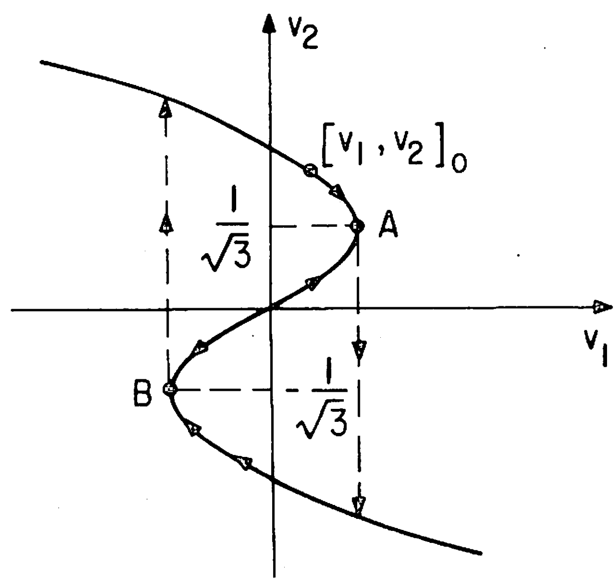

E
L_1
R_5
R_4
L_3
R_6
C
L_2
(a)
E
i_5
E
L_3
R_5
R_4
L_1
i_6
i_1
R_6
+
v_4
−
i_3
i_2
L_2
(b)
i_4
G
-2
O
1
-1
2
v_4
-G
(c)

Fig. 2.

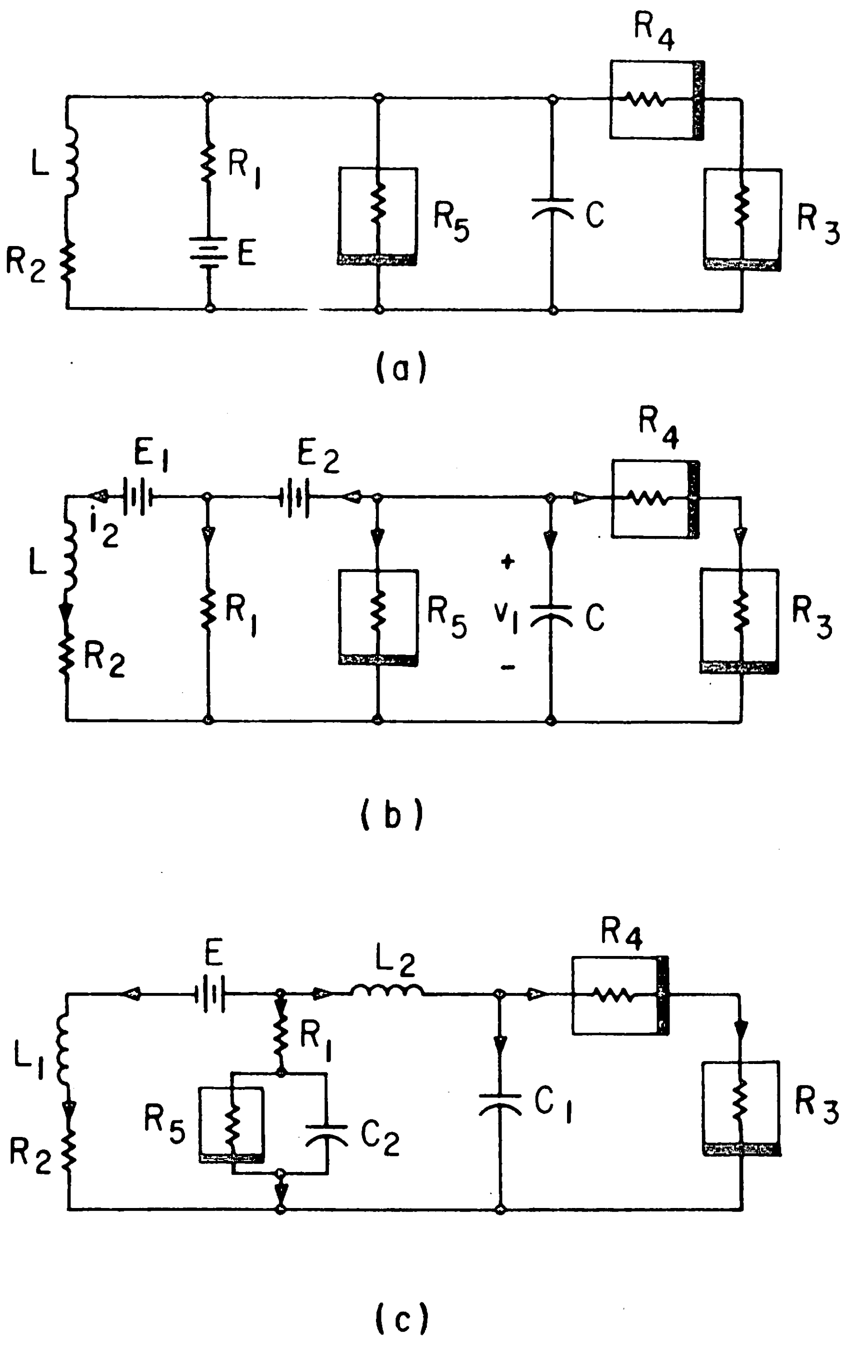


(a)



(b)

Fig. 3.



R_1, R_2, L, C: positive constants.

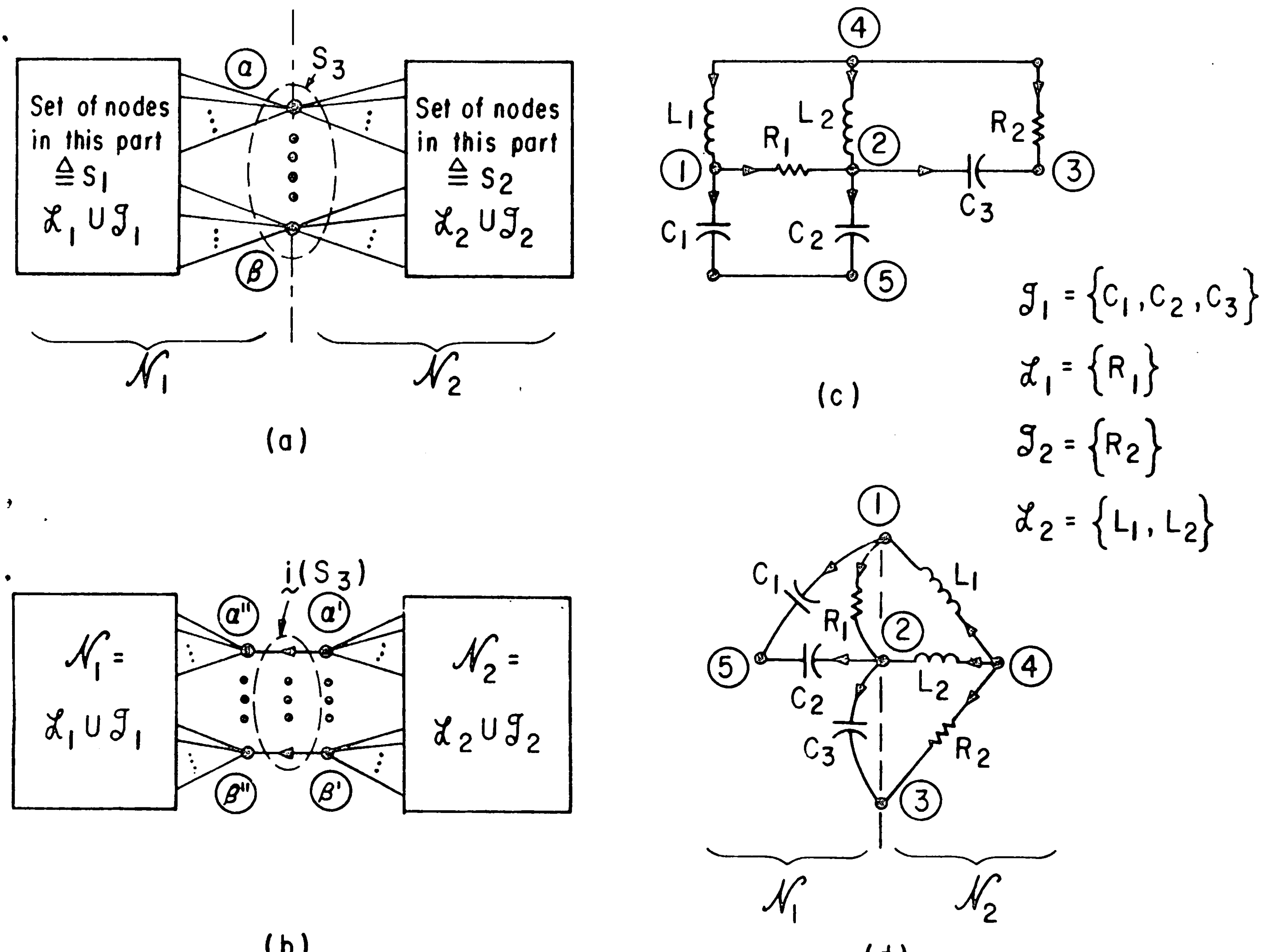
$E_1 = E_2 = E$

R_3, R_4, R_5 : v.c. nonlinear resistors.

R_1, R_2, L_1, L_2, C_1, C_2: positive constants.

R_3, R_4, R_5 : v.c. nonlinear resistors.

Fig. 4.



Set of nodes in this part $\triangleq S_1$
$\mathcal{L}_1 \cup \mathcal{J}_1$
Set of nodes in this part $\triangleq S_2$
$\mathcal{L}_2 \cup \mathcal{J}_2$
S_3
α
β
$\mathcal{N}_1$
$\mathcal{N}_2$
(a)
$\mathcal{N}_1 =$ $\mathcal{L}_1 \cup \mathcal{J}_1$
$\mathcal{N}_2 =$ $\mathcal{L}_2 \cup \mathcal{J}_2$
$\underset{\sim}{i}(S_3)$
α''
α'
β''
β'
(b)
L_1 L_2 R_1 R_2 C_1 C_2 C_3
1 2 3 4 5
(c)
$\mathcal{J}_1 = \{C_1, C_2, C_3\}$
$\mathcal{L}_1 = \{R_1\}$
$\mathcal{J}_2 = \{R_2\}$
$\mathcal{L}_2 = \{L_1, L_2\}$
(d)

Fig. 5.

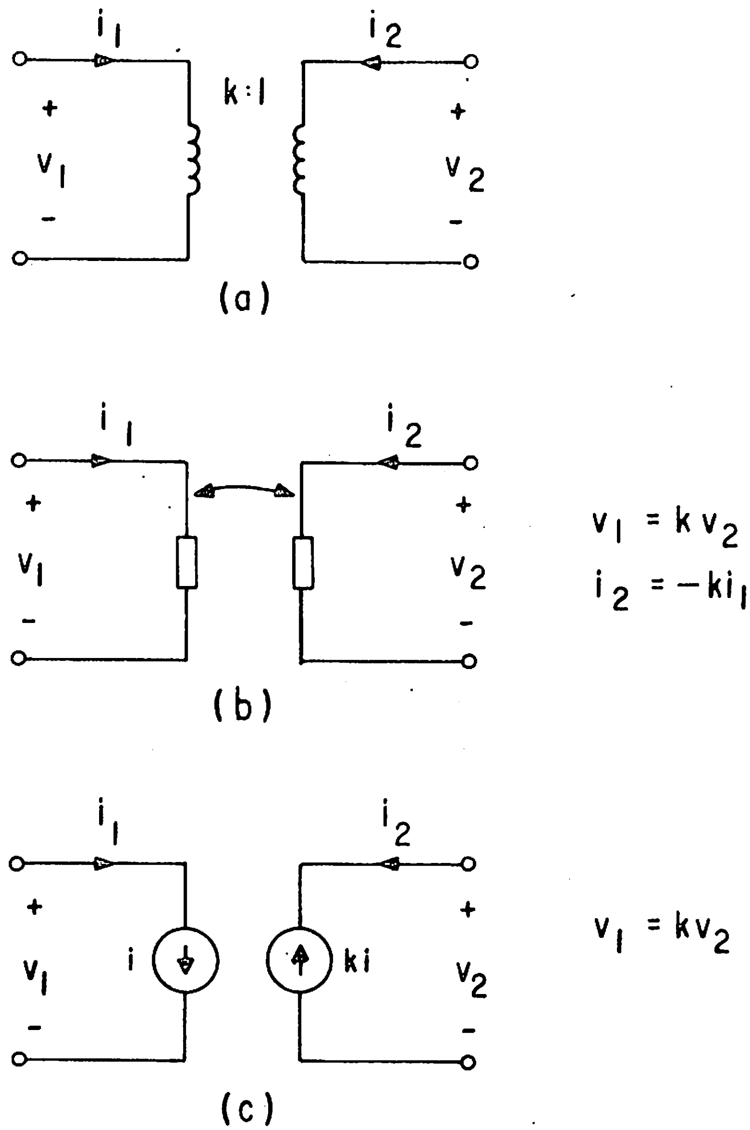


Fig. 6.