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THEORY OF FUZZY SETS

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The theory of fuzzy sets is a body of concepts and techniques for dealing in a systematic way with classes whose boundaries are not sharply defined -- that is, classes in which an object may have a grade of membership intermediate between full membership and nonmembership. The principal motivation for this theory rests on the premise that much of human thinking involves the manipulation of fuzzy rather than nonfuzzy sets and is approximate rather than precise in nature.

An important aspect of the theory of fuzzy sets relates to the fact that it provides a basis for a possibilistic framework for human information processing and natural language comprehension. Following a summary of the basic properties of fuzzy sets and operations on them, various types of translation rules for fuzzy propositions are described and illustrated by examples.

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THEORY OF FUZZY SETS

L.A. Zadeh*

The theory of fuzzy sets may be viewed as an attempt at developing a body of concepts and techniques for dealing in a systematic way with a type of imprecision which arises when the boundaries of a class of objects are not sharply defined. Among the very common examples of such classes are the classes of "bald men," "young women," "small cars," "narrow streets," "grammatical sentences," "funny jokes," etc. Membership in such classes or, as they are suggestively called, fuzzy sets is a matter of degree rather than an all or nothing proposition. Thus, informally, a fuzzy set may be regarded as a class in which there is a graduality of progression from membership to nonmembership or, more precisely, in which an object may have a grade of membership intermediate between unity (full membership) and zero (nonmembership). In this perspective, then, a set in the conventional mathematical sense of the term may be viewed as a degenerate case of a fuzzy set -- that is, a nonfuzzy set which admits of only two grades of membership: unity and zero.

Clearly, most of the classes of objects which we encounter in the real world are fuzzy sets in the informal sense defined above. And yet, the major focus of attention in mathematics, logic and the "hard" sciences has been and continues to be centered on classes which are sets in the traditional sense. In the main, this is due to the misconception that fuzziness is a form of randomness and as such can be adequately treated by the tools provided by probability theory. However, as we develop a better understanding of the different varieties of imprecision, it is becoming increasingly clear

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that (a) fuzziness is fundamentally different from randomness; (b) that fuzziness plays a much more basic role in human cognition than randomness; and (c) that to deal with fuzziness effectively, we may have to abandon many long-held beliefs and attitudes, and develop radically new conceptual frameworks for the analysis of humanistic as well as mechanistic systems.

In speaking of the varieties of imprecision, a point that is in need of clarification relates to the distinction between fuzziness and vagueness. Although to some the terms are coextensive, it is more accurate to view vagueness as a particular form of fuzziness. More specifically, a <u>fuzzy proposition</u>, e.g., "Jill is <u>quite</u> <u>tall</u>" is fuzzy by virtue of the fuzziness of the class labeled <u>quite</u> <u>tall</u>. A <u>vague</u> <u>proposition</u>, on the other hand, is one which is (i) fuzzy and (ii) ambiguous -- in the sense of providing insufficient information for a particular purpose. For example, the proposition "Jill is <u>quite</u> <u>tall</u>" may not be sufficiently specific for deciding which size jeans to buy for Jill. In this case, then, the proposition in question is both fuzzy and ambiguous -- and hence is vague. On the other hand, "Jill is <u>quite</u> <u>tall</u>" may provide sufficient information for choosing a necklace for Jill, in which case the proposition in question is fuzzy but not vague. In effect, vagueness is an application-dependent or context-dependent characteristic of a proposition, whereas fuzziness is not.

To understand the distinction between fuzziness and randomness it is helpful to interpret the grade of membership in a fuzzy set as a degree of <u>compatibility</u> (or <u>possibility</u>) rather than probability. As an illustration, consider the proposition "They got out of Roberta's car," (which is a Pinto). The question is: How many passengers got out of Roberta's car? -- assuming for simplicity that the individuals involved have the same dimensions.

Let n be the number in question. Then, with each n we can associate two numbers μ_n and p_n representing, respectively, the possibility and the probability that n passengers got out of the car. For example, we may have for μ_n and p_n:

n	1	2	3	4	5	6	7
μ_n	0	1	1	1	0.7	0.2	0
p_n	0	0.6	0.3	0.1	0	0	0

in which μ_n is interpreted as the degree of ease with which n passengers can squeeze into a Pinto. Thus, $\mu_5 = 0.7$ means that, by some specified or unspecified criterion, the degree of ease of squeezing five passengers into a Pinto is 0.7. On the other hand, the probability that Roberta may be carrying five passengers might be zero. Similarly, the possibility that a Pinto may carry 4 passengers is one; by contrast, the corresponding probability in the case of Roberta might be 0.1.

This simple example brings out three important points. First, that possibility is not an all or nothing property and may be present to a degree. Two, that the degrees of possibility are not the same as probabilities. And three, that possibilistic[1] information is more elementary and less context-dependent than probabilistic information. But, what is most important as a motivation for the theory of fuzzy sets is that much, perhaps most, of human reasoning is based on information that is possibilistic rather than probabilistic in nature. This basic issue will be discussed in greater detail at a later point, at which a connection between possibilities and probabilities will be stated as a <u>possibility/probability consistency principle</u>.

[1]The term <u>possibilistic</u> in the sense close to that used here was coined by B.R. Gaines and L. Kohout in connection with their analysis of so-called possible automata [69].

The theory of fuzzy sets has two distinct branches at this juncture. In one, a fuzzy set is treated as a mathematical construct concerning which one can make provable assertions. This "nonfuzzy" theory of fuzzy sets is in the spirit of traditional mathematics and is typified by the rapidly growing literature on fuzzy topological spaces, fuzzy switching functions, fuzzy orderings, applications to system analysis, etc. (See the appended bibliography.)

The other branch may be viewed as a "fuzzy" theory of fuzzy sets in which fuzziness is introduced into the logic which underlies the rules of manipulation of fuzzy sets and assertions about them. The genesis of this branch of the theory is related to the introduction of the so-called linguistic approach [245], [248] which in turn has led to the development of fuzzy logic [247], [18]. In this logic, the truth-values as well as the rules of inference are allowed to be imprecise, with the result that the assertions about fuzzy sets based on this logic are not, in general, provable as propositions in two-valued logic. For example, the proposition "Helen is very intelligent," may be "more or less true," which in turn may be an approximate consequence of the truth-values of other fuzzy propositions. Although the "fuzzy" theory of fuzzy sets is still in its initial stages of development, it is important as a foundation for <u>approximate</u> or, equivalently, <u>fuzzy</u> <u>reasoning</u>. Such reasoning permeates much of human thinking and is at the base of the remarkable human ability to attain imprecisely specified goals in an incompletely known environment.

In the following exposition of the theory of fuzzy sets, the accent is on the basic aspects of the theory. Expositions of such topics as the linguistic approach, fuzzy logic, fuzzy topological spaces, fuzzy languages, fuzzy algorithms and the applications to systems analysis, decision analysis, pattern classification and other fields may be found in the papers listed in the bibliography and in the comprehensive texts by Kaufmann [102] and Negoita-Ralescu [155].

Notation, Terminology and Basic Operation

A fuzzy set is generally assumed to be imbedded in a nonfuzzy *universe of discourse*, which may be any collection of objects, concepts or mathematical constructs. For example, a universe of discourse, U, may be the set of all real numbers; the set of integers $0,1,2,\ldots,100$; the set of all residents in a city; the set of all students in a course; the set of objects in a room; the set of all names in a telephone directory, etc. Universes of discourse are usually denoted by the symbols $U,V,W,\ldots,$ with or without subscripts and/or superscripts. A fuzzy set in U or, equivalently, a fuzzy subset of U, is usually denoted by one of the uppercase symbols $A, B, C, D, E, F, G, H,$ with or without subscripts and/or superscripts.

A fuzzy subset A of a universe of discourse U is characterized by a *membership function* $\mu_A: U \to [0,1]$ which associates with each element u of U a number $\mu_A(u)$ in the interval $[0,1]$ (or, more generally, a point in a partially ordered set [75]), with $\mu_A(u)$ representing the *grade of membership* of u in A. The *support* of A is the set of points in U at which $\mu_A(u)$ is positive. The *height* of A is the supremum of $\mu_A(u)$ over A. A *crossover point* of A is a point in U whose grade of membership in A is 0.5. A is *normal* if its height is unity and *subnormal* if this is not the case.

Example. Let the universe of discourse be the interval $[0,100]$, with u interpreted as *age*. A fuzzy subset of U labeled *old* may be defined by a membership function such as

$$\mu_A(u) = 0 \qquad \text{for} \quad 0 \le u \le 50 \qquad (1)$$
$$= \left(1 + \left(\frac{u-50}{5}\right)^{-2}\right)^{-1} \qquad \text{for} \quad 50 \le u \le 100 \ .$$

In this case, the support of old is the interval [50,100]; the height of old is effectively unity; and the crossover point of old is 55.

It should be remarked that in many applications the grade of membership $\mu_A(u)$ may be interpreted as the degree of compatibility of u with the concept represented by A. (For example, in the case of the fuzzy set old as defined by (1), the degree to which the numerical age 60 is compatible with the concept of old is $\mu_{\underline{old}}(60) = 0.8$.) In other cases, $\mu_A(u)$ may be interpreted as the degree of possibility of u given A. When $\mu_A(u)$ plays the role of a degree of compatibility or possibility, the function $\mu_A: U \to [0,1]$ may be referred to as the compatibility function. The less specific term membership function is generally used in situations in which the interpretation of μ_A is unspecified.

It is important to note that the meaning attached to a particular numerical value of the membership function is purely subjective in nature. For example, in stating that the degree of ease with which 5 passengers may be squeezed into a Pinto is 0.7, one may or may not be able to explain how this figure is arrived at. In some instances, the meaning of an anchor (i.e., a reference) point on the scale may be explained and the meaning of others might be defined in relative terms. As will be seen later, what matters in most cases is not the meaning attached to the grades of membership in a particular context, but the manner in which the membership function of a fuzzy set is related to those of other fuzzy sets.

To simplify the representation of fuzzy sets it is convenient to employ the following notation.

A nonfuzzy finite set such as

$$U = \{u_1, \ldots, u_n\}$$

is expressed as

$$U = u_1 + u_2 + \cdots + u_n \tag{2}$$

or

$$U = \sum_{i=1}^{n} u_i$$

with the understanding that (2) is a representation of U as the union of its constituent singletons, with $+$ playing the role of the union rather than the arithmetic sum. Thus,

$$u_i + u_j = u_j + u_i$$

and

$$u_i + u_i = u_i$$

for $i, j = 1, \ldots, n$.

As an extension of this notation, a finite fuzzy subset, A, of U is expressed as the linear form

$$A = \mu_1 u_1 + \cdots + \mu_n u_n \tag{3}$$

or

$$A = \sum_{i=1}^{n} \mu_i u_i$$

where μ_i, $i = 1, \ldots, n$, is the grade of membership of u_i in A. In cases where the u_i are numbers, there might be some ambiguity regarding the identity of the μ_i and u_i components of the string $\mu_i u_i$. In such cases, it is convenient to employ a separator symbol such as $/$ for disambiguation, writing

$$A = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{4}$$

or

$$A = \sum_{i=1}^{n} \mu_i / u_i \ .$$

<u>Example</u>. Let $U = \{a,b,c,d\}$ or, equivalently,

$$U = a + b + c + d \ .$$

In this case, a fuzzy subset A of U may be represented unambiguously as

$$A = 0.3a + b + 0.9c + 0.5d \ .$$

On the other hand, if

$$U = 1 + 2 + \cdots + 100$$

then A should be expressed as

$$A = 0.3/25 + 0.9/3$$

in order to avoid ambiguity.

<u>Example</u>. In the universe of discourse comprising the integers $1,2,\ldots,10$, i.e.,

$$U = 1 + 2 + \cdots + 10$$

the fuzzy subset labeled <u>several</u> may be defined as

$$\underline{\text{several}} = 0.5/3 + 0.8/4 + 1/5 + 1/6 + 0.8/7 + 0.5/8 \ . \tag{5}$$

<u>Example</u>. In the case of the countable universe of discourse

$$U = 0 + 1 + 2 + \cdots$$

the fuzzy set labeled <u>small</u> may be expressed as

$$\underline{\text{small}} = \sum_{0}^{\infty} \left(1 + \left(\frac{u}{10}\right)^2\right)^{-1} / u \ . \tag{6}$$

Like (2), (3) may be interpreted as a representation of a fuzzy set as the union of its constituent fuzzy singletons $\mu_i u_i$ (or μ_i/u_i). From the definition of the union (see (26)), it follows that if in the representation of A we have $u_i = u_j$, then we can make the substitution expressed by

$$\mu_i u_i + \mu_j u_i = (\mu_i \vee \mu_j)u_i \tag{7}$$

where $\vee$ is the symbol for max.

For example,

$$A = 0.3a + 0.8a + 0.5b$$

may be rewritten as

$$\begin{aligned} A &= (0.3 \vee 0.8)a + 0.5b \\ &= 0.8a + 0.5b \ . \end{aligned}$$

Consistent with the representation of a finite fuzzy set as a linear form in the u_i, an arbitrary fuzzy subset of U may be expressed in the form of an integral

$$A \triangleq \int_U \mu_A(u)/u \tag{8}$$

with the understanding that $\mu_A(u)$ is the grade of membership of u in A, and the integral denotes the union of the fuzzy singletons $\mu_A(u)/u$, $u \in U$. (The symbol $\triangleq$ stands for "is defined to be.")

<u>Example</u>. In the universe of discourse consisting of the interval $[0,100]$, with u = <u>age</u>, the fuzzy subset labeled <u>old</u> (whose membership function is given by (1)), may be expressed as

$$\underline{\text{old}} = \int_{50}^{100} \left(1 + \left(\frac{u-50}{5}\right)^{-2}\right)^{-1} / u \ . \tag{9}$$

In many cases, it is convenient to express the membership function of a fuzzy subset of the real line in terms of a standard function whose parameters may be adjusted to fit a specified membership function in an approximate fashion. Two such functions, the S-function and the π-function, are defined below.

$$
\begin{aligned}
S(u;\alpha,\beta,\gamma) &= 0 && \text{for} \quad u \leq \alpha \\
&= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \text{for} \quad \alpha \leq u \leq \beta \\
&= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for} \quad \beta \leq u \leq \gamma \\
&= 1 && \text{for} \quad u \geq \gamma
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\pi(u;\beta,\gamma) &= S(u;\gamma-\beta,\gamma-\tfrac{\beta}{2},\gamma) && \text{for} \quad u \leq \gamma \\
&= 1 - S(u;\gamma,\gamma+\tfrac{\beta}{2},\gamma+\beta) && \text{for} \quad u \geq \gamma\ .
\end{aligned} \tag{11}
$$

In $S(u;\alpha,\beta,\gamma)$, the parameter β, $\beta = \frac{\alpha+\gamma}{2}$, is the crossover point. In $\pi(u;\beta,\gamma)$, β is the bandwidth, that is the separation between the crossover points of π, while γ is the point at which π is unity.

In some cases, the assumption that μ_A is a mapping from U to $[0,1]$ may be too restrictive, and it may be desirable to allow μ_A to take values in a lattice or, more particularly, in a Boolean algebra. For most purposes, however, it is sufficient to deal with the first two of the following hierarchy of fuzzy sets.

<u>Definition</u>. A fuzzy subset, A, of U is of <u>Type</u> 1 if its membership function, μ_A, is a mapping from U to $[0,1]$; and A is of Type n, $n = 2,3,\ldots$, if μ_A is a mapping from U to the set of fuzzy subsets of Type n-1. For simplicity, it will always be understood that A is of Type 1 if it is not specified to be of a higher type.

Example. Suppose that U is the set of all nonnegative integers and A is a fuzzy subset of U labeled small integers. Then A is of Type 1 if the grade of membership of a generic element u in A is a number in the interval [0,1], e.g.,

$$\mu_{\underline{\text{small integers}}}(u) = \left(1 + \left(\frac{u}{5}\right)^2\right)^{-1} , \quad u = 0,1,2,\ldots \tag{12}$$

On the other hand, A is of Type 2 if for each u in U, $\mu_A(u)$ is a fuzzy subset of [0,1] of Type 1, e.g., for $u = 10$,

$$\mu_{\underline{\text{small integers}}}(10) = \underline{\text{low}} \tag{13}$$

where low is a fuzzy subset of [0,1] whose membership function is defined by, say,

$$\mu_{\underline{\text{low}}}(v) = 1 - S(v;0,0.25,0.5) , \quad v \in [0,1] \tag{14}$$

which implies that

$$\underline{\text{low}} = \int_0^1 \big(1 - S(v;0,0.25,0.5)\big)/v . \tag{15}$$

Containment

A fuzzy subset of U may be a subset of another fuzzy or nonfuzzy subset of U. More specifically, A is a subset of B or is contained in B if and only if $\mu_A(u) \le \mu_B(u)$ for all u in U. In symbols

$$A \subset B \Leftrightarrow \mu_A(u) \le \mu_B(u) , \quad u \in U . \tag{16}$$

Example. If $u = a + b + c + d$ and

$$A = 0.5a + 0.8b + 0.3d$$

$$B = 0.7a + b + 0.3c + d$$

then $A \subset B$.

Level-Sets of a Fuzzy Set

If A is a fuzzy subset of U, then an α-level set of A is a non-fuzzy set denoted by A_α which comprises all elements of U whose grade of membership in A is greater than or equal to α. In symbols

$$A_\alpha = \{u \mid \mu_A(u) \geq \alpha\} . \tag{17}$$

A fuzzy set A may be decomposed into its level-sets through the resolution identity [241], [248]

$$A = \int_0^1 \alpha A_\alpha \tag{18}$$

or

$$A = \sum_\alpha \alpha A_\alpha \tag{19}$$

where αA_α is the product of a scalar α with the set A_α (in the sense of (30) and $\int_0^1$ (or $\sum_\alpha$) is the union of the A_α, with α ranging from 0 to 1.

The resolution identity may be viewed as the result of combining together those terms in (3) which fall into the same level-set. More specifically, suppose that A is represented in the form

$$A = 0.1/2 + 0.3/1 + 0.5/7 + 0.9/6 + 1/9 . \tag{20}$$

Then by using (7), A can be rewritten as

$$\begin{aligned} A = 0.1/2 + 0.1/1 + 0.1/7 + 0.1/6 + 0.1/9 \\ + 0.3/1 + 0.3/7 + 0.3/6 + 0.3/9 \\ + 0.5/7 + 0.5/6 + 0.5/9 \\ + 0.9/6 + 0.9/9 \\ + 1/9 \end{aligned}$$

or

$$A = 0.1(1/2 + 1/1 + 1/7 + 1/6 + 1/9) + 0.3(1/1 + 1/7 + 1/6 + 1/9) + 0.5(1/7 + 1/6 + 1/9) + 0.9(1/6 + 1/9) + 1/9 \tag{21}$$

which is in the form (19), with the level-sets given by

$$\begin{aligned} A_{0.1} &= 2 + 1 + 7 + 6 + 9 \\ A_{0.3} &= 1 + 7 + 6 + 9 \\ A_{0.5} &= 7 + 6 + 9 \\ A_{0.9} &= 6 + 9 \\ A_1 &= 9 \end{aligned} \tag{22}$$

As will be seen in later sections, the resolution identity -- in combination with the extension principle -- provides a convenient way of generalizing various concepts associated with nonfuzzy sets to fuzzy sets. As an illustration, if U is a linear vector space, then A is <u>convex</u> if and only if for all $\lambda \in [0,1]$ and all u_1, u_2 in U,

$$\mu_A(\lambda u_1 + (1-\lambda)u_2) \geq \min(\mu_A(u_1), \mu_A(u_2)) \quad . \tag{23}$$

In terms of the level-sets of A, A is convex if and only if the A_α are convex for all $\alpha \in (0,1]$. Dually, A is <u>concave</u> if and only if

$$\mu_A(\lambda u_1 + (1-\lambda)u_2) \leq \max(\mu_A(u_1), \mu_A(u_2)) \quad . \tag{24}$$

Operations on Fuzzy Sets

Among the basic operations which can be performed on fuzzy sets are the following. (A, B are fuzzy subsets of U.)

1. The complement of A is denoted by A' and is defined by

$$A' \triangleq \int_U (1 - \mu_A(u))/u \ . \tag{25}$$

2. The union of fuzzy sets A and B is denoted by A + B (or, more conventionally, by $A \cup B$) and is defined by

$$A + B \triangleq \int_U (\mu_A(u) \vee \mu_B(u))/u \tag{26}$$

where $\vee$ is the symbol for max.

3. The intersection of A and B is denoted by $A \cap B$ and is defined by

$$A \cap B \triangleq \int_U (\mu_A(u) \wedge \mu_B(u))/u \tag{27}$$

where $\wedge$ is the symbol for min.

4. The product of A and B is denoted by AB and is defined by

$$AB \triangleq \int_U \mu_A(u)\mu_B(u)/u \ . \tag{28}$$

Thus, A^α, where α is any positive number, should be interpreted as

$$A^\alpha \triangleq \int_U (\mu_A(u))^\alpha/u \ . \tag{29}$$

Similarly, if α is any nonnegative real number such that $\alpha \sup_u \mu_A(u) \leq 1$, then

$$\alpha A \triangleq \int_U \alpha\mu_A(u)/u \ . \tag{30}$$

As a special case of (29), the operation of concentration is defined as

$$\mathrm{CON}(A) \triangleq A^2 \tag{31}$$

while that of dilation is expressed by

$$\mathrm{DIL}(A) \triangleq A^{0.5} \;. \tag{32}$$

5. The bounded-sum of A and B is denoted by $A \oplus B$ and is defined by

$$A \oplus B \triangleq \int_U 1 \wedge (\mu_A(u)+\mu_B(u))/u \tag{33}$$

where $+$ is the arithmetic sum.

6. The bounded-difference of A and B is denoted by $A \ominus B$ and is defined by

$$A \ominus B \triangleq \int_U 0 \vee (\mu_A(u)-\mu_B(u))/u \tag{34}$$

where $-$ is the arithmetic difference.

7. The left-square of A is denoted by ${}^2A$ and is defined by

$${}^2A \triangleq \int_V \mu_A(u)/u^2 \tag{35}$$

where $V \triangleq \{u^2 \mid u \in U\}$. More generally,

$${}^\alpha A \triangleq \int_V \mu_A(u)/u^\alpha \tag{36}$$

where $V \triangleq \{u^\alpha \mid u \in U\}$.

Example. If

$$U = 1 + 2 + \cdots + 10$$

$$A = 0.8/3 + 1/5 + 0.6/6$$

$$B = 0.7/3 + 1/4 + 0.5/6$$

then

$$A' = 1/1 + 1/2 + 0.2/3 + 1/4 + 0.4/6 + 1/7 + 1/8 + 1/9 + 1/10 \quad (37)$$

$$A + B = 0.8/3 + 1/4 + 1/5 + 0.6/6$$

$$A \cap B = 0.7/3 + 0.5/6$$

$$AB = 0.56/3 + 0.3/6$$

$$A^2 = 0.64/3 + 1/5 + 0.36/6$$

$$0.4A = 0.32/3 + 0.4/5 + 0.24/6$$

$$\mathrm{CON}(B) = 0.49/3 + 1/4 + 0.25/6$$

$$\mathrm{DIL}(B) = 0.84/3 + 1/4 + 0.7/6$$

$$A \oplus B = 1/3 + 1/4 + 1/5 + 1/6$$

$$A \ominus B = 0.1/3 + 1/5 + 0.1/6$$

$${}^2A = 0.8/9 + 1/25 + 0.6/36$$

$${}^3A = 0.8/27 + 1/125 + 0.6/216$$

8. If $A_1,\ldots,A_n$ are fuzzy subsets of U, and $w_1,\ldots,w_n$ are non-negative weights adding up to unity, then a <u>convex combination</u> of $A_1,\ldots,A_n$ is a fuzzy set A whose membership function is expressed by

$$\mu_A = w_1\mu_{A_1} + \cdots + w_n\mu_{A_n} \quad (38)$$

where $+$ denotes the arithmetic sum. The concept of a convex combination is useful in the representation of linguistic modifiers such as <u>essentially</u>, <u>typically</u>, etc. which modify the weights associated with the components of a fuzzy set [243].

9. If $A_1,\ldots,A_n$ are fuzzy subsets of $U_1,\ldots,U_n$, respectively, the <u>cartesian product</u> of $A_1,\ldots,A_n$ is denoted by $A_1 \times \cdots \times A_n$ and is defined as a fuzzy subset of $U_1 \times \cdots \times U_n$ whose membership function is expressed by

$$\mu_{A_1\times\cdots\times A_n}(u_1,\ldots,u_n) = \mu_{A_1}(u_1) \wedge \cdots \wedge \mu_{A_n}(u_n) . \quad (39)$$

Equivalently,

$$A_1 \times \cdots \times A_n = \int_{U_1\times\cdots\times U_n} (\mu_{A_1}(u_1) \wedge \cdots \wedge \mu_{A_n}(u_n))/(u_1,\ldots,u_n) \ . \tag{40}$$

<u>Example</u>. If $U_1 = U_2 = 3+5+7$, $A_1 = 0.5/3+1/5+0.6/7$ and $A_2 = 1/3+0.6/5$, then

$$\begin{aligned} A_1 \times A_2 = {} & 0.5/(3,3) + 1/(5,3) + 0.6/(7,3) \\ & + 0.5/(3,5) + 0.6/(5,5) + 0.6/(7,5) \ . \end{aligned} \tag{41}$$

<u>Fuzzy Relations</u>

If U is the cartesian product of n universes of discourse $U_1,\ldots,U_n$, then an n-ary <u>fuzzy relation</u>, R, <u>in</u> U is a fuzzy subset of U. As in (8), R may be expressed as the union of its constituent fuzzy singletons $\mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n)$, i.e.,

$$R = \int_{U_1\times\cdots\times U_n} \mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n) \tag{42}$$

where μ_R is the membership function of R.

Common examples of (binary) fuzzy relations are: <u>much greater than</u>, <u>resembles</u>, <u>is relevant to</u>, <u>is close to</u>, etc. For example, if $U_1 = U_2 = (-\infty,\infty)$, the relation <u>is close to</u> may be defined by

$$\underline{\text{is close to}} \triangleq \int_{U_1\times U_2} e^{-a|u_1-u_2|}/(u_1,u_2) \tag{43}$$

where a is a scale factor. Similarly, if $U_1 = U_2 = 1+2+3+4$ then the relation <u>much greater than</u> may be defined by the relation matrix

R	1	2	3	4
1	0	0.3	0.8	1
2	0	0	0	0.8
3	0	0	0	0.3
4	0	0	0	0

(44)

in which the $(i,j)^{th}$ element is the value of $\mu_R(u_1,u_2)$ for the i^{th} value of u_1 and j^{th} value of u_2.

If R is a relation from U to V (or, equivalently, a relation in $U\times V$) and S is a relation from V to W, then the composition of R and S is a fuzzy relation from U to W denoted by $R\circ S$ and defined by

$$R\circ S = \int_{U\times W} \vee_v (\mu_R(u,v) \wedge \mu_S(v,w))/(u,w) \quad . \tag{45}$$

If U, V and W are finite sets, then the relation matrix for $R\circ S$ is the max-min product of the relation matrices for R and S. For example, the max-min product of the relation matrices on the left-hand side of (46) is given by the right-hand member of (46):

$$\overset{R}{\begin{bmatrix} 0.3 & 0.8 \\ 0.6 & 0.9 \end{bmatrix}} \circ \overset{S}{\begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 1 \end{bmatrix}} = \overset{R\circ S}{\begin{bmatrix} 0.4 & 0.8 \\ 0.5 & 0.9 \end{bmatrix}} \quad . \tag{46}$$

Projections and Cylindrical Fuzzy Sets

If R is an n-ary fuzzy relation in $U_1\times\cdots\times U_n$, then its projection (shadow) on $U_{i_1}\times\cdots\times U_{i_k}$ is a k-ary fuzzy relation R_q in U which is defined by

$$R_q \triangleq \text{Proj } R \text{ on } U_{i_1}\times\cdots\times U_{i_k} \triangleq P_q R \tag{47}$$

$$\triangleq \int_{U_{i_1}\times\cdots\times U_{i_k}} (\vee_{u_{(q')}} \mu_R(u_1,\ldots,u_n))/(u_{i_1},\ldots,u_{i_k})$$

where q is the index sequence $(i_1,\ldots,i_k)$; $u_{(q)} \triangleq (u_{i_1},\ldots,u_{i_k})$; q' is the complement of q; and $\vee_{u_{(q')}}$ is the supremum of $\mu_R(u_1,\ldots,u_n)$ over the u's which are in $u_{(q')}$.

Example. For the fuzzy relation defined by the relation matrix (44), we have

$$R_1 = 1/2 + 0.8/2 + 0.3/3 \tag{48}$$

and

$$R_2 = 0.3/2 + 0.8/3 + 1/4 \ . \tag{49}$$

It is clear that distinct fuzzy relations in $U_1 \times \cdots \times U_n$ can have identical projections on $U_{i_1} \times \cdots \times U_{i_k}$. However, given a fuzzy relation R_q in $U_{i_1} \times \cdots \times U_{i_k}$, there exists a unique largest relation $\bar{R}_q$ in $U_1 \times \cdots \times U_n$ whose projection on $U_{i_1} \times \cdots \times U_{i_k}$ is R_q. In consequence of (47), the membership function of $\bar{R}_q$ is given by

$$\mu_{\bar{R}_q}(u_1,\ldots,u_n) = \mu_{R_q}(u_{i_1},\ldots,u_{i_k}) \tag{50}$$

with the understanding that (50) holds for all $u_1,\ldots,u_n$ such that the $i_1,\ldots,i_k$ arguments in $\mu_{\bar{R}_q}$ are equal, respectively, to the first,second,...,k^{th} arguments in μ_{R_q}. This implies that the value of $\mu_{\bar{R}_q}$ at the point $(u_1,\ldots,u_n)$ is the same as that at the point $(u_1',\ldots,u_n')$ provided that $u_{i_1} = u_{i_1}',\ldots,u_{i_k} = u_{i_k}'$. For this reason, $\bar{R}_q$ is referred to as the cylindrical extension of R_q, with R_q constituting the base of $\bar{R}_q$.

Suppose that R is an n-ary relation in $U_1 \times \cdots \times U_n$, R_q is its projection on $U_{i_1} \times \cdots \times U_{i_k}$, and $\bar{R}_q$ is the cylindrical extension of R_q. Since $\bar{R}_q$ is the largest relation in $U_1 \times \cdots \times U_n$ whose projection on $U_{i_1} \times \cdots \times U_{i_k}$ is R_q, it follows that R_q satisfies the containment relation

$$R \subset \bar{R}_q \tag{51}$$

for all q, and hence

$$R \subset \bar{R}_{q_1} \cap \bar{R}_{q_2} \cap \cdots \cap \bar{R}_{q_r} \tag{52}$$

for arbitrary $q_1,\ldots,q_r$ (index subsequences of $(1,2,\ldots,n)$).

In particular, if we set $q_1 = 1,\ldots,q_r = n$, then (52) reduces to

$$R \subset \bar{R}_1 \cap \bar{R}_2 \cap \cdots \cap \bar{R}_n \tag{53}$$

where $R_1,\ldots,R_n$ are the projections of R on $U_1,\ldots,U_n$, respectively, and $\bar{R}_1,\ldots,\bar{R}_n$ are their cylindrical extensions. But, from the definition of the cartesian product (see (40)) it follows that

$$\bar{R}_1 \cap \cdots \cap \bar{R}_n = R_1 \times \cdots \times R_n$$

which leads to the containment relation

$$R \subset R_1 \times \cdots \times R_n \; . \tag{54}$$

The concept of a cylindrical extension can also be used to provide an intuitively appealing interpretation of the composition of fuzzy relations. Thus, suppose that R and S are binary fuzzy relations in $U_1 \times U_2$ and $U_2 \times U_3$, respectively. Let $\bar{R}$ and $\bar{S}$ be the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Then, from the definition of $R \circ S$ (see (45)) it follows that

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \tag{55}$$

The Extension Principle

The extension principle for fuzzy sets is in essence a basic identity which allows the domain of the definition of a mapping or a relation to be extended from points in U to fuzzy subsets of U. More specifically, suppose that f is a mapping from U to V and A is a fuzzy subset of U expressed as

$$A = \mu_1 u_1 + \cdots + \mu_n u_n \ . \tag{56}$$

Then, the extension principle asserts that

$$f(A) = f(\mu_1 u_1 + \cdots + \mu_n u_n) \equiv \mu_1 f(u_1) + \cdots + \mu_n f(u_n) \ . \tag{57}$$

Thus, the image of A under f can be deduced from the knowledge of the images of $u_1, \ldots, u_n$ under f. When it is necessary to signify that $f(A)$ is to be evaluated by the use of (57), $f(A)$ is enclosed in angular brackets. Thus,

$$\langle f(A) \rangle \triangleq f(A) \triangleq \mu_1 f(u_1) + \cdots + \mu_n f(u_n) \ . \tag{58}$$

Example. Let

$$U = 1 + 2 + \cdots + 10$$

and let f be the operation of squaring. Let small be a fuzzy subset of U defined by

$$\underline{\text{small}} = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 \ . \tag{59}$$

Then, in consequence of (57) and (35), we have

$$^2\underline{\text{small}} = \langle \underline{\text{small}}^2 \rangle = 1/1 + 1/4 + 0.8/9 + 0.6/16 + 0.4/25 \ . \tag{60}$$

If the support of A is a continuum, that is

$$A = \int_U \mu_A(u)/u \tag{61}$$

then the statement of the extension principle assumes the following form

$$f(A) \triangleq f\left(\int_U \mu_A(u)/u\right) \triangleq \int_V \mu_A(u)/f(u) \tag{62}$$

with the understanding that $f(u)$ is a point in V and $\mu_A(u)$ is its grade of membership in $f(A)$, which is a fuzzy subset of V.

In some applications it is convenient to use a modified form of the extension principle which follows from (62) by decomposing A into its constituent level-sets rather than its fuzzy singletons (see the resolution identity (18)). Thus, on writing

$$A = \int_0^1 \alpha A_\alpha \tag{63}$$

where A_α is an α-level-set of A, the statement of the extension principle assumes the form

$$f(A) = f\left(\int_0^1 \alpha A_\alpha\right) \equiv \int_0^1 \alpha f(A_\alpha) \tag{64}$$

when the support of A is a continuum, and

$$f(A) = f\left(\sum_\alpha \alpha A_\alpha\right) = \sum_\alpha \alpha f(A_\alpha) \tag{65}$$

when either the support of A is a countable set or the distinct level-sets of A form a countable collection.

In many applications of the extension principle, one encounters the following problem. We have an n-ary function f, which is a mapping from a cartesian product $U_1 \times \cdots \times U_n$ to a space V, and a fuzzy set (relation) A in $U_1 \times \cdots \times U_n$ which is characterized by a membership function $\mu_A(u_1,\ldots,u_n)$, with u_i, $i = 1,\ldots,n$, denoting a generic point in U_i. A direct application of the extension principle (62) to this case yields

$$f(A) = f\left(\int_{U_1 \times \cdots \times U_n} \mu_A(u_1,\dots,u_n)/(u_1,\dots,u_n) \right) \tag{66}$$

$$= \int_V \mu_A(u_1,\dots,u_n)/f(u_1,\dots,u_n) \; .$$

However, in many instances what we know is not A but its projections $A_1,\dots,A_n$ on $U_1,\dots,U_n$, respectively (see (47)). The question that arises, then, is: What expression for μ_A should be used in (66)?

In such cases, unless otherwise specified it is assumed that the membership function of A is expressed by

$$\mu_A(u_1,\dots,u_n) = \mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \cdots \wedge \mu_{A_n}(u_n) \tag{67}$$

where μ_{A_i}, $i = 1,\dots,n$, is the membership function of A_i. In view of (39), this is equivalent to assuming that A is the cartesian product of its projections, i.e.,

$$A = A_1 \times \cdots \times A_n$$

which in turn implies that A is the largest set whose projections on $U_1,\dots,U_n$ are $A_1,\dots,A_n$, respectively.

Example. Suppose that

$$U_1 = U_2 = 1 + 2 + 3 + \cdots + 10$$

and

$$A_1 = \underset{\sim}{2} \triangleq \underline{\text{approximately}}\ 2 = 1/2 + 0.6/1 + 0.8/3 \tag{68}$$

$$A_2 = \underset{\sim}{6} \triangleq \underline{\text{approximately}}\ 6 = 1/6 + 0.8/5 + 0.7/7 \tag{69}$$

and

$$f(u_1,u_2) = u_1 \times u_2 = \text{arithmetic product of } u_1 \text{ and } u_2 .$$

Using (67) and applying the extension principle as expressed by (66) to this case, we have

$$\begin{aligned} \underset{\sim}{2} \times \underset{\sim}{6} &= (1/2 + 0.6/1 + 0.8/3) \times (1/6 + 0.8/5 + 0.7/7) \qquad (70) \\ &= 1/12 + 0.8/10 + 0.7/14 + 0.6/6 + 0.6/5 + 0.6/7 \\ &\quad + 0.8/18 + 0.8/15 + 0.7/21 \\ &= 0.6/5 + 0.6/6 + 0.6/7 + 0.8/10 + 1/12 + 0.7/14 + 0.8/15 \\ &\quad + 0.8/18 + 0.7/21 . \end{aligned}$$

Thus, the arithmetic product of the fuzzy numbers <u>approximately</u> 2 and <u>approximately</u> 6 is a fuzzy number given by (70).

More generally, let $*$ be a binary operation defined on $U \times V$ with values in W. Thus, if $u \in U$ and $v \in V$, then

$$w = u * v , \quad w \in W .$$

Now suppose that A and B are fuzzy subsets of U and V, respectively, with

$$A = \mu_1 u_1 + \cdots + \mu_n u_n \qquad (71)$$

and

$$B = \nu_1 v_1 + \cdots + \nu_m v_m .$$

By using the extension principle under the assumption (67), the operation $*$ may be extended to fuzzy subsets of U and V by the defining relation

$$\begin{aligned} A * B &= \Big(\sum_i \mu_i u_i\Big) * \Big(\sum_j \nu_j v_j\Big) \qquad (72) \\ &= \sum_{i,j} (\mu_i \wedge \nu_j)(u_i * v_j) . \end{aligned}$$

It is easy to verify that for the case where $A = \underset{\sim}{2}$, $B = \underset{\sim}{6}$ and $* = \times$, the application of (72) yields the expression for $\underset{\sim}{2} \times \underset{\sim}{6}$.

Fuzzy Sets with Fuzzy Membership Functions

Fuzzy sets with fuzzy membership functions play an important role in the linguistic approach [245], [248], in which the values of variables are not numbers but words or sentences in a natural or synthetic language. For example, if Age is treated as a linguistic variable, its values might be: young, not young, very young, more or less young, not very young, old, not old, not very young and not very old, etc. Each of these values represents a label of a fuzzy subset of a universe of discourse which is associated with Age -- e.g., the interval [0,100]. A fuzzy set which corresponds to a linguistic value of Age, say not very young, constitutes the meaning of not very young. The meaning of each possible value of a linguistic variable is defined by the semantic rule which is associated with the variable [248].

Frequently, the grade of membership in a fuzzy set is not well-defined. In such cases, it is natural to treat the grade of membership as a linguistic variable with the linguistic values: low, not low, very low, more or less low, medium, high, not high, very high, more or less high, not low and not high, etc. Each of these values represents a fuzzy subset of the interval [0,1], e.g.,

$$\mu_{\underline{low}}(v) = 1 - S(v;0,0.25,0.5) \;, \qquad v \in [0,1] \tag{73}$$

$$\mu_{\underline{very}\ \underline{low}}(v) = (1 - S(v;0,0.25,0.5))^2 \tag{74}$$

$$\mu_{\underline{medium}}(v) = \pi(v;0.5,0.2) \tag{75}$$

$$\mu_{\underline{high}}(v) = \mu_{\underline{low}}(1-v) \tag{76}$$

where S and π are the S- and π-functions defined by (10) and (11).

The fuzzy sets in question are of Type 2 (see (13)). Consequently, to manipulate the linguistic grades of membership, it is necessary to extend to fuzzy sets of Type 2 the definitions of complementation, intersection, union, product, etc. which were stated earlier for fuzzy sets of Type 1 (see (25)-(36)). The method used for this purpose is illustrated in the sequel by application to the computation of the intersection of fuzzy sets of Type 2. The same technique can be used to define other types of operations on fuzzy sets with fuzzy membership functions [138] and, in particular, to characterize the operations of negation, conjunction, disjunction and implication in fuzzy logic [18].

To extend the definition of intersection to fuzzy sets of Type 2, it is natural to make use of the extension principle. It is convenient, however, to accomplish this in two stages: First, by extending the Type 1 definition to fuzzy sets with interval-valued membership functions; and second, generalizing from intervals to fuzzy sets by the use of the level-set form of the extension principle (see (64)). More specifically, it will be recalled that the expression for the membership function of the intersection of A and B, where A and B are fuzzy subsets of Type 1, is given by

$$\mu_{A\cap B}(u) = \mu_A(u) \wedge \mu_B(u) , \qquad u \in U . \tag{77}$$

Now if $\mu_A(u)$ and $\mu_B(u)$ are intervals in $[0,1]$ rather than points in $[0,1]$, that is, for a fixed u

$$\mu_A(u) = [a_1,a_2]$$

$$\mu_B(u) = [b_1,b_2]$$

where a_1, a_2, b_1 and b_2 depend on u, then the application of the extension principle (64) to the function $\wedge$ (min) yields

$$[a_1,a_2] \wedge [b_1,b_2] = [a_1 \wedge b_1,\ a_2 \wedge b_2] \quad . \tag{78}$$

Thus, if A and B have interval-valued membership functions, then their intersection is an interval-valued function whose value for each u is given by (78).

Next, consider the case where, for each u, $\mu_A(u)$ and $\mu_B(u)$ are fuzzy subsets of the interval $[0,1]$. For simplicity, we shall assume that these subsets are <u>convex</u>, that is, have intervals as level-sets. In other words, we shall assume that, for each α in $(0,1]$, the α-level sets of μ_A and μ_B are interval-valued membership functions.

By applying the level-set form of the extension principle (64) to the α-level sets of μ_A and μ_B we are led to the following definition of the intersection of fuzzy sets of Type 2.

<u>Definition</u>. Let A and B be fuzzy subsets of Type 2 of U such that, for each $u \in U$, $\mu_A(u)$ and $\mu_B(u)$ are convex fuzzy subsets of Type 1 of $[0,1]$, which implies that, for each α in $(0,1]$, the α-level sets of the fuzzy membership functions μ_A and μ_B are interval-valued membership functions μ_A^α and μ_B^α.

Let the α-level-set of the fuzzy membership function of the intersection of A and B be denoted by $\mu_{A\cap B}^\alpha$, with the α-level-sets μ_A^α and μ_B^α defined for each u by

$$\mu_A^\alpha \triangleq \{v \mid \nu_A(v) \geq \alpha\} \tag{79}$$

$$\mu_B^\alpha \triangleq \{v \mid \nu_B(v) \geq \alpha\} \tag{80}$$

where $\nu_A(v)$ denotes the grade of membership of a point v, $v \in [0,1]$, in the fuzzy set $\mu_A(u)$, and likewise for μ_B. Then, for each u,

$$\mu^{\alpha}_{A\cap B} = \mu^{\alpha}_{A} \wedge \mu^{\alpha}_{B} . \tag{81}$$

In other words, the α-level-set of the fuzzy membership function of the intersection of A and B is the minimum (in the sense of (78)) of the α-level-sets of the fuzzy membership functions of A and B. Thus, using the resolution identity (18), we can express $\mu_{A\cap B}$ as

$$\mu_{A\cap B} = \int_0^1 \alpha\left(\mu^{\alpha}_{A} \wedge \mu^{\alpha}_{B}\right) . \tag{82}$$

For the case where μ_A and μ_B have finite supports, that is, μ_A and μ_B are of the form

$$\mu_A = \alpha_1 v_1 + \cdots + \alpha_n v_n , \qquad v_i \in [0,1], \quad i = 1,\ldots,n \tag{83}$$

and

$$\mu_B = \beta_1 w_1 + \cdots + \beta_m w_m , \qquad w_j \in [0,1], \quad j = 1,\ldots,m \tag{84}$$

where α_i and β_j are the grades of membership of v_i and w_j in μ_A and μ_B, respectively, the expression for $\mu_{A\cap B}$ can readily be derived by employing the extension principle in the form (72). Thus, by applying (72) to the operation $\wedge$, we obtain at once

$$\begin{aligned} \mu_{A\cap B} &= \mu_A \wedge \mu_B \\ &= (\alpha_1 v_1 + \cdots + \alpha_n v_n) \wedge (\beta_1 w_1 + \cdots + \beta_m w_m) \\ &= \sum_{i,j} (\alpha_i \wedge \beta_j)(v_i \wedge w_j) \end{aligned} \tag{85}$$

as the desired expression for $\mu_{A\cap B}$.

The Concept of a Fuzzy Restriction and Translation Rules for Fuzzy Propositions

The concept of a fuzzy restriction plays a basic role in the applications of the theory of fuzzy sets to logic, approximate reasoning, pattern classification, and many other fields. In what follows, a brief discussion of the basic aspects of this concept is presented and its application to the formulation of translation rules for fuzzy propositions is outlined.

Informally, by a fuzzy restriction is meant a fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable. More specifically, if X is a variable that takes values in a universe of discourse U, then a fuzzy restriction $R(X)$ on the values that may be assigned to X is a fuzzy relation in U such that the assignment of a value u to X requires a stretch of the restriction expressed by

$$\text{degree of stretch} = 1 - \mu_{R(X)}(u) \tag{86}$$

where $\mu_{R(X)}(u)$ is the grade of membership of u in $R(X)$. In symbols, this is expressed as the assignment equation

$$x = u\colon \mu_{R(X)}(u) \tag{87}$$

where x denotes a generic value of X and $\mu_{R(X)}(u)$ is the "degree of ease" with which u may be assigned to X.

As a simple illustration, suppose that $U = 0+1+2+\cdots$ and that X is a variable labeled "small integer." Assume that the fuzzy set small integer is defined by

$$\underline{\text{small}}\ \underline{\text{integer}} = 1/0 + 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5\ . \tag{88}$$

Then, if x is a generic value of the variable "small integer" and we assign the value 3 to this variable, we have

$$x = 3: 0.6 \tag{89}$$

which implies that the fuzzy restriction labeled <u>small integer</u> must be stretched to the degree 0.4 to allow the assignment of the value 3 to the variable "small integer."

More generally, if $X = (X_1,\ldots,X_n)$ is an n-ary variable taking values in the cartesian product space

$$U = U_1 \times \cdots \times U_n$$

then an n-ary fuzzy relation $R(X_1,\ldots,X_n)$ in U is a <u>fuzzy restriction</u> if it acts as an elastic constraint on the values that may be assigned to X. An n-ary variable which is associated with a fuzzy restriction on the values that may be assigned to it is said to be an <u>n-ary fuzzy variable</u>.

The concept of a fuzzy restriction provides a basis for the formulation of translation rules for <u>fuzzy propositions</u>, that is, propositions which contain names of fuzzy sets. Common examples of such propositions are the following. (Names of fuzzy sets are italicized.)

Karl is <u>very intelligent</u>.

Anneliese is <u>rather emotional</u>.

John is <u>tall</u> and Pat is <u>very kind</u>.

If X is <u>large</u> then Y is <u>small</u>.

X is <u>much smaller</u> than Y.

<u>Most tall</u> women are <u>well-built</u>.

X is <u>small</u> is <u>true</u>.

X is <u>small</u> is <u>likely</u>.

X is <u>small</u> is <u>possible</u>.

If X is <u>small</u> is <u>true</u> then Y is <u>large</u> is <u>very likely</u>.

By a translation of a fuzzy proposition is meant a representation of the meaning of a fuzzy proposition as a system of relational assignment equations, that is, a set of assignment equations whose right-hand members are fuzzy relations which are assigned to fuzzy restrictions on the variables associated with the proposition in question.

As a simple illustration, the translation of the proposition "John is tall" has the form

$$\text{John is } \underline{\text{tall}} \rightarrow R(\text{Height}(\text{John})) = \underline{\text{tall}} \tag{90}$$

where Height(John) is a variable, R(Height(John)) is a fuzzy restriction on the values that may be assigned to this variable, and tall is a unary fuzzy relation which is assigned to the fuzzy restriction R(Height(John)). More generally, the translation of a fuzzy proposition has the form

$$\begin{aligned} p \rightarrow R(X_1) &= F_1 \\ R(X_2) &= F_2 \\ \cdots & \cdot\cdot \\ R(X_n) &= F_n \end{aligned} \tag{91}$$

where $X_1,\ldots,X_n$ are variables which are implicit or explicit in p, $R(X_1),\ldots,R(X_n)$ are the fuzzy restrictions on these variables and $F_1,\ldots,F_n$ are fuzzy relations which are assigned to $R(X_1),\ldots,R(X_n)$, respectively. For brevity, the system of relational assignment equations associated with p is denoted by $R(p)$.

Example. The translation of "A tall man is blond" may be expressed as

$$\begin{aligned} \text{A } \underline{\text{tall}} \text{ man is } \underline{\text{blond}} \rightarrow R(\text{Color}(\text{Hair}(X))) &= \underline{\text{blond}} \\ R(\text{Height}(X)) &= \underline{\text{tall}} \end{aligned}$$

where X is an element of a population of men.

To deduce the translation of a given fuzzy proposition p it is convenient to treat p as the result of a sequence of operations on a set of kernel fuzzy propositions which play the role of generators. For example, attributional modification of the kernel propositions

Mike is intelligent

Zene is charming

results in

Mike is very intelligent

Zene is extremely charming

which upon conjunctive composition yield the composite fuzzy proposition

Mike is very intelligent and Zene is extremely charming

which upon truth-qualification results in

(Mike is very intelligent and Zene is extremely charming) is very true .

With each operation is associated a translation rule which describes the effect of the operation on the relational assignment equations associated with the operand proposition. Thus, for example, if $M(p)$ is the result of applying a modification M to a fuzzy proposition p and $\tilde{M}(R(p))$ is the modification induced by M in $R(p)$, then the associated translation rule has the general form

$$\begin{aligned} &\text{If} \quad p \rightarrow R(p) \\ &\text{then} \quad M(p) \rightarrow \tilde{M}(R(p)) \end{aligned} \tag{92}$$

which implies that R, viewed as a mapping, is a homomorphism.

In what follows, the translation process is described in greater detail for (i) a type of attributional modification (Type I); (ii) conjunctive composition (Type II); and (iii) likelihood and possibility-qualifications.

Translation Rules of Type I

Translation rules of this type pertain to operations involving attribute modification; more specifically, they apply to fuzzy propositions of the form $p \triangleq X$ is mF, where F is a fuzzy subset of $U = \{u\}$, m is a modifer such as not, very, more or less, slightly, somewhat, etc., and either X or $A(X)$ -- where A is an implied attribute of X -- is a fuzzy variable which takes values in U.

Translation rules of Type I may be subsumed under a general rule which, for convenience, is referred to as the modifier rule. In essence, this rule asserts that the translation of a fuzzy proposition of the form $p \triangleq X$ is mF is expressed by

$$X \text{ is } mF \rightarrow R(A(X)) = mF \tag{93}$$

where m is interpreted as an operator which transforms the fuzzy set F into the fuzzy set mF.

In particular, if $m \triangleq$ not, then the rule of negation asserts that the translation of $p \triangleq X$ is not F is expressed by

$$X \text{ is } \underline{\text{not}}\ F \rightarrow X \text{ is } F' \rightarrow R(A(X)) = F' \tag{94}$$

where F' is the complement of F, i.e.,

$$\mu_{F'}(u) = 1 - \mu_F(u) \ , \quad u \in U \ .$$

For example, if

$$\mu_{\underline{young}}(u) = 1 - S(u;20,30,40) \tag{95}$$

then $p \triangleq$ John is $\underline{not}$ $\underline{young}$ translates into

$$R(Age(John)) = \underline{young}' \tag{96}$$

where, in the notation of (25),

$$\underline{young}' = \int_0^\infty S(u;20,30,40)/u \ . \tag{97}$$

In general, m may be viewed as a restriction modifier which acts in a specified way on its operand. For example, the modifier $\underline{very}$ may be assumed to act -- to a first approximation -- as a concentrator which has the effect of squaring the membership function of its operand. Correspondingly, the $\underline{rule}$ $\underline{of}$ $\underline{concentration}$ asserts that the translation of the fuzzy proposition $p = X$ is $\underline{very}$ F is expressed by

$$X \text{ is } \underline{very}\ F \rightarrow X \text{ is } F^2 \rightarrow R(A(X)) = F^2 \tag{98}$$

where

$$\underline{very}\ F = F^2 = \int_U (\mu_F(u))^2/u \tag{99}$$

and $A(X)$ is an implied attribute of X.

As an illustration, on applying (98) one finds that "Jennifer is $\underline{very}$ $\underline{young}$" translates into

$$R(Age(Jennifer)) = \underline{young}^2 \tag{100}$$

where

$$\underline{young}^2 = \int_0^\infty (1 - S(u;20,30,40))^2/u \ . \tag{101}$$

The effect of the modifier more or less is less susceptible to simple approximation than that of very. In some contexts, more or less acts as a dilator, playing a role inverse to that of very. Thus, to a first approximation, we may assume that, in such contexts, more or less may be defined by

$$\text{more or less } F = \sqrt{F} \tag{102}$$

where

$$\sqrt{F} = \int_U (\mu_F(u))^{1/2}/u \ .$$

Based on this definition of more or less, the rule of dilation asserts that

$$X \text{ is more or less } F \rightarrow X \text{ is } \sqrt{F} \rightarrow R(A(X)) = \sqrt{F} \tag{103}$$

where $A(X)$ is an implied attribute of X. For example, "Pat is more or less young" translates into

$$R(\mathrm{Age(Pat)}) = \sqrt{\text{young}} = \int_0^\infty (1 - S(u;20,30,40))^{1/2}/u \ . \tag{104}$$

Translation Rules of Type II

Translation rules of this type apply to composite fuzzy propositions which are generated from fuzzy propositions of the form "X is F" through the use of various kinds of binary connectives such as the conjunction, and, the disjunction, or, the conditional if...then..., etc.

More specifically, let $U = \{u\}$ and $V = \{v\}$ be two possibly different universes of discourse, and let F and G be fuzzy subsets of U and V, respectively.

Consider the propositions "X is F" and "Y is G," and let q be their conjunction "X is F and Y is G." Then, the rule of noninteractive conjunctive composition or, for short, the rule of conjunctive composition asserts that the translation of q is expressed by

$$X \text{ is } F \text{ and } Y \text{ is } G \rightarrow (X,Y) \text{ is } F\times G \rightarrow R(A(X),B(Y)) = F\times G \tag{105}$$

where $A(X)$ and $B(Y)$ are implied attributes of X and Y, respectively; $R(A(X),B(Y))$ is a fuzzy restriction on the values of the binary fuzzy variable $(A(X),B(Y))$; and $F\times G$ is the cartesian product of F and G. Thus, under this rule, the fuzzy proposition "Eugene is tall and Cathleen is young" translates into

$$R(\text{Height(Eugene)),Age(Cathleen)}) = \underline{tall} \times \underline{young} \tag{106}$$

where tall and young are fuzzy subsets of the real line.

To differentiate between noninteractive and interactive conjunction, the latter is denoted by and*. With this understanding, the rule of interactive conjunction, in its general form, may be expressed as

$$X \text{ is } F \text{ and* } Y \text{ is } G \rightarrow R(A(X),B(Y)) = F\otimes G \tag{107}$$

where $\otimes$ is a binary operation which maps F and G into a subset of $U\times V$ and thus provides a definition of and* in a particular context.

A simple example of an interactive conjunction is provided by the translation rule

$$X \text{ is } F \text{ and* } Y \text{ is } G \rightarrow R(A(X),B(Y)) = FG \tag{108}$$

where

$$\mu_{FG} = \mu_F\mu_G \,. \tag{109}$$

Note that, in this case, an increase in the grade of membership in F can be compensated for by a decrease in the grade of membership in G, and vice-versa.

In general, interactive conjunction is strongly application-dependent and has no universally applicable definition.

The translation rule for conditional fuzzy propositions of the form "If X is F then Y is G" is referred to as the <u>rule of conditional composition</u> and may be expressed as

$$\text{If X is F then Y is G} \rightarrow R(A(X),B(Y)) = \bar{F}' \oplus \bar{G} \tag{110}$$

where $\oplus$ denotes the bounded sum and $\bar{F}'$ is the complement of the cylindrical extension of F.

As an illustration, assume that <u>tall</u> and <u>young</u> are defined by

$$\underline{\text{tall}} = \int_U S(u;160,170,180)/u$$

$$\underline{\text{young}} = \int_V (1 - S(v;20,30,40))/v$$

where U and V may be taken to be the real line and the height is assumed to be measured in centimeters. Then, the fuzzy proposition "If Eugene is <u>tall</u> then Cathleen is <u>young</u>" translates into

$$R(\text{Height(Eugene)},\text{Age(Cathleen)}) = \overline{\underline{\text{tall}}'} \oplus \overline{\underline{\text{young}}} \tag{111}$$

or, more explicitly,

$$\begin{aligned} &R(\text{Height(Eugene)},\text{Age(Cathleen)}) \\ &= \int_{U\times V} \Big(1 \wedge \big(1 - \mu_{\underline{\text{tall}}}(u) + \mu_{\underline{\text{young}}}(v)\big)\Big)/(u,v) \\ &= \int_{U\times V} \Big(1 \wedge \big(1 - S(u;160,170,180) + 1 - S(v;20,30,40)\big)\Big)/(u,v) \ . \end{aligned} \tag{112}$$

If the conditional fuzzy proposition "If X is F then Y is G else Y is H" is interpreted as the conjunction of the propositions "If X is F then

Y is G" and "If X is not F then Y is H," then by using in combination the rule of negation (94), the rule of conjunctive composition (105), and the rule of conditional composition (110), the translation of the proposition in question is found to be expressed by

$$\text{If X is F then Y is G else Y is H} \rightarrow R(A(X),B(Y)) = (F' \oplus \bar{G}) \cap (\bar{F} \oplus \bar{H}) \ . \tag{113}$$

Translation Rules for Likelihood- and Possibility-Qualified Propositions

An important mechanism for effecting a modification in a proposition p involves the use of a qualifier following or preceding p. In ordinary discourse, the most commonly used qualifiers are truth-values, likelihood-values and possibility-values. For example, if $p \triangleq$ X is small, then as modifications of p we may have propositions such as

X is small is quite true

X is small is very likely

X is small is possible

A discussion of translation rules for truth-qualified fuzzy propositions of the form "X is F is τ," where τ is a linguistic truth-value such as true, quite true, very true, not very true, etc. may be found in [18]. The translation rules for likelihood-qualified propositions of the form "X is F is λ," where λ is a linguistic likelihood-value such as likely, unlikely, very likely, etc., are quite similar to the rules described in [18] which apply to quantified propositions of the form "QX are F," where Q is a fuzzy quantifier such as most, many, few, etc.

As was stated earlier, the concept of possibility differ in essential ways from that of probability. Reflecting these differences, the translation rules for likelihood-qualified propositions are very different from the corresponding rules for possibility-qualified propositions. More specifically, the translation rule for a likelihood-qualified fuzzy proposition of the form "X is F is λ" is expressed by

$$X \text{ is } F \text{ is } \lambda \rightarrow R\left(\int_U p_{A(X)}(u)\mu_F(u)du\right) = \lambda \tag{114}$$

where $p_{A(X)}(u)du$ is the probability that the value of the implied attribute $A(X)$ falls in the interval $(u,u+du)$, and μ_F is the membership function of F as a subset of U. For example,

$$\text{Laura is } \underline{\text{young}} \text{ is } \underline{\text{likely}} \rightarrow R\left(\int_0^{100} p_{age}(u)\mu_{\underline{young}}(u)du\right) = \underline{\text{likely}} \tag{115}$$

where <u>likely</u> is a fuzzy subset of the unit interval $[0,1]$. In effect, (115) defines a fuzzy set of probability density functions $p_{age}(\cdot)$ which is induced by the proposition in question.

By contrast, the translation rule for the possibility-qualified fuzzy proposition "X is F is possible" is expressed by

$$X \text{ is } F \text{ is possible} \rightarrow R(A(X)) = F^+ \tag{116}$$

where $A(X)$ is an implied attribute of X and F^+ is a fuzzy set of Type 2 which is related to F by

$$\mu_{F^+}(u) = [1 - \mu_F(u), 1] \tag{117}$$

which signifies that μ_{F^+} is interval-valued, with the value of μ_{F^+} at u being the interval expressed by the right-hand member of (117).

As a simple illustration of (116), consider the nonfuzzy proposition "X is in [a,b]" where X is a real-valued variable. Applying (116) to this proposition, we obtain the translation

$$X \text{ is in } [a,b] \text{ is possible} \longrightarrow R(X) = [a,b]^{+} \tag{118}$$

which implies that

$$\begin{aligned} \mu_{R(X)}(u) &= 1 && \text{for } a \le u \le b \\ &= [0,1] && \text{elsewhere .} \end{aligned} \tag{119}$$

Intuitively, (118) signifies that, whereas "X is in [a,b]" implies that the degree of possibility that X is outside of the interval $[a,b]$ is zero, "X is in [a,b] is possible" implies that the degree of possibility that X is outside of the interval $[a,b]$ is unknown, i.e., is the interval $[0,1]$.

An interesting point that is worthy of note is that (118) provides a justification for an intuitively plausible implication, namely,

$$X \text{ is in } [a,b] \text{ is possible} \Rightarrow X \text{ is in } [c,d] \text{ is possible} \tag{120}$$

where

$$[c,d] \subset [a,b] \; . \tag{121}$$

By contrast,

$$X \text{ is in } [a,b] \Leftarrow X \text{ is in } [c,d] \; . \tag{122}$$

To verify (121) and (122), it is sufficient to demonstrate that the restriction associated with the antecedent is a subset of the restriction associated with the consequent. This is obvious in the case of (122) and is an immediate consequence of (118) in the case of (120).

What is particularly important about the concept of possibility is that much of the knowledge on which human decision-making is based is in reality possibilistic rather than probabilistic in nature. Thus, if X is a variable which takes the values $x_1,\ldots,x_n$ with respective probabilities $p_1,\ldots,p_n$ and possibilities $\mu_1,\ldots,\mu_n$, then, in practice, one is much more likely to know--or be given--the μ's rather than the p's. In many cases, the distinction between the two is not clearly understood, so that any collection of data, regardless of whether it is possibilistic or probabilistic, is treated as if it were probabilistic in nature. However, as the foregoing analysis shows, the manipulation of possibilities calls for rules that are quite different from those that apply to probabilities. Thus, in any realistic application of decision analysis, it is essential to differentiate between probabilities and possibilities and treat them by different methods.

Although in principle there is no connection between probabilities and possibilities, in practice the knowledge of possibilities conveys some information about the probabilities but not vice-versa. Certainly, if an event is impossible then it is also improbable. However, it is not true that an event which is possible is also probable. This rather weak connection between the two may be stated more precisely in the form of the _possibility/ probability consistency principle_, namely:

If X is a variable which takes the values $x_1,\ldots,x_n$ with probabilities $p_1,\ldots,p_n$ and possibilities $\mu_1,\ldots,\mu_n$, respectively, then the _degree of consistency_ of the probabilities $p_1,\ldots,p_n$ with the possibilities $\mu_1,\ldots,\mu_n$ is given by

$$\rho = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_n p_n \,. \tag{123}$$

Intuitively, (123) means that, in order to be consistent with μ's, high probabilities should not be assigned to those values of X which are associated with low degrees of possibility.

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