LINGUISTIC CHARACTERIZATION OF PREFERENCE RELATIONS

AS A BASIS FOR CHOICE IN SOCIAL SYSTEMS

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L.A. Zadeh*

1. Introduction

In assessing the applicability of the theories of choice to social systems, one cannot escape the fact that such systems are generally much too complex and much too ill-defined to be susceptible of analysis in precise, quantitative terms. For example, as discussed in [1], [2], there is considerable uncertainty in the valuation of interpersonal as well as individual utilities among a collection of individuals. Another complicating factor is that the preference relations are frequently conditioned on variables whose values are unknown or, at least, not well-defined; in addition, they may be, and frequently are, interdependent in the sense that the preference relation of an individual may be affected by his or her perception of the preference relations of other members of the collection. Furthermore, the underlying decision processes are, in most cases, multi-stage processes with poorly defined horizons, uncertain dynamics and vague constraints. These are but a few of the many considerations which suggest that, in their present form, the mathematical theories of choice may well be excessively precise in relation to the overwhelming complexity of real-world social systems.

A less precise alternative to the conventional methods of quantitative analysis is provided by the so-called linguistic approach [3], [4], in which words rather than numbers are employed to characterize approximately the values of

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variables as well as the relations between them. As a matter of fact, we frequently resort to such characterizations in everyday discourse, e.g., when we describe the age of an individual by a label such as young, not young, very young, not very young, etc. rather than by a number in the set $U = \{0,1,2,\ldots,100\}$. In this case, the labels in question may be interpreted as the names of fuzzy subsets of the set $U = [0,100]$, with a fuzzy subset such as <u>young</u>, characterized by its membership (or compatibility) function $\mu_{young}: U \to [0,1]$, which associates with each numerical age u the degree, $\mu_{young}(u)$, to which u is subjectively compatible (in a given context) with one's perception of the meaning of young.[1] For example, the value of the compatibility function μ_{young} at $u = 20$ might be 1; at $u = 25$: 0.9; at $u = 28$: 0.7; at $u = 30$: 0.5, etc., meaning that the subjective compatibilities of the numerical ages 20, 25, 28 and 30 with young are 1, 0.9, 0.7 and 0.5, respectively.

If <u>Age</u> is regarded as a <u>linguistic variable</u> whose <u>linguistic values</u> are young, not young, very young, etc., the meaning of each such value may be defined by specifying its compatibility function. However, a basic assumption behind the concept of a linguistic variable is that the meaning of its linguistic values may be computed in terms of the specified meaning of the <u>primary terms</u>, e.g., young in the case of <u>Age</u> [4]. For example, if the meaning of young is defined by its compatibility function μ_{young}, then the meaning of not young, very young, and not very young would be expressed as

$$\mu_{not\ young} = 1 - \mu_{young} \tag{1.1}$$

$$\mu_{very\ young} = \mu^2_{young} \tag{1.2}$$

$$\mu_{not\ very\ young} = 1 - \mu^2_{young} \tag{1.3}$$

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[1] For convenience of the reader, a summary of the pertinent properties of fuzzy sets is presented in the Appendix. More detailed discussions may be found in the books by Kaufmann [5] and Negoita-Ralescu [6].

in which the squaring operation has its usual meaning. Thus, if $\mu_{young}(28) = 0.7$, then $\mu_{not\ young}(28) = 0.3$; $\mu_{very\ young}(28) = 0.49$ and $\mu_{not\ very\ young}(28) = 0.51$.

To relate the linguistic approach to the characterization of preference relations in social systems, let $I = \{I_1,\ldots,I_N\}$ be a collection of individuals and let $A = \{a_1,\ldots,a_n\}$ be a set of alternatives. We assume that the preference relation for I_i, $i = 1,\ldots,N$, is a fuzzy relation R_i in A whose compatibility function, $\mu_i: A \times A \to [0,1]$, defines the "strength" of the preference of I_i for alternative a_k over alternative a_ℓ, $k,\ell = 1,\ldots,n$.

Our basic premise is that the available information about the preference relations $R_1,\ldots,R_N$ is fuzzy in nature and that it is expressed as a collection of propositions exemplified by:

1. Preference of I_1 for a_5 over a_3 is strong.
2. Preference of I_1 for a_2 over a_5 is not very strong.
3. Preference of I_1 for a_2 over a_3 is weak.
4. Preference of I_1 for a_5 over a_4 is much stronger than the preference of I_1 for a_5 over a_3.
5. Preference of I_1 for a_5 over a_3 is very strong is more or less true.
6. Preference of I_1 for a_5 over a_3 is very strong, is very probable.
7. Preference of I_1 for a_5 over a_3 is very strong is slightly possible.
8. Preference of most individuals for a_5 over a_3 is very strong.

9. If the preference of I_1 for a_5 over a_3 is strong then the preference of I_2 for a_5 over a_3 is very strong.

10. If the preference of many individuals for a_5 over a_3 is strong then the preference of I_3 for a_2 over a_3 is weak.

Examples 1, 2 and 3 are intended to indicate that in the linguistic characterization of preference relations the strength of preference is treated as a linguistic variable whose values are labeled strong, not strong, very strong, not very strong, weak, etc., with the understanding that each of these values denotes a fuzzy subset of the unit interval [0,1]. Furthermore, among these values strong plays the role of the primary term. Thus, if the compatibility function of strong is $\mu_{strong}: [0,1] \to [0,1]$, then

$$\mu_{not\ strong} = 1 - \mu_{strong} \tag{1.4}$$

$$\mu_{very\ strong} = \mu^2_{strong} \tag{1.5}$$

$$\mu_{not\ very\ strong} = 1 - \mu^2_{strong} \tag{1.6}$$

and

$$\mu_{weak}(v) = \mu_{strong}(1-v) , \quad v \in [0,1] \tag{1.7}$$

where (1.7) signifies that weak is the reverse -- rather than the negation -- of strong.

Example 4 illustrates a linguistic characterization of a relative strength of preference for the same individual, with the understanding that similar comparisons may be made for different individuals.[2]

Examples 5, 6 and 7 illustrate, respectively, the truth qualification, probability qualification and possibility qualification of the proposition "Preference of I_1 for a_5 over a_3 is very strong."

[2]The troublesome aspects of interpersonal comparisons are not at issue here [24].

Example 8 illustrates the use of linguistic quantifiers (e.g., most, few, many, all, some, not very many, etc.) to characterize the proportion of individuals who have a particular preference.

Example 9 illustrates the conditional composition of two propositions, namely, "Preference of I_1 for a_5 over a_3 is strong," and "Preference of I_2 for a_5 over a_3 is very strong."

Example 10, like Example 9, illustrates the conditional composition of two propositions, the first of which involves an assertion concerning the preference profile $\{R_1,\ldots,R_N\}$.

In aggregate, the above examples illustrate the manner in which the imprecise information concerning the preference relations of a collection of individuals may be expressed in the form of a set of linguistic propositions. The question, then, is: What is the meaning of such propositions and how can one infer other propositions from them?

To answer this question in general terms, it is necessary to specify, first, a grammar which can generate syntactically correct propositions of the type exemplified above. Second, a system of semantic rules for translating any proposition which can be generated by the grammar into a procedure for computing the compatibility function of the proposition in question. And third, a set of inference rules for deriving a consequent proposition from a set of premises.

In what follows, we shall employ a less formal approach which is adequate for the purposes of our analysis. As will be seen in the sequel, a key to the interpretation of linguistic propositions concerning preference relations and, more generally, fuzzy orderings, is provided by the concept of a possibility distribution of a fuzzy variable. We shall discuss this concept in Section 3, following a brief review of those aspects of fuzzy relations which will be needed in later sections.

The present paper has the limited objective of suggesting the possibility of applying the linguistic approach to the characterization of preference relations when the information about such relations is incomplete, imprecise and unreliable. We do not address ourselves to the important issue of how to derive a social preference relation from an imprecisely defined collection of linguistic preference relations, for this would require an extensive reformulation of the axiomatic basis of the theory of choice and collective behavior in the setting of the conceptual framework of fuzzy -- rather than two-valued -- logic [7].

2. Fuzzy Orderings

Our concern in this section is restricted to those aspects of fuzzy orderings which are of relevance to the linguistic characterization of preference relations. A more detailed discussion of the properties of various types of fuzzy orderings may be found in [8].

A fuzzy relation, R, in U is a fuzzy subset of $U \times U$. The membership (or compatibility) function of R is a mapping $\mu_R: U \times U \to [0,1]$, with $\mu_R(u,v)$, $(u,v) \in U \times U$, representing the strength of the relation between u and v. In the following definitions, the symbols $\vee$ and $\wedge$ denote max (or Sup) and min (or Inf), respectively, and $\triangleq$ stands for "is defined to be" or "is equal by definition."

The height of R is defined by

$$\text{Height}(R) \triangleq \bigvee_u \bigvee_v \mu_R(u,v) \ . \tag{2.1}$$

A fuzzy relation, R, is subnormal if $\text{Height}(R) < 1$ and normal if $\text{Height}(R) = 1$.

If R and Q are fuzzy relations in U, their composition, or more specifically, max-min composition is denoted by $R \circ Q$ and is defined by

$$\mu_{R \circ Q}(u,w) = \bigvee_v \left(\mu_R(u,v) \wedge \mu_Q(v,w)\right) , \quad u,v,w \in U . \tag{2.2}$$

Thus, if U is a finite set, $U = \{u_1,\ldots,u_n\}$ (e.g., $U \triangleq A$ = a finite set of alternatives) and R and Q are represented by their relation matrices in which the ij^{th} elements are $\mu_R(u_i,u_j)$ and $\mu_Q(u_i,u_j)$, respectively, then the relation matrix for $R \circ Q$ is the max-min product of the relation matrices for R and Q. An n-fold composition of R with itself is denoted by R^n.

In some cases it is desirable to employ an operation $*$ other than $\wedge$ in the definition of the composition. Assuming that $*$ is associative and monotone nondecreasing in each of its arguments, the definition of max-star composition becomes

$$\mu_{R \circ Q}(u,w) = \bigvee_v \left(\mu_R(u,v) * \mu_Q(v,w)\right) \tag{2.3}$$

and, in particular, if $*$ is taken to be the product, we have

$$\mu_{R \circ Q}(u,w) = \bigvee_v \left(\mu_R(u,v) \cdot \mu_Q(v,w)\right) . \tag{2.4}$$

Unless stated to the contrary, it will be assumed that $\circ$ is defined by (2.2).

A fuzzy relation is transitive iff

$$R \supset R \circ R \tag{2.5}$$

where the containment of fuzzy relations is defined by (A27). In more intuitive terms, R is transitive iff for any u, v, w in U

Strength of the relation between u and w

$\geq$ Strength of the relation between u and v or v and w. (2.6)

In the case of max-product transitivity, however, (2.6) becomes

Strength of the relation between u and w

$\geq$ Product of the strength of the relation between u (2.7)

and v and the strength of the relation between v and w.

Note that (2.7) is implied by (2.6).

The transitive closure, $\bar{R}$, of R is the smallest transitive relation which contains R. Equivalently, $\bar{R}$ may be expressed as

$$\bar{R} = R + R^2 + \cdots + R^n \tag{2.8}$$

where + denotes the union. The well-known Warshall's algorithm for the computation of the transitive closure of a nonfuzzy relation may readily be extended to the computation of the right-hand member of (2.8) [9], [10].

A fuzzy relation, R, is <u>reflexive</u> if

$$\mu_R(u,u) = 1 , \quad u \in U ; \tag{2.9}$$

it is <u>symmetric</u> if

$$\mu_R(u,v) = \mu_R(v,u) , \quad u,v \in U ; \tag{2.10}$$

and <u>antisymmetric</u> if

$$\mu_R(u,v) > 0 \text{ and } \mu_R(v,u) > 0 \Rightarrow u = v , \quad u,v \in U . \tag{2.11}$$

A fuzzy relation, R, is a <u>fuzzy ordering</u> if it is transitive. In particular, R is a <u>fuzzy preordering</u> if it is reflexive and transitive, and a <u>fuzzy partial ordering</u> if it is reflexive, transitive and antisymmetric.

A fuzzy ordering is a similarity relation if it is reflexive, transitive and symmetric. A similarity relation may be viewed as a generalization to fuzzy relations of the concept of an equivalence relation.

It should be noted that if transitivity is interpreted in the max-product sense, a similarity relation may serve as an indifference relation without entailing the usual difficulties associated with the notion of transitivity of indifference relations [11], [12], [13]. For example, if U is the real line, a transitive indifference relation may be defined by [8]

$$\mu_R(u,v) = e^{-\beta|u-v|} \tag{2.12}$$

where β is a positive constant.

In Section 3, our concern will be with preference relations in which the membership function ranges over the fuzzy subsets of $[0,1]$, that is, over fuzzy sets of Type 2 (see Appendix). Such subsets will be identified by the labels strong, not strong, very strong, not very strong, etc. and will be regarded as the values of the linguistic variable Strength. A simple example of a linguistic relation of this type is shown in Table 1.

R	a_1	a_2	a_3
a_1	1	strong	very strong
a_2	0	1	strong
a_3	0	0	1

Table 1. Relation matrix for a linguistic relation.

In this example, the entry in (1,2) signifies that the strength of preference for a_2 over a_1 is strong. Similarly, the strength of preference for a_3 over a_1 is very strong.

In what sense can a relation of this type be said to be transitive? To extend the definition of transitivity to linguistic relations it is convenient to employ the extension principle (see Appendix), which allows the domain of definition of a function or a relation to be extended to the set of fuzzy subsets of the space on which they are defined. For example, if $U = \{u_1, \ldots, u_n\}$ is a subset of points on the real line, and F and G are fuzzy subsets of U defined by

$$F = \mu_1/u_1 + \cdots + \mu_k/u_k \tag{2.13}$$

$$G = \nu_1/u_1 + \cdots + \nu_k/u_k \tag{2.14}$$

where the μ_i and ν_i are the grades of membership of u_i in F and G, respectively, then the extensions of $\wedge$ (min) and $\vee$ (max) to the fuzzy subsets of U may be expressed as

$$F \wedge G = \sum_{i,j} \mu_i \wedge \nu_j / u_i \wedge u_j \tag{2.15}$$

$$F \vee G = \sum_{i,j} \mu_i \wedge \nu_j / u_i \vee u_j \ . \tag{2.16}$$

These definitions entail the extension of the inequality $\geq$ which is expressed by

$$F \geq G \quad \text{iff} \quad F \wedge G = G \ . \tag{2.17}$$

In terms of (2.15) and (2.16), the composition of linguistic relations may be expressed, as before, by (2.2), with the understanding that the μ's in (2.2) are fuzzy sets and that $\wedge$ and $\vee$ are defined by (2.15) and (2.16), respectively. Likewise, the definitions of transitivity, (2.5) and (2.6), remain unchanged on the understanding that $\geq$ is defined by (2.17).

As a simple example, in the case of the linguistic relation defined by Table 1, it is easy to verify that (see (3.20) for the expression for very strong)

$$\text{strong} \wedge \text{very strong} = \text{strong} \quad (2.18)$$

and hence

$$\text{strong} \leq \text{very strong} . \quad (2.19)$$

Using (2.18) and (2.19) in forming the composition of R with itself, we find that $R^2 = R$ and hence that R, as defined by Table 1, is transitive.

In a similar fashion, it is possible to extend to linguistic relations many of the other basic concepts pertaining to fuzzy relations in which the membership function takes values in the interval [0,1]. We shall not dwell upon this subject, however, and, in the next section, will turn our attention to another important issue, namely, the translation of linguistic propositions concerning preference relations and their aggregates.

3. Translation Rules for Linguistic Propositions

Our main concern in this section is with the interpretation of linguistic propositions relating to a collection of fuzzy orderings. As was stated in the Introduction, a concept that plays a basic role in the translation of linguistic propositions is that of the possibility distribution of a fuzzy variable.[3] More specifically, let X be a variable which takes values in $U = \{u\}$ and let F be a fuzzy subset of U whose membership function is given by $\mu_F: U \rightarrow [0,1]$. Then, a proposition, p, of the form

$$X \text{ is } F \quad (3.1)$$

[3] A more detailed discussion of the concept of a possibility distribution may be found in [14].

has the effect of associating with X a <u>possibility distribution</u> Π_X which is equal to F, that is,

$$\Pi_X = F \,, \tag{3.2}$$

and a <u>possibility distribution function</u> π_X which is given by

$$\pi_X = \mu_F \,. \tag{3.3}$$

Thus, the proposition $p \triangleq X$ is F implies that the possibility that X can take a value $u \in U$ is $\mu_F(u)$, and, more generally, that the possibility that $X \in G$, where G is a subset of U, is given by

$$\text{Poss}\{X \in G\} = \sup_{u \in G} \mu_F(u) \,. \tag{3.4}$$

When G is a fuzzy subset of U, it is not meaningful to speak of the possibility of X belonging to G. In this case, $X \in G$ is replaced by the proposition X is G, and (3.4) becomes

$$\text{Poss}\{X \text{ is } G\} = \sup_{u \in U} \mu_F(u) \wedge \mu_G(u) \tag{3.5}$$

where μ_G is the membership function of G. Thus, we have

$$X \text{ is } F \Rightarrow \text{Poss}\{X \text{ is } G\} = \sup_{u \in U} \mu_F(u) \wedge \mu_G(u) \tag{3.6}$$

$$= \text{Height}(F \cap G) \,.$$

As a simple illustration, let U be the universe of positive integers and let F be the fuzzy subset of small integers defined by

$$\text{small integer} = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 + 0.2/6 \,. \tag{3.7}$$

Then, the proposition $p \triangleq X$ is a small integer associates with X the possibility distribution

$$X = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 + 0.2/6 \qquad (3.8)$$

in which a term such as 0.8/3 signifies that the possibility that X is 3, given that X is a small integer, is 0.8. Furthermore, the possibility that $X \in \{4,5\}$ is 0.6 and the possibility that X is a very small integer is 1.

In essence, the possibility distribution, Π_X, which is associated with X may be interpreted as an elastic restraint on the values that may be assigned to X, with $\pi_X(u)$ representing the degree of ease with which X can take the value u. In this sense, a variable which is associated with a possibility distribution is a fuzzy variable, with Π_X playing the role of a fuzzy restriction on the values of X.

To clarify the distinction between possibility and probability distributions, assume that X is the number of passengers that can be put in a given car, say a VW. Then, by some specified or unspecified criterion, the possibilities associated with the values of X might be as follows:

$$\pi_X(1) = \pi_X(2) = \pi_X(3) = \pi_X(4) = 1 \ ;$$
$$\pi_X(5) = 0.8 \ ;$$
$$\pi_X(6) = 0.4 \ ;$$
$$\pi_X(7) = 0.2 \ .$$

In general, the probability that X passengers might be carried in the car in question would be quite different from the possibility that X passengers could be put in it. For example, the probability that four passengers might be carried could be quite small, say 0.05, whereas the corresponding possibility is 1.

What is important about the concept of a possibility distribution is that much of human decision-making appears to be based on possibilistic rather than probabilistic information. In particular, as is pointed out in [14], the imprecision of natural languages is, for the most part, possibilistic in origin. Indeed, this is the main reason why the concept of a possibility distribution plays a basic role in the translation of linguistic propositions.

Translation Rules for Linguistic Propositions

Let X be a variable taking values in $U = \{u\}$, and let F be a fuzzy subset of U. By the translation of the proposition $p \triangleq X$ is F is meant the relation

$$X \text{ is } F \rightarrow \Pi_X = F \qquad (3.9)$$

whose right-hand member is the possibility association equation (3.2). Thus, the translation of a proposition has the form of a possibility association equation or, more generally, a set of such equations.

As an illustration, consider the proposition

$$p \triangleq \text{Preference of } I_1 \text{ for } a_3 \text{ over } a_5 \text{ is very strong} \qquad (3.10)$$

which for simplicity will be abbreviated to

$$p \triangleq \text{Strength is very strong} \qquad (3.11)$$

with Strength playing the role of a linguistic variable. Then by (3.9) the translation of p may be expressed as

$$\text{Strength is very strong} \rightarrow \Pi_{Strength} = \text{very strong} \qquad (3.12)$$

where very strong is a fuzzy subset of the unit interval $U = [0,1]$.

The translation rules of interest to us are conditional in nature in the sense that they are of the form

$$\begin{array}{ll} \text{If} & p \rightarrow \Pi_X = F \\ \text{then} & M(p) \rightarrow M^+(\Pi_X = F) \end{array} \qquad (3.13)$$

where $M(p)$ is a modification of p and M^+ is a corresponding modification of the possibility association equation $\Pi_X = F$.

A basic rule of this type is the modifier rule [7], which may be stated as follows.[4]

Modifier Rule

$$\begin{array}{ll} \text{If} & \text{Strength is } F \rightarrow \Pi_{\text{Strength}} = F \\ \text{then} & \text{Strength is } mF \rightarrow \Pi_{\text{Strength}} = F^+ \end{array} \qquad (3.14)$$

where m is a modifier such as not, very, more or less, etc., and F^+ is a modification of F induced by m.[4] More specifically,

$$\text{If } m = \text{not}, \text{ then } F^+ = F' \triangleq \text{complement of } F \qquad (3.15)$$

$$\text{If } m = \text{very}, \text{ then } F^+ = F^2 \qquad (3.16)$$

$$\text{If } m = \text{more or less}, \text{ then } F^+ = \sqrt{F}\ . \qquad (3.17)$$

As an illustration, assume that strong is a fuzzy subset of [0,1] which is characterized by

$$\mu_{strong} = S(0.7, 0.8, 0.9) \qquad (3.18)$$

where the S-function (with its argument suppressed) is defined by (A17). Then, by (3.16), we have

[4] A more detailed discussion of the effect of modifiers (or hedges) may be found in [15], [16], [17] and [18].

If $\quad$ Strength is strong $\longrightarrow \Pi_{Strength}$ = strong $\qquad$ (3.190

then $\quad$ Strength is very strong $\longrightarrow \Pi_{Strength}$ = strong^2

where

$$\mu_{strong^2} = (S(0.7, 0.8, 0.9))^2 \ . \qquad (3.20)$$

The implication of this translation of the proposition $p \triangleq$ Strength is very strong is the following. On evaluating the right-hand member of (3.20) for, say, $v = 0.85$, we find

$$\mu_{strong}(0.85) \cong 0.88 \qquad (3.21)$$

which implies that the possibility that the strength of preference of I_1 for a_3 over a_5 is 0.85 is 0.88. It is in this sense, then, that the proposition in question translates into a possibility distribution over the numerical values of the strength of preference of I_1 for a_3 over a_5.

In a similar fashion, it is readily seen that in virtue of (1.7), we have

If $\quad$ Strength is strong $\longrightarrow \Pi_{Strength}$ = strong $\qquad$ (3.22)

then $\quad$ Strength is weak $\longrightarrow \Pi_{Strength}$ = strong$^+$

where

$$\mu_{strong^+} = 1 - S(0.1, 0.2, 0.3) \ . \qquad (3.23)$$

Thus, in this case the possibility that the strength of preference of I_1 for a_3 over a_5 is 0.85 is zero.

Compositional Rules

Compositional rules apply to the translation of a proposition p which is a composition of propositions q and r. The most commonly employed modes of composition are: conjunction, disjunction and implication. The translation rules for these modes of composition are as follows.

Let X and Y be variables taking values in U and V, respectively, and let F and G be fuzzy subsets of U and V. If

$$X \text{ is } F \rightarrow \Pi_X = F \qquad (3.24)$$

and

$$Y \text{ is } G \rightarrow \Pi_Y = G \qquad (3.25)$$

then

(a) $$X \text{ is } F \text{ and } Y \text{ is } G \rightarrow \Pi_{(X,Y)} = F \times G \qquad (3.26)$$

(b) $$X \text{ is } F \text{ or } Y \text{ is } G \rightarrow \Pi_{(X,Y)} = \bar{F} + \bar{G} \qquad (3.27)$$

and

(c) $$\text{If } X \text{ is } F \text{ then } Y \text{ is } G \rightarrow \Pi_{(X,Y)} = \bar{F}' \oplus \bar{G} \qquad (3.28)$$

where $\Pi_{(X,Y)}$ is the possibility distribution of the binary variable (X,Y). Furthermore, in the <u>conjunctive rule</u> expressed by (a) $F \times G$ denotes the cartesian product of F and G (see (A56)); in the <u>disjunctive rule</u> expressed by (b), $\bar{F}$ and $\bar{G}$ are the cylindrical extensions (see (A59)) of F and G, respectively, and + denotes the union; and in the <u>conditional rule</u>, expressed by (c) $\bar{F}'$ is the cylindrical extension of the complement of F and $\oplus$ denotes the bounded sum (see (A30)).

As an illustration, by applying the modifier rule and the conjunctive rule in combination, we obtain the following result.

If $\quad$ Strength is strong $\rightarrow \Pi_{Strength}$ = strong $\quad$ (3.29)

and

$$\text{Strength is weak} \rightarrow \Pi_{Strength} = \text{weak} \tag{3.30}$$

then

$$\text{Strength is not very strong and not very weak} \rightarrow \Pi_{Strength} = (\text{strong}^2)' \cap (\text{weak}^2)' \tag{3.31}$$

where weak is the reverse of strong (see (1.7)) and $\cap$ denotes the intersection.

As a further illustration, consider the proposition "If the preference of I_1 for a_3 over a_5 is strong then the preference of I_2 for a_2 over a_3 is very strong." On abbreviating this proposition to "If Strength_1 is strong then Strength_2 is very strong," and applying (3.28), we deduce as its translation the possibility association equation

$$\Pi_{(\text{Strength}_1,\text{Strength}_2)}(v_1,v_2) = (1 - S(v_1;0.7,0.8,0.9)) \oplus (S(v_2;0.7,0.8,0.9))^2 \tag{3.32}$$

where v_1, $v_2 \in [0,1]$, the S-function is defined by (A17), and $\pi_{(\text{Strength}_1,\text{Strength}_2)}$ denotes the possibility distribution function of the linguistic variables Strength_1 and Strength_2.

Quantifier Rule

The quantifier rule applies to propositions of the general form

$$p = QX \text{ are } F \tag{3.33}$$

where Q is a linguistic quantifier (e.g., most, many, few, etc.), X is a variable taking values in U, and F is a fuzzy subset of U. In the context of preference relations, a typical instance of (3.33) might be:

$$p \triangleq \text{Most individuals have very strong preference for } a_5 \text{ over } a_3 \quad (3.34)$$

which for simplicity will be abbreviated to

$$p \triangleq \text{Most Strengths are very strong .} \quad (3.35)$$

In general terms, let μ_Q be the membership function of Q and let μ_F be that of F. It should be observed that when Q relates to a proportion, as in the case of most, μ_Q is a mapping from $[0,1]$ to $[0,1]$, while μ_F is a mapping from U to $[0,1]$. In the case of preference relations, however, $U = [0,1]$ and thus μ_F, like μ_Q, is a mapping from $[0,1]$ to $[0,1]$.

Since a fuzzy set does not have sharply defined boundaries, the concept of the cardinality of a fuzzy set does not have a unique natural meaning. For many purposes, however, the concept of the power of a fuzzy set [19] may be used as a suitable measure of the number of elements in such a set. Thus, if $U = \{u_1,\ldots,u_N\}$, then the power of F is defined by

$$|F| \triangleq \sum_{i=1}^{N} \mu_F(u_i) \quad (3.36)$$

where $\mu_F(u_i)$, $i = 1,\ldots,N$, is the grade of membership of u_i in F. For example, if

$$F = 0.8/u_1 + 0.9/u_2 + 0.6/u_3 + 0.8/u_4 \quad (3.37)$$

then

$$|F| = 3.1 \ . \quad (3.38)$$

For some applications, it is necessary to eliminate from the count those elements of F whose grade of membership falls below a specified threshold (which may be fuzzy). This is equivalent to replacing F in (3.36) with $F \cap \Gamma$, where Γ is a fuzzy or nonfuzzy set which induces the desired threshold.

Using (3.36), for simplicity, the quantifier rule may be expressed as follows.

$$\text{If} \quad U = \{u_1,\ldots,u_N\} \text{ and } X \text{ is } F \rightarrow \Pi_X = F \tag{3.39}$$

$$\text{then} \quad QX \text{ are } F \rightarrow \Pi_{|F|} = Q \;;$$

and if Q is a proportional quantifier

$$QX \text{ are } F \rightarrow \Pi_{|F|/N} = Q \;. \tag{3.40}$$

As a simple illustration, consider the proposition

$$p \triangleq \text{Most Strengths are very strong} \tag{3.41}$$

where

$$\mu_{most} = S(0.6,\ 0.7,\ 0.8) \tag{3.42}$$

and

$$\mu_{strong} = S(0.7,\ 0.8,\ 0.9) \;. \tag{3.43}$$

On applying (3.40), (3.36) and (3.20), we obtain as the translation of (3.41)

$$\Pi_{(S^2(v_1;0.7,0.8,0.9)+\cdots+S^2(v_N;0.7,0.8,0.9))/N} = S(0.6,\ 0.7,\ 0.8) \tag{3.44}$$

where $S^2(v_i;0.7,0.8,0.9)$ is the compatibility of the strength of preference of I_i with very strong, $S(0.6, 0.7, 0.8)$ is the membership function of most, and the argument of Π is the number of individuals whose preference is very strong.

Truth Qualification, Probability Qualification and Possibility Qualification

In natural languages, an important mechanism for the modification of the meaning of a proposition is provided by the adjunction of three types of qualifiers: (i) is τ, where τ is a linguistic truth-value, e.g., true, very true, more or less true, false, etc.; (ii) is λ, where λ is a linguistic probability-value (or likelihood), e.g., probable, very probable, very improbable, etc.; and (iii) is π, where π is a linguistic possibility-value, e.g., possible, quite possible, slightly possible, impossible, etc. The rules governing these qualifications may be stated as follows.

Truth qualification: If

$$X \text{ is } F \rightarrow \Pi_X = F \tag{3.45}$$

then

$$X \text{ is } F \rightarrow \Pi_X = F^+$$

where

$$\mu_{F^+}(u) = \mu_\tau(\mu_F(u)) \,, \qquad u \in U \; ; \tag{3.46}$$

μ_τ and μ_F are the membership functions of τ and F, respectively, and U is the universe of discourse associated with X. As an illustration, if strong is defined by (3.18); $\tau \triangleq$ very true is defined by

$$\text{very true} = S^2(0.6,\ 0.8,\ 1) \tag{3.47}$$

and

$$\text{Strength is strong} \rightarrow \Pi_{Strength} = \text{strong} \tag{3.48}$$

then

$$\mu_{strong^+}(u) = S^2(1 - S(u;\ 0.7,\ 0.8,\ 0.9);\ 0.6,\ 0.8,\ 1)\ , \qquad u \in U\ .$$

Probability qualification: If

$$X \text{ is } F \rightarrow \Pi_X = F \tag{3.49}$$

then

$$X \text{ is } F \text{ is } \lambda \rightarrow \Pi_{\int_U p(u)\mu_F(u)du} = \lambda \tag{3.50}$$

where $p(u)du$ is the probability that the value of X falls in the interval $(u,u+du)$; the integral

$$\int_U p(u)\mu_F(u)du \tag{3.51}$$

is the probability of the fuzzy event F [20]; and λ is a linguistic probability-value. Thus, (3.50) defines a possibility distribution of probability distributions, with the possibility of a probability density $p(\cdot)$ given by

$$\pi(p(\cdot)) = \mu_\lambda\left(\int_U p(u)\mu_F(u)du\right)\ . \tag{3.52}$$

As an illustration, consider the proposition $p \triangleq$ Strength is strong is very probable, in which strong is defined by (3.18) and

$$\mu_{very\ probable} = S^2(0.6,\ 0.8,\ 1)\ . \tag{3.53}$$

Then

$$\pi(p(\cdot)) = S^2\left(\int_0^{100} p(u)\left(1 - S(u;\ 0.7,\ 0.8,\ 0.9)\right)du;\ 0.6,\ 0.8,\ 1\right). \tag{3.54}$$

Possibility qualification: If

$$X \text{ is } F \rightarrow \Pi_X = F \tag{3.55}$$

then

$$X \text{ is } F \text{ is possible} \rightarrow \Pi_X = F^+$$

in which

$$F^+ = F \oplus \Pi \tag{3.56}$$

where Π is a fuzzy set of Type 2 defined by

$$\mu_\Pi(u) = [0,1] \ , \quad u \in U \ , \tag{3.57}$$

and $\oplus$ is the bounded sum defined by (A30). Equivalently,

$$\mu_{F^+}(u) = [\mu_F(u),1] \ , \quad u \in U \ , \tag{3.58}$$

which defines μ_{F^+} as an interval-valued membership function.

In effect, the rule in question signifies that possibility qualification has the effect of weakening the proposition which it qualifies through the addition to F of a possibility distribution Π which represents total indeterminacy in the sense that the degree of possibility which it associates with each point in U may be any number in the interval $[0,1]$.

The rules formulated above may be applied in combination, thus making it possible to translate fairly complex propositions regarding preference relations and their aggregates. More importantly, however, the translation of a linguistic proposition into a possibility association equation or a set of such equations provides a basis for inference from such propositions as well as the formulation of fuzzy algorithms or programs for the characterization of social welfare functions. These issues lie beyond the scope of the present paper and will not be considered here.

Appendix

Fuzzy Sets -- Notation, Terminology and Basic Properties

The symbols $U,V,W,\ldots,$ with or without subscripts, are generally used to denote specific universes of discourse, which may be arbitrary collections of objects, concepts or mathematical constructs. For example, U may denote the set of all real numbers; the set of all residents in a city; the set of all sentences in a book; the set of all colors that can be perceived by the human eye, etc.

Conventionally, if A is a fuzzy subset of U whose elements are $u_1,\ldots,u_n$, then A is expressed as

$$A = \{u_1,\ldots,u_n\} \ . \tag{A1}$$

For our purposes, however, it is more convenient to express A as

$$A = u_1 + \cdots + u_n \tag{A2}$$

or

$$A = \sum_{i=1}^{n} u_i \tag{A3}$$

with the understanding that, for all i, j,

$$u_i + u_j = u_j + u_i \tag{A4}$$

and

$$u_i + u_i = u_i \ . \tag{A5}$$

As an extension of this notation, a finite <u>fuzzy</u> subset of U is expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A6}$$

or, equivalently, as

$$F = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{A7}$$

where the μ_i, $i = 1,\ldots,n$, represent the <u>grades of membership</u> of the u_i in F. Unless stated to the contrary, the μ_i are assumed to lie in the interval [0,1], with 0 and 1 denoting <u>no</u> membership and <u>full</u> membership, respectively.

Consistent with the representation of a finite fuzzy set as a linear form in the u_i, an arbitrary fuzzy subset of U may be expressed in the form of an integral

$$F = \int_U \mu_F(u)/u \tag{A8}$$

in which $\mu_F\colon U \to [0,1]$ is the <u>membership</u> or, equivalently, the <u>compatibility function</u> of F; and the integral $\int_U$ denotes the union (defined by (A28)) of <u>fuzzy singletons</u> $\mu_F(u)/u$ over the universe of discourse U.

The points in U at which $\mu_F(u) > 0$ constitute the <u>support</u> of F. The points at which $\mu_F(u) = 0.5$ are the <u>crossover</u> points of F.

<u>Example A9</u>. Assume

$$U = a + b + c + d\ . \tag{A10}$$

Then, we may have

$$A = a + b + d \tag{A11}$$

and

$$F = 0.3a + 0.9b + d \tag{A12}$$

as nonfuzzy and fuzzy subsets of U, respectively.

If

$$U = 0 + 0.1 + 0.2 + \cdots + 1 \tag{A13}$$

then a fuzzy subset of U would be expressed as, say,

$$F = 0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1\ . \tag{A14}$$

If $U = [0,1]$, then F might be expressed as

$$F = \int_0^1 \frac{1}{1+u^2}/u \tag{A15}$$

which means that F is a fuzzy subset of the unit interval $[0,1]$ whose membership function is defined by

$$\mu_F(u) = \frac{1}{1+u^2} . \tag{A16}$$

In many cases, it is convenient to express the membership function of a fuzzy subset of the real line in terms of a standard function whose parameters may be adjusted to fit a specified membership function in an approximate fashion. Two such functions are defined below.

$$\begin{aligned} S(u;\alpha,\beta,\gamma) &= 0 && \text{for } u \le \alpha \\ &= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \text{for } \alpha \le u \le \beta \\ &= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for } \beta \le u \le \gamma \\ &= 1 && \text{for } u \ge \gamma \end{aligned} \tag{A17}$$

$$\begin{aligned} \pi(u;\beta,\gamma) &= S(u;\gamma-\beta,\gamma-\tfrac{\beta}{2},\gamma) && \text{for } u \le \gamma \\ &= 1 - S(u;\gamma,\gamma+\tfrac{\beta}{2},\gamma+\beta) && \text{for } u \ge \gamma . \end{aligned} \tag{A18}$$

In $S(u;\alpha,\beta,\gamma)$, the parameter β, $\beta = \frac{a+\gamma}{2}$, is the crossover point. In $\pi(u;\beta,\gamma)$, β is the bandwidth, that is the separation between the crossover points of π, while γ is the point at which π is unity.

In some cases, the assumption that μ_F is a mapping from U to $[0,1]$ may be too restrictive, and it may be desirable to allow μ_F to take values in a lattice or, more particularly, in a Boolean algebra. For most purposes, however, it is sufficient to deal with the first two of the following hierarchy of fuzzy sets.

Definition A19. A fuzzy subset, F, of U is of Type 1 if its membership function, μ_F, is a mapping from U to [0,1]; and F is of Type n, n = 2,3,..., if μ_F is a mapping from U to the set of fuzzy subsets of Type n-1. For simplicity, it will always be understood that F is of Type 1 if it is not specified to be of a higher type.

Example A20. Suppose that U is the set of all nonnegative integers and F is a fuzzy subset of U labeled small integers. Then F is of Type 1 if the grade of membership of a generic element u in F is a number in the interval [0,1], e.g.,

$$\mu_{\underline{\text{small integers}}}(u) = \left(1 + \left(\frac{u}{5}\right)^2\right)^{-1}, \quad u = 0,1,2,\dots \qquad \text{(A21)}$$

On the other hand, F is of Type 2 if for each u in U, $\mu_F(u)$ is a fuzzy subset of [0,1] of Type 1, e.g., for u = 10,

$$\mu_{\underline{\text{small integers}}}(10) = \underline{\text{low}} \qquad \text{(A22)}$$

where low is a fuzzy subset of [0,1] whose membership function is defined by, say,

$$\mu_{\underline{\text{low}}}(v) = 1 - S(v;0,0.25,0.5), \quad v \in [0,1] \qquad \text{(A23)}$$

which implies that

$$\underline{\text{low}} = \int_0^1 (1 - S(v;0,0.25,0.5))/v \,. \qquad \text{(A24)}$$

If F is a fuzzy subset of U, then its α-level-set, F_α, is a nonfuzzy subset of U defined by

$$F_\alpha = \{u \mid \mu_F(u) \geq \alpha\} \qquad \text{(A25)}$$

for $0 < \alpha \leq 1$.

If U is a linear vector space, the F is convex if and only if for all $\lambda \in [0,1]$ and all u_1, u_2 in U,

$$\mu_F(\lambda u_1 + (1-\mu)u_2) \geq \min(\mu_F(u_1), \mu_F(u_2)) \ . \qquad \text{(A26)}$$

In terms of the level-sets of F, F is convex if and only if the F_α are convex for all $\alpha \in (0,1]$.[5]

The relation of containment for fuzzy subsets F and G of U is defined by

$$F \subset G \Leftrightarrow \mu_F(u) \leq \mu_G(u) \ , \quad u \in U \ . \qquad \text{(A27)}$$

Thus, F is a fuzzy subset of G if (A27) holds for all u in U.

Operations on Fuzzy Sets

If F and G are fuzzy subsets of U, their union, $F \cup G$, intersection, $F \cap G$, bounded-sum, $F \oplus G$, and bounded-difference, $F \ominus G$, are fuzzy subsets of U defined by

$$F \cup G \triangleq \int_U \mu_F(u) \vee \mu_G(u)/u \qquad \text{(A28)}$$

$$F \cap G \triangleq \int_U \mu_F(u) \wedge \mu_G(u)/u \qquad \text{(A29)}$$

$$F \oplus G \triangleq \int_U 1 \wedge (\mu_F(u) + \mu_G(u))/u \qquad \text{(A30)}$$

$$F \ominus G \triangleq \int_U 0 \vee (\mu_F(u) - \mu_G(u))/u \qquad \text{(A31)}$$

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[5] This definition of convexity can readily be extended to fuzzy sets of Type 2 by applying the extension principle (see (A70)) to (A26).

where $\vee$ and $\wedge$ denote max and min, respectively. The complement of F is defined by

$$F' = \int_U (1 - \mu_F(u))/u \qquad \text{(A32)}$$

or, equivalently,

$$F' = U \ominus F \;. \qquad \text{(A33)}$$

It can readily be shown that F and G satisfy the identities

$$(F \cap G)' = F' \cup G' \qquad \text{(A34)}$$

$$(F \cup G)' = F' \cap G' \qquad \text{(A35)}$$

$$(F \oplus G)' = F' \ominus G \qquad \text{(A36)}$$

$$(F \ominus G)' = F' \oplus G \qquad \text{(A37)}$$

and that F satisfies the resolution identity

$$F = \int_0^1 \alpha F_\alpha \qquad \text{(A38)}$$

where F_α is the α-level-set of F; αF_α is a set whose membership function is $\mu_{\alpha F_\alpha} = \alpha \mu_{F_\alpha}$, and $\int_0^1$ denotes the union of the αF, with $\alpha \in (0,1]$.

Although it is traditional to use the symbol $\cup$ to denote the union of nonfuzzy sets, in the case of fuzzy sets it is advantageous to use the symbol $+$ in place of $\cup$ where no confusion with the arithmetic sum can result. This convention is employed in the following example, which is intended to illustrate (A28), (A29), (A30), (A31) and (A32).

Example A39. For U defined by (A10) and F and G expressed by

$$F = 0.4a + 0.9b + d \quad \text{(A40)}$$

$$G = 0.6a + 0.5b \quad \text{(A41)}$$

we have

$$F + G = 0.6a + 0.9b + d \quad \text{(A42)}$$

$$F \cap G = 0.4a + 0.5b \quad \text{(A43)}$$

$$F \oplus G = a + b + d \quad \text{(A44)}$$

$$F \ominus G = 0.4b + d \quad \text{(A45)}$$

$$F' = 0.6a + 0.1b + c \quad \text{(A46)}$$

The linguistic connectives and (conjunction) and or (disjunction) are identified with $\cap$ and $+$, respectively. Thus,

$$F \text{ and } G \triangleq F \cap G \quad \text{(A47)}$$

and

$$F \text{ or } G \triangleq F + G \ . \quad \text{(A48)}$$

As defined by (A47) and (A48), and and or are implied to be noninteractive in the sense that there is no "trade-off" between their operands. When this is not the case, and and or are denoted by and* and or* respectively, and are defined in a way that reflects the nature of the trade-off. For example, we may have

$$F \text{ and* } G \triangleq \int_U \mu_F(u)\mu_G(u)/u \quad \text{(A49)}$$

$$F \text{ or* } G \triangleq \int_U (\mu_F(u) + \mu_G(u) - \mu_F(u)\mu_G(u))/u \quad \text{(A50)}$$

whose $+$ denotes the arithmetic sum. In general, the interactive versions of and and or do not possess the simplifying properties of the connectives defined by (A47) and (A48), e.g., associativity, distributivity, etc.

If α is a real number, then F^α is defined by

$$F^\alpha \triangleq \int_U (\mu_F(n))^\alpha/u \; . \tag{A51}$$

For example, for the fuzzy set defined by (A40), we have

$$F^2 = 0.16a + 0.81b + d \tag{A52}$$

and

$$F^{1/2} = 0.63a + 0.95b + d \; . \tag{A53}$$

These operations may be used to approximate, very roughly, the effect of the linguistic modifiers <u>very</u> and <u>more or less</u>. Thus,

$$\underline{\text{very}}\ F \triangleq F^2 \tag{A54}$$

and

$$\underline{\text{more or less}}\ F \triangleq F^{1/2} \; . \tag{A55}$$

If $F_1,\ldots,F_n$ are fuzzy subsets of $U_1,\ldots,U_n$, then the <u>cartesian product</u> of $F_1,\ldots,F_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$F_1 \times \cdots \times F_n = \int_{U_1\times\cdots\times U_n} (\mu_{F_1}(u_1) \wedge \cdots \wedge \mu_{F_n}(u_n))/(u_1,\ldots,u_n) \; . \tag{A56}$$

As an illustration, for the fuzzy sets defined by (A40) and (A41), we have

$$\begin{aligned} F \times G &= (0.4a + 0.9b + d) \times (0.6a + 0.5b) \\ &= 0.4/(a,a) + 0.4/(a,b) + 0.6/(b,a) \\ &\quad + 0.5/(b,b) + 0.6/(d,a) + 0.5/(d,b) \end{aligned} \tag{A57}$$

which is a fuzzy subset of $(a+b+c+d) \times (a+b+c+d)$.

Fuzzy Relations

An n-ary fuzzy relation R in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$. The projection of R on $U_{i_1} \times \cdots \times U_{i_k}$, where $(i_1,\ldots,i_k)$ is a subsequence of $(1,\ldots,n)$, is a relation in $U_{i_1} \times \cdots \times U_{i_k}$ defined by

$$\text{Proj } R \text{ on } U_{i_1} \times \cdots \times U_{i_k} \triangleq \int_{U_{i_1} \times \cdots \times U_{i_k}} \vee_{u_{j_1},\ldots,u_{j_\ell}} \mu_R(u_1,\ldots,u_n)/(u_1,\ldots,u_n) \quad \text{(A58)}$$

where $(j_1,\ldots,j_\ell)$ is the sequence complementary to $(i_1,\ldots,i_k)$ (e.g., if $n = 6$ then $(1,3,6)$ is complementary to $(2,4,5)$), and $\vee_{u_{j_1},\ldots,u_{j_\ell}}$ denotes the supremum over $U_{j_1} \times \cdots \times U_{j_\ell}$.

If R is a fuzzy subset of $U_{i_1},\ldots,U_{i_k}$, then its cylindrical extension in $U_1 \times \cdots \times U_n$ is a fuzzy subset of $U_1 \times \cdots \times U_n$ defined by

$$\bar{R} = \int_{U_1 \times \cdots \times U_n} \mu_R(U_{i_1},\ldots,U_{i_k})/(u_1,\ldots,u_n) \ . \quad \text{(A59)}$$

In terms of their cylindrical extensions, the composition of two binary relations R and S (in $U_1 \times U_2$ and $U_2 \times U_3$, respectively) is expressed by

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3 \quad \text{(A60)}$$

where $\bar{R}$ and $\bar{S}$ are the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Similarly, if R is a binary relation in $U_1 \times U_2$ and S is a unary relation in U_2, their composition is given by

$$R \circ S = \text{Proj } R \cap \bar{S} \text{ on } U_1 \ . \quad \text{(A61)}$$

Example A62. Let R be defined by the right-hand member of (A57) and

$$S = 0.4a + b + 0.8d \,. \tag{A63}$$

Then

$$\text{Proj } R \text{ on } U_1 \ (\triangleq a+b+c+d) = 0.4a + 0.6b + 0.6d \tag{A64}$$

and

$$R \circ S = 0.4a + 0.5b + 0.5d \,. \tag{A65}$$

Linguistic Variables

Informally, a linguistic variable, χ, is a variable whose values are words or sentences in a natural or artificial language. For example, if age is interpreted as a linguistic variable, then its term-set, $T(\chi)$, that is, the set of linguistic values, might be

$$\begin{aligned} T(\text{age}) = {} & \text{young} + \text{old} + \text{very young} + \text{not young} \\ & + \text{very old} + \text{very very young} \\ & + \text{rather young} + \text{more or less young} + \cdots \end{aligned} \tag{A66}$$

where each of the terms in $T(\text{age})$ is a label of a fuzzy subset of a universe of discourse, say $U = [0,100]$.

A linguistic variable is associated with two rules: (a) a syntactic rule, which defines the well-formed sentences in $T(\chi)$; and (b) a semantic rule, by which the meaning of the terms in $T(\chi)$ may be determined. If X is a term in $T(\chi)$, then its meaning (in a denotational sense) is a subset of U. A primary term in $T(\chi)$ is a term whose meaning is a primary fuzzy set, that is, a term whose meaning must be defined a priori, and which serves as a basis for the computation of the meaning of the non-primary terms in $T(\chi)$. For example, the primary terms in (A66) are young and old, whose meaning might be defined by their respective compatibility functions μ_{young} and μ_{old}. From these, then, the meaning -- or, equivalently, the compatibility functions -- of the non-primary terms in (A66) may be computed by the application of a semantic rule. For example, employing (A54) and (A55) we have

$$\mu_{very\ young} = (\mu_{young})^2 \tag{A67}$$

$$\mu_{more\ or\ less\ old} = (\mu_{old})^{1/2} \tag{A68}$$

$$\mu_{not\ very\ young} = 1 - (\mu_{young})^2 \ . \tag{A69}$$

The Extension Principle

Let g be a mapping from U to V. Thus,

$$v = g(u) \tag{A70}$$

where u and v are generic elements of U and V, respectively.

Let F be a fuzzy subset of U expressed as

$$F = \mu_1 u_1 + \cdots + \mu_n u_n \tag{A71}$$

or, more generally,

$$F = \int_U \mu_F(u)/u \ . \tag{A72}$$

By the extension principle, the image of F under g is given by

$$g(F) = \mu_1 g(u_1) + \cdots + \mu_n g(u_n) \tag{A73}$$

or, more generally,

$$g(F) = \int_U \mu_F(u)/g(u) \ . \tag{A74}$$

Similarly, if g is a mapping from $U \times V$ to W, and F and G are fuzzy subsets of U and V, respectively, then

$$g(F,G) = \int_W (\mu_F(u) \wedge \mu_G(v))/g(u,v) \ . \tag{A75}$$

Example A76. Assume that g is the operation of squaring. Then, for the set defined by (A14), we have

$$g(0.3/0.5 + 0.6/0.7 + 0.8/0.9 + 1/1) \tag{A77}$$
$$= 0.3/0.25 + 0.6/0.49 + 0.8/0.81 + 1/1 \ .$$

Similarly, for the binary operation $\vee$ ($\triangleq$ max), we have

$$(0.9/0.1 + 0.2/0.5 + 1/1) \vee (0.3/0.2 + 0.8/0.6) \tag{A78}$$
$$= 0.3/0.2 + 0.2/0.5 + 0.8/1 + 0.8/0.6 + 0.2/0.6 \ .$$

It should be noted that the operation of squaring in (A77) is different from that of (A51) and (A52).

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