A GLOBAL REPRESENTATION OF MULTI-DIMENSIONAL PIECEWISE-LINEAR FUNCTION

by

S. M. Kang and L. O. Chua

Memorandum No. UCB/ERL M77/51

9 August 1977

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A GLOBAL REPRESENTATION OF MULTI-DIMENSIONAL PIECEWISE-LINEAR FUNCTIONS

S. M. Kang[†] and L. O. Chua

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

An analytical representation is introduced for m-dimensional piecewise-linear functions which are affine over convex polyhedral sets. Explicit formulas are presented to compute the coefficients associated with this representation along with an example.

Research sponsored in part by the Office of Naval Research Contract N00014-76-0572 and by the Miller Institute which supported the second author during the 1976-77 Academic Year as a Miller Research Professor.

[†]S. M. Kang was formerly with Rutgers University. He is now with Bell Laboratories at Holmdel, New Jersey 07733.

Piecewise-linear techniques have been used extensively in Circuits and Systems theory to model dc nonlinear characteristics of electronic devices [1-3] and to study a large class of nonlinear resistive networks [4-8]. In a recent paper [9], Chua and Kang introduced new analytical representations for one-dimensional piecewise-linear functions and multi-dimensional section-wise piecewise-linear functions. These representations led to the possibility of deriving explicit closed form expressions for system parameters and design formulas. They also allowed standard mathematical operations and manipulations to be carried out in various theoretical studies.

The objective of this paper is to present a new compact representation for m-dimensional piecewise-linear functions. This representation is global in the sense that a single equation can be used to completely characterize piecewise-linear surfaces. Simple formulas are derived in this paper for the identification of the coefficients associated with this explicit representation.

Let L non-degenerate[†] linear partition vectors in m-dimensional Euclidean space $\mathbb{R}^m$ be defined as [10]

$$\{(\alpha_{k_1},\alpha_{k_2},\ldots,\alpha_{k_m};\beta_k),\ k=1,2,\ldots,L\} \tag{1}$$

[†] A linear partition is said to be "non-degenerate" if the intersection of any k of the hyperplanes $\underset{\sim}{\alpha}_i^T\underset{\sim}{x} = \beta_i$ has a dimension $\leq$ m-k.

Then associated with this partition, there corresponds a closed convex polyhedral set S_I defined by[2]

$$S_I \triangleq \{\underset{\sim}{x} \in \mathbb{R}^m | \underset{\sim}{\alpha}_i^T \underset{\sim}{x} \geq \beta_i,\ i \in I \text{ and } \underset{\sim}{\alpha}_i^T \underset{\sim}{x} < \beta_i,\ i \notin I\} \tag{2}$$

where I denotes a subset of $\{1,2,\ldots,L\}$ and the superscript T denotes a matrix transpose. Hence, S_I is simply the intersection of a finite family of closed half-spaces of $\mathbb{R}^m$. Under the above-mentioned conditions, we will show that any single-valued m-dimensional piecewise-linear function $f:\mathbb{R}^m \to \mathbb{R}^1$ with jump discontinuities can be represented globally by the following explicit analytical expression:

$$f(\underset{\sim}{x}) = a_0 + \underset{\sim}{a}_1^T \underset{\sim}{x} + \sum_{i=1}^{L} \{g_i |\underset{\sim}{\alpha}_i^T \underset{\sim}{x} - \beta_i| + h_i \operatorname{sgn}(\underset{\sim}{\alpha}_i^T \underset{\sim}{x} - \beta_i)\} \tag{3}$$

where a_0, g_i, h_i are real numbers and $\underset{\sim}{a}_1, \underset{\sim}{\alpha}_i, \underset{\sim}{x} \in \mathbb{R}^m$. It will be obvious later that when $f(\underset{\sim}{x})$ is continuous, $h_i = 0$ for all i in Eq. (3).

To prove the above representation, observe that $f(\underset{\sim}{x})$ restricted to any polyhedral set S_I is an affine function of $\underset{\sim}{x}$, i.e.,

$$f|_{S_I}(\underset{\sim}{x}) = \underset{\sim}{a}^T \underset{\sim}{x} + b \tag{4}$$

Hence, it remains for us to show how the coefficients, $a_0, \underset{\sim}{a}_1, g_i$, and h_i, $i = 1,2,\ldots,L$ associated with Eq. (3) can be identified.

Identification of Coefficients

We first observe that for any $\underset{\sim}{x}$ lying in any open region $S_I^\circ \subset S_I$, the gradient $\partial f(\underset{\sim}{x})/\partial \underset{\sim}{x}$ is well-defined and is given by

---

[†]It is easy to prove that S_I represents a closed and convex polyhedral set. The class of piecewise-linear functions defined on convex polyhedral sets is important specially in the representation of nonlinear circuits containing 2-terminal nonlinear elements characterized by piecewise-linear v-i curves [1-3].

$$\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}} = \underset{\sim}{a}_1 + \sum_{i=1}^{L} g_i \underset{\sim}{\alpha}_i \; \mathrm{sgn}(\underset{\sim}{\alpha}_i^T \underset{\sim}{x} - \beta_i) \tag{5}$$

Since the domain of $f(x)$ is the entire space $\mathbb{R}^m$, there exists an open region, say S_+°, where $\mathrm{sgn}(\underset{\sim}{\alpha}_i^T \underset{\sim}{x} - \beta_i) = 1$ for all i and an open region S_-°, where $\mathrm{sgn}(\underset{\sim}{\alpha}_i^T \underset{\sim}{x} - \beta_i) = -1$ for all i. Hence it follows that

$$a_1 = \frac{1}{2}\left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_+^\circ} + \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_-^\circ}\right) \tag{6}$$

In order to determine the coefficient g_k, $k = 1,2,\ldots,L$, we use the information on the slopes of $f(\underset{\sim}{x})$ for points lying on two "adjacent" open regions S_α° and S_β° whose associated "sign sequences" $(\mathrm{sgn}(\underset{\sim}{\alpha}_i^T x - \beta_i)$, $i = 1,2,\ldots,L)$ differ only in the kth position. If we evaluate Eq. (5) at any point lying the the two adjacent open regions S_α° and S_β°, then g_k is easily seen to be given by:

$$g_k = \pm \frac{1}{2}\, \alpha_k^T\left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_\alpha^\circ} - \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_\beta^\circ}\right) \Big/ \underset{\sim}{\alpha}_k^T \underset{\sim}{\alpha}_k \tag{7}$$

where the "+" sign is chosen if the sign sequence associated with S_α° contains a "+" sign in the kth position. Otherwise, the "−" sign is chosen. The coefficients h_i's are determined from the amount of jump discontinuity of $f(\underset{\sim}{x})$ along the ith hyperplane; namely

$$h_i = \frac{1}{2}\left\{\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_i^+} f(\underset{\sim}{x}) - \lim_{\underset{\sim}{x} \to \underset{\sim}{x}_i^-} f(\underset{\sim}{x})\right\} \tag{8}$$

where $\underset{\sim}{x}_i$ is any point lying on the i^{th} hyperplane $\underset{\sim}{\alpha}_i^T \underset{\sim}{x} = \beta_i$. Finally, having computed $\underset{\sim}{a}_1, g_k$, $k = 1,2,\ldots,L$, and h_i, the remaining coefficient a_0 is obtained by substituting $\underset{\sim}{x} = \underset{\sim}{0}$ in Eq. (3); namely;

$$a_0 = f(\underset{\sim}{0}) - \sum_{i=1}^{L} \{g_i|\beta_i| - h_i \operatorname{sgn}(\beta_i)\} \tag{9}$$

To illustrate the above coefficient identification procedure, consider a typical continuous piecewise-linear function $f(\underset{\sim}{x})$ as shown in Fig. 1. Since $f(\underset{\sim}{x})$ is continuous, it follows immediately from Eq. (8) that $h_i = 0$, $i = 1,2,3$. The notation $(-,+,+)$ in Fig. 1 identifies the associated polyhedral set S_k by specifying the "direction" of the inequalities; namely,

$$(-,+,+) \leftrightarrow (\underset{\sim}{\alpha}_1^T < \beta_1,\ \underset{\sim}{\alpha}_2^T\underset{\sim}{x} \geq \beta_2,\ \underset{\sim}{\alpha}_3^T\underset{\sim}{x} \geq \beta_3).$$

Since there are three hyperplanes in Fig. 1, the function $f(\underset{\sim}{x})$ can be represented by

$$f(\underset{\sim}{x}) = a_0 + \underset{\sim}{a}_1^T\underset{\sim}{x} + \sum_{i=1}^{L=3} g_i|\underset{\sim}{\alpha}_i^T\underset{\sim}{x}-\beta_i| \tag{10}$$

To determine the associated coefficients, the following set of derivative information can be used:

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_1^\circ} = \underset{\sim}{a}_1 - \sum_{i=1}^{3} g_i\underset{\sim}{\alpha}_i \tag{11.a}$$

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_4^\circ} = \underset{\sim}{a}_1 + g_1\underset{\sim}{\alpha} - g_2\underset{\sim}{\alpha}_2 - g_3\underset{\sim}{\alpha}_3 \tag{11.b}$$

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_5^\circ} = \underset{\sim}{a}_1 + g_1\alpha_1 + g_2\underset{\sim}{\alpha}_2 - g_3\underset{\sim}{\alpha}_3 \tag{11.c}$$

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_7^\circ} = a_1 + g_1\underset{\sim}{\alpha}_1 + g_2\underset{\sim}{\alpha}_2 + g_3\underset{\sim}{\alpha}_3 \tag{11.d}$$

Using Eqs. (6), (7), and (9), we obtain

$$a_1 = \frac{1}{2}\left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_1^\circ} + \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_7^\circ} \right) \tag{12.a}$$

$$g_1 = \frac{1}{2}\, \underset{\sim}{\alpha}_1^T \left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_4^\circ} - \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_1^\circ} \right) \Big/ \underset{\sim}{\alpha}_1^T \underset{\sim}{\alpha}_1 \tag{12.b}$$

$$g_2 = \frac{1}{2}\, \underset{\sim}{\alpha}_2^T \left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_5^\circ} - \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_4^\circ} \right) \Big/ \underset{\sim}{\alpha}_2^T \underset{\sim}{\alpha}_2 \tag{12.c}$$

$$g_3 = \frac{1}{2}\, \underset{\sim}{\alpha}_3^T \left(\left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_7^\circ} - \left.\frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}}\right|_{S_5^\circ} \right) \Big/ \underset{\sim}{\alpha}_3^T \underset{\sim}{\alpha}_3 \tag{12.d}$$

and

$$a_0 = f(\underset{\sim}{0}) - \sum_{i=1}^{3} g_i |\beta_i| \tag{12.e}$$

Remarks: To determine all coefficients, it is necessary for the path (denoted by dotted lines in Fig. 1) to cross the bounding hyperplanes "L" times. Such a path need not be unique however. For example, other paths such as S_1-S_2-S_6-S_7 or S_1-S_4-S_6-S_7 may also be chosen.

To demonstrate the compactness of our new representation, we close this correspondence by considering an example taken from [11].

Example. Consider the two-dimensional continuous piecewise-linear function $f(x_1,x_2)$ shown in Fig. 2. It is described by six affine equations and those straight-line boundaries defining the six convex polyhedral sets shown in Fig. 2. Using our new representation, this piecewise-linear function is defined compactly by the following explicit analytical expression:

$$f(x_1,x_2) = \frac{1}{2\sqrt{3}}\,|-\sqrt{3}\,x_1+x_2| - \frac{2}{\sqrt{3}}\,|x_2| + \frac{1}{2\sqrt{3}}\,|\sqrt{3}\,x_1+x_2| \tag{13}$$

It is clear from this example that our piecewise-linear representation not only greatly simplifies the computer programming efforts and storage requirements, but it also allows algebraic operations and equation manipulations to be carried out in symbolic form.

REFERENCES

[1] L. O. Chua, Introduction to Nonlinear Network Theory, McGraw-Hill, New York, 1969.

[2] L. O. Chua, "Modeling of Three-Terminal Devices: A Black-Box Approach," IEEE Trans. on Circuit Theory, CT-19, No. 6, pp. 555-562, November 1972.

[3] L. O. Chua and P.-M. Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, Prentice-Hall, Englewood Cliffs, N. J., 1975.

[4] J. Katzenelson, "An Algorithm for Solving Nonlinear Resistive Networks," Bell System Tech. J., 44, pp. 1605-1620, 1965.

[5] T. Fujisawa, E. S. Kuh and T. Ohtsuki, "A Sparse Matrix Method for Analysis of Piecewise-Linear Networks," IEEE Trans. on Circuit Theory, CT-19, No. 6, pp. 571-584, November 1973.

[6] M. J. Chien and E. S. Kuh, "Solving Piecewise-Linear Equations for Resistive Networks," Inter. J. Circuit Theory and Applications, 4, No. 1, pp. 3-24, January 1976.

[7] T. Ohtsuki, T. Fujisawa and S. Kumagai, "Existence Theorems and a Solution Algorithm for Piecewise-Linear Resistor Networks," SIAM J. Math. Anal., 8, No. 1, February 1977.

[8] M.-J. Chien and E. S. Kuh, "Solving Nonlinear Resistive Networks Using Piecewise-Linear Analysis and Simplicial Subdivision," IEEE Trans. on Circuits and Systems, CAS-24, No. 6, pp. 305-317, June 1977.

[9] L. O. Chua and S. M. Kang, "Section-Wise Piecewise-Linear Functions: Canonical Representation, Properties and Applications," Proc. IEEE (Special Issue on Multidimensional Systems), 65, No. 6, pp. 915-929, June 1977.

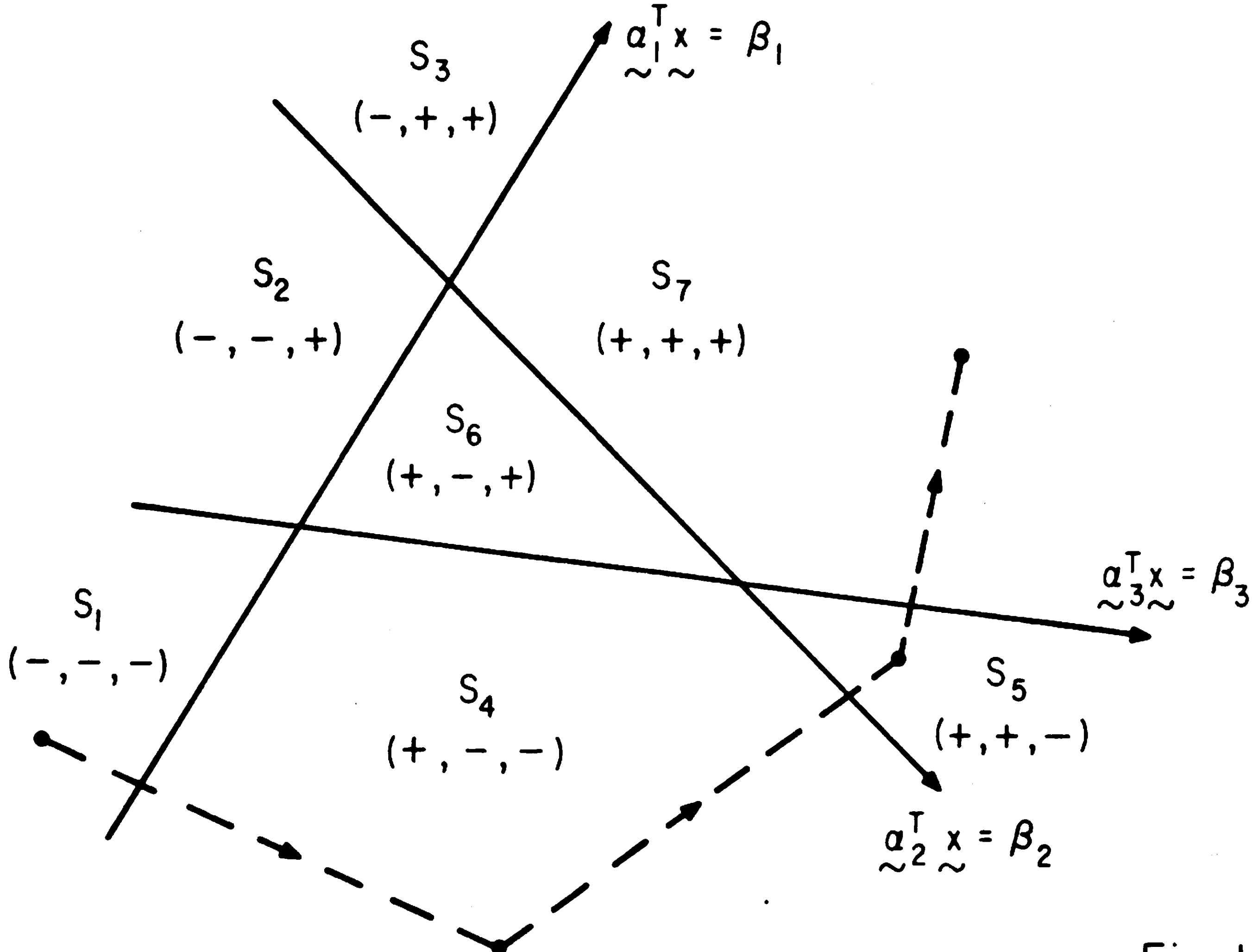
[10] P. B. Zwart, "Multivariate Splines with Nondegenerate Partitions," SIAM J. Numerical Analysis, 10, No. 4, pp. 665-673, September 1973.

[11] T. Fujisawa and E. S. Kuh, "Piecewise-Linear Theory of Nonlinear Networks," SIAM J. Appl. Math., 22, No. 2, pp. 307-328, March 1972.

FIGURE CAPTIONS

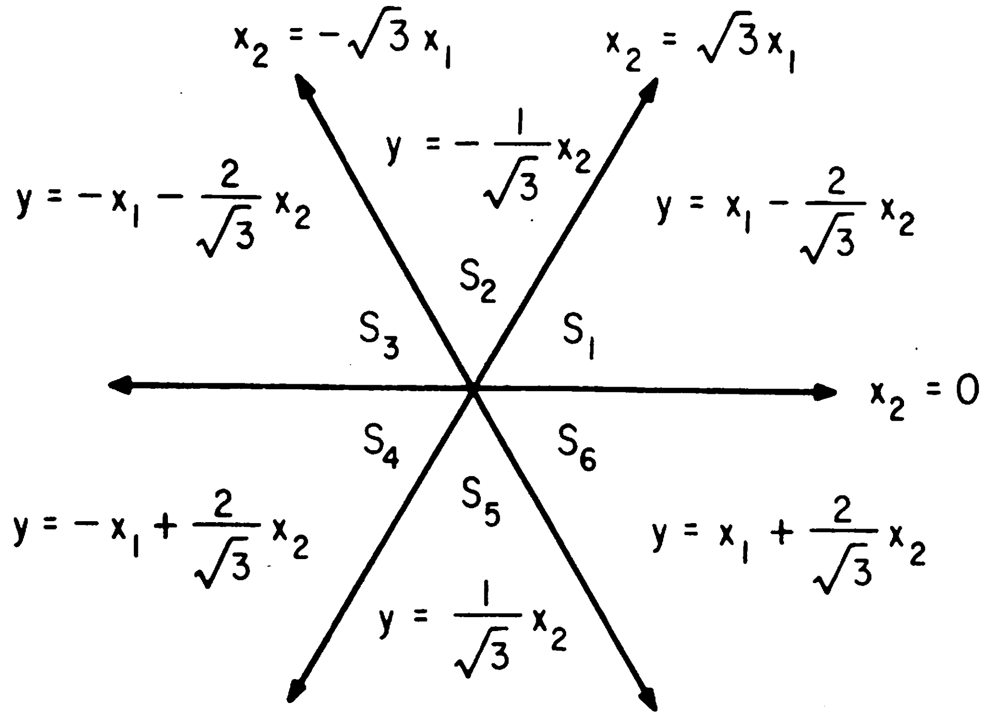
Fig. 1. A two-dimensional piecewise-linear function f: $\mathbb{R}^2 \to \mathbb{R}^1$ defined on a plane which has been partitioned into seven convex polyhedral sets.

Fig. 2. Example of a two-dimensional piecewise-linear function f: $\mathbb{R}^2 \to \mathbb{R}^1$ as described by conventional representation requiring six affine equations associated with six convex polyhedral sets.



$\underset{\sim}{a}_1^T \underset{\sim}{x} = \beta_1$
$\underset{\sim}{a}_2^T \underset{\sim}{x} = \beta_2$
$\underset{\sim}{a}_3^T \underset{\sim}{x} = \beta_3$
S_1 $(-,-,-)$
S_2 $(-,-,+)$
S_3 $(-,+,+)$
S_4 $(+,-,-)$
S_5 $(+,+,-)$
S_6 $(+,-,+)$
S_7 $(+,+,+)$

Fig. 1



$x_2 = -\sqrt{3}\,x_1$
$x_2 = \sqrt{3}\,x_1$
$x_2 = 0$
$y = -\frac{1}{\sqrt{3}}x_2$
$y = -x_1 - \frac{2}{\sqrt{3}}x_2$
$y = x_1 - \frac{2}{\sqrt{3}}x_2$
$y = -x_1 + \frac{2}{\sqrt{3}}x_2$
$y = x_1 + \frac{2}{\sqrt{3}}x_2$
$y = \frac{1}{\sqrt{3}}x_2$
S_1
S_2
S_3
S_4
S_5
S_6

Fig. 2