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by

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ABSTRACT

This paper presents a multiplier method for solving optimization problems with equality and inequality constraints. The method realizes all the good features that were foreseen by R. Fletcher for this type of algorithm in the past, but which suffers from none of the drawbacks of the earlier attempts.

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1. INTRODUCTION

This paper presents an algorithm which is at the end of a fairly long evolutionary chain. Originally, as proposed by Hestenes [8], Powell [12] and Haarhoff and Buys [4], multiplier methods for solving a problem of the form $\underline{P}$: $\min\{f(x) | h(x) = 0\}$ proceeded as follows. They constructed recursively, a sequence of augmented Lagrangians

$$F(x,c_i) \triangleq f(x) + \langle \psi_i, h(x)\rangle + \frac{c_i}{2} \|h(x)\|^2, \quad i = 0,1,2,\ldots$$

with $c_i > 0$ monotonically increasing. Beginning with a ψ_0, they minimized $F(x,c_i)$ to get an x_i and then updated the multiplier to ψ_{i+1} by some formula, for the next augmented Lagrangian. Since it soon became known (see e.g. [3],[1]) that the x_i thus constructed may converge to a solution $\hat{x}$ of $\underline{P}$ even when the c_i do not converge to infinity, it was proposed by Fletcher in a series of papers [3,4,5] to combine the infinite sequence of unconstrained minimizations of the $F(x,c_i)$ into a single problem by using a formula $\psi(x)$ for ψ_i. This idea was very good, except for two shortcomings. The first was that he did not know how to find automatically a satisfactory value of the penalty c, while the other was that his extension of his formula to problems with inequalities [6] resulted in discontinuous derivatives in the augmented Lagrangian, which caused algorithms to jam.

In [10], Mukai and Polak proposed a test for limiting the growth of the penalty for Fletcher's scheme for the case of equality constraints. In this paper, we proposed a new formula for the multiplier as well as a test for limiting the growth of the penalty for problems of the form $\min\{f(x) | g(x) \leq 0,\ h(x) = 0\}$. Our formula results in twice continuously differentiable multipliers and in a twice continuously differentiable augmented

Lagrangian. To simplify exposition, for the augmented Lagrangian we have used the quadratic form proposed by Buys [2] and Rockaffelar [13]. If one wishes to get more differentiability, one can use the slightly more complex, cubic form proposed by Kort and Bertsekas [9]. Our test for limiting the growth of the penalty conforms to the requirements of the abstract algorithm model for exact penalty methods proposed in Polak [11]. As a result, if the sequence $\{x_i\}$, which the algorithm constructs, stays bounded then the penalty stays bounded. Mostly, this is the case in practice.

In summary, we present in this paper a multiplier method for solving optimization problems with equality and inequality constraints. This method realizes all the good features that were foreseen by Fletcher for this type of algorithm in the past, but which suffers from none of the drawbacks of the earlier attempts.

2. THE PROBLEM AND BUILDING BLOCKS FOR AN ALGORITHM

Consider the problem

$$\min\{f(x) \mid g(x) \leq 0,\ h(x) = 0\} \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}^1$, $g:\mathbb{R}^n \to \mathbb{R}^m$, $h:\mathbb{R}^n \to \mathbb{R}^\ell$.

Notation: We shall denote by $\underline{m}$ the set $\{1,2,\dots,m\}$, by $\underline{\ell}$ the set $\{1,2,\dots,\ell\}$ and by $B(x,\varepsilon)$ the set $\{x' \mid \|x'-x\| \leq \varepsilon\}$. ¤

Assumption 1: The functions f, g and h are three times continuously differentiable. ¤

Assumption 2: For any $x \in \mathbb{R}^n$, let $I(x) \triangleq \{j \in \underline{m} \mid g^j(x) \geq 0\}$. We assume that for any $x \in \mathbb{R}^n$, the vectors $\nabla g^j(x)$, $j \in I(x)$ together with the vectors $\nabla h^j(x)$, $j \in \underline{\ell}$ are linearly independent. ¤

We note that Assumption 2 is somewhat stronger than the positive linear independence condition usually required for the Kuhn-Tucker constraint qualification to hold.

For problem (1) we shall seek Kuhn Tucker points, which we define as follows.

Definition 1: A point $\bar{x} \in \mathbb{R}^n$ is a K-T (Kuhn-Tucker) point if

$$g(\bar{x}) \leqq 0, \ h(\bar{x}) = 0 \tag{2}$$

and there exist vectors $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\psi} \in \mathbb{R}^\ell$ such that

$$\nabla f(\bar{x}) + \frac{\partial g(\bar{x})^T}{\partial x} \bar{\lambda} + \frac{\partial h(\bar{x})^T}{\partial x} \bar{\psi} = 0 \tag{3}$$

$$\bar{\lambda} \geqq 0, \ \langle \bar{\lambda}, g(x) \rangle = 0 \tag{4}$$

We shall denote the set of all K-T points for (1) by Δ. ¤

Note that because of assumption 2, $\bar{\lambda}$ and $\bar{\psi}$ are uniquely defined by (3).

Assumption 3: Let $\bar{x}$ be any K-T point for problem (1), with associated multipliers $\bar{\lambda}$, $\bar{\psi}$, and let

$$L(x,\lambda,\psi) = f(x) + \langle \lambda, g(x) \rangle + \langle \psi, h(x) \rangle \tag{5}$$

Then (i) $\bar{\lambda}^j > 0$ for all $j \in I(\bar{x})$ (strict complementarity) and (ii) the Hessian matrix $\frac{\partial^2 L(\bar{x},\bar{\lambda},\bar{\psi})}{\partial x^2}$ is nonsingular on the subspace

$$M = \{y \mid \frac{\partial h(\bar{x})}{\partial x} y = 0;\ \langle \nabla g^j(\bar{x}|,h\rangle = 0,\ j \in I(\bar{x})\} \tag{6}$$

¤

We shall shortly define the augmented Lagrangian, as in Buys and Rockafellar [13], and substitute into it multipliers obtained by a modification of Fletcher's formula [6]. It was necessary to modify the formula because the multiplier it yields can have a discontinuous gradient.

Thus, we define $\lambda: \mathbb{R}^n \to \mathbb{R}^m$, $\psi: \mathbb{R}^n \to \mathbb{R}^\ell$ by

$$(\lambda(x),\psi(x)) \triangleq \arg\min_{(\lambda,\psi)} \{\|\nabla f(x) + \frac{\partial g(x)}{\partial x}^T \lambda + \frac{\partial h(x)}{\partial x}^T \psi\|^2 + \langle \lambda, G(x)\lambda\rangle \} \tag{7}$$

where

$$G(x) \triangleq \mathrm{diag}((g^j(x))^2) \tag{8}$$

Proposition 1: The functions $\lambda(\cdot)$, $\psi(\cdot)$ are well defined and are twice continuously differentiable. Furthermore, for any K-T point $\bar{x}$, $\lambda(\bar{x})$, $\psi(\bar{x})$ satisfy (3) and (4).

Proof: First we show that the second order term in (7) is positive definite. Since it is obviously at least positive semidefinite, it is sufficient to show that

$$\left|\left|\frac{\partial g(x)}{\partial x}^T \lambda + \frac{\partial h(x)}{\partial x}^T \psi\right|\right|^2 + \langle \lambda, G(x)\lambda\rangle = 0 \tag{9}$$

implies that $\lambda = 0$ and $\psi = 0$. We can rewrite (9) as

$$\sum_{j=1}^{m} (\lambda^j)^2 (g^j(x))^2 + \left|\left|\sum_{j=1}^{m} \lambda^j \nabla g^j(x) + \sum_{j=1}^{\ell} \psi^j \nabla h^j(x)\right|\right|^2 = 0 \tag{10}$$

Then we must have $\lambda^j = 0$ for all $j \in \underline{m}$ such that $g^j(x) \neq 0$. Consequently

$$\sum_{j\in I(x)} \lambda^j \nabla g^j(x) + \sum_{j=1}^{\ell} \psi^j \nabla h^j(x) = 0 \tag{11}$$

and this implies that $\lambda = 0$ and $\psi = 0$ because of Assumption 2. Since λ and ψ are given by

$$(\frac{\partial}{\partial \lambda}) : \frac{\partial g(x)}{\partial x} (\nabla f(x) + \frac{\partial g(x)}{\partial x}^T \lambda + \frac{\partial h(x)}{\partial x}^T \psi) + G(x)\lambda = 0 \tag{12}$$

$$(\frac{\partial}{\partial \psi}) : \frac{\partial h(x)}{\partial x} (\nabla f(x) + \frac{\partial g(x)}{\partial x}^T \lambda + \frac{\partial h(x)}{\partial x}^T \psi) = 0 \tag{13}$$

we conclude, because of Assumption 1, that they are twice continuously differentiable.

Now suppose that $\bar{x}$ is a K-T point. Then there exist $(\bar{\lambda},\bar{\psi})$ satisfying (3) and (4) and hence also (12) and (13). Since the solutions of (12) and (13) are unique, we conclude that $\lambda(\bar{x}) = \bar{\lambda}$, $\psi(\bar{x}) = \bar{\psi}$, which completes our proof. ◻

We can now define an augmented Lagrangian $F:\mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^1$ by

$$F(x,c) \triangleq f(x) + \langle \psi(x),h(x)\rangle + \frac{1}{2c} \{\|(cg(x)+\lambda(x))_+\|^2 - \|\lambda(x)\|^2\} + \frac{c}{2}\|h(x)\|^2 \tag{14}$$

where for any vector $y \in \mathbb{R}^m$, $y_+ \in \mathbb{R}^m$ is a vector with components $y_+^j \triangleq \max\{0,y^j\}$, $j = 1,2,\ldots,m$. We shall denote the set of points $x \in \mathbb{R}^n$ at which the gradient of F vanishes by Δ_c, i.e.

$$\Delta_c \triangleq \{x \in \mathbb{R}^n | \nabla_x F(x,c) = 0\} \tag{15}$$

To construct an algorithm for solving (1) based on the unconstrained minimization of a finite number of augmented Lagrangians, we follow the scheme described as Algorithm Model 4 in [11]. This scheme requires that we have an algorithm for minimizing $F(x,c)$ for any value of $c > 0$ and a real valued test function $(x,c)\longmapsto t(x,c)$. Denoting the iteration function for the unconstrained optimization algorithm by A

$(A:\mathbb{R}^n \times \mathbb{R}^+ \to 2^{\mathbb{R}^n})^\dagger$ we can restate algorithm Model 4 in [11] in terms of the quantities in this paper. Let $\phi:\mathbb{R}^+ \to \mathbb{R}^+$ be such that

$$\phi(c) > c \quad \forall c \in \mathbb{R}^+ \tag{16}$$

$$\phi(c) \to \infty \text{ as } c \to \infty \tag{17}$$

e.g. $\phi(c) \triangleq 2c$.

Algorithm Model

Data: $x_0 \in \mathbb{R}^n$, $c_0 > 0$.

Step 0: Set $i=0$, $j=0$.

Step 1: If $t(x_i,c_j) > 0$, go to step 2; else go to step 3.

Step 2: Set $\bar{x}_j = x_i$, set $c_{j+1} = \phi(c_j)$, set $j = j+1$ and go to step 1.

Step 3: If $x_i \in \Delta_{c_j}$, stop; else compute an $x_{i+1} \in A(x_i,c_j)$ set $i = i+1$ and go to step 1. ◻

The following convergence result specifies the properties of the test function t, which we shall need to construct, as well as the properties of the resulting algorithm (see [11]).

Theorem 1: Consider the sequences $\{x_i\}$, $\{\bar{x}_j\}$ and $\{c_j\}$ constructed by the Algorithm Model. Suppose that

(i) for each j, $A(\cdot,c_j)$ is such that any accumulation point $\hat{x}$ of a sequence $\{x_i\}_{i=0}^{\infty}$ constructed according to $x_{i+1} \in A(x_i,c_j)$, $j = 0,1,2,\ldots$, satisfies $\hat{x} \in \Delta_{c_j}$.

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†For example, $A(x,c) \triangleq x-\lambda\nabla_x F(x,c)$ for the Armijo gradient method, and $A(x,c) = x - (\frac{\partial^2 F}{\partial x^2}(x,c))^{-1}\nabla F(x,c)$ for Newton's method. When the unconstrained algorithm is a multipoint method, e.g. a quasi-Newton method, the notation $A(x,c)$ is not correct, but will be used for simplicity of exposition. Theorem 1 is not sensitive to whether A represents a single or multipoint method.

(ii) For any $c > 0$, $t(\cdot,c)$ is continuous.

(iii) For $j = 0,1,2,3,\ldots$, $\{x \in \Delta_{c_j} | t(x,c_j) \leq 0\} \subset \Delta$.

(iv) For every $\hat{x} \in \mathbb{R}^n$ there exists a $\hat{c} \geq 0$ and an $\hat{\varepsilon} > 0$ such that $t(x,c) \leq 0$ for all $c \geq \hat{c}$ for all $x \in \{x' | \|x'-\hat{x}\| \leq \hat{\varepsilon}\}$.

Under these assumptions:

(i) If the sequence $\{x_i\}$ is finite (so that $\{\bar{x}_j\}$ is also finite) then the last element, say x_k, is in Δ.

(ii) If the sequence $\{\bar{x}_j\}$ is finite and the sequence $\{x_i\}$ is infinite, then every accumulation point $\hat{x}$ of $\{x_i\}$ is in Δ.

(iii) If the sequence $\{\bar{x}_j\}$ is infinte, then this sequence has no accumulation points.

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We see from this theorem that if we combine any convergent method for unconstrained minimization $A(\cdot,\cdot)$ with an appropriate test function $t(\cdot,\cdot)$ as in the Algorithm Model, we obtain a convergent algorithm with the property that it will not drive the penalty c to infinity whenever the constructed sequence $\{x_i\}$ stays bounded. For example, this will always be the case when the level sets (for $\alpha \in [0,\infty)$) $\{x | f(x) \leq \alpha\}$ are compact.

Since there is no shortage of unconstrained minimization algorithms, we see that the main task in constructing an algorithm of the form of our Algorithm Model, is the construction of the test function $t(\cdot,\cdot)$. We shall do this in the next section.

3. THE TEST FUNCTION t.

Our construction of the test function, for the algorithm we are developing in this paper, is based on ideas which also were used in

[10] and [9a] to construct test functions for other exact penalty type algorithms. In this spirit, we propose to develop a test function of the form

$$t(x,c) \triangleq -\|\nabla_x F(x,c)\|^2 + c\{\|a(x,c)\|^2 + \|h(x)\|^2\} \tag{18}$$

so that when $\nabla_x F(x,c) = 0$, $t(x,c) \le 0$ if and only if $h(x) = 0$ and $a(x,c) = 0$. Obviously, to ensure that, in that case, $x \in \Delta$, $a(\cdot,\cdot)$ has to be chosen so that $a(x,c) = 0$ if and only if $g(x) \leqq 0$, $\lambda(x) \geqq 0$ and $\langle \lambda(x), g(x) \rangle = 0$. We note that

$$\begin{aligned} \nabla_x F(x,c) &= \nabla f(x) + \frac{\partial h(x)}{\partial x}^T \psi(x) + \frac{\partial \psi(x)}{\partial x}^T h(x) \\ &+ \frac{1}{c}\{c(\frac{\partial g(x)}{\partial x} + \frac{\partial \lambda(x)}{\partial x})^T (cg(x)+\lambda(x))_+ \\ &- \frac{\partial \lambda(x)}{\partial x}^T \lambda(x)\} + c\,\frac{\partial h(x)}{\partial x}^T h(x) \end{aligned} \tag{19}$$

If we define $a(\cdot,\cdot)$ by

$$a(x,c) \triangleq (g(x) + \frac{1}{c}\lambda(x))_+ - \frac{1}{c}\lambda(x) \tag{20}$$

then we get

$$\begin{aligned} \nabla_x F(x,c) &= \nabla f(x) + \frac{\partial h(x)}{\partial x}^T (\psi(x) + ch(x)) + \frac{\partial \psi(x)}{\partial x}^T h(x) \\ &+ \frac{\partial g(x)}{\partial x}^T \lambda(x) + c(\frac{\partial g(x)}{\partial x} + \frac{1}{c}\frac{\partial \lambda(x)}{\partial x})^T a(x,c) \end{aligned} \tag{21}$$

<u>Lemma 1</u>: If, for some $c > 0$, $a(x,c) = 0$, then $g(x) \leqq 0$, $\lambda(x) \geqq 0$ and $\langle \lambda(x), g(x) \rangle = 0$.

<u>Proof</u>: Suppose $a(x,c) = 0$. Then, from (20)

$$(g(x) + \frac{1}{c}\lambda(x))_+ = \frac{1}{c}\lambda(x) \tag{22}$$

and hence $\lambda(x) \geqq 0$. Next, for any i such that $\lambda^i(x) = 0$, we must have (by (20)) that $g^i(x) < 0$ and for any i such that $\lambda^i(x) > 0$ we must have $g^i(x) + \frac{1}{c}\lambda^i(x) = \frac{1}{c}\lambda(x)$, i.e., $g^i(x) = 0$. We therefore conclude that

$g(x) \leq 0$ and $\langle \lambda(x), g(x) \rangle = 0$. ⌑

The next result is obvious.

Corollary: Suppose $a(x,c) = 0$ for some $c \in (0,\infty)$. Then $a(x,c) = 0$ for all $c \in (0,\infty)$. ⌑

Lemma 2: $\hat{x}$ is a K-T point if and only if for any $c \in (0,\infty)$ $\nabla_x F(\hat{x},c) = 0$, $a(\hat{x},c) = 0$ and $h(\hat{x}) = 0$.

Proof: Suppose $\hat{x}$ is a K-T point. Then $h(\hat{x}) = 0$, $g(\hat{x}) = 0$ and $\psi(\hat{x})$, $\lambda(\hat{x})$ satisfy (3) and (4). Hence we must also have $\nabla_x F(\hat{x},c) = 0$, $a(\hat{x},c) = 0$ for any $c \in (0,\infty)$.

Now suppose that $\nabla_x F(\hat{x},c) = 0$, $a(\hat{x},c) = 0$ and $h(\hat{x}) = 0$. Then, by Lemma 1, $g(\hat{x}) \leq 0$, $\lambda(\hat{x})$ satisfies (4) and from (21), $\lambda(\hat{x})$ and $\psi(\hat{x})$ satisfy (3), i.e., $\hat{x}$ is a K-T point. ⌑

Corollary: Let $t(\cdot,\cdot)$ be defined as in (18), with $a(\cdot,\cdot)$ as in (20). Then $t(\cdot,c)$ is continuous for any $c \in (0,\infty)$ and $\{x \in \Delta_c \,|\, t(x,c) \leq 0\} \subset \Delta$.

Proof: The fact that $t(\cdot,c)$ is continuous follows from the continuity of the constituting functions. Now suppose that for some $c \in (0,\infty)$ $x \in \Delta_c$ and $t(x,c) \leq 0$. Then $\nabla_x F(x,c) = 0$ and therefore $t(x,c) \leq 0$ implies that $a(x,c) = 0$ and $h(x) = 0$. Hence x is a K-T point (i.e., $x \in \Delta$) by Lemma 2. ⌑

The above corollary shows that our test functions satisfies the assumptions (ii) and (iii) of Theorem 1. It will be much more difficult to show that it also satisfies assumption (iv) of Theorem 1. This will require the following two lemmas.

Lemma 3: Suppose that Assumption 3 holds. Then, for every compact set $S \subset \mathbb{R}^n$ which does not contain a K-T point there exist a $c_0 > 0$ and a

$\delta > 0$ such that

$$\|\nabla_x F(x,c)\| \geq \delta, \qquad \forall c \geq c_0, \quad \forall x \in S \tag{23}$$

Proof: Suppose the lemma is false. Then there exist sequences $\{x_i\}_{i=0}^{\infty}$, $\{c_i\}_{i=0}^{\infty}$ such that $x_i \in S$ for all i, $x_i \to \hat{x} \in S$, as $i \to \infty$, $c_i \to \infty$ as $i \to \infty$ and $\|\nabla_x F(x_i,c_i)\| \to 0$ as $i \to \infty$. Since $c_i \to \infty$, it follows from (20) and (21) and Assumption 1 that

$$\left[\frac{\partial h(x_i)}{\partial x}^T h(x_i) + \frac{\partial g(x_i)}{\partial x}^T (g(x_i) + \frac{1}{c_i}\lambda(x_i))_+\right] \to \frac{\partial h(\hat{x})}{\partial x}^T h(\hat{x}) + \frac{\partial g(\hat{x})}{\partial x}^T g(\hat{x})_+ = 0. \tag{24}$$

It now follows from

Assumption 2 that $h(\hat{x}) = 0$, $g(\hat{x}) \leq 0$, i.e. $\hat{x}$ is feasible. If S contains no feasible points, we have a contradiction and we are done. To explore the other possibility, suppose that S does contain feasible points. In that case, there is an i_0 such that $c_i g^j(x_i) + \lambda^j(x_i) \leq 0$, and therefore $(c_i g(x_i) + \lambda(x_i))_+^j = 0$, for all $i \geq i_0$ and $j \in I^c(\hat{x}) \triangleq \{j \in \underline{m} | g^j(\hat{x}) < 0\}$. Hence, for all $i \geq i_0$,

$$\begin{aligned}\nabla_x F(x_i,c_i) &= \nabla f(x_i) + \frac{\partial h(x_i)}{\partial x}^T [\psi(x_i) + c_i h(x_i)] + \frac{\partial \psi(x_i)}{\partial x}^T h(x_i) \\ &+ \frac{\partial \tilde{g}(x_i)}{\partial x}^T [c_i \tilde{g}(x_i) + \tilde{\lambda}(x_i))_+ \\ &+ \frac{\partial \tilde{\lambda}(x_i)}{\partial x}^T [g(x_i) + \frac{1}{c_i}\tilde{\lambda}(x_i)]_+ - \frac{1}{c_i}\frac{\partial \lambda(x_i)}{\partial x}^T \lambda(x_i)\end{aligned} \tag{25}$$

where $\tilde{g}$, $\tilde{\lambda}$ are vectors whose components are the components g^j, λ^j of g and λ, respectively, with $j \in I(\hat{x})$. Let

$$\psi_i \triangleq \psi(x_i) + c_i h(x_i), \ \tilde{\lambda}_i = (c_i \tilde{g}(x_i) + \tilde{\lambda}(x_i))_+, \ i = 1,2,\ldots, \tag{26}$$

Then, since $\nabla_x F(x_i,c_i) \to 0$, we must have $\psi_i \to \hat{\psi}$ and $\tilde{\lambda} \to \hat{\tilde{\lambda}}$. Furthermore, since $\tilde{\lambda}_i > 0$ for all i, $\hat{\tilde{\lambda}} \geq 0$, and

$$\nabla f(\hat{x}) + \frac{\partial \tilde{g}(\tilde{x})^T}{\partial x} \hat{\lambda} + \frac{\partial h(\hat{x})^T}{\partial x} \hat{\psi} = 0 \tag{27}$$

which implies that $\hat{x}$ is a K-T point.† But this contradicts our assumption that there are no K-T points in S. Hence we are done. ¤

<u>Lemma 4</u>: Suppose that Assumptions 1-3 are satisfied and that $\hat{x}$ is a K-T point. Then there exist a $\hat{c} > 0$, a $\hat{\delta} > 0$ and an $\hat{\varepsilon} > 0$ such that

$$\|\nabla_x F(x,c)\| \geq \hat{\delta}\|x-\hat{x}\|, \; \forall c \geq \hat{c}, \; \forall x \in B(\hat{x},\hat{\varepsilon}) \tag{28}$$

<u>Proof</u>: Because of Assumption 3, there exists an $\hat{\varepsilon} \in (0,1]$ such that $\hat{x}$ is the only K-T point in $B(\hat{x},\hat{\varepsilon})$ and, in addition (by continuity) $g^i(x) < 0$ for all $x \in B(\hat{x},\hat{\varepsilon})$ and all $i \in I^c(\hat{x}) \triangleq \{i \in \underline{m} | g^j(\hat{x}) < 0\}$. Hence there exists a c_0 such that $cg^i(x) + \lambda^i(x) \leq 0$ for all $x \in B(\hat{x},\hat{\varepsilon})$, for all $c \geq c_0$ and all $i \in I^c(\hat{x})$. For any $x \in B(\hat{x},\hat{\varepsilon})$, and $c \geq c_0$ let $J(x,c)$ be defined by

$$J(x,c) \triangleq \{i \in \underline{m} | cg^i(x) + \lambda^i(x) > 0\} \tag{29}$$

Then $J(x,c) \subset I(\hat{x})$ for all $x \in B(\hat{x},\hat{\varepsilon})$, $c \geq c_0$.

First we show that there exist a $c_1 \geq c_0$ and a $\delta_1 > 0$ such that for all $x \in B(\hat{x},\hat{\varepsilon})$ and $c \geq c_1$ satisfying $J(x,c) \subset I(\hat{x})$, $J(x,c) \neq I(\hat{x})$, we must have

$$\begin{aligned}\|\nabla_x F(x,c)\| = \|\nabla f(x) &+ \frac{\partial h(x)^T}{\partial x}(\psi(x) + ch(x)) + \frac{\partial \psi(x)^T}{\partial x} h(x) \\ &+ \sum_{i \in J(x,c)} \{[cg^i(x) + \lambda^i(x)]\nabla g^i(x) + g^i(x)\nabla\lambda^i(x)\} \\ &+ \frac{1}{c} \sum_{i \in J^c(x,c)} \lambda^i(x)\nabla\lambda^i(x)\| \geq \delta_1\end{aligned} \tag{29}$$

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†We conclude from this discussion that if the x_i are minimizers of $F(x,c_i)$, and $x_i \to \hat{x}$ while $c_i \to \infty$, then $\hat{x}$ is a K-T point for (1) and, furthermore, that $\lambda_i \to \lambda(\hat{x})$, $\psi_i \to \psi(\hat{x})$, which implies that $c_i\|\tilde{g}(x_i)\| \to 0$ and $c_i\|h(x_i)\| \to 0$ as $i \to \infty$, c.f. Bertsekas [1], i.e. that we get superlinear convergence to feasibility. With suitable strong convexity assumptions, this gives superlinear convergence of $x_i \to \hat{x}$ in the sense that $c_i\|x_i-\hat{x}\| \to 0$ as $i \to \infty$.

For suppose this is false. Then there exist sequences $\{x_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=0}^{\infty}$ such that $x_k \in B(\hat{x},\hat{\varepsilon})$, $k = 0,1,2,\ldots$, $x_k \to \bar{x}$ and $c_k \to \infty$ as $k \to \infty$, $J(x_k,c_k) \subset I(\hat{x})$ and $J(x_k,c_k) \neq I(\hat{x})$, for $k = 0,1,2,\ldots$, and $\nabla_x F(x_k,c_k) \to 0$ as $k \to \infty$. Since $J(x_k,c_k)$ is an element of the finite set of subsets of $I(\hat{x})$, there exists a $J \neq I(\hat{x})$, $J \subset I(\hat{x})$ and an infinite subsequence indexed by $K \subset \{0,1,2,\ldots\}$ such that $J(x_k,c_k) = J$ for all $k \in K$. Now, since $\nabla_x F(x_k,c_k) \to 0$ as $k \to \infty$ while $c_k \to \infty$ and $x_k \to \bar{x}$, we must have, because of Assumption 2 and because $(c_k g(x_k) + \lambda(x_k))_+ = \sum_{i \in J(x_k,c_k)} (c_k g^i(x_k) + \lambda^i(x_k)) e_i$† converges as $k \to \infty$, $k \in K$, that $g(\bar{x}) \leqq 0$ and $h(\bar{x}) = 0$, i.e. that $\bar{x}$ is feasible. Now for $k \in K$,

$$\|\nabla_x F(x_k,c_k)\| = \|\nabla f(x_k) + \frac{\partial h(x_k)}{\partial x}^T (\psi(x_k) + c_k h(x_k)) + \frac{\partial \psi(x_k)}{\partial x}^T h(x_k) + \sum_{i \in J} \{[c_k g^i(x_k) + \lambda^i(x_k)]\nabla g^i(x_k) + g^i(x_k)\nabla\lambda^i(x_k)\} - \frac{1}{c_k}\sum_{i \in J^c} \lambda^i(x_k)\nabla\lambda^i(x_k)\| \tag{30}$$

Since $\nabla_x F(x_k,c_k) \overset{K}{\to} 0$ as $k \to \infty$, and $J(x_k,c_k) = J$ for all $k \in K$, and $x_k \to \bar{x}$ as $k \to \infty$ and $c_k g^i(x_k) + \lambda^i(x_k) \geqq 0$ for all $k \in K$, $i \in J$, we find that $[c_k g^i(x_k) + \lambda^i(x_k)] \overset{K}{\to} \bar{\lambda}^i \geqq 0$, as $k \to \infty$, for $i \in J$, and $[c_k h(x_k) + \psi(x_k)] \overset{K}{\to} \bar{\psi}$ as $k \to \infty$. Furthermore, since for $i \in J$, $c_k g^i(x_k) + \lambda^i(x_k)$ converges as $k \to \infty$, $k \in K$, we must have $g^i(\bar{x}) = 0$ for $i \in J$. But then we have

$$\nabla f(\bar{x}) + \frac{\partial h(\bar{x})}{\partial x}\bar{\psi} + \sum_{i \in J} \bar{\lambda}^i \nabla g^i(\bar{x}) = 0 \tag{31}$$

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†We denote by e_i the i^{th} column of the m x m identity matrix.

with $\bar{x}$ feasible and $\bar{\lambda}^i \geq 0$ for all $i \in J$, i.e., $\bar{x}$ is a K-T point. But there is only one K-T point in $B(\hat{x},\hat{\varepsilon})$ and hence we must have $x = \hat{x}$, $\psi = \psi(\hat{x})$ and $\bar{\lambda}^i = \lambda^i(\hat{x})$ for all $i \in J$. But then (31) contradicts the strict complimentarity assumption at $\hat{x}$ since J is a proper subset of I(x). Consequently, there must exist a $c_1 \geq c_0$ and a $\delta_1 > 0$ such that (29) holds for all $x \in B(\hat{x},\hat{\varepsilon})$ and $c \geq c_1$ such that $J(x,c)$ is strictly contained in $I(\hat{x})$.

Next we consider all $x \in B(\hat{x},\hat{\varepsilon})$ such that $J(x,c) = I(\hat{x})$, with $c \geq c_1$. Suppose that (28) does not hold for all such $x \in B(\hat{x},\hat{\varepsilon})$. Then there exist sequences $x_k \to \bar{x}$, $x_k \in B(\hat{x},\hat{\varepsilon})$ and $c_k \to \infty$ such that $J(x_k,c_k) = I(\hat{x})$ and

$$\|\nabla_x F(x_k,c_k)\|/\|x_k-\hat{x}\| \to 0 \text{ as } k \to \infty \tag{32}$$

Let $b:\mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ be defined by

$$b(x,c) \triangleq \nabla f(x) + \frac{\partial h(x)}{\partial x}^T \psi(x) + \frac{\partial \psi(x)}{\partial x}^T h(x) + \sum_{i \in I(x)} \{\lambda^i(x)\nabla g^i(x) + g^i(x)\nabla\lambda^i(x)\} - \frac{1}{c} \sum_{i \in I^c(\hat{x})} \lambda^i(x)\nabla\lambda^i(x) \tag{33}$$

Then for our sequences $\{x_k\}$, $\{c_k\}$, we obtain,

$$\nabla_x F(x_k,c_k) = b(x_k,c_k) + c_k\{\frac{\partial h(x_k)}{\partial x}^T h(x_k) + \sum_{i \in I(\hat{x})} g^i(x_k)\nabla g^i(x_k)\} \tag{34}$$

Now, suppose (32) holds. Then $\nabla_x F(x_k,c_k) \to 0$ and we conclude that $\bar{x}$ is a K-T point. But $\hat{x}$ is the only K-T point in $B(\hat{x},\hat{\varepsilon})$ and hence we must have $\bar{x} = \hat{x}$. We recall that by Lemma 2, because $\hat{x}$ is a K-T point, we must have $\nabla_x F(\hat{x},c_k) = 0$ for all c_k and hence (since $h(\hat{x}) = 0$, $g^i(\hat{x}) = 0$ for $i \in I(\hat{x})$), $b(\hat{x},c_k) = 0$ for all c_k.

Hence, expanding (34) by the Taylor formula, we obtain

$$\|\nabla F(x_k,c_k)\|/\|x_k-\hat{x}\| = \|B_k(x_k-\hat{x}) + c_kG_k(x_k-\hat{x})\|/\|x_k-\hat{x}\| \tag{35}$$

where

$$B_k \triangleq \int_0^1 \frac{\partial b}{\partial x}(\hat{x}+s(x_k-\hat{x}),c_k)ds \tag{36a}$$

$$\begin{aligned} G_k \triangleq \int_0^1 \Bigg(\sum_{i \in I(\hat{x})} \{g^i(\hat{x}+s(x_k-\hat{x})) \frac{\partial^2 g^i}{\partial x^2}(\hat{x}+s(x_k-\hat{x})) \\ + \nabla g^i(\hat{x}+s(x_k-\hat{x}))\nabla g^j(\hat{x}+s(x_k-\hat{x}))^T\} + \frac{\partial h(x+s(x_k-x))^T}{\partial x} \frac{\partial h(\hat{x}+s(x_k-\hat{x}))}{\partial x} \\ + \sum_{i \in \underline{\ell}} h_i(\hat{x}+s(x_k-\hat{x})) \frac{\partial^2 h^i(\hat{x}+s(x_k-\hat{x}))}{\partial x^2}\Bigg) \end{aligned} \tag{36b}$$

Hence we get that

$$\|[B_k(x_k-\hat{x})/\|x_k-\hat{x}\| + c_kG_k(x_k-\hat{x})/\|x_k-\hat{x}\|]\| \to 0 \text{ as } k \to \infty \tag{37}$$

Let $K \subset \{0,1,2,...\}$ define a subsequence such that $(x_k-\hat{x})/\|x_k-\hat{x}\| \to d$ as $k \to \infty$, $k \in K$. Then, since G_k converges (see (42) below) and $c_k \to \infty$ as $k \to \infty$, it follows from (37) that $\lim G_k d = 0$, i.e., that

$$\sum_{i \in I(\hat{x})} \nabla g^i(\hat{x})\langle \nabla g^i(\hat{x}),d\rangle + \frac{\partial h(\hat{x})^T}{\partial x}(\frac{\partial h(\hat{x})}{\partial x} d) = 0 \tag{38}$$

Because of the linear independence assumption (ssumption 2) we conclude from (38) that

$$\langle \nabla g^i(\hat{x}),d\rangle = 0 \ \forall i \in I(\hat{x}), \text{ and } \frac{\partial h(\hat{x})}{\partial x} d = 0 \tag{39}$$

Now, from (37) again, we get

$$\lim_{h\to\infty} \{\langle d,B_k(x_k-\hat{x})/\|x_h-\hat{x}\|\rangle + c_k\langle d,G_k(x_k-\hat{x})/\|x_k-\hat{x}\|\rangle\} = 0 \tag{40}$$

Since $B_k \to \frac{\partial \bar{b}(\hat{x})}{\partial x}$ as $k \to \infty$, where

$$\bar{b}(x) \triangleq \nabla f(x) + \frac{\partial h(x)}{\partial x}^T \psi(x) + \sum_{i \in I(\hat{x})} [\lambda^i(x)\nabla g^i(x) + g^i(x)\nabla\lambda^i(x)]$$
$$+ \frac{\partial \psi(x)}{\partial x}^T h(x) \tag{41}$$

and

$$G_k \to \sum_{i \in I(\hat{x})} \nabla g^i(\hat{x})\nabla g^i(\hat{x})^T + \frac{\partial h(\hat{x})}{\partial x}^T \frac{\partial h(\hat{x})}{\partial x}, \text{ as } k \to \infty \tag{42}$$

we conclude from (37) that

$$\lim_{k\to\infty} \{\langle d, \frac{\partial \bar{b}(\hat{x})}{\partial x} d\rangle + c_k \sum_{i \in I(\hat{x})} \langle \nabla g^i(\hat{x}) d\rangle^2 + \|\frac{\partial h(\hat{x})}{\partial x} d\|^2\} = 0 \tag{43}$$

i.e. that $\langle d, \frac{\partial \bar{b}(\hat{x})}{\partial x} d\rangle = 0$. Now (see (5)), we obtain from (41) that

$$\frac{\partial \bar{b}(\hat{x})}{\partial x} = \frac{\partial^2 L}{\partial x^2}(\hat{x}, \lambda(\hat{x}), \psi(\hat{x})) + \frac{\partial h(\hat{x})}{\partial x}^T \frac{\partial \psi(\hat{x})}{\partial x} + \frac{\partial \psi(\hat{x})}{\partial x}^T h(\hat{x})$$
$$+ \sum_{i \in I(\hat{x})} (\nabla g^i(\hat{x})\nabla\lambda^i(\hat{x})^T + \nabla\lambda(\hat{x})\nabla g^i(\hat{x})^T) \tag{44}$$

Making use of (39), we now obtain that for any $c > 0$

$$0 = \langle d, \frac{\partial \bar{b}(\hat{x})}{\partial x} d\rangle = \langle d, \frac{\partial^2 L(\hat{x},\lambda(\hat{x}),\psi(\hat{x}))}{\partial x^2} d\rangle$$

But this contradicts Assumption 3. Hence there exists a $\delta_2 > 0$, and a $\hat{c} \geq c_1$ such that (28) holds for all $x \in B(\hat{x},\hat{\varepsilon})$ for all $c \geq \hat{c}$ such that $J(x,c) = I(\hat{x})$. Letting $\hat{\delta} = \min\{\delta_1,\delta_2\}$ and recalling that $0 < \hat{\varepsilon} \leq 1$ by construction, we conclude that $\delta_1 \geq \hat{\delta}\|x-\hat{x}\|$ for all $x \in B(\hat{x},\hat{\varepsilon})$ and hence (making use of (29)) that (28) holds with $\hat{c} > 0$ $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, as defined.

□

We can now collect our results.

<u>Theorem 2</u>: Suppose that Assumptions 1-3 hold. Then the function $t:\mathbb{R} + \mathbb{R}^+ \to \mathbb{R}$ defined by

$$t(x,c) \triangleq -\|\nabla_x F(x,c)\|^2 + \frac{1}{c}\{\|a(x,c)\|^2 + \|h(x)\|^2\} \tag{45}$$

with

$$a(x,c) \triangleq (g(x) + \frac{1}{c}\lambda(x))_+ - \frac{1}{c}\lambda(x) \tag{46}$$

satisfies the assumptions (ii)-(iv) of Theorem 1.

Proof: (i) For any $c > 0$, $t(\cdot,c)$ is continuous because of Assumption 1 and Proposition 1. (ii) Suppose $\hat{x} \in \{x \in \Delta_{c_j} | t(x,c_j) \leq 0\}$. Then by the corollary to Lemma 2, we have that $\hat{x} \in \Delta$. (iii) Suppose $\hat{x}$ is arbitrary. If $\hat{x} \notin \Delta$, i.e., $\hat{x}$ is not a K-T point, then, by Lemma 3, and assumption 3, there exist a $c_0 > 0$, a $\delta_0 > 0$ and an $\hat{\varepsilon} > 0$ such that there is no K-T point in $B(\hat{x},\hat{\varepsilon})$ and

$$\|\nabla_x F(x,c)\| \geq \delta_0 \quad \forall c \geq c_0, \quad \forall x \in B(\hat{x},\hat{\varepsilon}) \tag{47}$$

Since $\|h(x)\|^2$ and $\|a(x,c)\|^2$ are bounded on $B(\hat{x},\hat{\varepsilon})$ for all $c \geq c_0$, there exists a $\hat{c} \geq c_0$ such that $t(x,c) \leq 0$ for all $x \in B(\hat{x},\hat{\varepsilon})$ and all $c \geq \hat{c}$.

Next, suppose that $\hat{x}$ is a K-T point. Then, by Lemma 4, there exist a $c_1 > 0$ a $\hat{\delta} > 0$ and an $\hat{\varepsilon} > 0$ such that

$$\|\nabla_x F(x,c)\| \geq \hat{\delta}\,\|x-\hat{x}\| \quad \forall c \geq c_1, \ \forall x \in B(\hat{x},\hat{\varepsilon}) \tag{48}$$

Now, because of Assumption 1 and Proposition 1, $h(x)$ and $a(x,c)$ are both Lipschitz continuous on $B(\hat{x},\hat{\varepsilon})$, with constant, say, K, uniformly in $c \in [c,\infty)$. Since $h(\hat{x}) = 0$ and $a(\hat{x},c) = 0$, for all $c > 0$, we obtain

$$\|h(x)\| \leq K\|x-\hat{x}\| \tag{49}$$

$$\|a(x,c)\| \; K\|x-\hat{x}\| \tag{50}$$

Consequently, for any $x \in B(\hat{x},\hat{\varepsilon})$ and $c \geq c_1$ we obtain that

$$t(x,c) \leq (-\hat{\delta} + \frac{2K}{c}) \|x-\hat{x}\| \tag{51}$$

Obviously, if we set $\hat{c} = 2K/\hat{\delta}$, we find that $t(x,c) \leq 0$ for all $x \in B(\hat{x},\hat{\varepsilon})$ and all $c \geq \hat{c}$. This completes the proof. ¤

Thus, we have established that the proposed test function $t(\cdot,\cdot)$ can be used in a scheme conforming to the Algorithm Model. It remains to say a few words about the selection of an unconstrained optimization algorithm for minimizing $F(x,c)$. Since by (21), $\nabla_x F(x,c)$ involves terms in $\frac{\partial\lambda(x)}{\partial x}$ and $\frac{\partial\psi(x)}{\partial x}$, it is clear that to compute $\nabla_x F(x,c)$ we neeed to compute second derivatives of g and h. To justify this extra work, it would be nice to use an optimization algorithm with quadratic convergence properties. Fortunately, we find such an algorithm described in [10]. It is of the Gauss-Newton type and does not require eventual computation of third derivatives.

4. CONCLUSION

We have demonstrated that the general constrained optimization problem can be converted into an unconstrained problem having a continuously differentiable objective function. In addition we give a scheme for updating the penalty parameter c. If our algorithm produces a sequence that remains bounded, we show that the penalty parameter remains bounded and the algorithm is globally convergent in the sense that all accumulation points are Kuhn-Tucker points.

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