ADIABATIC AND STOCHASTIC MOTION OF CHARGED PARTICLES IN THE FIELD OF A SINGLE WAVE

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ABSTRACT

A unified treatment of particle motion in a wave field, is presented both for propagation oblique to a magnetic field and propagation across a magnetic field. It is shown that both cases are related to the dynamical motion of two nonlinear coupled oscillators whose frequencies are harmonically related at some values of their actions. The oblique propagation corresponds to accidental degeneracy of the coupled oscillators for which the oscillator frequencies are functions of the action in the absence of the perturbation. The motion with perpendicular propagation corresponds to intrinsic degeneracy for which the nonlinearity occurs only in the coupling term. For the former case islands in the phase space trajectories are formed around the resonant actions with the ratio of the bounce to cyclotron frequency $\Omega_B/\Omega \propto \varepsilon^{1/2}$ and the island amplitude $\Delta p_M \propto \varepsilon^{1/2}$, where ε is the coupling parameter. In the latter case $\Omega_B/\Omega \propto \varepsilon$ and $\Delta p_M \propto 1$. For both cases coupled oscillator theory[1] predicts overlap ($2\Delta p_M/\delta p = 1$, where δp is the resonance separation) at $\Omega_B/\Omega = 1/4$. With similar parameters this implies overlap at smaller coupling for accidental degeneracy, although the use of practical parameters, corresponding to real plasma waves, may reverse this situation. The importance of 2nd order resonances

in leading to stochasticity is demonstrated by showing that the 2nd order islands are exponentially small for small Ω_B/Ω, but that the ratio of the 2nd order island width to their separation becomes comparable to the 1st order island ratio near overlap. The result is a nearly complete randomization of the trajectories in the phase plane near $\Omega_B/\Omega = \frac{1}{4}$ leading to stochasticity and heating.

I. Introduction

Recently there has been renewed interest in the conditions under which the dynamical motion of a phase trajectory of two nonlinear coupled oscillators can lead to stochastic motion in the phase plane, that is, motion which tends to fill a three dimensional phase volume, rather than being restricted by a constant of the motion to a two dimensional torus within that volume. This renewed interest has arisen from numerical observations that waves propagating either obliquely[1] or perpendicularly[2,3] to a magnetic field can give rise to stochastic heating of particles gyrating in that field, providing appropriate field and dynamical variables are chosen. In both cases these results have been interpreted in terms of the breakdown of the invariants arising from system resonances[1,2,3].

The purpose of this paper is twofold. First, we show the relation between particle motion in a wave field in which the wave is propagating obliquely to a d.c. magnetic field[1] and particle motion for a wave propagating perpendicularly to the magnetic field.[2,3] In both cases we consider the wave field as a perturbation on the particle motion during a single gyroperiod rather than the dominant behavior.[4] Second, we demonstrate, in a more systematic way than previously[5] the mechanism by which the second and higher order islands lead to ergodic motion over portions of the available phase space. We shall show that the resonant forms of the basic Hamiltonian's for the oblique and perpendicular wave propagation, correspond to the two basic forms of the Hamiltonian for two nonlinearly coupled oscillators, which we have treated previously,[5] that of accidental and intrinsic degeneracy.

II. First Order Resonances

For an oblique wave, Smith and Kaufman[1] find the Hamiltonian for a particle as measured in the wave frame $v_z = \frac{\omega}{k_z}$

$$H = \frac{p_z^2}{2M} + \Omega p_\phi + e\Phi_o \sum_m J_m(k_\perp \rho) \sin(k_z z - m\phi) \qquad (1)$$

where p_z is the axial momentum, p_ϕ the magnetic moment, Ω the gyrofrequency, ϕ the gyrophase, k_z and $k_\perp$ the componants of wave vector along and perpendicular to the static field B_o, ρ the gyroradius, $e\Phi_o$ the magnitude of the perturbing potential, and J_ℓ the Bessel functions of the 1st kind. The Bessel function summation arises from the nonlinear phase shift resulting from the Larmor orbit extend over a spatially varying wave phase. For $\Phi_o = 0$, p_z and p_ϕ are constants of the motion.

For $k \perp B_o$ Karny and Bers[2] and Fukuyama et al[3] obtain the Hamiltonian

$$H = \Omega p_\phi + e\Phi_o \sum_m J_m(k\rho) \sin(\omega t - m\phi) \qquad (2)$$

One difference between the Hamiltonians in Eqs. (1) and (2) is that the 2nd is explicitly time dependent. However, introducing the near-identity canonical transformation with the generating function

$$F_2 = \phi \tilde{p}_\phi + \omega t \tilde{p}_\psi \qquad (3)$$

the new Hamiltonian given by

$$\tilde{H} = H + \frac{\partial F_2}{\partial t} \qquad (4)$$

becomes

$$\tilde{H} = \Omega\tilde{p}_\phi + \omega\tilde{p}_\psi + e\,\Phi_o \sum J_m(k_\rho) \sin(\psi - m\phi) \qquad (5)$$

which does not contain time explicitly.

Eqs. (1) and (5) can now be considered as the Hamiltonians of two-dimensional oscillators which are coupled through a perturbation term of coupling strength $e\Phi_o$, considered small. There remains, however, a fundamental difference between the forms of the two equations. In (5) both momenta in the unperturbed terms are linear and thus in action-angle form, while the z-momentum in (1) is quadratic. These two cases have been previously treated for a pair of weakly coupled oscillators in which a resonance existed between some harmonics of the two degrees of freedom.[5] A resonance in Eq. (1) represents an "accidental degeneracy" for a given value of $m=\ell$ and

$$p_z = M\ell\ \Omega/k_z \tag{6}$$

The existance of the perturbation causes p_z to vary, moving the particles' momentum away from resonance, and thus limiting the effect of the resonant perturbation. A resonance in Eq. (5) represents an "intrinsic degeneracy" for which the frequency shift with momentum occurs only due to the non-linearity within the purturbation itself, thus allowing much larger variations in momentum before shifting the particle away from resonance. We calculate these effects, explicitly, below.

Assume Eq. (6) is satisfied for some ℓ we transform Eq. (1) with a generating function

$$F_2 = (k_z z - \ell\phi)\ \hat{p}_z + \phi\hat{p}_\phi \tag{8}$$

to obtain the Hamiltonian in terms of the new canonical variables

$$\hat{z} = \frac{\partial F_2}{\partial \hat{p}_z} = k_z z - \ell\phi \qquad p_z = \frac{\partial F_2}{\partial z} = k_z \hat{p}_z \tag{9}$$

$$\hat{\phi} = \frac{\partial F_2}{\partial \hat{p}_\phi} = \phi \qquad p_\phi = \frac{\partial F_2}{\partial \phi} = (\hat{p}_\phi - \ell \hat{p}_z) \tag{10}$$

as

$$\hat{H} = \frac{k_z}{2M} \hat{p}_z^{\,2} + \Omega(\hat{p}_\phi - \ell\hat{p}_z) + e\Phi_o \sum_m J_m(k_\perp \rho) \sin[\hat{z} - (m-\ell)\hat{\phi}] \tag{11}$$

where ρ is implicity a function of the actions. Sufficiently close to a resonance $\hat{z}$ is slowly varying; we can average over a period of the $(m-\ell)\hat{\phi}$ coordinate, obtaining zero for the perturbation term except for $m = \ell$, for which Eq. (11) becomes

$$\hat{H} = \frac{k_z^{\,2}}{2M} \hat{p}_z^{\,2} + \Omega(\hat{p}_\phi - \ell\hat{p}_z) + \varepsilon M v_z^{\,2} J_\ell(k_\perp \rho) \sin \hat{z}. \tag{12}$$

Here we have written $\varepsilon = \frac{e\Phi_o}{Mv_z^{\,2}}$ which is assumed to be a small quantity. The motion is singular at $\hat{p}_{zo}$, $\hat{z}_o$ obtained as in Eq. (13):

$$\begin{aligned} \frac{\partial H}{\partial \hat{p}_z} = 0 \quad &\Rightarrow \quad \hat{p}_{zo} = \frac{\ell\Omega}{k_z^{\,2}/M} + 0(\varepsilon) \\ \frac{\partial H}{\partial \hat{z}} = 0 \quad &\Rightarrow \quad \hat{z}_o = \pm \frac{\pi}{2} . \end{aligned} \tag{13}$$

Expanding around the singularity as $\hat{p}_z = \hat{p}_{zo} + \Delta\hat{p}_z$ $\hat{z} = \hat{z}_o + \Delta\hat{z}$ we obtain the linearized Hamiltonian for the perturbed motion

$$\Delta\hat{H} = g^{(a)} \frac{(\Delta\hat{p}_z)^2}{2} + f \frac{(\Delta\hat{z})^2}{2} = \text{Const.} \tag{14}$$

where the superscript (a) refers to the accidentally degenerate case. Here

$$g^{(a)} = \frac{\partial^2 H}{\partial \hat{p}_z^2} = \frac{k_z^2}{M} + O(\varepsilon) \tag{15}$$

and

$$f = \frac{\partial^2 H}{\partial \hat{z}^2} = \pm\, \varepsilon M v_z^2\, J_\ell(k_\perp \rho) \ . \tag{16}$$

To lowest order in ε, the frequency near the elliptic singular point for the perturbed oscillation, corresponding to the Hamiltonian of Eq. (14), is [6]

$$\hat{\Omega}_{Bo}^{(a)} = (f g^{(a)})^{1/2} = \left[\varepsilon\, \ell^2\, J_\ell(k_\perp \rho)\right]^{1/2} \Omega \tag{18}$$

and the peak amplitude at the separatrix, as obtained from Eq. (12), is [6]

$$\Delta \hat{p}_{z_{Max}} = 2\left(\frac{f}{g^{(a)}}\right)^{1/2} = 2\left[\frac{\varepsilon M v_z^2\, J_\ell(k_\perp \rho) M}{k_z^2}\right]^{1/2} = \frac{2\, \hat{\Omega}_{Bo}}{g^{(a)}} \ . \tag{19}$$

Both Ω_B and $\Delta\hat{p}_z$ are proportional to the square root of the small perturbation. The spearation of adjecent resonances is given from Eqs. (6) and (9) as

$$\delta \hat{p}_z = \frac{M\Omega}{k_z^2} \tag{20}$$

such that the ratio of the momentum oscillation to momentum separation is, from Eqs. (15) and (19) and (20),

$$\frac{2\, \Delta \hat{p}_{zmax}}{\delta \hat{p}_z} = \frac{4\, \hat{\Omega}_{Bo}}{\Omega} \ . \tag{21}$$

A simple overlap condition, $\Delta\hat{p}/\delta\hat{p} \geq 1$ is then

$$\frac{\hat{\Omega}^{(a)}_{Bo}}{\Omega} = \frac{1}{4} \tag{22}$$

i.e. the frequency of the perturbed resonant oscillation is one fourth of the lowest fundamental frequency. These results, although not presented in the same way, have been obtained by Smith and Kaufman.

We now compare the results obtained from Eq. (1) with those for the intrinsicly degenerate system given by Eq. (5). Applying the generating function $F_2 = (\psi - \ell\phi)\,\hat{p}_\psi + \phi\hat{p}_\phi$, assuming sufficient closeness to resonance, as previously, to keep a single term in Eq. (5) after averaging, transformation equations analogous to Eqs. (9) and (10) then transform Eq. (5) to

$$\hat{H} = \Omega(\hat{p}_\phi - \ell\,\hat{p}_\psi) + \omega\,\hat{p}_\psi + \varepsilon M v_\perp^2 J_\ell(k\rho)\ \sin\hat{\psi} \tag{23}$$

where $\varepsilon = \frac{e\Phi_o}{Mv_\perp^2}$. The gyroradius ρ is a function of $\hat{p}_\psi$ and $\hat{p}_\phi$ through the transformation Eq. $p_\phi = \hat{p}_\phi - \ell\,\hat{p}_\psi)$ and the definition of ρ

$$\rho = [2\ p_\phi/M\Omega]^{1/2}\ .$$

Expanding around the singular point as previously, we obtain

$$\Delta\hat{H} = g^{(i)}(\Delta\hat{p}_\psi)^2 + f(\Delta\hat{\psi})^2 = \text{Const.}, \tag{24}$$

$$g^{(i)} = \varepsilon M v_\perp^2\ \frac{d^2 J_\ell(k\rho)}{d\hat{p}_\psi^{\,2}}\ , \tag{25}$$

and

$$f = \varepsilon M v_\perp^2 J_\ell, \tag{25}$$

where the superscript (i) is for the intrinsicly degenerate case. To lowest order in ε the frequency and momentum excursion are

$$\hat{\Omega}_{Bo}^{(i)} = (fg^{(i)})^{\frac{1}{2}} = \varepsilon M v_\perp^2 \left(J_\ell \frac{d^2 J_\ell}{d\hat{p}_\psi^2} \right)^{\frac{1}{2}} \tag{26}$$

and

$$\Delta\hat{p}_{max}^{(i)} = 2 \frac{f}{g^{(i)}} = 2 \left(J_\ell \Big/ \frac{d^2 J_\ell}{d\hat{p}_\psi^2} \right)^{\frac{1}{2}} = \frac{2\hat{\Omega}_{Bo}^{(i)}}{g^{(i)}} \ . \tag{27}$$

Comparing Eqs. (26) and (27) with Eqs. (18) and (19) we observe that for intrinsic degeneracy the frequency of the beat oscillation is of order ε, $\varepsilon^{\frac{1}{2}}$ slower than for accidental degeneracy, while the excursion in momentum of order unity, $\varepsilon^{-\frac{1}{2}}$ larger than for accidental degeneracy.

Unlike the situation for wave propagation at an angle to the magnetic field, the higher freqeuncy is fixed at ω, and thus there are no resonances at higher harmonics of Ω. For the Hamiltonian of Eq. (23), it is still possible to obtain resonance at a succession of values of $v_\perp$. This can be seen by setting the derivative of (23) equal to zero

$$\frac{\partial\hat{H}}{\partial\hat{p}_\theta} = \omega - \ell\Omega + \varepsilon M v_\perp^2 \frac{dJ_\ell(k_\rho)}{d\hat{p}_\psi} = 0 \tag{28}$$

which give the values of $v_\perp$ at the singularities of the motion. We note that these zeros can occur over a range of values of $k\rho$. In particular, for $\omega - \ell\Omega = 0$ they occur for

$$J_\ell'(k_\rho) = 0 \tag{29}$$

Although no first order resonant overlap exists, to lead to stochastic motion, resonances can occur between the island frequency and the fundamental cyclotron frequency, leading to 2nd order island formation and stochasticity. We shall show below, that this behavior if similar in character to the second order island formation for the accidentally degenerate motion.

II. Second Order Resonances

Second order islands play an important part in the development of the ergodic motion. Considering the linearized perturbed motion about the resonance given either by Eq. (14) or Eq. (24) the generating function

$$F_1 = \frac{1}{2} R \, \Delta p \cot \theta, \quad R = (f/g)^{\frac{1}{2}} \tag{31}$$

generates the canonical transformation to new variables

$$\begin{aligned} \Delta p &= (2\, I\, R)^{\frac{1}{2}} \cos \theta \\ \Delta q &= (2\, I/R)^{\frac{1}{2}} \sin \theta \end{aligned} \tag{32}$$

where Δp and Δq represent the perturbation momentum and position coordinates for either problem. The transformed Hamiltonian is

$$K_o = \hat{\Omega}_{Bo} \, I \tag{33}$$

which is in action-angle form. We can extend this result to the nonlinear region by expanding $\Delta\hat{H}$ to fourth order in $\Delta\hat{p}_z$ and $\Delta\hat{z}$ to obtain

$$K_o = K_o + K_2 \tag{34}$$

and

$$\hat{\Omega}_B = \hat{\Omega}_{Bo} + \frac{\partial K_2}{\partial I} \quad (35)$$

We have suppressed the rather lengthy details of the calculation of the K_2's, from 4th order perturbation theory, as they are not required for the following argument.

In localized regions of the phase space harmonics of the slow oscillation $\hat{\Omega}_B$ resonate with the slowest fundamental frequency Ω to produce new local distortions of the phase plane. We exhibit these resonances by explicitly reintroducing the oscillatory terms into the perturbation Hamiltonian

$$K = K_o + K_2 + \tilde{\Lambda} \cdot \quad (36)$$

Here

$$\tilde{\Lambda} = \varepsilon M v^2 \sum_{m(m\neq\ell)} J_m(k_\perp\rho) \sin\left[\hat{q}_o + (2I/R)^{\frac{1}{2}} \sin\theta - (m-\ell)\phi\right]$$

$$= \varepsilon M v^2 \sum_{m(m\neq\ell)} J_m(k_\perp\rho) \sum_n J_n\left[(2I/R)^{\frac{1}{2}}\right] e^{i(\hat{q}_o - (m-\ell)\phi + in\theta)} \quad (37)$$

and v^2 is either v_z^2 or $v_\perp^2$ for accidental or intrinsic resonance, respectively. Taking only the lowest order resonant term, as previously,

$$\tilde{\Lambda} \cong \Lambda_n \sin(n\theta-\phi) = \varepsilon J_{\ell+1}(k_\perp\rho) J_n\left[(2I/R)^{\frac{1}{2}}\right] \sin(n\theta-\phi), \quad (38)$$

transforming to a new locally slow variable

$$\hat{\theta} = n\theta - \phi \quad , \tag{39}$$

by a generating function as in Eq. (8), and expanding around the singularity, as in (19), we obtain the Hamiltonian for the secondary islands

$$\Delta K = g_s \frac{(\Delta \hat{I})^2}{2} + f_s \frac{(\Delta \hat{\theta})^2}{2} \tag{40}$$

where

$$g_s = \frac{\partial^2 K_2}{\partial \hat{I}^2} \; , \quad f_s = \Lambda_n , \quad \text{and} \quad \hat{I} = I/n. \tag{41}$$

The frequency and peak to peak momentum excursion of the second order islands are then given, as previously, by

$$\hat{\Omega}_{Bs} = (f_s g_s)^{\frac{1}{2}} \tag{42}$$

and

$$\Delta \hat{I}_m = \frac{2\,\Omega_{Bs}}{g_s} \tag{43}$$

Substituting for f_s and g_s in Eqs. (42) and (43) from Eqs. (41) and (38), we find that $\hat{\Omega}_{Bs}$ and ΔI_m are proportional through f_s to

$$\left\{ J_n \left[(2I/R)^{\frac{1}{2}} \right] \right\}^{\frac{1}{2}} = 0 \left[(1/n!)^{\frac{1}{2}} \right] \tag{44}$$

where the last relationship can be shown by expanding the Bessel function for large n.[5] Thus for large n (small ε) the factorial dominates and the islands become vanishingly small. However, the secondary resonances are also close together. Without calculating the perturbation Hamiltonian for the secondary resonances in detail we can compare the island width to the distance between islands, as previously. We calculate

$$\delta \hat{\Omega}_B = \frac{\partial^2 K_2}{\partial I^2} \delta I = \frac{\hat{\Omega}_B}{n} \tag{45}$$

for the distance between adjacent resonances. In terms of the hat variables the distance between resonances is given by

$$\delta \hat{\Omega}_B = \frac{\partial^2 K_2}{\partial (n\hat{I})^2} \delta (n\hat{I}) = \frac{g_s}{n} \delta \hat{I} \tag{46}$$

or

$$\delta \hat{I} = \frac{\hat{\Omega}_B}{g_s} \tag{47}$$

Substituting for ΔI_m from Eq. (43) we obtain, for overlap,

$$\frac{2\Delta \hat{I}_m}{\delta \hat{I}} = \frac{4\hat{\Omega}_{Bs}}{\hat{\Omega}_B} \geq 1 \tag{48}$$

which is identical in form to the overlap condition obtained for the primary resonances in Eq. (21) and (22). By induction, higher order resonances would also have the same form. Note that the secondary and higher order resonances are always accidentally degenerate. To determine when second order overlap occurs for increasing size of the purturbation, we muct explicitly calculate the secondary bounce frequency $\hat{\Omega}_{Bs}$ in terms of the perturbation amplitude. We calculate g_s in Appendix B, obtaining to lowest order in ε

$$g_s \simeq g/8 \tag{49}$$

Substituting g_s from Eq. (49) and f_s from Eq. (41) into Eq. (42), we obtain

$$\hat{\Omega}_{Bs} \simeq \left[\varepsilon M v^2 \; J_{\ell+1}(k_\perp \rho) \; J_n\left[(2I/R)^{\frac{1}{2}}\right] \; g/8\right]^{\frac{1}{2}} \tag{50}$$

For simplicity we take $J_{\ell+1} \simeq J_\ell$, and taking J_n at its maximum value at the separatrix, $J_n(\pi)$, we obtain using either Eqs. (16) and (18) or Eqs. (25) and (26)

$$\frac{\hat{\Omega}_{Bs}}{\Omega_B} = \frac{1}{n} \simeq \left(\frac{J_n(\pi)}{8}\right)^{\frac{1}{2}} . \tag{51}$$

For accidental degeneracy secondary islands would overlap first if Eq. (48) is satisfied before (22); i.e. with n = 4

$$\left(\frac{J_4(\pi)}{8}\right)^{\frac{1}{2}} \gtrsim \frac{1}{4} \tag{52}$$

which is marginally not satisfied. The important point, however, is that for either type of primary degeneracy the second order islands rapidly become important as the first order islands become large. For smaller values of perturbation it has been shown[3,8] that overlap exists near the island separatrices, giving bands of ergodicity in the phase plane that grow in area with the strength of the perturbation. The regions of ergodicity near secondary island resonances are very small until the first order resonances become large; they then increase rapidly leading to an ergodic phase plane with isolated adiabatic islands. This behavior has been confirmed numerically, for both oblique[1] and perpendicular[2] waves.

The basic results obtained here for perpendicular propagation, including the calculation of second order island amplitudes, have also been obtained by Fukuyama et al.,[3] but in a form emphasizing the stochastic regions near separatrices. It should also be noted that second order island calculations, in the neighborhood of an elliptic singularity, are in the same form for all coupled oscillator problems[5,9].

III. Discussion of Numerical Examples

In particular numerical examples for the two cases, Smith and Kaufman[1] found that stochasticity occurred for a perturbation amplitude considerably larger than that found by Karney and Bers.[2] This appears to be a surprising result in that $\hat{\Omega}_B/\Omega \propto \varepsilon^{1/2}$ for accidental degeneracy as seen from Eq. (18) while $\hat{\Omega}_B/\Omega \propto \varepsilon$ for intrinsic degeneracy as seen from Eq. (26). We would therefore expect the opposite result, that for ε small, the accidentally degenerate case should exhibit stochasticity first. To resolve this problem we analytically estimate the perturbation amplitude for overlap for the two cases in which comparable parameters are used. In Appendix A we calculate an approximate value of g in Eq. (25) for the intrinsic resonance problem, for the parameters that give maximum perturbation, as

$$g \cong \varepsilon M v_\perp^2 J_\ell(k\rho) \frac{\ell^4 \Omega^2}{M^2 v_\perp^4} \tag{53}$$

Substituting this result and f, from Eq. (25) into the expression for $\hat{\Omega}_B$ we obtain

$$\hat{\Omega}_B^{(i)} = \varepsilon \ell^2 J_\ell(k_\perp\rho) \Omega \tag{54}$$

Comparing this result with Eq. (18) for accidental degeneracy we find that the resonance conditions are

$$\varepsilon\, \ell^2\, J_\ell(k_\perp \rho) = \begin{cases} \dfrac{1}{n^2} & \text{accidental degeneracy} \\[2ex] \dfrac{1}{n} & \text{intrinsic degeneracy} \end{cases} \tag{55}$$

Thus, as expected, for identical ℓ and $k_\perp\rho$, and assuming $v_z = v_\perp$, the value of perturbation field for a given harmonic resonance is smaller in the case of accidental resonance. This situation reversed in the numerical examples studied, because a large value of ℓ=30 was used for the intrinsic resonance[2] (wave propagation perpendicular to B) while ℓ=1 was used to study the accidental resonance[(2)] (wave propagation at an angle to B) which corresponded to the physical plasma waves being studied. In both cases $k_\perp\rho$ was chosen to put J_ℓ near its maximum value; otherwise n remains large for all reasonable values of the perturbing field. In the intrinsicly degenerate case the stochasticity occurred due to the large amplitude of the 2nd and higher order islands, while interaction of both first and second order islands were important for accidental degeneracy.

IV. Transition Between Accidental and Intrinsic Degeneracy

There remains the question of the transition from accidental to intrinsic degeneracy as the wave direction approaches a normal to the magnetic field. The transition can be found by keeping the $O(\varepsilon)$ term in Eq. (15) for g

$$g = \frac{k_z^2}{M} + \varepsilon M v_z^2 \frac{\partial^2 J_\ell(k_\perp\rho)}{d^2 \hat{p}_z} \tag{56}$$

Approximating the Bessel function derivative as in Appendix A, we obtain

$$g \simeq \frac{k_z^2}{M} + \frac{\varepsilon M v_z^2 J_\ell \ell^2 (k_\perp \rho)^2 \Omega^2}{(M v_\perp^2)}$$

which, after substituting $k_z v_z = \ell\Omega$ and $k_\perp \rho = \ell$, becomes

$$g \simeq \frac{\ell^2 \Omega^2}{M v_z^2} \left[1 + \varepsilon \left(\frac{M v_z^2}{M v_\perp^2} \right)^2 \ell^2 J_\ell \right] . \quad (57)$$

The two terms are equal for

$$\frac{v_z^2}{v_\perp^2} = 1/(\varepsilon \ell^2 J_\ell)^{1/2} = n^{(a)} . \quad (58)$$

Assuming that $v_\perp$ is the characteristic velocity, we are considering resonance with an axial velocity class on the wings of the distribution function. In terms of the propagation vector

$$\frac{k_z}{k_\perp} = \left(\frac{1}{n^{(a)}} \right)^{1/2} . \quad (59)$$

For overlap we would put $n^{(a)} = 4$ in Eqs. (58) and (59).

Appendix A - Evaluation of $g^{(i)}$

We wish to evaluate

$$g^{(i)} = \epsilon M v_{\perp}^{2} \frac{d^2 J_{\ell}(k_{\perp}\rho)}{d\hat{p}_{\psi}^{2}} \qquad \text{A-1}$$

near its maximum value. Carying out the derivative implicitly we obtain

$$\frac{d^2 J_{\ell}(k\rho)}{d\hat{p}_{\psi}} = \frac{d^2 J_{\ell}(x)}{dx} \left(\frac{d(k\rho)}{d\hat{p}_{\psi}}\right)^2 + \frac{dJ_{\ell}(x)}{dx} \frac{d^2(k\rho)}{d\hat{p}_{\psi}^{2}} \; . \qquad \text{A-2}$$

To order the terms we assume that the Bessel function derivatives are chosen near their maximum values such that we can approximately set

$$\frac{d^2 J_{\ell}(x)}{dx} = \frac{dJ_{\ell}(x)}{dx} = J_{\ell}(x) \qquad \text{A-3}$$

obtaining for A-2

$$\frac{d^2 J_{\ell}(k\rho)}{d\hat{p}_{\psi}} \cong J_{\ell}(x)\left[\left(\frac{d(k\rho)}{d\hat{p}_{\psi}}\right)^2 + \frac{d^2(k\rho)}{d\hat{p}_{\psi}^{2}}\right] \; . \qquad \text{A-4}$$

We also have

$$\frac{d(k\rho)}{d\hat{p}_{\psi}} = \frac{k}{\rho} \frac{\ell}{M\Omega} \; .$$

Taking $k\rho \simeq \ell$ at the maximum of the Bessel Function we find the first term in A-4 is larger than the 2nd by ℓ, and assuming $\ell >> 1$ keep only the 1st term to obtain

$$\frac{d^2 J_\ell(k\rho)}{d\hat{p}_\psi^{\ 2}} = J_\ell \frac{\ell^4 \Omega^2}{M^2 v_\perp^{\ 4}} \quad . \qquad \text{A-5}$$

Substituting A-5 in A-1, Eq. (53) is obtained.

Appendix B - Evaluation of g_s

We wish to calculate g_s. From the Hamiltonian in Eq. (12) or Eq. (23) we calculate the fourth order terms in the expansion about the elliptic singularity

$$\hat{H}_2 = \varepsilon M v^2 \left[\frac{1}{4!} J_\ell (\Delta\hat{z})^4 - \frac{1}{4} \frac{\partial^2 J_\ell}{\partial \hat{p}_z^2} (\Delta\hat{p}_z)^2 (\Delta\hat{z})^2 + \frac{1}{4!} \frac{\partial^4 J_\ell}{\partial \hat{p}_z^4} (\Delta\hat{p}_z)^4 \right]$$

and trnasforming to action-angle form by using Eq. (32) and averaging, we have

$$K_2 = \langle \hat{H}_2 \rangle = \varepsilon M v^2 \left[\frac{1}{64} J_\ell \left(\frac{2J}{R} \right)^2 - \frac{1}{32} \frac{\partial^2 J_\ell}{\partial \hat{p}_z^2} (2J)^2 + \frac{1}{64} \frac{\partial^4 J_\ell}{\partial \hat{p}_z^4} (2RJ)^2 \right] .$$

By differentiating twice with respect to J we obtain

$$g_s = \varepsilon M v^2 \left[\frac{1}{8} J_\ell \frac{1}{R^2} - \frac{1}{4} \frac{\partial^2 J_\ell}{\partial \hat{p}_z^2} + \frac{1}{8} \frac{\partial^4 J_\ell}{\partial \hat{p}_z^4} R^4 \right] .$$

For accidental degeneracy $R^2 = f/g = 0(\varepsilon)$, then to lowest order in ε, $0(\varepsilon^\circ)$, we keep only the leading term giving

$$g_s^{(a)} = \frac{\varepsilon M v^2}{8} \frac{J_\ell}{f} g^{(a)} = \frac{g^{(a)}}{8}$$

where the superscript (a) stands for accidental degeneracy and the last equality was obtained by substituting for f from Eq. (16) or (25) For intrinsic degeneracy $f/g = 0(1)$, and all of the terms in g_s must be kept. Using the results from Appendix A we find that all terms scale in the same manner, such that for intrinsic degeneracy

$$g_s^{(i)} = 0\left[\frac{g^{(i)}}{8}\right]$$

Appendix C - Regions of validity for resonant and non-resonant transformations

In karney and Bers numerical example, treating particles moving in a perpendicularly propagating wave, a slightly off harmonic resonance frequency was chosen. This was required to compare the results with their analytic trajectories that were calculated from a first order invariant obtained from a non-resonant transformation.[7] Their transformation generated an invariant whose maximum rate of change can be estimated as

$$\tilde{p}_\phi = \frac{\varepsilon \ell \ J_\ell(\ell)}{\ell-\nu} \dot{p}_\phi \tag{56}$$

where $\nu = \omega/\Omega$ and the argument of the Bessel function was taken. Thus for near resonant fields ε must be at its maximum sufficiently small that

$$\left| \frac{\varepsilon \ell J_\ell(\ell)}{\ell-\nu} \right| << 1 \tag{57}$$

This restricts, the region of validity of their transformation near a resonance. The resonant transformation, on the other hand, requires that the applied frequency be sufficiently close to a resonance that the resonant term dominates the summation. The perturbation term can then shift the harmonic of the gyration frequency sufficiently to bring it into resonance with the applied frequency. This value can be estimated from Eq. (23) by noting that the singularity (resonance) occurs at

$$\frac{\partial \hat{H}}{\partial \hat{p}_0} = \omega - \ell\Omega + \varepsilon M v_\perp^2 \frac{\partial J_\ell(k\rho)}{\partial \hat{p}_0} = 0 \tag{58}$$

where the plus and minus correspond to the stable and unstable singular points, respectively. The condition for resonance is then that Eq. (29a) can be satisfied for some value of $\hat{p}_\theta$, and using the same approximations as in Eq. (29) we obtain

$$\left| \frac{\epsilon \ell^2 J_\ell(\ell)}{\ell - \nu} \right| > 1 \cdot \qquad (59)$$

For ℓ=1 the regions of validity of the two transformations are mutually exclusive. For large ℓ there can be considerable overlap between them. For an example in Karney and Bers $\ell-\nu \doteq .1$ ℓ = 30 and $\epsilon\ell^2$ = 1-3 we find that the left hand side of Eq. (57) lies between 0.1 and 0.3 and thus their transformation is well satisfied for the smaller perturbation and marginally for the larger. For the resonant transformation the inequality in Eq. (59) is well satisfied, the left hand side lying between 3 and 10.

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