A THEORY OF APPROXIMATE REASONING (AR)

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Summary

The theory of approximate reasoning outlined in this paper is concerned with the deduction of possibly imprecise conclusions from a set of imprecise premises.

The theory is based on a fuzzy logic, FL, in which the truth-values are linguistic, i.e., of the form true, not true, very true, more or less true, false, not very false, etc., and the rules of inference are approximate rather than exact. Furthermore, the premises are assumed to have the form of fuzzy propositions, e.g., "(X is much smaller than Y) is quite true," "If X is small is possible then Y is very large is very likely," etc. By using the concept of a possibility -- rather than probability -- distribution, such propositions are translated into expressions in PRUF (Possibilistic Relational Universal Fuzzy), which is a meaning representation language for natural languages.

An expression in PRUF is a procedure for computing the possibility distribution which is induced by a proposition in a natural language. By applying the rules of inference in PRUF to such distributions, other distributions are obtained which upon retranslation and linguistic approximation yield the conclusions deduced from the fuzzy premises.

The principal rules of inference in fuzzy logic are the projection principle, the particularization/conjunction principle, and the entailment principle. The application of these rules to approximate reasoning is described and illustrated by examples.

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1. Introduction

Informally, by approximate or, equivalently, fuzzy reasoning we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. Such reasoning is, for the most part, qualitative rather than quantitative in nature and almost all of it falls outside of the domain of applicability of classical logic.[1]

Approximate reasoning underlies the remarkable human ability to understand natural language, decipher sloppy handwriting, play games requiring mental and/or physical skill and, more generally, make rational decisions in complex and/or uncertain environments. In fact, it is the capability to reason in qualitative, imprecise terms that distinguishes human intelligence from machine intelligence. And yet, approximate reasoning has received little if any attention within psychology, philosophy, logic, artificial intelligence and other branches of cognitive sciences, largely because it is not consonant with the deeply entrenched tradition of precise reasoning in science and contravenes the widely held belief that precise, quantitative reasoning has the capability of solving the extremely complex and ill-defined problems which pervade the analysis of humanistic systems.

In earlier papers (see Zadeh 1973,1975,1976,1977), we have outlined a conceptual framework for approximate reasoning based on the notions of

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**Computer Science Division, Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720. Research supported by the National Science Foundation Grant MCS76-06693, the Naval Electronics Systems Command Contract N00039-77-C-0022, and the U.S. Army Research Office Grant DAHC04-76-G0056.

[1] A thorough exposition of the foundations of fuzzy reasoning may be found in Gaines (1976).

linguistic variable and fuzzy logic. In the present paper, a novel direction involving the concept of a possibility distribution will be described.[2] As will be seen in the sequel, the concept of a possibility distribution provides a natural basis for the representation of the meaning of propositions expressed in a natural language, and thereby serves as a convenient point of departure for the translation of imprecise premises into expressions in a language PRUF[3] to which the rules of inference associated with this language can be applied.

The theory of approximate reasoning which is outlined in the following sections is still in its initial stages of development. Consequently, our exposition of it in the present paper is informal in nature and our simple examples are intended merely to aid the reader in the understanding of the basic concepts and their applications within the theory. However, approximate reasoning and fuzzy logic appear to have the potential for many significant applications in the analysis of both humanistic and mechanistic systems, as is evidenced by the applications to control theory, pattern recognition and related fields which have already been reported in the literature. (See the appended bibliography.)

In what follows, our exposition of approximate reasoning begins with a brief discussion of the concept of a possibility distribution and its role in the translation of fuzzy propositions expressed in a natural language. In Section 3, the concept of a linguistic variable is introduced as a device for an approximate characterization of the values of variables and their interrelations. In Sections 4 and 5, we shall discuss some of the

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[2]An exposition of a theory of possibility based on the theory of fuzzy sets may be found in Zadeh (1977).

[3]PRUF is an acronym for Possibilistic Relational Universal Fuzzy. A brief discussion of some of the relevant aspects of PRUF is contained in Section 2.

basic aspects of fuzzy logic -- the logic that serves as a foundation for approximate reasoning -- and introduce the concepts of semantic equivalence and semantic entailment. Finally, in Section 6, we formulate the basic rules of inference in fuzzy logic and illustrate their application to approximate reasoning by a number of simple examples.

2. The Concept of a Possibility Distribution

A basic assumption which underlies our approach to approximate reasoning is that the imprecision which is intrinsic in natural languages is, in the main, possibilistic[4] rather than probabilistic in nature.

To illustrate the point, consider the proposition $p \triangleq X$ is an integer in the interval $[0,8]$.[5] Clearly, such a proposition does not associate a unique integer with X; rather, it indicates that any integer in the interval $[0,8]$ could possibly be a value of X, and that any integer not in the interval could not be a value of X.

This obvious observation suggests the following interpretation of p. The proposition "X is an integer in the interval [0,8]" induces a possibility distribution Π_X which associates with each integer n the possibility that n could be a value of X. Thus, for the proposition in question

$$\mathrm{Poss}\{X=n\} = 1 \quad \text{for} \quad 0 \leq n \leq 8$$

and

$$\mathrm{Poss}\{X=n\} = 0 \quad \text{for} \quad n < 0 \quad \text{or} \quad n > 8$$

where $\mathrm{Poss}\{X=n\}$ is an abbreviation for "The possibility that X may assume the value n." Note that the possibility distribution induced by p is uniform in the sense that the possibility values are equal to unity for n in $[0,8]$ and zero elsewhere.

Next, consider the fuzzy proposition $q \triangleq X$ is a small integer, in which small integer is a fuzzy set defined by, say,

$$\text{small integer} = 1/0 + 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5 \qquad (2.1)$$

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[4] The term "possibilistic" was coined by B.R. Gaines and L.J. Kohout in their paper on possible automata (1975).

[5] The symbol $\triangleq$ stands for "is defined to be," or "denotes."

in which + denotes the union rather than the arithmetic sum and a singleton of the form 0.8/2 signifies that the grade of membership of the integer 2 in the fuzzy set <u>small integer</u> is 0.8.[6]

As an extension of our interpretation of the nonfuzzy proposition p, we shall interpret q as follows. The proposition $q \triangleq X$ is a small integer induces a possibility distribution Π_X which equates the possibility of X taking a value n to the grade of membership of n in the fuzzy set <u>small integer</u>. Thus

$$\text{Poss}\{X=0\} = 1$$

$$\text{Poss}\{X=2\} = 0.8$$

$$\text{Poss}\{X=5\} = 0.2$$

and

$$\text{Poss}\{X=6\} = 0$$

More generally, we shall say that a fuzzy proposition of the form $p \triangleq X$ is F, where X is a variable taking values in a universe of discourse U and F is a fuzzy subset of U, <u>induces a possibility distribution</u> Π_X <u>which is equal to</u> F, i.e.,

$$\Pi_X = F \tag{2.2}$$

Thus, in essence, the possibility distribution of X is a fuzzy set which serves to define the possibility that X could assume any specified value in U. Stated more concretely, if $u \in U$ and $\mu_F: U \rightarrow [0,1]$ is the membership function of F, then the possibility that $X=u$ given "X is F" is

$$\text{Poss}\{X=u \mid X \text{ is } F\} = \mu_F(u) \ , \quad u \in U \ . \tag{2.3}$$

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[6]Expositions of the relevant aspects of the theory of fuzzy sets may be found in the books and papers noted in the bibliography, especially A. Kaufmann (1975), L. Negoita and D. Ralescu (1975), and L.A. Zadeh, K.S. Fu, K. Tanaka and M. Shimura (1975).

Since the concept of a possibility distribution coincides with that of a fuzzy set, possibility distributions may be manipulated by the rules governing the manipulation of fuzzy sets and, more particularly, fuzzy restrictions.[7] In what follows, we shall focus our attention only on those aspects of possibility distributions which are of relevance to approximate reasoning.

Possibility vs. Probability

What is the difference between possibility and probability? Intuitively, possibility relates to our perception of the degree of feasibility or ease of attainment, whereas probability is associated with the degree of belief, likelihood, frequency or proportion. Thus, what is possible may not be probable and what is improbable need not be impossible.[8] More importantly, however, the distinction between possibility and probability manifests itself in the different rules which govern their combinations, especially under the union. More specifically, if A is a nonfuzzy subset of U, and Π_X is the possibility distribution induced by the proposition "X is F," then the possibility measure, $\Pi(A)$, of A is defined as[9]

$$\Pi(A) \triangleq \mathrm{Poss}\{X \in A\} \triangleq \mathrm{Sup}_{u \in A}\, \mu_F(u) \tag{2.4}$$

and, more generally, if A is a fuzzy subset of U, then

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[7] A fuzzy restriction is a fuzzy set which serves as an elastic constraint on the values that may be assigned to a variable. A variable which is associated with a fuzzy restriction or, equivalently, with a possibility distribution, is a fuzzy variable.

[8] A more concrete statement of this relation is embodied in the possibility/probability consistency principle (see Zadeh, 1977).

[9] The possibility measure defined by (2.4) is a special case of the more general concept of a fuzzy measure defined by Sugeno (1974) and Terano and Sugeno (1975).

$$\Pi(A) \triangleq \text{Poss}\{X \text{ is } A\} \triangleq \text{Sup}_u(\mu_F(u) \wedge \mu_A(u)) \tag{2.5}$$

where μ_A is the membership function of A and $\wedge \triangleq$ min.

From the definition of possibility measure, it follows at once that, for arbitrary subsets A and B of U, the possibility measure of the union of A and B is given by

$$\Pi(A \cup B) = \Pi(A) \vee \Pi(B) \tag{2.6}$$

where $\vee \triangleq$ max. Thus, the possibility measure does not have the basic additivity property of probability measure, namely,

$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \text{ and } B \text{ are disjoint} \tag{2.7}$$

where $P(A)$ and $P(B)$ denote the probability measures of A and B, respectively.

Unlike probability, the concept of possibility in no way involves the notion of repeated experimentation. Thus, the concept of possibility is nonstatistical in character and, as such, is a natural concept to use when the imprecision or uncertainty in the phenomena under study are not susceptable of statistical analysis or characterization.

Possibility Assignment Equations

The reason why the concept of a possibility distribution plays such an important role in approximate reasoning relates to our assumption that a proposition in a natural language may be interpreted as an assignment of a fuzzy set to a possibility distribution. More specifically, if p is a proposition in a natural language, we shall say that p translates into a possibility assignment equation:

$$p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F \tag{2.8}$$

where $X_1,\ldots,X_n$ are variables which are explicit or implicit in p; $\Pi_{(X_1,\ldots,X_n)}$ is the possibility distribution of the n-ary variable $X \triangleq (X_1,\ldots,X_n)$; and F is a fuzzy relation, i.e., a fuzzy subset of the cartesian product $U_1 \times \cdots \times U_n$, where U_i, $i = 1,\ldots,n$, is the universe of discourse associated with X_i. In this context, the possibility assignment equation

$$\Pi_{(X_1,\ldots,X_n)} = F \tag{2.9}$$

will be referred to as the <u>translation</u> of p and, conversely, p will be said to be a <u>retranslation</u> of (2.9), in which case its relation to (2.9) will be represented as

$$p \leftarrow \Pi_{(X_1,\ldots,X_n)} = F\ . \tag{2.10}$$

In general, a proposition of the form $p \triangleq X$ is F, where X is the name of an object or a proposition, translates not into

$$p \dashrightarrow \Pi_X = F \tag{2.11}$$

but into

$$p \rightarrow \Pi_{A(X)} = F \tag{2.12}$$

where $A(X)$ is an implied attribute of X. For example,

$$\text{Joe is young} \rightarrow \Pi_{\text{Age(Joe)}} = \text{young} \tag{2.13}$$

$$\text{Maria is blond} \rightarrow \Pi_{\text{Color(Hair(Maria))}} = \text{blond} \tag{2.14}$$

$$\begin{gathered}\text{Max is about as tall as Jim} \rightarrow \\ \Pi_{\text{(Height(Max),Height(Jim))}} = \text{approximately equal}\end{gathered} \tag{2.15}$$

where young, blond and approximately equal are specified fuzzy relations (unary and binary) in their respective universes of discourse. More concretely, if u is a numerical value of the age of Joe, then (2.13) implies that

$$\text{Poss}\{\text{Age(Joe)} = u\} = \mu_{\text{young}}(u) \tag{2.16}$$

Similarly, if u is an identifying label for the color of hair, then (2.14) implies that

$$\text{Poss}\{\text{Color(Hair(Maria))} = u\} = \mu_{\text{blond}}(u) \tag{2.17}$$

while (2.15) signifies that

$$\text{Poss}\{\text{Height(Max)} = u, \text{Height(Jim)} = v\} = \mu_{\text{approximately equal}}(u,v) \tag{2.18}$$

where u and v are the generic values of the variables Height(Max) and Height(Jim), respectively.

Projection and Particularization

Among the operations that may be performed on a possibility distribution, there are two that are of particular relevance to approximate reasoning: projection and particularization.

Let $\Pi_{(X_1,\ldots,X_n)}$ denote an n-ary possibility distribution which is a fuzzy relation in $U_1 \times \cdots \times U_n$, with the possibility distribution function of $\Pi_{(X_1,\ldots,X_n)}$ (i.e., the membership function of $\Pi_{(X_1,\ldots,X_n)}$) denoted by $\pi_{(X_1,\ldots,X_n)}$ or, more simply, as π_X.

Let $s \triangleq (i_1,\ldots,i_k)$ be a subsequence of the index sequence $(1,\ldots,n)$ and let s' denote the complementary subsequence $s' \triangleq (j_1,\ldots,j_m)$ (e.g., for $n = 5$, $s = (1,3,4)$ and $s' = (2,5)$). In terms of such sequences, a k-tuple of the form $(A_{i_1},\ldots,A_{i_k})$ may be expressed in an abbreviated form as $A_{(s)}$. In particular, the variable $X_{(s)} = (X_{i_1},\ldots,X_{i_k})$ will be referred to as a k-ary subvariable of $X \triangleq (X_1,\ldots,X_n)$, with $X_{(s')} = (X_{j_1},\ldots,X_{j_m})$ being a subvariable complementary to $X_{(s)}$.

The projection of $\Pi_{(X_1,\ldots,X_n)}$ on $U_{(s)} \triangleq U_{i_1}\times\cdots\times U_{i_k}$ is a k-ary possibility distribution denoted by

$$\Pi_{X_{(s)}} \triangleq \mathrm{Proj}_{U_{(s)}}\ \Pi_{(X_1,\ldots,X_n)} \tag{2.19}$$

and defined by

$$\pi_{X_{(s)}}(u_{(s)}) \triangleq \mathrm{Sup}_{u_{(s')}}\ \pi_X(u_1,\ldots,u_n) \tag{2.20}$$

where $\pi_{X_{(s)}}$ is the possibility distribution function of $\Pi_{X_{(s)}}$. For example, for $n = 2$,

$$\pi_{X_1}(u_1) \triangleq \mathrm{Sup}_{u_2}\ \pi_{(X_1,X_2)}(u_1,u_2)$$

is the expression for the possibility distribution function of the projection of $\Pi_{(X_1,X_2)}$ on U_1. By analogy with the concept of a marginal probability distribution, $\Pi_{X_{(s)}}$ will be referred to as a marginal possibility distribution.

The importance of the concept of a marginal possibility distribution derives from the fact that $\Pi_{X_{(s)}}$ may be regarded as the possibility distribution of the subvariable $X_{(s)}$. Thus, stated as the projection principle (in Section 6), the relation between $X_{(s)}$ and $\Pi_{X_{(s)}}$ may be expressed as:

From the possibility distribution, $\Pi_{(X_1,\ldots,X_n)}$, of the variable $X \triangleq (X_1,\ldots,X_n)$, the possibility distribution $\Pi_{X_{(s)}}$ of the subvariable

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[10] Note that our use of the symbol $\Pi_{X_{(s)}}$ in (2.19) to denote the projection of Π_X on $U_{(s)}$ anticipates (2.21).

$X_{(s)} \triangleq (X_{i_1},\ldots,X_{i_k})$ may be inferred by projecting $\Pi_{(X_1,\ldots,X_n)}$ on $U_{(s)}$, i.e.,

$$\Pi_{X_{(s)}} = \mathrm{Proj}_{U_{(s)}} \Pi_{(X_1,\ldots,X_n)} \quad . \tag{2.21}$$

As a simple illustration, assume that $n = 3$, $U_1 = U_2 = U_3 = a+b$ or, more conventionally $\{a,b\}$, and $\Pi_{(X_1,X_2,X_3)}$ is expressed as a linear form

$$\Pi_{(X_1,X_2,X_3)} = 0.8aaa + 1aab + 0.6baa + 0.2bab + 0.5bbb \tag{2.22}$$

in which a term of the form 0.6baa signifies that

$$\mathrm{Poss}\{X_1 = b,\ X_2 = a,\ X_3 = a\} = 0.6 \tag{2.23}$$

To derive $\Pi_{(X_1,X_2)}$ from (2.22) it is sufficient to replace the value of X_3 in each term in (2.22) by the null string Λ. This yields

$$\begin{aligned}\Pi_{(X_1,X_2)} &= 0.8aa + 1aa + 0.6ba + 0.2ba + 0.5bb \\ &= 1aa + 0.6ba + 0.5bb\end{aligned} \tag{2.24}$$

and similarly

$$\begin{aligned}\Pi_{X_1} &= 1a + 0.6b + 0.5b \\ &= 1a + 0.6b\end{aligned} \tag{2.25}$$

Turning to the operation of particularization, let $\Pi_{(X_1,\ldots,X_n)} = F$ denote the possibility distribution of $X = (X_1,\ldots,X_n)$, and let $\Pi_{X_{(s)}} = G$ denote a specified possibility distribution (not necessarily the marginal distribution) of the subvariable $X_{(s)} = (X_{i_1},\ldots,X_{i_k})$.

Informally, by the particularization of $\Pi_{(X_1,\ldots,X_n)}$ is meant the modification of $\Pi_{(X_1,\ldots,X_n)}$ resulting from the stipulation that the possibility distribution of $\Pi_{X_{(s)}}$ is G. More specifically,

$$\Pi_{(X_1,\ldots,X_n)}[\Pi_{X_{(s)}} = G] \triangleq F \cap \bar{G} \tag{2.26}$$

where the left-hand member places in evidence the X_i (i.e., the attributes) which are particularized in $\Pi_{(X_1,\ldots,X_n)}$, while the right-hand member defines the effect of particularization, with $\bar{G}$ denoting the cylindrical extension of G, i.e., the cylindrical fuzzy set in $U_1 \times \cdots \times U_n$ whose projection on $U_{(s)}$ is G. Thus,

$$\mu_{\bar{G}}(u_1,\ldots,u_n) \triangleq \mu_G(u_{i_1},\ldots,u_{i_k}) \ , \quad (u_1,\ldots,u_n) \in U_1 \times \cdots \times U_n \tag{2.27}$$

As a simple illustration, consider the possibility distribution defined by (2.22) and assume that

$$\Pi_{(X_1,X_2)} = 0.4aa + 0.9ba + 0.1bb \tag{2.28}$$

In this case,

$$\bar{G} = 0.4aaa + 0.4aab + 0.9baa + 0.9\ bab + 0.1bba + 0.1bbb$$

$$F \cap \bar{G} = 0.4aaa + 0.4aab + 0.6baa + 0.2bab + 0.1bbb$$

and hence

$$\Pi_{(X_1,X_2,X_3)}[\Pi_{(X_1,X_2)} = G] = 0.4aaa + 0.4aab + 0.6baa + 0.2bab + 0.1bbb \tag{2.29}$$

In general, some of the variables in a particularized possibility distribution (or a fuzzy relation) are assigned fixed values in their respective universes of discourse, while others are associated with possibility distributions. For example, in the case of a fuzzy relation which characterizes the fuzzy set of men who are tall, blond and named Smith, the particularized relation has the form[11]

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[11] Note that the label of a relation is capitalized when it is desired to stress that it denotes a relation.

$$\text{MAN}[\text{Name} = \text{Smith};\ \Pi_{\text{Height}} = \text{TALL};\ \Pi_{\text{Color(Hair)}} = \text{BLOND}] \tag{2.30}$$

Similarly, the fuzzy set of men who have the above characteristics and, in addition, are approximately 30 years old, would be represented as

$$\text{MAN}[\text{Name} = \text{Smith};\ \Pi_{\text{Height}} = \text{TALL};\ \Pi_{\text{Color(Hair)}} = \text{BLOND};\ \Pi_{\text{Age}} = \text{APPROXIMATELY EQUAL } [\text{Age} = 30]] \tag{2.31}$$

In this case, the possibility distribution which is associated with the variable Age is in itself a particularized possibility distribution.

It should be noted that the representations exemplified by (2.30) and (2.31) are somewhat similar in appearance to those that are commonly employed in semantic network and higher order predicate calculi representations of propositions in a natural language.[12] An essential difference, however, lies in the use of possibility distributions in (2.30) and (2.31) for the characterization of values of fuzzy variables, and in the concrete specification of the manner in which a possibility distribution is modified by particularization.

Meaning and Information

Particularization as defined by (2.26) plays a particularly important role in PRUF[13] -- a language intended for the representation of the meaning of fuzzy propositions.

Briefly, an expression, P, in PRUF is, in general, a procedure for computing a possibility distribution. More specifically, let U be a

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[12] Expositions of such representations may be found in Newell and Simon (1972), Miller and Johnson-Laird (1976), Bobrow and Collins (1975), Minsky (1976), and other books and papers listed in the bibliography.

[13] A brief description of PRUF appears in Zadeh (1977). A more detailed exposition of PRUF will be provided in a forthcoming paper.

universe of discourse and let $\mathcal{R}$ be a set of relations in U. Then, the pair

$$D \triangleq (U,\mathcal{R}) \tag{2.32}$$

constitutes a database,[14] with P defined on a subset of relations in $\mathcal{R}$.

If p is an expression in a natural language and P is its translation in PRUF, i.e.,

$$p \rightarrow P ,$$

then the procedure P may be viewed as defining the meaning, M(p), of p, with the possibility distribution computed by P constituting the information, I(p), conveyed by p.[15]

As a simple illustration, consider the proposition

$$p \triangleq \text{John resides near Berkeley} \tag{2.33}$$

which in PRUF translates into

$$\text{RESIDENCE}[\text{Subject} = \text{John};\ \Pi_{\text{location}} = \text{Proj}_{\mu \times \text{City1}}\ \text{NEAR}[\text{City2} = \text{Berkeley}]] \tag{2.34}$$

where NEAR is a fuzzy relation with the frame[16] NEAR ‖ City1 | City2 | μ |

and the expression $\text{Proj}_{\mu \times \text{City1}}\ \text{NEAR}[\text{City2} = \text{Berkeley}]$ represents the fuzzy set of cities which are near Berkeley.

The expression in PRUF represented by (2.34)) is, in effect, a procedure for computing the possibility distribution of the location of

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[14] As defined here, the concept of a database is related to that of a possible world in modal logic (see Hughes and Cresswell, 1968; Miller and Johnson-Laird, 1976).

[15] The procedure defined by an expression in PRUF and the possibility distribution which it yields are analogous to the intension and extension of a predicate in two-valued logic. (See Cresswell, 1975.) When meaning is used loosely, no differentiation between M(p) and I(p) is made.

[16] The frame of a fuzzy relation exhibits its name together with the names of its variables (i.e., attributes) and μ -- the grade of membership of each tuple in the relation.

residence of John. Thus, given a relation NEAR, it will return a possibility distribution of the form ($\pi \triangleq$ possibility-value)

RESIDENCE	Subject	Location	π
	John	Oakland	1
	John	Palo Alto	0.6
	John	San Jose	0.2
	John	Orinda	0.8
	...	...	...

which may be regarded as the information conveyed by the proposition "John resides near Berkeley."

PRUF plays an essential role in approximate reasoning because it serves as a basis for translating the fuzzy premises expressed in a natural language into possibility assignment equations to which the rules of inference in approximate reasoning can be applied in a systematic fashion. In Section 4, we shall discuss in greater detail some of the basic translation rules in fuzzy logic which constitute a small subset of the translation rules in PRUF. This brief exposition of PRUF will suffice for our purposes in the present paper.

We turn next to the concept of a linguistic variable -- a concept that plays a basic role in approximate reasoning, fuzzy logic and the linguistic approach to systems analysis.

3. The Concept of a Linguistic Variable

In describing the behavior of humanistic -- that is, human-centered -- systems, we generally use words rather than numbers to characterize the values of variables as well as the relations between them. For example, the age of a person may be described as very young, intelligence as quite high, the relation with another person as not very friendly, and appearance as quite attractive.

Clearly, the use of words in place of numbers implies a lower degree of precision in the characterization of the values of a variable. In some instances, we elect to be imprecise because there is no need for a higher degree of precision. In most cases, however, the imprecision is forced upon us by the fact that there are no units of measurement for the attributes of an object and no quantitative criteria for representing the values of such attributes as points on an anchored scale.

Viewed in this perspective, the concept of a linguistic variable may be regarded as a device for systematizing the use of words or sentences in a natural or synthetic language for the purpose of characterizing the values of variables and describing their interrelations. In this role, the concept of a linguistic variable serves a basic function in approximate reasoning both in the representation of values of variables and in the characterization of truth-values, probability-values and possibility-values of fuzzy propositions.

In this section, we shall focus our attention only on those aspects of the concept of a linguistic variable which have a direct bearing on approximate reasoning. More detailed discussions of the concept of a linguistic variable and its applications may be found in Zadeh (1973,1975), Wenstop (1975,1976), Mamdani and Assilian (1975,1976), Procyk (1976), and other papers listed in the bibliography.

As a starting point for our discussion, it is convenient to consider a variable such as Age, which may be viewed both as a numerical variable ranging over, say, the interval [0,150], and as a linguistic variable which can take the values young, not young, very young, not very young, quite young, old, not very young and not very old, etc. Each of these values may be interpreted as a label of a fuzzy subset of the universe of discourse $U = [0,150]$, whose base variable, u, is the generic numerical value of Age.

Typically, the values of a linguistic variable such as Age are built up of one or more primary terms (which are the labels of primary fuzzy sets[17]), together with a collection of modifiers and connectives which allow a composite linguistic value to be generated from the primary terms. Usually, the number of such terms is two, with one being an antonym of the other. For example, in the case of Age, the primary terms are young and old, with old being the antonym of young.

A basic assumption underlying the concept of a linguistic variable is that the meaning of the primary terms is context-dependent whereas the meaning of the modifiers and connectives is not. Furthermore, once the meaning of the primary terms is specified (or "calibrated") in a given context, the meaning of composite terms such as not very young, not very young and not very old, etc., may be computed by the application of a semantic rule.

Typically, the term-set, that is, the set of linguistic values of a linguistic variable, comprises the values generated from each of the primary terms together with the values generated from various combinations of the

[17] In the case of humanistic systems, primary fuzzy sets play a role that is somewhat analogous to that of physical units in the case of mechanistic systems.

primary terms. For example, in the case of Age, a partial list of the linguistic values of Age is the following.

young	old	not young nor old
not young	not old	not very young and not very old
very young	very old	young or old
not very young	not very old	not young or not old
quite young	quite old	- - -
more or less young	more or less old	
extremely young	extremely old	
- - -	- - -	

What is important to observe is that most linguistic variables have the same basic structure as Age. For example, on replacing young with tall and old with short, we obtain the list of linguistic values of the linguistic variable Height. The same applies to the linguistic variables Weight (heavy and light), Appearance (beautiful and ugly), Speed (fast and slow), Truth (true and false), etc., with the words in parentheses representing the primary terms.

As is shown in Zadeh (1973,1975), a linguistic variable may be characterized by an attributed grammar (see Knuth, 1968; Lewis et al, 1974) which generates the term-set of the variable and provides a simple procedure for computing the meaning of a composite linguistic value in terms of the primary fuzzy sets which appear in its constituents.

As an illustration, consider the attributed grammar shown below in which S, B, C, D and E are nonterminals; not, and, a and b are terminals; a and b are the primary terms (and also the primary fuzzy sets); subscripted symbols are the fuzzy sets which are labeled by the corresponding nonterminals, with $L \triangleq$ left (i.e., pertaining to the antecedent), $R \triangleq$ right (i.e., pertaining to consequent); and a production of the form

$$S \rightarrow S \text{ and } B \quad : \quad S_L = S_R \cap B_R \tag{3.1}$$

signifies that the fuzzy set which is the meaning of the antecedent, S, is the intersection of S_L, the fuzzy set which is the meaning of the consequent S , and B_R, the fuzzy set which is the meaning of the consequent B.

$$\begin{array}{lcll}
S \rightarrow B & : & S_L = B_R & \\
S \rightarrow S \text{ and } B & : & S_L = S_R \cap B_R & \\
B \rightarrow C & : & B_L = C_R & \\
B \rightarrow \text{not } C & : & B_L = C'_R & (\triangleq \text{ complement of } C_R) \\
C \rightarrow S & : & C_L = S_R & \\
C \rightarrow D & : & C_L = D_R & \\
C \rightarrow E & : & C_L = E_R & \\
D \rightarrow \text{very } D & : & D_L = D_R^2 & (\triangleq \text{ square of } D_R) \\
E \rightarrow \text{very } E & : & D_L = E_R^2 & (\triangleq \text{ square of } E_R) \\
D \rightarrow a & : & D_L = a & \\
E \rightarrow b & : & E_L = b &
\end{array} \tag{3.2}$$

The grammar in question generates the linguistic values exemplified by the list:

a	b	a and b
not a	not b	not a and b
very a	very b	not a and not b
not very a	not very b	not very a and not very b
not very very a	not very very b	- - - - -
- - - - -	- - - - -	

In general, to compute the meaning of a linguistic value, ℓ, generated by the grammar, the meaning of each node of the syntax tree of ℓ is computed -- by the use of equations (3.2) -- in terms of the meanings of its immediate descendants. In most cases, however, this can be done by inspection -- which involves a straightforward application of the translation rules which will be formulated in Section 4. Thus, we readily obtain, for example:

$$\text{not very a} \rightarrow (a^2)' \qquad (3.3)$$

$$\text{not very a and not very b} \rightarrow (a^2)' \cap (b^2)'$$

where a' is the complement of a and a^2 is defined by

$$\mu_{a^2}(u) = (\mu_a(u))^2 , \quad u \in U . \qquad (3.4)$$

To characterize the primary fuzzy sets a and b, it is frequently convenient to employ standardized membership functions with adjustable parameters. One such function is the S-function, $S(u;\alpha,\beta,\gamma)$, defined by

$$\begin{aligned} S(u;\alpha,\beta,\gamma) &= 0 && \text{for } u \leq \alpha \\ &= 2\left(\frac{u-\alpha}{\gamma-\alpha}\right)^2 && \text{for } \alpha \leq u \leq \beta \\ &= 1 - 2\left(\frac{u-\gamma}{\gamma-\alpha}\right)^2 && \text{for } \beta \leq u \leq \gamma \\ &= 1 && \text{for } u \geq \gamma \end{aligned} \qquad (3.5)$$

where the parameter $\beta \triangleq \frac{\alpha+\gamma}{2}$ is the crossover point, that is, the value of u at which $S(u;\alpha,\beta,\gamma) = 0.5$. For example, if $a \triangleq$ young and $b \triangleq$ old, we may have (see Fig. 1)

$$\mu_{young} = 1 - S(20,30,40) \tag{3.6}$$

and

$$\mu_{old} = S(40,55,70) \tag{3.7}$$

in which the argument u is suppressed for simplicity. Thus, in terms of (3.6), the translation of the proposition $p \triangleq$ Joe is young (see (2.13)), may be expressed more concretely as

$$\text{Joe is young} \rightarrow \pi_{Age(Joe)} = 1 - S(20,30,40) \tag{3.8}$$

where $\pi_{Age(Joe)}$ is the possibility distribution function of the linguistic variable Age(Joe). Similarly,

$$\text{Joe is not very young} \rightarrow \pi_{Age(Joe)} = 1 - \left(1 - S(20,30,40)\right)^2 \tag{3.9}$$

An important aspect of the concept of a linguistic variable relates to the fact that, in general, the term-set of such a variable is not closed under the various operations that may be performed on fuzzy sets, e.g., union, intersection, product, etc. For example, if ℓ is a linguistic value of a variable X, then, in general, ℓ^2 is not in the term-set of X.

The problem of finding a linguistic value of X whose meaning approximates to a given fuzzy subset of U is called the problem of <u>linguistic approximation</u> (<u>see</u> Zadeh, 1975; Wenstop, 1975; Procyk, 1976). We shall not discuss in the present paper the ways in which this nontrivial problem can be approached, but will assume that linguistic approximation is implicit in the retranslation of a possibility distribution (see (2.10)) into a proposition expressed in a natural language.

4. Fuzzy Logic (FL)

In a broad sense, fuzzy logic is the logic of approximate reasoning; that is, it bears the same relation to approximate reasoning that two-valued logic does to precise reasoning.

In this section, we shall focus our attention on a particular fuzzy logic, FL, whose truth-values are linguistic, i.e., are expressible as the values of a linguistic variable Truth whose base variable takes values in the unit interval. In this sense, the base logic for FL is Lukasiewicz's L_{Aleph_1} logic whose truth-value set is the interval [0,1].

The principal constituents of FL are the following: (i) Translation rules; (ii) Valuation rules and (iii) Inference rules.

By translation rules is meant a set of rules which yield the translation of a modified or composite proposition from the translations of its constituents. For example, if p and q are fuzzy propositions which translate into (see (2.8))

$$p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F \tag{4.1}$$

and

$$q \rightarrow \Pi_{(Y_1,\ldots,Y_m)} = G \tag{4.2}$$

respectively, then the rule of conjunctive composition -- which will be stated at a later point in this section -- yields the translation of the composite proposition "p and q."

By valuation rules is meant the set of rules which yield the truth-value (or the probability-value or the possibility-value) of a modified or composite proposition from the specification of the truth-values (or probability-values or possibility-values) of its constituents. A typical example of a valuation rule is the conjunctive valuation rule which expresses the truth-value of the composite proposition "p and q" as a function of the truth-values of p and q -- e.g., <u>not very true</u> and <u>quite true</u>, respectively.

The principal rules of inference in FL are: (a) The projection principle; (b) The particularization/conjunction principle; and (c) The entailment principle. In combination, these rules lead to the <u>compositional rule of inference</u> which may be viewed as a generalization of the <u>modus ponens</u>.

In what follows, we shall discuss briefly only those aspects of fuzzy logic which are of direct relevance to approximate reasoning. A more detailed discussion of FL may be found in Zadeh (1975) and Bellman and Zadeh (1977).

<u>Translation Rules</u>

The translation rules in FL may be divided into several basic categories. Among these are:

Type I. Rules pertaining to modification

Type II. Rules pertaining to composition

Type III. Rules pertaining to quantification

Type IV. Rules pertaining to qualification

Simple examples of propositions to which the rules in question apply are the following:

Type I. X is very small.
Therese is highly intelligent.

Type II. X is small and Y is large.
If X is small then Y is large.

Type III. Most Swedes are tall.
Many men are taller than most men.

Type IV. John is tall is very true.
John is tall is not very likely.
John is tall is quite possible.

In combination, the rules in question may be applied to the translation of more complex propositions exemplified by:

If ((X is small and Y is large) is very likely) then (Z is very large is not very likely).

((Many men are taller than most men) is very true) is quite possible.

Rules of Type I

A basic rule of Type I is the modifier rule, which may be stated as follows.

Let X be a variable taking values in $U = \{u\}$, let F be a fuzzy subset of U, and let p be a proposition of the form "X is F." If the translation of p is expressed by

$$X \text{ is } F \rightarrow \Pi_X = F \tag{4.3}$$

then the translation of the modified proposition "X is mF," where m is a modifier such as not, very, more or less, etc., is given by

$$X \text{ is } mF \rightarrow \Pi_X = F^+ \tag{4.4}$$

where F^+ is a modification of F induced by m. More specifically,

[18]More detailed discussions of various types of modifiers may be found in Zadeh (1972,1975), Lakoff (1973,1975), Wenstop (1975), McVicar-Whalen (1975), Hersh and Caramazza (1976), and other papers listed in the bibliography.

$$\text{If } m = \text{not}, \text{ then } F^{+} = F' \triangleq \text{complement of } F \tag{4.5}$$

$$\text{If } m = \text{very}, \text{ then } F^{+} = F^{2} \tag{4.6}$$

where[19]

$$F^{2} = \int_{U} \mu_F^{2}(u)/u \tag{4.7}$$

$$\text{If } m = \text{more or less}, \text{ then } F^{+} = \sqrt{F} \tag{4.8}$$

where

$$\sqrt{F} = \int_{U} \sqrt{\mu_F(u)}/u \tag{4.9}$$

or, alternatively,

$$F^{+} = \int_{U} \mu_F(u) K(u) \tag{4.10}$$

where $K(u)$ is the kernel of more or less (see Zadeh, 1972) .

As a simple illustration, consider the proposition "X is small," where small is defined by

$$\text{small} = 1/0 + 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5 \tag{4.11}$$

Then

$$X \text{ is very small} \rightarrow \Pi_X = F^{+} \tag{4.12}$$

where

$$F^{+} = F^{2} = 1/0 + 1/1 + 0.64/2 + 0.36/3 + 0.16/4 + 0.04/5 \tag{4.13}$$

It is important to note that (4.6) and (4.8) should be regarded merely as standardized default definitions which may be replaced by other definitions

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[19]The "integral" representation of a fuzzy set in the form $F = \int_U \mu_F(u)/u$ signifies that F is a union of the fuzzy singletons $\mu_F(u)/u$, $u \in U$, where μ_F is the membership function of F. Thus, (4.7) means that the membership function of F^2 is the square of that of F.

whenever they do not fit the desired sense of the modifier m. Another point that should be noted is that X in (4.3) need not be a unary variable. Thus, (4.3) subsumes propositions of the form "X and Y are F," as in "X and Y are close," where CLOSE is a fuzzy binary relation in $U \times U$. Thus, if

$$\text{X and Y are close} \rightarrow \Pi_{(X,Y)} = \text{CLOSE} \tag{4.14}$$

then

$$\text{X and Y are very close} \rightarrow \Pi_{(X,Y)} = \text{CLOSE}^2 \tag{4.15}$$

Rules of Type II

Compositional rules of Type II pertain to the translation of a proposition p which is a composition of propositions q and r. The most commonly exployed modes of composition are: conjunction, disjunction and conditional composition (or implication). The translation rules for these modes of composition are as follows.[20]

Let X and Y be variables taking values in U and V, respectively, and let F and G be fuzzy subsets of U and V. If

$$\text{X is F} \rightarrow \Pi_X = F \tag{4.16}$$

$$\text{Y is G} \rightarrow \Pi_Y = G \tag{4.17}$$

then

(a) $$\text{X is F and Y is G} \rightarrow \Pi_{(X,Y)} = \bar{F} \cap \bar{G} = F \times G \tag{4.18}$$

(b) $$\text{X is F or Y is G} \rightarrow \Pi_{(X,Y)} = \bar{F} + \bar{G} \tag{4.19}$$

and (c_1) $$\text{If X is F then Y is G} \rightarrow \Pi_{(X,Y)} = \bar{F}' \oplus \bar{G} \tag{4.20}$$

or (c_2) $$\text{If X is F then Y is G} \rightarrow \Pi_{(X,Y)} = F \times G + F' \times V \tag{4.21}$$

where $\Pi_{(X,Y)}$ is the possibility distribution of the binary variable (X,Y),

[20] We are tacitly assuming that the compositions in question are noninteractive in the sense defined in Zadeh (1975).

$\bar{F}$ and $\bar{G}$ are the cylindrical extensions of F and G, respectively, i.e.,

$$\bar{F} = F \times V \tag{4.22}$$

$$\bar{G} = U \times G \;; \tag{4.23}$$

$F \times G$ is the cartesian product of F and G, which may be expressed as $\bar{F} \cap \bar{G}$ and is defined by

$$\mu_{F \times G}(u,v) = \mu_F(u) \wedge \mu_G(v) \;, \quad u \in U, \quad v \in V \;; \tag{4.24}$$

$+$ is the union, and $\oplus$ is the bounded sum, i.e.,

$$\mu_{\bar{F}' \oplus \bar{G}}(u,v) = 1 \wedge (1 - \mu_F(u) + \mu_G(v)) \tag{4.25}$$

where $+$ and $-$ denote the arithmetic sum and difference. Note that there are two interpretations of the conditional composition, (c_1) and (c_2). Of these, (c_1) is consistent with the definition of implication in L_{Aleph_1} logic, while (c_2) corresponds to the table

X	Y
F	G
F'	V

As a very simple illustration, assume that $U = V = 1 + 2 + 3$ [21]

$$F \triangleq \text{small} \triangleq 1/1 + 0.6/2 + 0.1/3 \tag{4.26}$$

$$G \triangleq \text{large} \triangleq 0.1/1 + 0.6/2 + 1/3$$

Then (4.18), (4.19), (4.20) and (4.21) yield

[21] To be consistent with our notation for fuzzy sets, a finite nonfuzzy set $U = \{u_1, \ldots, u_n\}$ may be expressed as $U = u_1 + \cdots + u_n$.

$$X \text{ is small and } Y \text{ is large} \rightarrow \Pi_{(X,Y)} = 0.1/(1,1) + 0.6/(1,2) + 1/(1,3) + 0.1/(2,1) + 0.6/(2,2) + 0.6/(2,3) + 0.1/(3,1) + 0.1/(3,2) + 0.1/(3,3)$$

$$X \text{ is small or } Y \text{ is large} \rightarrow \Pi_{(X,Y)} = 1/(1,1) + 1/(1,2) + 1/(1,3) + 0.6/(2,1) + 0.6/(2,2) + 1/(2,3) + 0.1/(3,1) + 0.6/(3,2) + 1/(3,3)$$

$$\text{If } X \text{ is small then } Y \text{ is large} \rightarrow \Pi_{(X,Y)} = 0.1/(1,1) + 0.6/(1,2) + 1/(1,3) + 0.5/(2,1) + 1/(2,2) + 1/(2,3) + 1/(3,1) + 1/(3,2) + 1/(3,3)$$

$$\text{If } X \text{ is small then } Y \text{ is large} \rightarrow \Pi_{(X,Y)} = 0.1/(1,1) + 0.6/(1,2) + 1/(1,3) + 0.4/(2,1) + 0.6/(2,2) + 0.6/(2,3) + 0.9/(3,1) + 0.9/(3,2) + 0.9/(3.3)$$

Rules of Type III

Quantificational rules of Type III apply to propositions of the general form

$$p \triangleq QX \text{ are } F \tag{4.27}$$

where Q is a fuzzy quantifier (e.g., most, many, few, some, almost all, etc.), X is a variable taking values in U, and F is a fuzzy subset of U. Simple examples of (4.27) are: "Most X's are small," "Some X's are small," "Many X's are very small." A somewhat less simple example is: "Most large X's are much smaller than α," where α is a specified number.

In general, a fuzzy quantifier is a fuzzy subset of the real line. However, when Q relates to a proportion, as is true of most, it may be represented as a fuzzy subset of the unit interval. Thus, the membership function of $Q \triangleq$ most may be represented as, say,

$$\mu_{most} = S(0.5, 0.7, 0.9) \tag{4.28}$$

where the S-function is defined by (3.5).

In order to be able to translate propositions of the form (4.27), it is necessary to define the cardinality of a fuzzy set, i.e., the number (or the proportion) of elements of U which are in F. When U is a finite set $\{u_1,\ldots,u_N\}$, a possible extension of the concept of cardinality of a nonfuzzy set -- to which we shall refer as fuzzy cardinality -- is the following. Let

$$F = \sum_{\alpha} \alpha F_{\alpha} \tag{4.29}$$

be the resolution[22] of F into its level-sets, that is,

$$F_{\alpha} \triangleq \{u | \mu_F(u) \geq \alpha\} \tag{4.30}$$

where αF_{α} is a fuzzy set defined by

$$\mu_{\alpha F_{\alpha}} = \alpha \mu_{F_{\alpha}} \tag{4.31}$$

and $\sum_{\alpha}$ denotes the union of the αF_{α} over $\alpha \in [0,1]$. Let $|F_{\alpha}|$ denote the cardinality of the nonfuzzy set F_{α}. Then, the fuzzy cardinality of F is denoted by $|F|_f$ and is defined to be the fuzzy subset of $\{0,1,2,\ldots\}$ expressed by

$$|F|_f = \sum_{\alpha} \alpha / |F_{\alpha}| \tag{4.32}$$

As a simple example, consider the fuzzy subset small defined by (2.1). In this case,

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[22] A discussion of the resolution of fuzzy sets may be found in Zadeh (1971).

$$F_1 = 0+1 \quad , \quad |F_1| = 2$$
$$F_{0.8} = 0+1+2 \quad , \quad |F_{0.8}| = 3$$
$$F_{0.6} = 0+1+2+3 \quad , \quad |F_{0.6}| = 4$$
$$F_{0.4} = 0+1+2+3+4 \quad , \quad |F_{0.4}| = 5$$
$$F_{0.2} = 0+1+2+3+4+5 \quad , \quad |F_{0.2}| = 6$$

and

$$|F|_f = 1/2 + 0.8/3 + 0.6/4 + 0.4/5 + 0.2/6 \tag{4.33}$$

Frequently, it is convenient or necessary to express the cardinality of a fuzzy set as a nonfuzzy real number (or an integer) rather than as a fuzzy number. In such cases, the concept of the power of a fuzzy set (DeLuca and Termini, 1972) may be used as a numerical summary of the fuzzy cardinality of a fuzzy set. Thus, the power of a fuzzy subset, F, of $U = \{u_1,\ldots,u_N\}$ is defined by

$$|F| \triangleq \sum_{i=1}^{N} \mu_F(u_i) \tag{4.34}$$

where $\mu_F(u_i)$ is the grade of membership of u_i in F and $\sum$ denotes the arithmetic sum. For example, for the fuzzy set small defined by (2.1), we have

$$|F| = 1+1+0.8+0.6+0.4+0.2 = 4$$

For some applications, it is necessary to eliminate from the count those elements of F whose grade of membership falls below a specified threshold. This is equivalent to replacing F in (4.34) with $F \cap \Gamma$, where Γ is a fuzzy or nonfuzzy set which induces the desired threshold.

As N increases and U becomes a continuum, the concept of the power of F gives way to that of a measure of F (Zadeh, 1968; Sugeno, 1974) , which may be regarded as a limiting form of the expression for the proportion of the elements of U which are in F. More specifically, if ρ is a density function defined on U, the measure in question is defined by

$$\|F\| \triangleq \int_U \rho(u)\mu_F(u)du \tag{4.35}$$

where μ_F is the membership function of F. For example, if $\rho(u)du$ is the proportion of men whose height lies in the interval $[u,u+du]$, then the proportion of men who are tall is given by

$$\|tall\| = \int_0^\infty \rho(u)\mu_{tall}(u)du \tag{4.36}$$

Making use of the above definitions, the <u>quantifier rule</u> for propositions of the form "QX are F" may be stated as follows.

If $U = \{u_1,\ldots,u_N\}$ and

$$X \text{ is } F \rightarrow \Pi_X = F \tag{4.37}$$

then

$$QX \text{ are } F \rightarrow \Pi_{|F|} = Q \tag{4.38}$$

and, if U is a continuum,

$$QX \text{ are } F \rightarrow \Pi_{\|F\|} = Q \tag{4.39}$$

which implies the more explicit rule

$$QX \text{ are } F \rightarrow \pi(\rho) = \mu_Q\left(\int_U \rho(u)\mu_F(u)du\right) \tag{4.40}$$

where $\rho(u)du$ is the proportion of X's whose value lies in the interval $[u,u+du]$, $\pi(\rho)$ is the possibility of ρ, and μ_Q and μ_F are the membership functions of Q and F, respectively.

As a simple illustration, if <u>most</u> and <u>tall</u> are defined by (4.28) and $\mu_{tall} = S(160,170,180)$, respectively, then

$$\text{Most men are tall} \rightarrow \pi(\rho) = S\left(\int_0^{200} \rho(u)S(u;160,170,180)du;0.5,0.7,0.9\right) \tag{4.41}$$

where $\rho(u)du$ is the proportion of men whose height (in cm) is in the interval $[u,u+du]$. Thus, the proposition "Most men are tall" induces a possibility distribution of the height density function ρ which is expressed by the right-hand member of (4.41).

<u>Rules of Type IV</u>

Among the many ways in which a proposition, p, may be qualified there are three that are of particular relevance to approximate reasoning. These are: (a) by a linguistic truth-value, as in "p is very true;" (b) by a linguistic probability-value, as in "p is highly probable;" and (c) by a linguistic possibility-value, as in "p is quite possible." Of these, we shall discuss only (a) in the sequel. Discussions of (b) and (c) may be found in Zadeh (1977).

As a preliminary to the formulation of translation rules pertaining to truth qualification, it is necessary to understand the role which a truth-value plays in modifying the meaning of a proposition. Thus, in FL, the truth-value of a proposition, p, is defined as the compatibility of a reference proposition, r, with p. More specifically, let

$$p \triangleq X \text{ is } F$$

where F is a subset of U, and let r be a reference proposition of the special form

$$r \triangleq X \text{ is } u \tag{4.42}$$

where u is an element of U. Then, the compatibility of r with p is defined as

$$\mathrm{Comp}(X \text{ is } u/X \text{ is } F) \triangleq \mu_F(u) \tag{4.43}$$

or, equivalently (in view of (2.3)),

$$\mathrm{Comp}(X \text{ is } u/X \text{ is } F) \triangleq \mathrm{Poss}\{X=u \mid X \text{ is } F\} \tag{4.44}$$

To extend (4.43) to the case where r is a fuzzy proposition of the form

$$r \triangleq X \text{ is } G\ , \quad G \subset U \tag{4.45}$$

we apply the extension principle[23] to the evaluation of the expression $\mu_F(G)$, yielding

$$\mathrm{Comp}\{X \text{ is } G/X \text{ is } F\} \triangleq \mu_F(G) \triangleq \int_{[0,1]} \mu_G(u)/\mu_F(u) \tag{4.46}$$

in which the right-hand member is the union over the unit interval of the fuzzy singletons $\mu_G(u)/\mu_F(u)$. Thus, the compatibility of "X is G" with "X is F" is a fuzzy subset of $[0,1]$ defined by (4.46).

In FL, the truth-value, τ, of the proposition $p \triangleq X$ is F relative to the reference proposition $r \triangleq X$ is G is defined as the compatibility of r with p. Thus, by definition,

[23]The extension principle (Zadeh (1975)) serves to extend the definition of a mapping $f\colon U \rightarrow V$ to the set of fuzzy subsets of U. Thus, $f(F) \triangleq \int_U \mu_F(u)/f(u)$, where $f(F)$ and $f(u)$ are, respectively, the images of F and u in V.

$$\tau \triangleq \mathrm{Tr}\{X \text{ is } F/X \text{ is } G\} \triangleq \mathrm{Comp}\{X \text{ is } G/X \text{ is } F\} \tag{4.47}$$

$$= \mu_F(G)$$

$$= \int_{[0,1]} \mu_G(u)/\mu_F(u)$$

which implies that the truth-value, τ, of the proposition "X is F" relative to "X is G" is a fuzzy subset of the unit interval defined by (4.47). In this sense, then, a linguistic truth-value may be regarded as a linguistic approximation to the fuzzy subset, τ, represented by (4.47). (See Fig. 2.)

A more explicit expression for τ which follows at once from (4.47) is the following. Let μ_τ denote the membership function of τ and let $v \in [0,1]$. Then

$$\mu_\tau(v) = \mathrm{Max}_u\ \mu_G(u) \tag{4.48}$$

subject to

$$\mu_F(u) = v \tag{4.49}$$

In particular, if μ_F is 1-1, then (4.48) and (4.49) yield

$$\mu_\tau(v) = \mu_G(\mu_F^{-1}(v)) \ , \quad v \in [0,1] \ . \tag{4.50}$$

As a simple illustration, consider the propositions (see Fig. 3)

$$p \triangleq X \text{ is } F \tag{4.51}$$

$$r \triangleq X \text{ is } G \quad \text{where} \quad G = [a,b]$$

In this case, it follows from (4.50) that τ is the interval given by

$$\tau = [\mu_F(b), \mu_F(a)]$$

The definition of the truth-value of p as the compatibility of a reference proposition r with p provides us with a basis for the translation of truth-qualified propositions of the form "p is τ" when τ is a fuzzy subset of [0,1]. Specifically, from the relation

$$\tau = \mu_F(G) \tag{4.52}$$

which defines τ as the image of G under the mapping $\mu_F: U \to [0,1]$, it follows that the membership function of G may be expressed in terms of those of τ and μ_F by (see Fig. 4)

$$\mu_G(u) = \mu_\tau(\mu_F(u)) \tag{4.53}$$

Now, if $r \triangleq X$ is G is the reference proposition for $p \triangleq X$ is F, we interpret the truth-qualified proposition

$$q \triangleq X \text{ is } F \text{ is } \tau \tag{4.54}$$

as the reference proposition $r \triangleq X$ is G. This leads us, then, to the following rule for truth qualification:

If

$$X \text{ is } F \longrightarrow \Pi_X = F \tag{4.55}$$

then

$$X \text{ is } F \text{ is } \tau \longrightarrow \Pi_X = F^+ \tag{4.56}$$

where

$$\mu_{F^+}(u) = \mu_\tau(\mu_F(u)) \tag{4.57}$$

In particular, if τ is the unitary truth-value, that is

$$\tau \triangleq \text{u-true} \tag{4.58}$$

where

$$\mu_{\text{u-true}}(v) \triangleq v \;, \quad v \in [0,1] \tag{4.59}$$

then

$$X \text{ is } F \text{ is u-true} \longrightarrow X \text{ is } F \tag{4.60}$$

As an illustration of (4.56), consider the proposition

$$p \triangleq \text{Lucia is young is very true} \tag{4.61}$$

in which

$$\mu_{young} = 1 - S(25;35,45) \tag{4.62}$$

$$\mu_{true} = S(0.6,0.8,1.0)$$

and

$$\mu_{very\ true} = S^2(0.6,0.8,1.0)$$

On applying (4.56) to p, we obtain

$$p \rightarrow \pi_{Age(Lucia)}(u) = S^2(1 - S(u;25,35,45);0.6,0.8,1.0) \tag{4.63}$$

which may be roughly approximated by the proposition

$$p^+ \triangleq \text{Lucia is very young} \tag{4.64}$$

Similarly, for the proposition

$$q \triangleq \text{Lucia is not young is very false} \tag{4.65}$$

where false $\triangleq$ ant true, i.e.,

$$\mu_{false}(v) \triangleq \mu_{true}(1-v) \ , \quad v \in [0,1] \tag{4.66}$$

$$= 1 - S(v;0,0.2,0.4)$$

we obtain

$$q \rightarrow \pi_{Age(Lucia)} = \left(1 - S\left(S(u;25,35,45);0,0.2,0.4\right)\right)^2 \tag{4.67}$$

which, as can readily be verified, defines the same possibility distribution as (4.63).

The translation rules described above provide us with the necessary basis for the formulation of the rules of inference in FL and the related notions of semantic equivalence and semantic entailment. We turn to these issues in the following section.

5. Semantic Equivalence and Semantic Entailment

In this section, we shall consider two related concepts in fuzzy logic that play an important role in approximate reasoning. These are the concepts of semantic equivalence and semantic entailment.

Informally, two propositions p and q are semantically equivalent if and only if the possibility distributions induced by p and q are equal. More specifically, if

$$p \rightarrow \Pi^p_{(X_1,\ldots,X_n)} = F$$

and

$$q \rightarrow \Pi^q_{(X_1,\ldots,X_n)} = G$$

where Π^p and Π^q are the possibility distributions induced by p and q, respectively, and $X_1,\ldots,X_n$ are the variables that are implicit or explicit in p and q, then

$$p \leftrightarrow q \quad \text{iff} \quad \Pi^p_{(X_1,\ldots,X_n)} = \Pi^q_{(X_1,\ldots,X_n)} \tag{5.1}$$

where $\leftrightarrow$ denotes semantic equivalence.

When (5.1) holds for all fuzzy sets in p and q that have a context-dependent meaning, the semantic equivalence will be said to be strong.

For example, the semantic equivalence

Adrienne is intelligent is true $\leftrightarrow$ Adrienne is not intelligent is false (5.2)

holds for all definitions of intelligent and true (false $\triangleq$ antonym of true) and hence is a strong equivalence. On the other hand, the semantic equivalence

[24]The concept of strong semantic equivalence as defined here reduces to that of semantic equivalence in predicate logic (see Lyndon, 1966) when p and q are nonfuzzy propositions.

$$\text{Lucia is young is very true} \leftrightarrow \text{Lucia is very young} \tag{5.3}$$

is not a strong equivalence because it holds only for some particular definitions of young and true. (See (4.64) and (4.65) et seq.) Usually, a semantic equivalence which is not strong is approximate in nature, as is true of (5.3).

Generally, it is clear from the context whether a semantic equivalence is or is not strong. Where it is necessary to place in evidence that a semantic equivalence is strong, it will be denoted by $s\leftrightarrow$, while approximate semantic equivalence will be denoted by $a\leftrightarrow$.

The concept of semantic entailment is weaker than that of semantic equivalence in that p semantically entails q (or q is semantically entailed by p) if and only if $\Pi^p_{(X_1,\ldots,X_n)} \subset \Pi^q_{(X_1,\ldots,X_n)}$. Thus, in symbols,

$$p \mapsto q \quad \text{iff} \quad \Pi^p_{(X_1,\ldots,X_n)} \subset \Pi^q_{(X_1,\ldots,X_n)} \tag{5.4}$$

where $\mapsto$ denotes semantic entailment and $\Pi^p_{(X_1,\ldots,X_n)}$ and $\Pi^q_{(X_1,\ldots,X_n)}$ are the possibility distributions induced by p and q, respectively.

As in the case of semantic equivalence, semantic entailment is strong if (5.4) holds for all fuzzy sets in p and q that have a context-dependent meaning. As an illustration, the semantic entailment expressed by

$$X \text{ is very small} \mapsto X \text{ is small} \tag{5.5}$$

is strong since it holds for all definitions of small. On the other hand, the validity of the semantic entailment expressed by

$$X \text{ is large} \mapsto X \text{ is not small} \tag{5.6}$$

depends on the way in which large and small are defined, and hence (5.6) is not an instance of strong semantic entailment.

In the case of propositions of the form $p \triangleq X$ is F and $q \triangleq X$ is G, it is evident that

$$X \text{ is } F \mapsto X \text{ is } G \quad \text{iff} \quad F \subset G \tag{5.7}$$

From this and the definition of conditional composition (4.20), it follows at once that

$$X \text{ is } F \mapsto X \text{ is } G \quad \text{iff} \quad \text{If } X \text{ is } F \text{ then } X \text{ is } G \rightarrow \Pi_X = U \tag{5.8}$$

or equivalently

$$X \text{ is } F \mapsto X \text{ is } G \quad \text{iff} \quad \text{If } X \text{ is } F \text{ then } X \text{ is } G \leftrightarrow X \text{ is } U \tag{5.9}$$

where Π_X is the possibility distribution of X and U is the universe of discourse associated with X. Similarly, from the definition of conjunctive composition, it follows that

$$X \text{ is } F \mapsto X \text{ is } G \quad \text{iff} \quad X \text{ is } F \text{ and } X \text{ is } G \rightarrow \Pi_X = F \tag{5.10}$$

or equivalently

$$X \text{ is } F \mapsto X \text{ is } G \quad \text{iff} \quad X \text{ is } F \text{ and } X \text{ is } G \leftrightarrow X \text{ is } F \tag{5.11}$$

An intuitively appealing interpretation of (5.11) is that p semantically entails q if the information conveyed by "p and q" is the same as the information conveyed by p alone.

As a preliminary to applying the concepts of semantic equivalence and semantic entailment to approximate reasoning -- which we shall do in Section 6 -- it will be helpful to formulate several rules pertaining to the transformation of a given proposition, p, into other propositions that have the same meaning as p, i.e., are strongly semantically equivalent to p.

A general rule governing such transformations may be stated informally as follows.

If m is a modifier and p is a proposition, than mp is semantically equivalent to the proposition which results from applying m to the possibility distribution which is induced by p.

Thus, on applying this rule to the case where $m \triangleq$ not and making use of the translation rules (4.5), (4.56) and (4.40), we arrive at the following specific rules governing the negation of a proposition

a) $$\text{not}(X \text{ is } F) \leftrightarrow X \text{ is not } F \tag{5.12}$$

e.g.,

$$\text{not}(X \text{ is small}) \leftrightarrow X \text{ is not small} \ ; \tag{5.13}$$

b) $$\text{not}(X \text{ is } F \text{ is } \tau) \leftrightarrow X \text{ is } F \text{ is not } \tau \tag{5.14}$$

e.g.,

$$\text{not}(X \text{ is small is very true}) \leftrightarrow X \text{ is small is not very true} \ ; \tag{5.15}$$

c) $$\text{not}(QX \text{ are } F) \leftrightarrow (\text{not } Q)X \text{ are } F \tag{5.16}$$

e.g.,

$$\text{not}(\text{many men are tall}) \leftrightarrow (\text{not many})\text{men are tall} \tag{5.17}$$

Similarly, for $m \triangleq$ very, we obtain

a) $$\text{very}(X \text{ is } F) \leftrightarrow X \text{ is very } F \tag{5.18}$$

b) $$\text{very}(X \text{ is } F \text{ is } \tau) \leftrightarrow X \text{ is } F \text{ is very } \tau \tag{5.19}$$

c) $$\text{very}(QX \text{ are } F) \leftrightarrow (\text{very } Q)X \text{ are } F \tag{5.20}$$

In addition, from the translation formulas (4.5), (4.40) and (4.56), it follows at once that

$$X \text{ is } F \text{ is } \tau \leftrightarrow X \text{ is not } F \text{ is ant } \tau \tag{5.21}$$

and

$$QX \text{ are } F \leftrightarrow (\text{ant } Q)X \text{ are not } F \tag{5.22}$$

where ant τ and ant Q denote the antonyms of τ and Q, respectively. (See (4.66).) Similarly, for m = very, we have

$$X \text{ is } F \text{ is } \tau \leftrightarrow X \text{ is very } F \text{ is } {}^2\tau \tag{5.23}$$

where the "left-square" operation on τ is defined by

$${}^2\tau = \int_0^1 \mu_\tau(v)/v^2 \ , \quad v \in [0,1] \tag{5.24}$$

or equivalently

$$\mu_{{}^2\tau}(v) = \mu_\tau(\sqrt{v}) \tag{5.25}$$

where μ_τ is the membership function of τ. However, as will be seen later, when F is modified to <u>very</u> F in "QX are F," we can assert only the semantic entailment -- rather than the semantic equivalence -- expressed by

$$QX \text{ are } F \mapsto ({}^2Q)F \text{ are very } F \tag{5.26}$$

where

$${}^2Q = \int_0^1 \mu_Q(v)/v^2 \tag{5.27}$$

or equivalently

$$\mu_{{}^2Q}(v) = \mu_Q(\sqrt{v}) \tag{5.30}$$

It should be noted in closing that the negation rule expressed by (5.16) appears to differ in form from the familiar negation rule in predicate calculus (see Lyndon, 1966) , which, when F is interpreted as a nonfuzzy predicate, may be expressed as

$$\text{not(all X are F)} \leftrightarrow \text{some X are not F} \tag{5.31}$$

However, by the use of (5.22), it is easy to show that the right-hand member of (5.31) is semantically equivalent to that of (5.16). Specifically, from (5.22) it follows that

$$\text{(not all)X are F} \leftrightarrow \text{(ant(not all)X)are not F}$$

and if some is defined as

$$\text{some} \triangleq \text{ant(not all)} \tag{5.32}$$

then

$$\text{(not all)X are F} \leftrightarrow \text{some X are not F} \tag{5.33}$$

in agreement with (5.31).

Remark. It should be observed that most of the definitions made in this and the preceding sections -- especially in regard to the semantic equivalence and semantic entailment of fuzzy propositions -- are nonfuzzy and, for the most part, quite precise. What should be understood, however, is that all such definitions may be fuzzified, if necessary, by the use of the following general convention.

Let U be a universe of discourse, with u denoting a generic element of U. A concept, C, in U is a subset, A, of U (or U^n, $n > 1$) which is defined by a predicate P such that $P(u)$ is true if $u \in A$, i.e., u is an instance of C, and false otherwise. Assume that

$P(u)$ is of the form $P(f(u))$, where $P(f(u))$ is true if $f(u) = 0$ and false if $f(u) > 0$. Then A -- and hence the concept C which is associated with it -- may be fuzzified by defining the grade of membership of u in A as a monotone function of $f(u)$ which assumes the value unity when $f(u) = 0$. (The definition of such a function is, in general, application-dependent rather than universal in nature.) In this sense, any definition which has the format stated above may be viewed as providing a mechanism for a fuzzification of the concept which it serves to define.

As an illustration of this convention, consider the concept of semantic equivalence as defined by (5.1). In this case, the concept of semantic equivalence may be fuzzified by defining the degree to which p and q are semantically equivalent as a monotone function of the "distance" between Π^p and Π^q, with the distance function defined in a way that reflects the specific nature of the domain of application. It should be understood, of course, that the concept in question may also be fuzzified in other ways which do not stem directly from its nonfuzzy definition.

6. Rules of Inference and Approximate Reasoning

As in any other logic, the rules of inference in FL govern the deduction of a proposition, q, from a set of premises $\{p_1,\ldots,p_n\}$. However, in FL both the premises and the conclusion are allowed to be fuzzy propositions. Furthermore, because of the use of linguistic approximation in the process of retranslation, the final conclusion drawn from the premises $p_1,\ldots,p_n$ is, in general, an approximate rather than exact consequence of $p_1,\ldots,p_n$.

The principal rules of inference in FL are the following.

1. Projection Principle

Let p be a fuzzy proposition whose translation is expressed as

$$p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F$$

Let $X_{(s)}$ denote a subvariable of the variable $X \triangleq (X_1,\ldots,X_n)$, i.e.,

$$X_{(s)} = (X_{i_1},\ldots,X_{i_k}) \tag{6.1}$$

where the index sequence $s \triangleq (i_1,\ldots,i_k)$ is a subsequence of the sequence $(1,\ldots,n)$.

Let $\Pi_{X_{(s)}}$ denote the marginal possibility distribution of $X_{(s)}$; that is,

$$\Pi_{X_{(s)}} = \mathrm{Proj}_{U_{(s)}} F \tag{6.2}$$

where U_i, $i = 1,\ldots,n$, is the universe of discourse associated with X_i;

$$U_{(s)} = U_{i_1} \times \cdots \times U_{i_k} \tag{6.3}$$

and the projection of F on $U_{(s)}$ is defined by the possibility distribution function

$$\pi_{X_{(s)}}(u_{i_1},\ldots,u_{i_k}) = \operatorname{Sup}_{u_{j_1},\ldots,u_{j_m}} \mu_F(u_1,\ldots,u_n) \tag{6.4}$$

where $s' \triangleq (j_1,\ldots,j_m)$ is the index subsequence which is complementary to s, and μ_F is the membership function of F.

Let q be a retranslation of the possibility assignment equation

$$\Pi_{X_{(s)}} = \mathrm{Proj}_{U_{(s)}} F \tag{6.5}$$

Then, the projection principle asserts that q may be inferred from p. In a schematic form, this assertion may be expressed more transparently as

$$\begin{array}{l} p \longrightarrow \Pi_{(X_1,\ldots,X_n)} = F \\ \qquad\qquad\downarrow \\ q \longleftarrow \Pi_{X_{(s)}} = \mathrm{Proj}_{U_{(s)}} F \end{array} \tag{6.6}$$

The statement of the projection principle assumes a particularly simple form for $n = 2$. In this case, writing X, Y, U, V for X_1, X_2, U_1, U_2, respectively, we have

$$p \longrightarrow \Pi_{(X,Y)} = F \tag{6.7}$$

$$q \longleftarrow \Pi_X = \mathrm{Proj}_U F \tag{6.8}$$

and likewise for the projection on V.

A special case of (6.6) obtains when $\Pi_{(X,Y)}$ is the cartesian product of normal fuzzy sets. Thus, if

$$p \longrightarrow \Pi_{(X,Y)} = G \times H \tag{6.9}$$

then from p we can infer q and r, where

$$q \leftarrow \Pi_X = G \tag{6.10}$$

and

$$r \leftarrow \Pi_Y = H \tag{6.11}$$

As a simple illustration, if

$$p \triangleq \text{John is tall and fat}$$

then from p we can infer

$$q \triangleq \text{John is tall}$$

and

$$r \triangleq \text{John is fat}$$

2. <u>Particularization/Conjunction Principle</u>

Let p be a fuzzy proposition whose translation is expressed as

$$p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F \;, \quad F \subset U_1 \times \cdots \times U_n \tag{6.12}$$

Then from p we can infer r, where r is a retranslation of a particularization of $\Pi_{(X_1,\ldots,X_n)}$, i.e.,

$$r \leftarrow \Pi_{(X_1,\ldots,X_n)}[\Pi_{X_{(s)}} = G] = F \cap \bar{G} \tag{6.13}$$

where $X_{(s)}$ is a subvariable of X, $\bar{G}$ is a cylindrical extension of G, $G \subset U$, and $\Pi_{(X_1,\ldots,X_n)}[\Pi_{X_{(s)}} = G]$ denotes an n-ary possibility distribution which results from particularizing $X_{(s)}$ to G. Equivalently, the principle in question may be expressed in the schematic form

$$\frac{\begin{array}{l} p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F \\ q \rightarrow \Pi_{(X_{i_1},\ldots,X_{i_k})} = G \end{array}}{r \leftarrow \Pi_{(X_1,\ldots,X_n)} = F \cap \bar{G}} \tag{6.14}$$

For the special case of $n = 2$, the particularization principle may be stated more simply as:

From

$$p \triangleq (X,Y) \text{ is } F \tag{6.15}$$

and

$$q \triangleq X \text{ is } G$$

we can infer

$$r \triangleq (X,Y) \text{ is } F \cap \bar{G} \tag{6.16}$$

Thus, for example, from

$$p \triangleq X \text{ and } Y \text{ are approximately equal} \tag{6.17}$$

and

$$q \triangleq X \text{ is small}$$

we can infer (without the application of linguistic approximation)

$$r \triangleq X \text{ and } Y \text{ are (approximately equal} \cap (\text{small} \times V)) \tag{6.18}$$

As stated above, the particularization principle may be viewed as a special case of a somewhat more general principle which will be referred to as the <u>conjunction principle</u>. Specifically, assume that

$$p \rightarrow \Pi^p_{(Y_1,\ldots,Y_k,X_{k+1},\ldots,X_n)} = F \tag{6.19}$$

$$q \rightarrow \Pi^q_{(Y_1,\ldots,Y_k,Z_{k+1},\ldots,Z_m)} = G \tag{6.20}$$

where $Y_1,\ldots,Y_k$ are variables which appear in both Π^p and Π^q, and U_i, V_j and W_ℓ are the universes of discourse associated with X_i, Y_j and Z_ℓ; let S be the smallest cartesian product of the U_i, V_j and W_ℓ which contains the cartesian products $V_1 \times \cdots \times V_k \times U_{k+1} \times \cdots \times U_n$ and $V_1 \times \cdots \times V_k \times W_{k+1} \times \cdots \times W_m$; and let $\bar{F}$ and $\bar{G}$ be, respectively, the cylindrical extensions of F and G in S. Then, from p and q we can infer r, where (in schematic form)

$$\begin{array}{l} p \rightarrow \Pi^p_{(Y,X)} = F \\ q \rightarrow \Pi^q_{(Y,Z)} = G \\ \hline r \leftarrow \Pi_{(X,Y,Z)} = \bar{F} \cap \bar{G} \end{array} \tag{6.21}$$

and $Y \triangleq (Y_1,\ldots,Y_k)$, $X \triangleq (X_{k+1},\ldots,X_n)$ and $Z \triangleq (Z_{k+1},\ldots,Z_m)$.

A particular but important case of (6.21) which we shall use at a later point results when $n = 3$, and $k = 1$. For this case, (6.21) may be expressed as

$$\begin{array}{l} p \rightarrow \Pi^p_{(X,Y)} \quad = F \\ q \leftarrow \Pi^q_{(Y,Z)} \quad = G \\ \hline r \leftarrow \Pi_{(X,Y,Z)} = (F \times W) \cap (U \times G) \end{array} \tag{6.22}$$

Although the particularization principle is subsumed by the conjunction principle, it is simpler than the latter, is employed more frequently, and has a somewhat greater intuitive appeal. For this reason, we use the designation "particularization/conjunction principle" to refer to the principle which, in most applications, is the particularization principle and, in some, the conjunction principle.[25]

[25] It should be noted that, in predicate logic (Lyndon, 1966), this principle implies the generalization rule.

3. Entailment Principle

Stated informally, the entailment principle asserts that from any fuzzy proposition p we can infer a fuzzy proposition q if the possibility distribution induced by p is contained in the possibility distribution induced by q. Thus, schematically, we have

$$\begin{array}{l} p \rightarrow \Pi_{(X_1,\ldots,X_n)} = F \\ \qquad\downarrow \\ q \leftarrow \Pi_{(X_1,\ldots,X_n)} = G \supset F \end{array} \qquad (6.23)$$

For example, from $p \triangleq X$ is very large we can infer $q \triangleq X$ is large.

The Compositional Rule of Inference

In general, the inference principles stated above are used in sequence or in combination. A combination that is particularly effective involves an application of the particularization/conjunction principle followed by that of the projection principle. This combination will be referred to as the compositional rule of inference (Zadeh, 1973). As will be seen later, the compositional rule of inference includes as a special case a generalization of the modus ponens.

For our purposes, it will be convenient to state the compositional rule of inference in the following schematic form

$$\begin{array}{l} p \rightarrow \Pi_{(X,Y)} = F \\ q \rightarrow \Pi_{(Y,Z)} = G \\ \hline r \leftarrow \Pi_{(X,Z)} = F \circ G \end{array} \qquad (6.24)$$

where X, Y and Z take values in U, V and W, respectively; F is a fuzzy subset of $U \times V$, G is a fuzzy subset of $V \times W$ and $F \circ G$ is the composition of F and G defined by

$$\mu_{F \circ G}(u,w) = \operatorname{Sup}_v(\mu_F(u,v) \wedge \mu_G(v,w)) \tag{6.25}$$

where $u \in U$, $v \in V$, $w \in W$ and μ_F and μ_G are the membership functions of F and G, respectively; and the dotted line signifies that, because of the use of linguistic approximation in retranslation, r is, in general, an approximate rather than exact consequence of p and q.[26]

It is easy to demonstrate that the compositional rule of inference may be regarded as a result of applying the particularization/conjunction principle followed by the application of the projection principle. Specifically, on applying (6.21) to (6.24), we obtain

$$\begin{array}{l} p \rightarrow \Pi_{(X,Y)} = F \\ q \rightarrow \Pi_{(Y,Z)} = G \\ \hline s \leftarrow \Pi_{(X,Y,Z)} = (F \times W) \cap (U \times G) \end{array} \tag{6.26}$$

where

$$\mu_{(F\times W)\cap(U\times G)}(u,v,w) = \mu_F(u,v) \wedge \mu_G(v,w) \tag{6.27}$$

Next, on applying the projection principle to s and projecting $\Pi_{(X,Y,Z)}$ on $U \times W$, we have

$$\operatorname{Proj}_{U\times W}((F\times W)\cap(U\times G)) = \int_{U\times W} \operatorname{Sup}_v(\mu_F(u,v) \wedge \mu_G(v,w))/(u,w) \tag{6.28}$$

which upon comparison with (6.25) shows that the resulting proposition may be expressed -- in agreement with (6.24) -- as

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[26] It should be noted that the compositional rule of inference is analogous to the rule which yields the probability distribution of Y from the probability distribution of X and the conditional probability distribution of Y given X.

$$r \leftarrow \Pi_{(X,Z)} = F \circ G \tag{6.29}$$

An important special case of the compositional rule of inference obtains when p and q are of the form $p \triangleq X$ is F, $q \triangleq$ If X is G then Y is H. For this case, (4.20) and (6.24) yield the <u>compositional modus ponens</u>:

$$\begin{array}{l} p \rightarrow \Pi_X = F \\ q \rightarrow \Pi_{(X,Y)} = \bar{G}' \oplus \bar{H} \\ \hdashline r \leftarrow F \circ (\bar{G}' \oplus \bar{H}) \end{array} \tag{6.30}$$

which may be regarded as a generalization of the classical <u>modus ponens</u>, with the latter corresponding to the special case of (6.30) in which F, G and H are nonfuzzy and F = G. For this case, (6.30) reduces to

$$\begin{array}{l} p \rightarrow \Pi_X = F \\ q \rightarrow \Pi_{(X,Y)} = \bar{F}' \oplus \bar{H} \\ \hline r \leftarrow \Pi_Y = F \circ (\bar{F}' \oplus \bar{H}) \end{array} \tag{6.31}$$

and since

$$F \circ (\bar{F}' \oplus \bar{H}) = H$$

it follows that

$$r \leftrightarrow Y \text{ is } H$$

which means that from $p \triangleq X$ is F and $q \triangleq$ If X is F then Y is H we can infer $r \triangleq Y$ is H, in agreement with the statement of the <u>modus ponens</u>.

The rules of inference presented in the foregoing discussion provide us with a basis for employing approximate reasoning for the purpose of question-answering and inference from fuzzy propositions. We shall illustrate the use of the methods based on these rules by applying them to several typical problems.

Semantic Equivalence

As a simple example, assume that from the premise

$$p \triangleq \text{Ellen is not very tall}$$

we wish to deduce the answer to the question "Is Ellen tall $?\tau$," where the symbol $?\tau$ signifies that the answer to the question is expected to be of the form

$$q \triangleq \text{Ellen is tall is } \tau$$

where τ is a linguistic truth-value.

To obtain the answer to the question, we shall require that p and q be semantically equivalent, implying that the possibility distribution induced by p is equal to that induced by q.

Thus, by using the translation rules (4.5), (4.6) and (4.56), we obtain

$$\text{Ellen is not very tall} \rightarrow \pi_{\text{Height(Ellen)}}(u) = 1 - \mu^2_{\text{tall}}(u) \qquad (6.32)$$

$$\text{Ellen is tall is } \tau \rightarrow \pi_{\text{Height(Ellen)}}(u) = \mu_\tau(\mu_{\text{tall}}(u)) \qquad (6.33)$$

where μ_{tall}, the membership function of tall, is assumed to be given. From (6.32) and (6.33), then, it follows that the desired membership function μ_τ satisfies the identity

$$1 - \mu^2_{\text{tall}}(u) \equiv \mu_\tau(\mu_{\text{tall}}(u)) \,, \quad u \in [0,200] \qquad (6.34)$$

from which we can conclude at once that μ_τ is given by (see Fig. 5)

$$\mu_\tau(v) = 1 - v^2 \tag{6.35}$$

to which a rough linguistic approximation may be expressed as

$$\tau \equiv \text{not very true} \tag{6.36}$$

It is instructive to obtain the same result by a successive use of the rules governing the application of negation, truth qualification and modification (by very). Thus, we can assert that

John is not very tall
$\leftrightarrow$ John is not very tall is u-true (by (4.60))

John is not very tall is u-true
$\leftrightarrow$ John is very tall is ant(u-true) (by (5.21))

John is very tall is ant(u-true)
$\leftrightarrow$ John is tall is $^{1/2}$(ant(u-true)) (by (5.23))

which implies that

$$\tau = {}^{1/2}(\text{ant(u-true)}) \tag{6.37}$$

i.e., τ is the "left-square root" of (ant(u-true)), and since

$$\mu_{\text{u-true}}(v) = v \tag{6.38}$$

we have

$$\mu_\tau = 1 - v^2 \tag{6.39}$$

in agreement with (6.35).

Semantic Entailment

Assume that we wish to deduce from the premise

$$p \triangleq \text{Most Swedes are tall}$$

the answer to the question "How many Swedes are very tall?"

Translating p by the use of (4.40), we have

$$\text{Most Swedes are tall} \longrightarrow \pi_p(\rho) = \mu_{most}\left(\int_0^{200} \rho(u)\mu_{tall}(u)du\right) \qquad (6.40)$$

where $\rho(u)du$ is the proportion of Swedes whose height is in the interval $[u,u+du]$ and π_p is the possibility distribution function of ρ. (Note that height is expressed in centimeters.)

Now, by (4.30) the proportion of Swedes who are very tall is given by

$$\gamma = \int_0^{200} \rho(u)\mu_{tall}^2(u)du \qquad (6.41)$$

Thus, our problem is to find the possibility distribution of γ from the knowledge of the possibility distribution of ρ -- which is given by the right-hand member of (6.40). In a variational formulation (which follows from (4.48)), this problem may be expressed as

$$\pi(\gamma) = \text{Max}_\rho \; \mu_{most}\left(\int_0^{200} \rho(u)\mu_{tall}(u)du\right) \qquad (6.42)$$

subject to

$$\gamma = \int_0^{200} \rho(u)\mu_{tall}^2(u)du \qquad (6.43)$$

The maximizing ρ for this problem is of the form (see Bellman and Zadeh, 1977)

$$\rho(u) = \delta(u-\alpha) \tag{6.44}$$

where δ is a δ-function and α is a point in the interval $[0,200]$. Thus, from (6.43) we have

$$\gamma = \mu^2_{tall}(\alpha) \tag{6.45}$$

and hence

$$\begin{aligned}\pi(\gamma) &= \mu_{most}(\mu_{tall}(\alpha)) \\ &= \mu_{most}(\sqrt{\gamma})\end{aligned} \tag{6.46}$$

or equivalently (see (5.30))

$$\pi(\gamma) = \mu_{^2most}(\gamma) \tag{6.47}$$

and hence the desired answer to the question "How many Swedes are very tall?" is (see Fig. 6)

$$q \triangleq {}^2\text{most Swedes are very tall} \tag{6.48}$$

To verify that p semantically entails q, we note that

$$\begin{aligned}q \rightarrow \pi_q(\rho) &= \mu_{^2most}\left(\int_0^{200} \rho(u)\mu^2_{tall}(u)du\right) \\ &= \mu_{most}\left(\sqrt{\int_0^{200} \rho(u)\mu^2_{tall}(u)du}\right)\end{aligned} \tag{6.49}$$

But, by Schwarz's inequality

$$\int_0^{200} \rho(u)\mu_{tall}(u)du \leq \sqrt{\int_0^{200} \rho(u)\mu^2_{tall}(u)du} \tag{6.50}$$

---

[27] The δ-function density implies that all elements of the population have the same value of the attribute in question.

and since μ_{most} is a monotone function, it follows that

$$\pi_p(\rho) \leq \pi_q(\rho) \quad \text{for all } \rho \text{ and } \mu_{tall}$$

which implies that p semantically entails q, strongly.

Particularization and Projection Principles

An illustration of the application of the particularization and projection principles is provided by the solution to the following simple problem.

Suppose that the premises are

$$p \triangleq \text{John is very big}$$
$$q \triangleq \text{John is tall}$$

where big is a given fuzzy subset of $U \times V$ (i.e., values of Height (in cms) $\times$ values of Weight (in kg)) and tall is a given fuzzy subset of U. The question is: "What is John's weight?"

Let us assume that the answer to the question is to be of the form $r \triangleq$ John is w, where w is a linguistic value of the weight of John (e.g., heavy, very heavy, not very heavy, etc.). Then, by employing the translation rule (4.6), the particularization principle and the projection principle, we arrive at the retranslation relation

$$r \leftarrow \text{Proj}_{\mu \times \text{Weight}} \text{BIG}^2[\pi_{\text{Height}} = \text{TALL}] \qquad (6.51)$$

which expresses the answer to the posed question.

In more concrete terms, assume that the (incompletely tabulated) tables defining the fuzzy sets BIG, TALL and HEAVY are of the form

BIG	Height	Weight	μ
	165	60	0.5
	170	60	0.6
	175	60	0.7
	170	65	0.75
	175	65	0.8
	180	65	0.85
	170	70	0.8
	175	70	0.85
	180	70	0.9
	170	75	0.85
	175	75	0.9
	180	75	0.95
	180	80	1

TALL	Height	μ
	165	0.7
	170	0.8
	175	0.9
	180	1
	185	1

HEAVY	Weight	μ
	60	0.7
	65	0.8
	70	0.9
	75	0.95
	80	1
	85	1

On substituting these tables in (6.51), we obtain for the attribute Weight a possibility distribution of the (approximate) form

$$\Pi_{Weight} = 0.5/60 + 0.7/65 + 0.8/70 + 0.9/75 + 1/80 \qquad (6.52)$$

which upon retranslation (and linguistic approximation) yields the answer

$$r \triangleq \text{John is very heavy}$$

As an additional illustration, consider the following premises

$$p \triangleq \text{Romy lives near a small city}$$

$$q \triangleq \text{Arnold lives near Romy}$$

from which we wish to deduce an answer to the question "Where does Arnold live?"

Assume that the relations entering in p and q have the frames shown below.

NEARp	City1	City2	μ

NEARq	City1	City2	μ

SMALL CITY	City	μ

in which NEARp and NEARq refer to the relations NEAR in p and q, respectively. In terms of these relations, the translations of p and q may be expressed as

$$p \longrightarrow \Pi_{\mathrm{Location(Residence(Romy))}} = \mathrm{Proj}_{\mu \times \mathrm{City1}}\ \mathrm{NEARp}[\Pi_{\mathrm{City2}} = \mathrm{SMALL\ CITY}] \qquad (6.53)$$

$$q \longrightarrow \Pi_{\mathrm{(Location(Residence(Romy)),Location(Residence(Arnold))}} = \mathrm{NEARq} \qquad (6.54)$$

On substituting (6.53) in (6.54) and projecting on the attribute Location(Residence(Arnold)), we obtain

$$r \longleftarrow \mathrm{Proj}_{\mu\times\mathrm{City2}}\ \mathrm{NEARq}[\Pi_{\mathrm{City1}} = \mathrm{Proj}_{\mu\times\mathrm{City1}}\ \mathrm{NEARp}[\Pi_{\mathrm{City2}} = \mathrm{SMALL\ CITY}]] \qquad (6.55)$$

as an expression for the answer to the posed question.

Compositional Rule of Inference

The compositional rule of inference is particularly convenient to use when the variables involved in the premises range over finite sets or can be approximated by variables ranging over such sets.

As a simple illustration, consider the premises

$p \triangleq X$ is small

$q \triangleq X$ and Y are approximately equal

in which X and Y range over the set $U = 1+2+3+4$, and small and approximately equal are defined by

$$\text{small} = 1/1 + 0.6/2 + 0.2/3$$

$$\begin{aligned}\text{approximately equal} = 1/\big((1,1) + (2,2) + (3,3) + (4,4)\big) \\ + 0.5/\big((1,2) + (2,1) + (2,3) + (3,2) \\ + (3,4) + (4,3)\big)\end{aligned}$$

In terms of these sets, the translations of p and q may be expressed as

$$\begin{aligned} p &\rightarrow \Pi_X = \text{small} \\ q &\rightarrow \Pi_{(X,Y)} = \text{approximately equal} \end{aligned} \tag{6.56}$$

and thus from p and q we may infer r, where

$$r \leftarrow \Pi_X \circ \Pi_{(X,Y)} = \text{small} \circ \text{approximately equal} \tag{6.57}$$

The composition of small and approximately equal can readily be performed by computing the max-min product of the relation matrices corresponding to small and approximately equal. Thus, we obtain

$$[1 \quad 0.6 \quad 0.2 \quad 0] \circ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} = [1 \quad 0.6 \quad 0.5 \quad 0.2]$$

which implies that

$$\Pi_Y = \Pi_X \circ \Pi_{(X,Y)} = 1/1 + 0.6/2 + 0.5/3 + 0.2/4$$

and which upon retranslation yields the linguistic approximation

$$r \triangleq Y \text{ is more or less small} \leftarrow \Pi_Y = 1/1 + 0.6/2 + 0.5/3 + 0.2/4$$

Thus, from $p \triangleq X$ is small and $q \triangleq X$ and Y are approximately equal, we can infer, approximately, that $r \triangleq Y$ is more or less small.

As a simple illustration of the compositional <u>modus ponens</u>, assume that, as in Bellman and Zadeh (1977),

$$U = V = 1 + 2 + 3 + 4$$

$$F = 0.2/2 + 0.6/3 + 0.5/4$$

$$G = 0.6/2 + 1/3 + 0.5/4$$

$$H = 1/2 + 0.6/3 + 0.2/4$$

with

$$p \triangleq X \text{ is } F \rightarrow \Pi_X = F \tag{6.58}$$

$$q \triangleq \text{If } X \text{ is } G \text{ then } Y \text{ is } H \rightarrow \Pi_{(X,Y)} = \bar{G}' \oplus \bar{H}$$

and

$$r \leftarrow \Pi_Y = F \circ (\bar{G}' \oplus \bar{H})$$

In this case,

$$\bar{G}' \oplus \bar{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 1 & 0.6 \\ 0 & 1 & 0.6 & 0.6 \\ 0.5 & 1 & 1 & 0.7 \end{bmatrix}$$

and

$$\bar{F} \circ (\bar{G}' \oplus \bar{H}) = [0 \quad 0.2 \quad 0.6 \quad 1] \circ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 1 & 0.6 \\ 0 & 1 & 0.6 & 0.6 \\ 0.5 & 1 & 1 & 0.7 \end{bmatrix}$$

$$= [0.5 \quad 1 \quad 1 \quad 0.7]$$

Thus, from p and q we can infer that

$$r \triangleq Y \text{ is } 0.5/1 + 1/2 + 1/3 + 0.7/4$$

The above example is intended merely to illustrate the computations involved in the application of the compositional modus ponens when X and Y range over finite sets. Detailed discussions of practical applications of the compositional rule of inference in the design of so-called fuzzy logic controllers may be found in the papers by Mamdani and Assilian (1975), Mamdani (1976), Kickert and van Nauta Lemke (1976), Rutherford and Bloore (1975), and others. (See the appended bibliography.)

7. Concluding Remark

The theory of approximate reasoning outlined in this paper may be viewed as an attempt at an accommodation with the pervasive imprecision of the real world.

Based as it is on fuzzy logic, approximate reasoning lacks the depth of universality of precise reasoning. And yet it may well prove to be more effective than precise reasoning in coming to grips with the complexity and ill-definedness of humanistic systems and thus may contribute to the conception and development of intelligent systems which could approach the remarkable capability of the human mind to make rational decisions in the face of uncertainty and imprecision.

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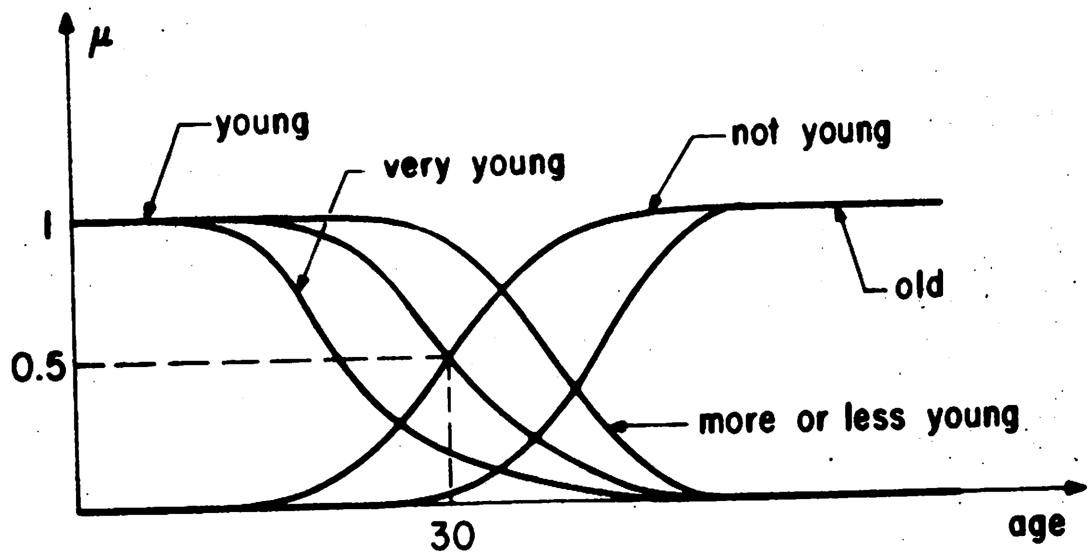

Fig. 1. Graphical representation of linguistic values of Age.

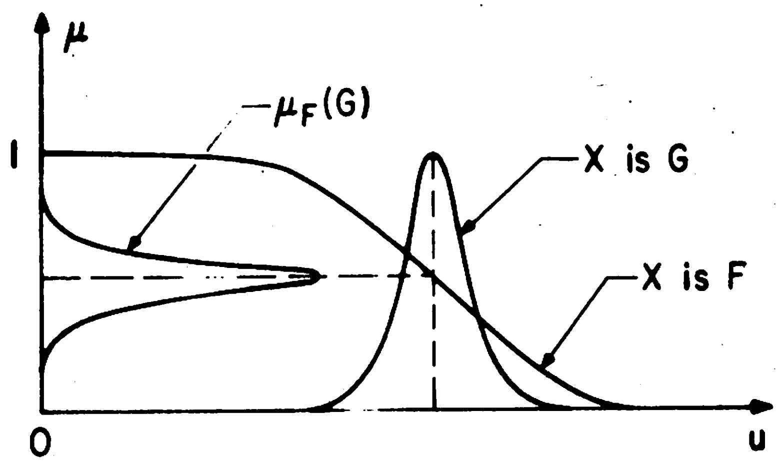

Fig. 2. Graphical illustration of the concept of relative truth.

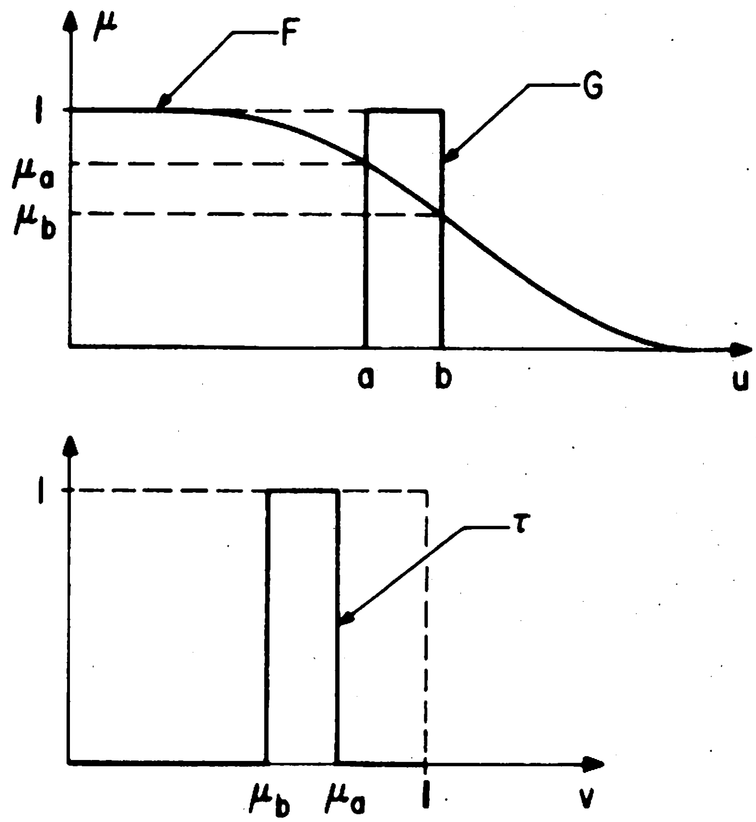

Fig. 3. Interval-valued truth-value for an interval-valued reference proposition.

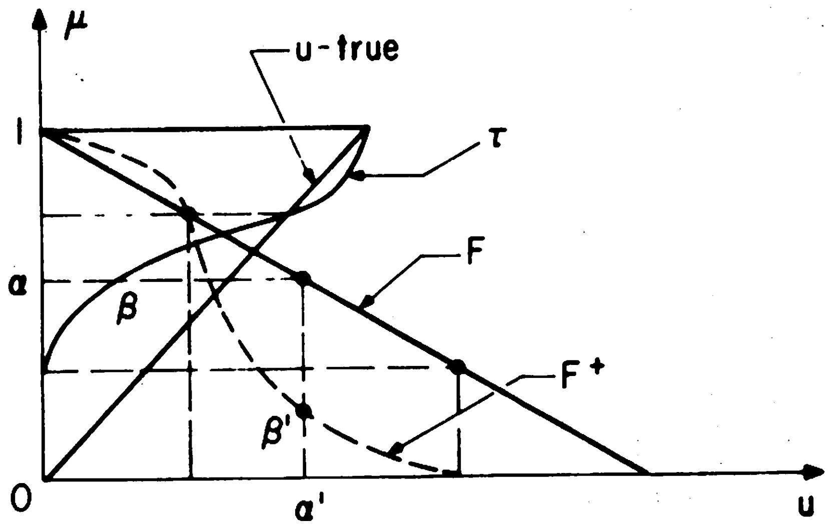

Fig. 4. Effect of truth qualification on F.
(β is mapped into β'.)

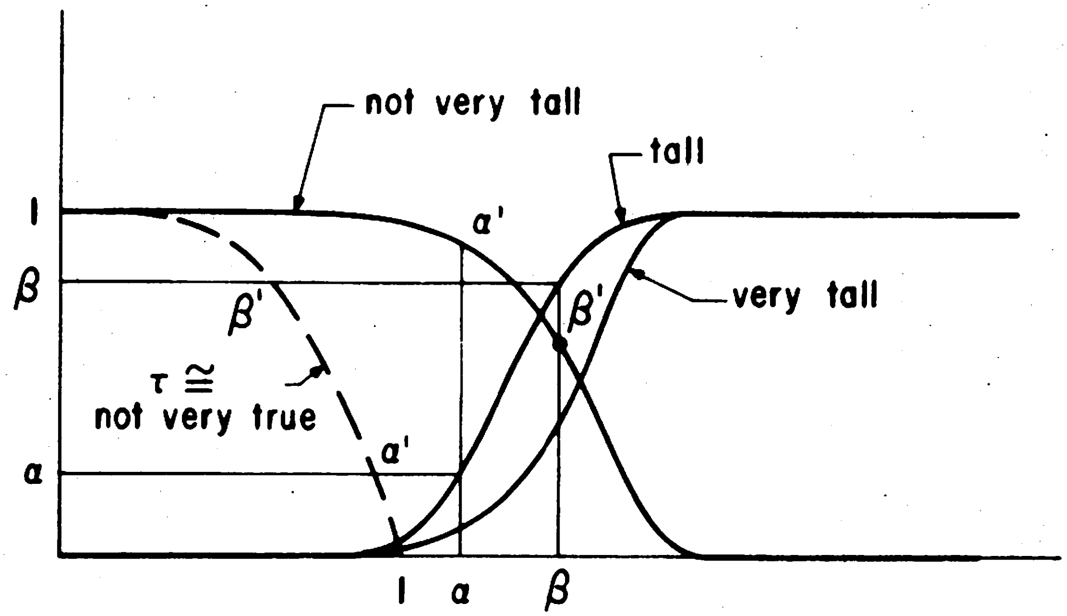

Fig. 5. Extraction of an answer by the use of semantic equivalence.

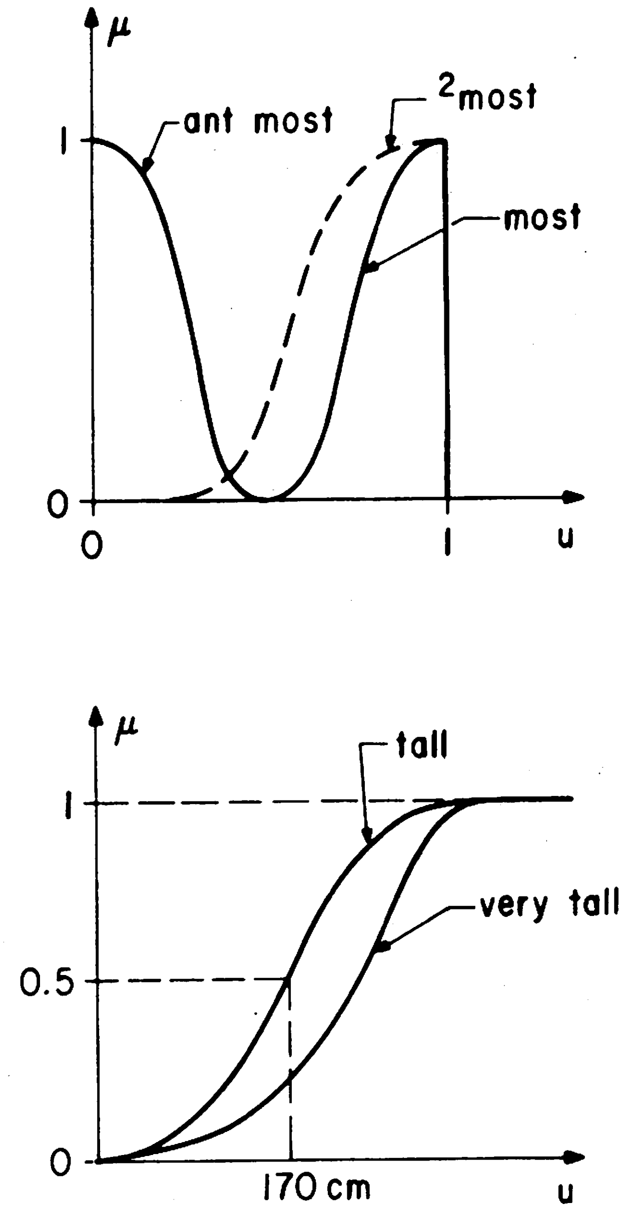

Fig. 6. Representation of <u>most</u>, <u>tall</u> and their modifications.