ON THE IMPLICATIONS OF CAPACITOR-ONLY CUT SETS AND INDUCTOR-ONLY LOOPS IN NONLINEAR NETWORKS

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ON THE IMPLICATIONS OF CAPACITOR-ONLY CUT SETS AND INDUCTOR-ONLY LOOPS IN NONLINEAR NETWORKS†

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ABSTRACT

Let $\mathcal{N}$ be an autonomous dynamic nonlinear network. Let $\mathcal{N}_{RG}$ be the associated resistive subnetwork obtained by open-circuiting all capacitors and short-circuiting all inductors. The following main results are proved: (i) Suppose that $\mathcal{N}_{RG}$ has only isolated operating points. Then $\mathcal{N}$ has only isolated equilibria <u>if, and only if</u>, "there are no capacitor-only cut sets and inductor-only loops." (<u>Condition A</u>) (ii) If <u>Condition A</u> is violated, then there are a continuum of equilibria even if the operating points are isolated. (iii) Let M be the set of equilibria. Then each trajectory is constrained to lie on an affine submanifold M*, which depends on the initial state, such that $M \cap M^*$ has only isolated points. Hence each trajectory behaves as if it has only isolated equilibria. The space M*, because of its nature, can be considered as the <u>minimal</u> state space of the dynamics.

It is shown that the results can be generalized to nonautonomous networks. Finally an application of the results to eventually passive networks is given.

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1. Introduction

Consider an autonomous network (i.e., no time-dependent sources) described by

$$\dot{x} = F(x,u), \quad F: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \tag{1}$$

For a fixed "dc" bias u, a point $x \in \mathbb{R}^n$ is called an equilibrium if

$$F(x,u) = 0. \tag{2}$$

Now, open all capacitors and short all inductors. Call the resulting resistor-independent source network $\mathcal{N}_{RG}$. The purpose of this paper is to show the following.

(i) Suppose that $\mathcal{N}_{RG}$ has only isolated operating points. Then (1) has only isolated equilibria if, and only if, the following holds:

Condition A. There are no capacitor-only cut sets and no inductor-only loops.

(ii) if condition A is violated, then there are a continuum of equilibria even if operating points are isolated. Call the set of equilibria M.

(iii) Each trajectory is constrained to lie on an affine submanifold[1] M*, which depends on the initial state, such that M ∩ M* consists only of isolated points. Hence each trajectory behaves as if it has only isolated equilibria. (See Fig. 1) The dimension of M* is the dimension of the state space minus the number of linearly independent capacitor-only cut sets and inductor-only loops. In this sense M* might be considered as the minimal state space.

The above results are generalizations of the phenomena depicted in the following example:

Example 1.

Consider the circuit of Fig. 2(a) with the resistor constitutive relation as shown in Fig. 2(b). Let the capacitors be linear. Now at equilibria (i=0), KVL implies

$$v_1 + v_2 = E \tag{3}$$

or

$$\frac{q_1}{C_1} + \frac{q_2}{C_2} = E \tag{4}$$

---

[1] An affine submanifold is a translate of a linear subspace. An affine submanifold does not necessarily contain the origin of the ambient space, whereas a linear subspace does contain the origin. A function $f(x) \triangleq Ax + b$ is called affine if A is a matrix and b is a constant vector. It is called linear if, in addition, $b = 0$. In electrical networks, the term "affine" is usually more appropriate than "linear" because the constant vector b is usually present in view of dc sources or initial conditions.

Hence, in the (q_1,q_2)-space, the set of equilibria M defined by (4) constitutes an affine submanifold. (See Fig. 3). Applying, next, KCL at the capacitor-only cut set, we obtain

$$q_1 - q_2 = q_1(0) - q_2(0) \triangleq Q(0) \tag{5}$$

so that the trajectory must stay in the set M* defined by (5), where M* is parameterized by Q(0). Corresponding to any value Q(0), the intersection $M \cap M^*$ is necessarily a single point if $C_1, C_2 > 0$. Thus, once the initial state is specified, the trajectory behaves like it has a unique equilibrium.

A dual example involving an inductor-only loop can be found in [1]. Several arguments concerning the significance of capacitor-only cut sets and inductor-only loops have appeared in the literature [2-4]. None of the authors, however, has examined this subject from the geometrical point of view which we believe is essential in obtaining a clear understanding of the many hidden subtleties.

Remark. In the following, we will sometimes be inconsistent in the use of our notation for a vector and its transpose, in order to save space. There will be no confusion, however.

2. Relation between Equilibria of a Dynamic Nonlinear Network and the Operating Points of the Associated Resistive Subnetwork

Given a network $\mathcal{N}$ let us form its resistive subnetwork $\mathcal{N}_{RG}$ by open-circuiting all capacitors and short-circuiting all inductors. We will assume the following standing hypotheses throughout this paper:

Assumption 1. There are no couplings among elements of different kinds.

Assumption 2. In $\mathcal{N}$, independent current sources do not form cut sets with themselves and/or with capacitors, while independent voltage sources do not form loops with themselves and/or with inductors. Or equivalently, in $\mathcal{N}_{RG}$, independent current sources do not form cut sets with themselves, and independent voltage sources do not form loops with themselves.

By Assumption 2 we can choose a C-normal tree T_C for $\mathcal{N}$ such that the following decomposition of variables is possible: (T_C^* denotes cotree)

	elements	voltage	current	number of elements
Cotree	capacitors in T_C^*	$\mathbf{v}_S$	$\mathbf{i}_S$	n_S
	resistors in T_C^*	$\mathbf{v}_R$	$\mathbf{i}_R$	n_R
	inductors in T_C^*	$\mathbf{v}_L$	$\mathbf{i}_L$	n_L
	independent current sources in T_C^*	$\mathbf{v}_I$	$\mathbf{i}_I$	n_I
Tree	independent voltage sources in T_C	$\mathbf{v}_V$	$\mathbf{i}_V$	n_V
	capacitors in T_C	$\mathbf{v}_C$	$\mathbf{i}_C$	n_C
	resistors in T_C	$\mathbf{v}_G$	$\mathbf{i}_G$	n_G
	inductors in T_C	$\mathbf{v}_\Gamma$	$\mathbf{i}_\Gamma$	n_Γ

The KVL and KCL equations associated with T_C assume the following well-known form:

$$\left[\begin{array}{cccc|cccc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{SV} & \mathbf{B}_{SC} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{RV} & \mathbf{B}_{RC} & \mathbf{B}_{RG} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{B}_{LV} & \mathbf{B}_{LC} & \mathbf{B}_{LG} & \mathbf{B}_{L\Gamma} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{B}_{IV} & \mathbf{B}_{IC} & \mathbf{B}_{IG} & \mathbf{B}_{I\Gamma} \end{array}\right] \left[\begin{array}{c} \mathbf{v}_S \\ \mathbf{v}_R \\ \mathbf{v}_L \\ \mathbf{v}_I \\ \hline \mathbf{v}_V \\ \mathbf{v}_C \\ \mathbf{v}_G \\ \mathbf{v}_\Gamma \end{array}\right] = \mathbf{0} \qquad \begin{array}{c} (6) \\ (7) \\ (8) \\ (9) \end{array}$$

$$
\left[\begin{array}{cccc|cccc}
-\underset{\sim}{B}_{SV}^{T} & -\underset{\sim}{B}_{RV}^{T} & -\underset{\sim}{B}_{LV}^{T} & -\underset{\sim}{B}_{IV}^{T} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\
-\underset{\sim}{B}_{SC}^{T} & -\underset{\sim}{B}_{RC}^{T} & -\underset{\sim}{B}_{LC}^{T} & -\underset{\sim}{B}_{IC}^{T} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} \\
\underset{\sim}{0} & -\underset{\sim}{B}_{RG}^{T} & -\underset{\sim}{B}_{LG}^{T} & -\underset{\sim}{B}_{IG}^{T} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} \\
\underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{B}_{L\Gamma}^{T} & -\underset{\sim}{B}_{I\Gamma}^{T} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}
\end{array}\right]
\left[\begin{array}{c}
\underset{\sim}{i}_S \\ \underset{\sim}{i}_R \\ \underset{\sim}{i}_L \\ \underset{\sim}{i}_I \\ \hline \underset{\sim}{i}_V \\ \underset{\sim}{i}_C \\ \underset{\sim}{i}_G \\ \underset{\sim}{i}_\Gamma
\end{array}\right] = \underset{\sim}{0}
\qquad
\begin{array}{r}(10)\\(11)\\(12)\\(13)\end{array}
$$

Let the capacitors be characterized by the constitutive relation

$$
(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S) \in \Lambda_C \subset \mathbb{R}^{2(n_C+n_S)} \tag{14}
$$

where Λ_C is an n_C+n_S dimensional C^1 submanifold, and let the inductors be characterized by the constitutive relation

$$
(\underset{\sim}{i}_L, \underset{\sim}{i}_\Gamma, \underset{\sim}{\phi}_L, \underset{\sim}{\phi}_\Gamma) \in \Lambda_L \subset \mathbb{R}^{2(n_L+n_\Gamma)} \tag{15}
$$

where Λ_L is an n_L+n_Γ dimensional C^1 submanifold. By Assumption 1, one can assume that the resistors are characterized by the constitutive relation

$$
(\underset{\sim}{v}_R, \underset{\sim}{v}_G, \underset{\sim}{i}_R, \underset{\sim}{i}_G) \in \Lambda_{RG} \subset \mathbb{R}^{2(n_R+n_G)} \tag{16}
$$

where Λ_{RG} is an n_R+n_G dimensional C^1 submanifold. Observe that (14), (15) and (16) imply that elements of the same type may be coupled to each other. Hence, multiterminal elements and multiports are allowed since they can be represented as coupled two-terminal elements.

Theorem 1. Suppose that the dynamics of (1) is defined in terms of $\underset{\sim}{x} = (\underset{\sim}{v}_C, \underset{\sim}{i}_L)$ and suppose that $\mathcal{N}_{RG}$ has only isolated operating points. Then $\mathcal{N}$ has only isolated equilibria if, and only if, Condition A holds.

Remark. Note the "if and only if" nature of the result. Hence, if there are capacitor-only cut sets and/or inductor-only loops, then there are a continuum of equilibria even if $\mathcal{N}_{RG}$ has a unique operating point.

Proof.

Necessity. Suppose there is a capacitor-only cut set. Let $(V_1,\ldots,V_\gamma)$ and $(I_1,\ldots,I_\gamma)$ be the voltages and the currents of the capacitors in the cut set. Decompose $\mathcal{N}$ into three parts as in Fig. 4(a). At an equilibrium

$$I_1 = I_2 = \ldots I_\gamma = 0$$

Hence $\mathcal{N}_{RG}$ will look like as in Fig. 4(b), where $\mathcal{N}_{1RG}$ and $\mathcal{N}_{2RG}$ consist only of memoryless elements. Let $(V^*_{1,2},\ldots,V^*_{2\gamma-1,2\gamma})$ be the voltages across the external terminal of $\mathcal{N}_1$ and $\mathcal{N}_2$ at an operating point of $\mathcal{N}_{RG}$. Then the equivalent network outside of the capacitors is as shown in Fig. 5(a). Insert two independent voltage sources E as in Fig. 5(b). Apply v-shift theorem [5] to obtain the network of Fig. 5(c). Since the network of Fig. 5(c) is equivalent to that of Fig. 5(a), if

$$(V_1,\ldots,V_\gamma)$$

is a solution, so is

$$(V_1+E, V_2+E,\ldots,V_\gamma+E)$$

for all $E \in \mathbb{R}$. Since, at least one of $V_1,\ldots,V_\gamma$ belongs to T_C, one gets a continuum of equilibria. A dual argument implies the existence of a continuum of equilibria when there are inductor-only loops.

Sufficiency. Suppose that Condition A holds. Let b_k be the branch denoting the k-th capacitor and let A be the remaining set of capacitor branches and let B be the rest of the branches of the network. By assumption, b_k does not form a cut set exclusively with branches of set A. Hence, by Colored Arc Lemma [6], b_k forms a loop exclusively with branches of set B, i.e., b_k forms a loop exclusively with resistors, inductors and independent sources. At an equilibrium, the voltages across inductors are zero so that the voltages across capacitors are uniquely determined by an operating point of $\mathcal{N}_{RG}$. A similar argument applies to inductors. ◻

Corollary 1. Under the same setting as Theorem 1 suppose $\mathcal{N}_{RG}$ has a unique operating point. Then $\mathcal{N}$ has a unique equilibrium if, and only if, Condition A holds.

Corollary 2. Under the same setting as Theorem 1 except that the dynamics is linear; namely,

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u} \tag{17}$$

Then

$$\det \underset{\sim}{A} \neq 0 \tag{18}$$

if, and only if, Condition A holds.

Remark. If $\underset{\sim}{y}$ is a set of variables in the network such that $\underset{\sim}{y} = \underset{\sim}{\psi}(\underset{\sim}{x})$, where $\underset{\sim}{\psi}$ is a diffeomorphism, i.e., $\underset{\sim}{y}$ is any other coordinate system, then the theorem still holds. There are, however, cases that cannot be taken care of by Theorem 1. The following is a case in point.

Example 2. Consider the circuit of Fig. 6(a) which consists of a 1 ohm linear resistor and a Josephson Junction device, characterized by

$$i_L = k_1 \sin k_2\phi_L \triangleq \hat{i}_L(\phi_L)$$

where k_1 and k_2 are constants. The state equation in this case cannot be written in terms of i_L. But it can be written in terms of ϕ_L; namely

$$\dot{\phi}_L = -\hat{i}_L(\phi_L).$$

At equilibria, $v_L = i_L = 0$ and

$$\phi_L = 0, \pm\frac{\pi}{k_2}, \pm\frac{2\pi}{k_2}, \ldots$$

are the set of isolated equilibria. The circuit clearly satisfies Condition A.

Now, if the inductor constitutive relation is replaced by that of Fig. 6(c), then the set

$$\phi_L = \{0, [a,b], c\}$$

of equilibria constitutes a continuum of points, even though Condition A holds.

Observe that this phenomenon occurs because the curve $\hat{i}_L(\phi_L)$ contains a flat portion which coincides with part of the ϕ_L-axis.

Now, let the constitutive relation of the Josephson Junction device be defined by a one-dimensional curve Λ_L and let π_i be its natural projection onto the i_L-axis. Then, roughly speaking, π_i is transversal to {0} if Λ_L has no points like those of [a,b]. This is a sort of non-tangency condition. We will formalize this concept and prove results with capacitor charges and inductor fluxes chosen as the state variables. The mathematical tool for this purpose turns out, not surprisingly, to be the transversality theory of functions and manifolds which will be introduced in the following section.

3. Relevance of the Transversality Theory

The main objective of this section is to provide a quick review of the basic results from transversality theory [7] which are relevant to the proof of the results of section 4. We will also present a self-contained introduction to transversality theory for surfaces, i.e., submanifolds in $\mathbb{R}^n$ in order to emphasize its geometrical interpretation. Examples will be given to introduce the somewhat unconventional notations used in transversality theory and to illustrate the application of some of the main results.

First of all, a submanifold of $\mathbb{R}^n$ is nothing but a higher dimensional version of a surface in $\mathbb{R}^3$ and a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Transversality of two surfaces (submanifolds) in $\mathbb{R}^n$ is essentially a <u>non-tangency</u> condition. Consider two surfaces X and Y as in Fig. 7(a) where Y is a plane. The intersection $X \cap Y$ defines a nice 1-dimensional curve Γ. This essentially comes from the fact that X and Y intersect in "the right" manner. If, however, the two submanifolds intersect in "a wrong" manner, the intersection can be a complicated object. Because of difficulties in drawing pictures in $\mathbb{R}^3$, let us see what might happen in $\mathbb{R}^2$. Consider X and Y of Fig. 7(b). The intersection has two parts; one is a finite line segment, while the other is a single point. Hence $X \cap Y$ is not a submanifold. This comes from the fact that X and Y meet tangentially. So, in higher dimensions, many bizarre situations could arise, if two objects meet tangentially <u>even if</u> each of them is a nice smooth surface. Now, how can we express the non-tangency condition mathematically? Return to Fig. 7(a) and consider a point $\underset{\sim}{x} \in X \cap Y$. Let $T_{\underset{\sim}{x}} X$ denote the plane tangent to X at $\underset{\sim}{x}$. (Called the <u>tangent space</u> of X at $\underset{\sim}{x}$). Let $T_{\underset{\sim}{x}}Y$ be defined similarly. Since Y is a plane $T_{\underset{\sim}{x}}Y = Y$. Now, it is easy to see that

$$T_{\underset{\sim}{x}}X + T_{\underset{\sim}{x}}Y = \mathbb{R}^3 = T_{\underset{\sim}{x}}\mathbb{R}^3 \tag{19}$$

i.e., the tangent spaces span the tangent space of the ambient space $\mathbb{R}^3$. Note that (19) is a <u>vector sum</u> and <u>not</u> a direct sum. The left hand side means the set of all vectors of the form $\underset{\sim}{v}+\underset{\sim}{w}$, where $\underset{\sim}{v} \in T_{\underset{\sim}{x}}X$, $\underset{\sim}{w} \in T_{\underset{\sim}{x}}Y$. Hence vectors belonging to $T_{\underset{\sim}{x}}X$ and $T_{\underset{\sim}{x}}Y$ need not be linearly independent. Next, consider Fig. 7(b). It is clear that

$$T_{\underset{\sim}{x}}X + T_{\underset{\sim}{x}}Y = T_{\underset{\sim}{x}}Y \neq T_{\underset{\sim}{x}}\mathbb{R}^2.$$

These observations naturally lead us to the following definition.

<u>Definition 1</u>. Let X and Y be two C^1 submanifolds of another manifold Z. Then X and Y are said to be <u>transversal</u> and is abbreviated as

$$X \pitchfork Y \tag{20}$$

if

$$T_{\underset{\sim}{x}}X + T_{\underset{\sim}{x}}Y = T_{\underset{\sim}{x}}Z \tag{21}$$

for all $\underset{\sim}{x} \in X \cap Y$.

<u>Fact A</u>. If $X \pitchfork Y$, then $X \cap Y$ is a submanifold and

$$\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y \tag{22}$$

where

$$\operatorname{codim} X = \dim Z - \dim X \tag{23}$$

and similarly for other symbols. ¤

Codimension of a submanifold is simply the complementary dimension of the manifold with respect to the ambient space. For example, X and Y in Fig. 7(a)-(c) have codimension one.

Since codimension is defined by (23), if we take intersection of X and Y, we should expect the codimension of $X \cap Y$ to be no less than codim X and codim Y. Why are, then, the codimensions additive as in (22)? Now, codimension of a submanifold in $\mathbb{R}^n$ is roughly speaking, the number of redundant coordinates that can be eliminated from the equations describing the submanifold in $\mathbb{R}^n$. For example, consider Fig. 7(c) where the unit sphere S^2 intersects with a plane P. Since S^2 is described by

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{24}$$

one coordinate x_i can be determined, given the other two, and is therefore redundant and can be eliminated. Similarly, the plane P is described by

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0, \quad \alpha,\beta,\gamma \in \mathbb{R} \tag{25}$$

and hence one coordinate is redundant and can be eliminated, provided, of course, that at least one of α, β and γ is nonzero. Hence both have codimension one relative to $\mathbb{R}^3$. Since $S^2 \cap P$ is described by (24) and (25), if (24) and (25) do not overlap each other, then, there should be two variables that can be eliminated, and hence $S^2 \cap P$ has codimension two. But $S^2 \pitchfork P$ precisely means that the two surfaces described by (24) and (25) do not overlap each other.

Taking into account the above geometrical interpretations, one can show the following [8, APPENDIX], which provides us with an easy way of checking transversality.

<u>Fact B</u>. Let $\underset{\sim}{F}(\underset{\sim}{x})$ be a C^1 function on $\mathbb{R}^n$ taking values in $\mathbb{R}^{n-m}$; i.e., $\underset{\sim}{F}: \mathbb{R}^n \to \mathbb{R}^{n-m}$. If

$$X = \{\underset{\sim}{x} \in \mathbb{R}^n | \underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{0},\ \text{rank}\ (\underset{\sim}{D}\underset{\sim}{F})_{\underset{\sim}{x}} = n-m\}$$

is nonempty, then X is an m-dimensional submanifold. Let

$$Y = \{\underset{\sim}{x} \in \mathbb{R}^n | \underset{\sim}{G}(\underset{\sim}{x}) = \underset{\sim}{0},\ \text{rank}\ (\underset{\sim}{D}\underset{\sim}{G})_{\underset{\sim}{x}} = n-\ell\}$$

be nonempty, where $\underset{\sim}{G}$ is a C^1 function taking values in $\mathbb{R}^{n-\ell}$. If

$$\text{rank} \begin{bmatrix} (\underset{\sim}{D}\underset{\sim}{F})_{\underset{\sim}{x}} \\ (\underset{\sim}{D}\underset{\sim}{G})_{\underset{\sim}{x}} \end{bmatrix} = n-m + n-\ell$$

for all $\underset{\sim}{x} \in X \cap Y$, then $X \pitchfork Y$. ¤

Let us check transversality for Fig. 7(c), by using <u>Fact B</u>. Let

$$F(\underset{\sim}{x}) \triangleq x_1^2 + x_2^2 + x_3^2 - 1.$$

For simplicity, let P be the (x_1,x_2)-plane, i.e., let

$$G(\underset{\sim}{x}) \triangleq x_3.$$

Note that $(x_1,x_2,x_3) \in S^2 \cap P$ implies $x_3 = 0$ and $x_1^2 + x_2^2 = 1$. Hence for $\underset{\sim}{x} \in S^2 \cap P$

$$\text{rank} \begin{bmatrix} (\underset{\sim}{D}F)_{\underset{\sim}{x}} \\ (\underset{\sim}{D}G)_{\underset{\sim}{x}} \end{bmatrix} = \text{rank} \begin{bmatrix} 2x_1 & 2x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\underset{\sim}{x}} = 2$$

so that $S^2 \pitchfork P$.

The concept of transversality can be extended to a more general situation than that depicted by (21).

Definition 2. Let X and Z be C^1 manifolds and let

$$\underset{\sim}{F}: X \to Z$$

be a C^1 function. Let Y be a submanifold of Z. $\underset{\sim}{F}$ is said to be transversal to Y and is abbreviated as

$$\underset{\sim}{F} \pitchfork Y$$

if

$$\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}} + T_{\underset{\sim}{y}}Y = T_{\underset{\sim}{y}}Z \tag{26}$$

for all $\underset{\sim}{y} = \underset{\sim}{F}(\underset{\sim}{x}) \in Y$, where $(\underset{\sim}{dF})_{\underset{\sim}{x}}$ denotes the differential of $\underset{\sim}{F}$ at $\underset{\sim}{x}$ and $\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}}$ denotes its image.

We use the symbol $\underset{\sim}{dF}$ when the domain of $\underset{\sim}{F}$ is a general manifold while we use $\underset{\sim}{DF}$ when the domain is an euclidean space. Observe that Def. 2 is a generalization of Def. 1 because the subset $\underset{\sim}{F}(X) \subset Z$ need not be a submanifold even though both X and Z are submanifolds. For example, if we take X to be the one-dimensional submanifold $\{\phi_L, k_1 \sin k_2 \phi_L\}$ in Fig. 6(b) and choose $F = \pi_i$, the projection of this curve into the i_L-axis, then $F(X) = [-k_1, k_1]$, which is not a (boundaryless) submanifold since it includes its end points. Transversality of $\underset{\sim}{F}$ is a natural generalization of (21) in the sense that $T_{\underset{\sim}{x}}X$ is replaced by

$$\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}} = (\underset{\sim}{dF})_{\underset{\sim}{x}}\,(T_{\underset{\sim}{x}}X) \tag{27}$$

where the right side means the map $(\underset{\sim}{dF})_{\underset{\sim}{x}}$ acting on the set $T_{\underset{\sim}{x}}X$. Observe that since $\underset{\sim}{F}(X) \subset Z$, $\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}}$ is a linear subspace of $T_{\underset{\sim}{F}(\underset{\sim}{x})}Z$. In particular if we take $\underset{\sim}{F}$ to be the inclusion map:

$$\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x}$$

then (26) coincides with (21), because $\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}} = T_{\underset{\sim}{y}}X$ is just the tangent space at $\underset{\sim}{y}$. Let us give simple examples. Consider the function $\underset{\sim}{F}: \mathbb{R} \to \mathbb{R}^2$ defined by (see Fig. 7(d))

$$\underset{\sim}{F}(x) = (x, \sin x) \tag{28}$$

Hence $X = \mathbb{R}$, $Z = \mathbb{R}^2$. Take $Y = \{(y_1, y_2) \mid y_2 = 0\}$. Then

$$\underset{\sim}{F}(x) \cap Y = \{(0,0), (\pm\pi, 0), (\pm 2\pi, 0), \ldots\}$$

For example, at $(y_1, y_2) = (0,0)$,

$$\mathrm{Im}(d\underset{\sim}{F})_x = \mathrm{Im}(d\underset{\sim}{F})_0 = \mathrm{Im}([1 \ \cos x]^T)_0 = \mathrm{Im}([1 \ 1]^T)$$
$$= \{[1 \ 1]^T x \mid x \in \mathbb{R}\} = \{[x \ x]^T \mid x \in \mathbb{R}\} \tag{29}$$

Hence $\mathrm{Im}(d\underset{\sim}{F})_x$ is just the tangent line to the curve $y_2 = \sin y_1$ at the point (0,0) as shown in Fig. 7(d). On the other hand

$$T_{(0,0)}Y = \{(y_1,y_2) \mid y_2 = 0\} . \tag{30}$$

Hence (29) and (30) imply (26). But if we let

$$\underset{\sim}{F}(x) = (x, \sin x + 1)$$

then

$$\underset{\sim}{F}(x) \cap Y = \{(\tfrac{(2k+1)\pi}{2}, 0) \ \mid k = \pm 1, \pm 2, \ldots \}$$

and at $(y_1,y_2) = (\frac{3\pi}{2}, 0)$,

$$\mathrm{Im}(d\underset{\sim}{F})_{\frac{3\pi}{2}} = \mathrm{Im}\,[1 \ \cos x\,]^T_{\frac{3\pi}{2}} = \mathrm{Im}\,[1 \ 0]^T = \{(y_1,y_2) \mid y_2 = 0\}.$$

In this case, the tangent line is just the y_1-axis itself and hence (26) does not hold.

<u>Fact C</u>. If $\underset{\sim}{F} \pitchfork Y$, then the preimage $\underset{\sim}{F}^{-1}(Y)$ is a submanifold and

$$\mathrm{codim}_X \underset{\sim}{F}^{-1}(Y) = \mathrm{codim}_Z Y \tag{31}$$

where the left hand side denotes the codimension with respect to X and the right hand side denotes the codimension with respect to Z. ¤

Let us explain (31) by an example. Consider $\underset{\sim}{F}$ of (28). Y is described by

$$f(y_1,y_2) = 0$$

where $f(y_1,y_2) = y_2$ and hence it has codimension one in $Z = \mathbb{R}^2$. Now, since

$$\underset{\sim}{F}^{-1}(Y) = \{x \mid f \circ \underset{\sim}{F}(x) = 0\} = \{0, \pm\pi, \pm 2\pi, \ldots\}$$

it has zero dimension and hence $\underset{\sim}{F}^{-1}(Y)$ has codimension one in $X = \mathbb{R}$.

<u>Remark</u>. In this paper, whenever we write $X \pitchfork Y$, we assume that $X \cap Y$ is nonempty. Similarly, if we write $\underset{\sim}{F} \pitchfork Y$, then we assume that $\underset{\sim}{F}(X) \cap Y$ is nonempty. The empty cases are trivial.

4. The Minimal State Space of Dynamic Nonlinear Networks

Let M be the set of equilibria. As we have seen, if there are capacitor-only cut sets and inductor-only loops, then M is a set of continuum even if operating points are isolated. In this section, we will show that each trajectory is constrained to stay on an affine submanifold M*, henceforth called the minimal state space, such that $M \cap M^*$ consists only of isolated points. In other words, the minimal state space M* is an invariant submanifold and hence, each trajectory behaves as if there are only isolated equilibria.

To do this we will have to impose stronger conditions on the capacitor and inductor constitutive relations. We will also need to investigate some geometric properties of nonlinear networks which we will give before stating our results.

Note first that under the standing assumptions 1 and 2, there is a tree T_L containing a maximum number of inductors and a minimum number of capacitors with the following properties:

(i) All independent voltage sources are contained in T_L and all independent current sources are contained in T_L^* (cotree of T_L).

(ii) KVL and KCL can be written in the following form[2]:

$$\left[\begin{array}{cccc:cccc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{11} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{21} & \underset{\sim}{B}_{22} & \underset{\sim}{B}_{23} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{31} & \underset{\sim}{B}_{32} & \underset{\sim}{B}_{33} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{41} & \underset{\sim}{B}_{42} & \underset{\sim}{B}_{43} & \underset{\sim}{B}_{44} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{v}_{L\mathcal{L}} \\ \underset{\sim}{v}_{R\mathcal{L}} \\ \underset{\sim}{v}_{I} \\ \underset{\sim}{v}_{C\mathcal{L}} \\ \hline \underset{\sim}{v}_{L\mathcal{J}} \\ \underset{\sim}{v}_{G\mathcal{J}} \\ \underset{\sim}{v}_{V} \\ \underset{\sim}{v}_{C\mathcal{J}} \end{array}\right] = \underset{\sim}{0} \qquad \begin{array}{r} (32) \\ (33) \\ (34) \\ (35) \end{array}$$

$$\left[\begin{array}{cccc:cccc} -\underset{\sim}{B}_{11}^T & -\underset{\sim}{B}_{21}^T & -\underset{\sim}{B}_{31}^T & -\underset{\sim}{B}_{41}^T & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{B}_{22}^T & -\underset{\sim}{B}_{32}^T & -\underset{\sim}{B}_{42}^T & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{B}_{23}^T & -\underset{\sim}{B}_{33}^T & -\underset{\sim}{B}_{43}^T & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{B}_{44}^T & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{i}_{L\mathcal{L}} \\ \underset{\sim}{i}_{R\mathcal{L}} \\ \underset{\sim}{i}_{I} \\ \underset{\sim}{i}_{C\mathcal{L}} \\ \hline \underset{\sim}{i}_{L\mathcal{J}} \\ \underset{\sim}{i}_{G\mathcal{J}} \\ \underset{\sim}{i}_{V} \\ \underset{\sim}{i}_{C\mathcal{J}} \end{array}\right] = \underset{\sim}{0} \qquad \begin{array}{r} (36) \\ (37) \\ (38) \\ (39) \end{array}$$

[2] Note that our present $\underset{\sim}{B}$ matrix differs from that used in Section 2 because a different tree is used here.

where the variables are decomposed as follows:

	elements	voltage	current	number of elements
Cotree	inductors in T_L^*	$\mathbf{v}_{L\mathcal{L}}$	$\mathbf{i}_{L\mathcal{L}}$	$n_{L\mathcal{L}}$
	resistors in T_L^*	$\mathbf{v}_{R\mathcal{L}}$	$\mathbf{i}_{R\mathcal{L}}$	$n_{R\mathcal{L}}$
	independent current sources in T_L^*	$\mathbf{v}_I$	$\mathbf{i}_I$	n_I
	capacitors in T_L^*	$\mathbf{v}_{C\mathcal{L}}$	$\mathbf{i}_{C\mathcal{L}}$	$n_{C\mathcal{L}}$
Tree	inductors in T_L	$\mathbf{v}_{L\mathcal{T}}$	$\mathbf{i}_{L\mathcal{T}}$	$n_{L\mathcal{T}}$
	resistors in T_L	$\mathbf{v}_{G\mathcal{T}}$	$\mathbf{i}_{G\mathcal{T}}$	$n_{G\mathcal{T}}$
	independent voltage sources in T_L	$\mathbf{v}_V$	$\mathbf{i}_V$	n_V
	capacitors in T_L	$\mathbf{v}_{C\mathcal{T}}$	$\mathbf{i}_{C\mathcal{T}}$	$n_{C\mathcal{T}}$

In order to see that (i) is valid, observe that by Assumption 2, it is possible to choose a tree such that it contains all independent voltage sources and the associated cotree contains all independent current sources. Next maximize the number of inductors and minimize the number of capacitors in the tree. We claim that this is the desired tree, i.e., the requirement of (i) does not destroy the maximality and the minimality of inductors and capacitors, respectively. Pick a link inductor L_1. We claim that the fundamental loop defined by L_1 consists only of inductors and hence the maximality property is retained. This is because by Assumption 2 $\mathcal{N}_{RG}$ has no voltage-source loops and hence $\mathcal{N}$ has no loops consisting only of inductors <u>and</u> independent voltage sources. Thus if this fundamental loop contains an independent voltage source, it must contain at least one capacitor or resistor. This contradicts the maximality hypothesis. Similarly, minimality of capacitors is retained. To see (ii) note that $\mathbf{B}_{12}$, $\mathbf{B}_{14}$ and $\mathbf{B}_{24}$ are zero submatrices because of the choice of the tree. Again, by Assumption 2, $\mathbf{B}_{13}$ and $-\mathbf{B}_{34}^T$ are zero submatrices.

Next let $\bar{n}_C$ and $\bar{n}_L$ be the number of linearly independent capacitor-only cut sets and the number of linearly independent inductor-only loops, respectively.

Since fundamental cut sets and fundamental loops are always linearly independent, it follows from (32) and (39) that

$$\bar{n}_C = n_{C\mathcal{J}} \tag{40}$$

$$\bar{n}_L = n_{L\mathcal{L}}. \tag{41}$$

<u>Remark</u>. The tree T_L is called an <u>L-normal tree</u> [4]. It is useful for deriving some interesting properties concerning the equilibrium points and invariant submanifold as we will see in the sequel.

Let

$$\underset{\sim}{\pi}_v,\ \underset{\sim}{\pi}_q : \Lambda_C \to \mathbb{R}^{n_C+n_S}$$

be the projection maps defined by

$$\underset{\sim}{\pi}_v(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S) = (\underset{\sim}{v}_C, \underset{\sim}{v}_S) \tag{42}$$

and

$$\underset{\sim}{\pi}_q(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S) = (\underset{\sim}{q}_C, \underset{\sim}{q}_S) \tag{43}$$

respectively. Similarly, let

$$\underset{\sim}{\pi}_i,\ \underset{\sim}{\pi}_\phi : \Lambda_L \to \mathbb{R}^{n_L+n_T}$$

be defined by

$$\underset{\sim}{\pi}_i(\underset{\sim}{i}_L, \underset{\sim}{i}_T, \underset{\sim}{\phi}_L, \underset{\sim}{\phi}_T) = (\underset{\sim}{i}_L, \underset{\sim}{i}_T) \tag{44}$$

and

$$\underset{\sim}{\pi}_\phi(\underset{\sim}{i}_L, \underset{\sim}{i}_T, \underset{\sim}{\phi}_L, \underset{\sim}{\phi}_T) = (\underset{\sim}{\phi}_L, \underset{\sim}{\phi}_T) \tag{45}$$

respectively.

Recall (8) and (11). In this section, we will choose the <u>generalized charge</u>

$$\underset{\sim}{q} = \underset{\sim}{q}_C - \underset{\sim}{B}_{SC}^T \underset{\sim}{q}_S = [\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^T] \begin{bmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{q}_S \end{bmatrix} \tag{46}$$

and the generalized flux

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}_L + \underset{\sim}{B}_{L\Gamma}\underset{\sim}{\phi}_\Gamma = [\underset{\sim}{1} \quad \underset{\sim}{B}_{L\Gamma}] \begin{bmatrix} \underset{\sim}{\phi}_L \\ \underset{\sim}{\phi}_\Gamma \end{bmatrix} \tag{47}$$

as the state variables [4].

Lemma 1. Let $(\underset{\sim}{v}^*, \underset{\sim}{i}^*)$ denote an operating point of $\mathcal{N}_{RG}$. Then the set of equilibria for $\mathcal{N}$ is given by

$$M = \bigcup_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)} M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)}, \tag{48}$$

where

$$M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)} \triangleq \left[\begin{array}{cc|cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^T & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right] \begin{bmatrix} \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1} \; (\underset{\sim}{v}^\perp + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]) \\ \underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1} \; (\underset{\sim}{i}^\perp + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]) \end{bmatrix} \tag{49}$$

where $\underset{\sim}{\pi}_v^{-1}(\cdot)$ and $\underset{\sim}{\pi}_i^{-1}(\cdot)$ denote preimages, $\underset{\sim}{v}^\perp$ is an $(n_C+n_S) = (n_{C\mathcal{J}} + n_{C\mathcal{L}})$-dimensional vector orthogonal to the subspace $\mathrm{Ker}[\underset{\sim}{B}_{44} \; \underset{\sim}{1}]$ and $\underset{\sim}{i}^\perp$ is an $(n_L+n_T) = (n_{L\mathcal{J}} + n_{L\mathcal{L}})$-dimensional vector orthogonal to the subspace $\mathrm{Ker}[\underset{\sim}{1} \; -\underset{\sim}{B}_{11}^T]$. ($\underset{\sim}{v}^\perp$ and $\underset{\sim}{i}^\perp$ depend on $(\underset{\sim}{v}^*, \underset{\sim}{i}^*)$.)

Proof. Note that (35) gives

$$[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{v}_{C\mathcal{J}} \\ \underset{\sim}{v}_{C\mathcal{L}} \end{bmatrix} = [\underset{\sim}{0} \quad -\underset{\sim}{B}_{41}] \begin{bmatrix} \underset{\sim}{v}_{L\mathcal{L}} \\ \underset{\sim}{v}_{L\mathcal{J}} \end{bmatrix} + [-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V \\ \underset{\sim}{v}_{G\mathcal{J}} \end{bmatrix} \tag{50}$$

Let $(\underset{\sim}{v}_V^*, \underset{\sim}{v}_{G\mathcal{J}}^*)$ be the value at an operating point. Since we are considering equilibria, let $(\underset{\sim}{v}_{L\mathcal{L}}, \underset{\sim}{v}_{L\mathcal{J}}) = (\underset{\sim}{0}, \underset{\sim}{0})$. Then (50) gives

$$[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{v}_{C\mathcal{J}} \\ \underset{\sim}{v}_{C\mathcal{L}} \end{bmatrix} = [-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix} \tag{51}$$

It follows from a standard result from linear algebra [9,p.159] that there is an element

$$\underset{\sim}{v}^\perp \in (\mathrm{Ker}[\underset{\sim}{B}_{44} \; \underset{\sim}{1}])^\perp \tag{52}$$

such that

$$\begin{bmatrix} \underset{\sim}{v}_{C\mathcal{J}} \\ \underset{\sim}{v}_{C\mathcal{L}} \end{bmatrix} = \underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}] \tag{53}$$

is the solution set of (51), where the set in (52) is the orthogonal complement of $\mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]$. Note that (39) implies that if there are capacitor-only cut sets, then $\underset{\sim}{B}_{44}$ is not a null matrix and hence (53) defines an affine submanifold of dimension $n_{C\mathcal{J}} = \bar{n}_C$, which is the number of linearly independent capacitor-only cut sets. If there are no capacitor-only cut sets, then $\underset{\sim}{B}_{44}$ is a null matrix and (53) degenerates into a single point $\underset{\sim}{v}^{\perp}$.

Now, without loss of generality, possibly by relabelling the branches, we can assume that

$$(\underset{\sim}{v}_{C\mathcal{J}}, \underset{\sim}{v}_{C\mathcal{L}}) = (\underset{\sim}{v}_C, \underset{\sim}{v}_S) \tag{54}$$

where $\underset{\sim}{v}_C$ and $\underset{\sim}{v}_S$ are the tree and cotree capacitor voltages relative to the C-normal tree T_C defined in Section 2. See APPENDIX 2 for an example illustrating this point.

It follows from (46), (53) and (54) that the "generalized charge" $\underset{\sim}{q}$ at equilibria are expressed by

$$\underset{\sim}{q} = [\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^{T}]\ \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\ (\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]) \tag{55}$$

Similarly the "generalized flux" $\underset{\sim}{\phi}$ at equilibria are expressed by

$$\underset{\sim}{\phi} = [\underset{\sim}{1} \quad \underset{\sim}{B}_{L\Gamma}]\ \underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1}\ (\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^{T}]) \tag{56}$$

where

$$\underset{\sim}{i}^{\perp} \in (\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^{T}])^{\perp} \tag{57}$$

Here, we have assumed without loss of generality that (see APPENDIX 2)

$$(\underset{\sim}{i}_{L\mathcal{L}}, \underset{\sim}{i}_{L\mathcal{J}}) = (\underset{\sim}{i}_L, \underset{\sim}{i}_\Gamma) \tag{58}$$

¤

Lemma 2. There is an affine submanifold M* such that $\left(\underset{\sim}{q}(t),\ \underset{\sim}{\phi}(t)\right) \in$ M* for all t and M* is defined by

$$M^* = \begin{bmatrix} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^T & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{bmatrix} \begin{bmatrix} \underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T] \\ \underset{\sim}{\phi}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}\ ] \end{bmatrix} \tag{59}$$

where $\underset{\sim}{q}^{\perp}$ is an $(n_{C\mathcal{J}} + n_{C\mathcal{L}})$-dimensional vector orthogonal to the subspace Ker$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T]$ and $\underset{\sim}{\phi}^{\perp}$ is an $(n_{L\mathcal{J}} + n_{L\mathcal{L}})$-dimensional vector orthogonal to the subspace Ker$[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}]$. $\left(\underset{\sim}{q}^{\perp}\right.$ depends on $(\underset{\sim}{q}_C(0),\ \underset{\sim}{q}_S(0))$ and $\underset{\sim}{\phi}^{\perp}$ depends on $(\underset{\sim}{\phi}_L(0),\ \underset{\sim}{\phi}_\Gamma(0))$.$\left.\right)$

Proof. Equation (39) gives

$$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T] \begin{bmatrix} \underset{\sim}{i}_{C\mathcal{J}} \\ \underset{\sim}{i}_{C\mathcal{L}} \end{bmatrix} = \underset{\sim}{0}$$

Integrating this equation with respect to time and making use of (54) we have[3]

$$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T] \begin{bmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{q}_S \end{bmatrix} = \underset{\sim}{Q}_0 \tag{60}$$

where $\underset{\sim}{Q}_0$ is uniquely determined by the initial condition $\left(\underset{\sim}{q}_C(0), \underset{\sim}{q}_S(0)\right)$. It follows from (60) that there is an element [9]

$$\underset{\sim}{q}^{\perp} \in \left(\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T]\right)^{\perp}$$

such that the set

$$\begin{bmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{q}_S \end{bmatrix} = \underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T] \tag{61}$$

is the solution set for (60), i.e., $\left(\underset{\sim}{q}_C(t), \underset{\sim}{q}_S(t)\right)$ is constrained to (61) for all t. Similarly, there is an element

[3] Observe that since the two trees T_C and T_L are distinct, the number of capacitors belonging to T_C is generally different from that belonging to T_L. However, $(T_C \cup T_C^*) = (T_L \cup T_L^*)$. Hence although the matrix multiplication indicated in (60) is compatible, the dimensions of $\underset{\sim}{q}_C$ and $\underset{\sim}{q}_S$, in general, do not correspond to that of $\underset{\sim}{1}$ and $-\underset{\sim}{B}_{44}^T$, respectively.

$$\underset{\sim}{\phi}^{\perp} \in (\mathrm{Ker}[\underset{\sim}{B}_{11} \quad \underset{\sim}{1})^{\perp}$$

such that

$$\begin{bmatrix} \underset{\sim}{\phi}_L \\ \\ \underset{\sim}{\phi}_\Gamma \end{bmatrix} = \underset{\sim}{\phi}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}] \tag{62}$$

is the solution set of

$$[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{\phi}_L \\ \\ \underset{\sim}{\phi}_\Gamma \end{bmatrix} = \underset{\sim}{\Phi}_0 \tag{63}$$

where $\underset{\sim}{\Phi}_0$ is uniquely determined by $\left(\underset{\sim}{\phi}_L(0), \underset{\sim}{\phi}_\Gamma(0)\right)$. Hence, (46), (47), (61) and (62) imply (59). □

The following is the main result of this section.

<u>Theorem 2</u>. Assume the following:

(i) The dynamics of (1) is described in terms of $\underset{\sim}{x} = (\underset{\sim}{q}, \underset{\sim}{\phi})$, where $\underset{\sim}{q}$ and $\underset{\sim}{\phi}$ denote the "generalized charge" and the "generalized flux," respectively.

(ii) $\mathcal{N}_{RG}$ has only isolated operating points.

(iii) For each operating point $(\underset{\sim}{v}^*, \underset{\sim}{i}^*)$ of $\mathcal{N}_{RG}$

$$\text{(a)} \quad \underset{\sim}{\pi}_v \pitchfork \left(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\right) \tag{64}$$

$$\underset{\sim}{\pi}_i \pitchfork \left(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\right) \tag{65}$$

where $\underset{\sim}{\pi}_v$ and $\underset{\sim}{\pi}_i$ are projection maps defined in (42) and (44), respectively.

(b) $\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\right)$ and $\underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1}\left(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\right)$ are $n_{C\mathcal{J}}$ and $n_{L\mathcal{L}}$ dimensional submanifolds, respectively, and

$$\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\right) \pitchfork \left(\underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T]\right) \tag{66}$$

$$\underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1}\left(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\right) \pitchfork \left(\underset{\sim}{\phi}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}]\right) \tag{67}$$

where $\underset{\sim}{\pi}_q$ and $\underset{\sim}{\pi}_\phi$ are projection maps defined in (43) and (45), respectively.

Then

(1) $\mathcal{N}$ has only isolated equilibria if, and only if, Condition A holds.

(2) There is a minimal state space $M^* \subset \mathbb{R}^{n_C+n_L}$ which is parameterized by the initial state $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ such that the trajectory $\underset{\sim}{x}(t;t_0,\underset{\sim}{x}_0) \in M^*$ for all t.

(3) The minimal state space M^* is an affine submanifold of $\mathbb{R}^{n_C+n_L}$ of dimension $m = (n_C-\bar{n}_C) + (n_L-\bar{n}_L)$, where $\bar{n}_C$ is the number of linearly independent capacitor-only cut sets, and $\bar{n}_L$ is the number of linearly independent inductor-only loops.

(4) $M \cap M^*$ consists only of isolated points. (See Fig. 1)

Remarks: 1. Note that relative to $\mathbb{R}^{n_C+n_L}$, M^* is merely a translated copy of $\mathbb{R}^m$. Hence once the initial state $\underset{\sim}{x}_0$ is fixed, the network behaves as if it is in an m-dimensional state space. Note also that (4) says the network behaves as if it has only isolated equilibria.

2. Condition (iii) is automatically satisfied if the capacitor and inductor constitutive relations are represented by uniformly-increasing functions. We will show this later (see Theorem 3).

We need the following lemma which relates some of the properties of T_C to those of T_L.

Lemma 3.

(i) $\mathrm{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T] \supseteq \mathrm{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]$ (68)

(ii) $\mathrm{Ker}[\underset{\sim}{B}_{11} \;\; \underset{\sim}{1}] \supseteq \mathrm{Ker}[\underset{\sim}{1} \;\; \underset{\sim}{B}_{L\Gamma}]$ (69)

Proof (i) Open-circuit all elements except capacitors. Then (10) and (11) give

$$\begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{B}_{SV}^T \\ \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{i}_S \end{bmatrix} = \underset{\sim}{0} \qquad \begin{matrix} (10') \\ (11') \end{matrix}$$

whereas (36)-(39) give

$$\begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{B}_{41}^T \\ \underset{\sim}{0} & -\underset{\sim}{B}_{42}^T \\ \underset{\sim}{0} & -\underset{\sim}{B}_{43}^T \\ \underset{\sim}{1} & -\underset{\sim}{B}_{44}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_{C\mathcal{J}} \\ \underset{\sim}{i}_{C\mathcal{L}} \end{bmatrix} = \underset{\sim}{0} \qquad \begin{matrix} (36') \\ (37') \\ (38') \\ (39') \end{matrix}$$

Since the independent voltage sources are contained in $T_C \cap T_L$, (10') and (38') describe the same equations provided that (54) holds. Hence

$$\text{Ker}[\underset{\sim}{0} \quad -\underset{\sim}{B}_{SV}^T] = \text{Ker}[\underset{\sim}{0} \quad -\underset{\sim}{B}_{43}^T]$$

and the remaining equations must define the same set; namely,

$$\text{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^T] = \text{Ker}[\underset{\sim}{0} \quad -\underset{\sim}{B}_{41}^T] \cap \text{Ker}[\underset{\sim}{0} \quad -\underset{\sim}{B}_{42}^T] \cap \text{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T]$$

The last equation implies (68). (ii) Short-circuit all the elements except inductors. Then a dual argument implies (69). ¤

<u>Proof of Theorem 2</u>.

(1) To prove <u>sufficiency</u>, assume <u>Condition A</u>. Then $\underset{\sim}{B}_{44}$ in (49) becomes a null matrix and

$$\text{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}] = \{\underset{\sim}{0}\}$$

so that the set defined by (53) degenerates into a single point

$$\underset{\sim}{v}^{\perp} = [-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix}$$

Hence, $\underset{\sim}{v}^{\perp}$ is a zero-dimensional submanifold of $\mathbb{R}^{n_C+n_S}$. By condition (iii-a) and by <u>Fact C</u>,

$$\underset{\sim}{\pi}_v^{-1}\left([-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}} \end{bmatrix}\right) \subset \Lambda_C \tag{71}$$

is a submanifold and

$$\text{codim}_{\Lambda_C} \underset{\sim}{\pi}_v^{-1}\left([-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix}\right) = \text{codim}_{\mathbb{R}^{n_C+n_S}}\left([-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix}\right) = n_C + n_S \tag{72}$$

Hence the set defined in (71) is a zero-dimensional submanifold, i.e., it is a set of isolated points. Hence

$$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^T]\, \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left([-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix}\right) \tag{73}$$

is a set of isolated points. A similar argument holds for inductors, by using

(36). Since each $M_{(\underset{\sim}{v}^*,\underset{\sim}{i}^*)}$ is a set of isolated points, the set defined by (48) is also a set of isolated points.

Proof of <u>necessity</u> is slightly involved although the idea is fairly simple. Here, we will give the idea of the proof with a figure (Fig. 8) which makes the situation clear. A rigorous proof is given in APPENDIX 3.

Suppose that <u>Condition A</u> is violated. First suppose that there is at least one capacitor only cut set. Then $\underset{\sim}{B}_{44}$ is not a null matrix so that $\underset{\sim}{v}^{\perp} + \text{Ker}[\underset{\sim}{B}_{44} \;\; \underset{\sim}{1}]$ is an $n_{C\mathcal{J}}$-dimensional affine submanifold, where $n_{C\mathcal{J}} \geq 1$. Let

$$M_C \triangleq \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1} \left(\underset{\sim}{v}^{\perp} + \text{Ker}[\underset{\sim}{B}_{44} \;\; \underset{\sim}{1}]\right) .$$

Our initial step in the proof depends on the fact that the projection $\hat{M}_C$ of M_C onto $(\text{Ker}[1 \;\; -B_{44}^T])^{\perp}$ consists of a continuum of points. This follows from condition (iii-b), as is shown in APPENDIX 3. Moreover, by (i) of <u>Lemma 3</u>,

$$\left(\text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T]\right)^{\perp} \subseteq \left(\text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]\right)^{\perp} . \tag{74}$$

Next, the matrix $[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]$ maps $\left(\text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]\right)^{\perp}$ onto its image space in a one-to-one manner [9]. Hence $[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]\,\hat{M}_C$ has a continuum of points. Hence $M_{(\underset{\sim}{v}^*,\underset{\sim}{i}^*)}$ in (49) has a continuum of points. A similar argument holds when there are inductor-only loops.

(2) This property follows from <u>Lemma 2</u>.

(3) First observe from (39) and (32) that

$$\dim \text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T] = n_{C\mathcal{L}} \tag{75}$$

$$\dim \text{Ker}[\underset{\sim}{B}_{11} \;\; \underset{\sim}{1}] = n_{L\mathcal{J}} \tag{76}$$

Moreover (59) implies

$$\dim M^* = \dim\left\{[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T]\ \text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T]\right\} + \dim\left\{[\underset{\sim}{1} \;\; \underset{\sim}{B}_{L\Gamma}]\ \text{Ker}[\underset{\sim}{B}_{11} \;\; \underset{\sim}{1}]\right\} .$$

Hence applying <u>Fact E</u> of APPENDIX 1, we obtain

$$\begin{aligned}\dim M^* &= \dim \text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T] - \dim \text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{44}^T] \cap \text{Ker}[\underset{\sim}{1} \;\; -\underset{\sim}{B}_{SC}^T] \\ &\quad + \dim \text{Ker}[\underset{\sim}{B}_{11} \;\; \underset{\sim}{1}] - \dim \text{Ker}[\underset{\sim}{B}_{11} \;\; \underset{\sim}{1}] \cap \text{Ker}[\underset{\sim}{1} \;\; \underset{\sim}{B}_{L\Gamma}] \\ &= (n_{C\mathcal{L}} - n_S) + (n_{L\mathcal{J}} - n_{\Gamma})\end{aligned} \tag{77}$$

where we have used Lemma 3, and the facts $\dim \operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{SC}^{T}] = n_S$, $\dim \operatorname{Ker}[\underset{\sim}{1} \ \ \underset{\sim}{B}_{L\Gamma}] = n_\Gamma$. Now since $n_C + n_S = n_{C\mathcal{J}} + n_{C\mathcal{L}}$ and $n_L + n_\Gamma = n_{L\mathcal{J}} + n_{L\mathcal{L}}$, we can solve for

$$n_{C\mathcal{L}} - n_S = n_C - n_{C\mathcal{J}}\,,\quad n_{L\mathcal{J}} - n_\Gamma = n_L - n_{L\mathcal{L}} \tag{78}$$

Substituting (78) into (77) and making use of (40) and (41), we obtain

$$\dim M^* = n_C - n_{C\mathcal{J}} + n_L - n_{L\mathcal{L}} = (n_C-\bar{n}_C) + (n_L-\bar{n}_L)$$

which is precisely what is to be proved

(4) By condition (iii-b) and by Fact A, the set

$$\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{44} \ \ \underset{\sim}{1}]\right) \cap \left(\underset{\sim}{q}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{44}^{T}]\right) \tag{79}$$

is a submanifold. Moreover, since

$$\operatorname{codim} \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{44} \ \ \underset{\sim}{1}]\right) = n_{C\mathcal{L}}$$

$$\operatorname{codim} \left(\underset{\sim}{q}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{44}^{T}]\right) = n_{C\mathcal{J}}$$

we have, by Fact A, that

$$\operatorname{codim}\left\{\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{44} \ \ \underset{\sim}{1}]\right) \cap \left(\underset{\sim}{q}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{44}^{T}]\right)\right\} = n_{C\mathcal{L}} + n_{C\mathcal{J}} \tag{80}$$

i.e., (79) is a zero-dimensional submanifold. Hence it consists of isolated points. Similarly,

$$\underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1}\left(\underset{\sim}{i}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{11}^{T}]\right) \cap \left(\underset{\sim}{\phi}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{11} \ \ \underset{\sim}{1}]\right) \tag{81}$$

is a set of isolated points. Recall (49) and (59). Then, clearly

$$M_{(\underset{\sim}{v}^*,\underset{\sim}{i}^*)} \cap M^* = \left[\begin{array}{cc|cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^{T} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right] \begin{bmatrix} \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1}\left(\underset{\sim}{v}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{44} \ \ \underset{\sim}{1}]\right) \cap \left(\underset{\sim}{q}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{44}^{T}]\right) \\ \underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1}\left(\underset{\sim}{i}^{\perp}+\operatorname{Ker}[\underset{\sim}{1} \ \ -\underset{\sim}{B}_{11}^{T}]\right) \cap \left(\underset{\sim}{\phi}^{\perp}+\operatorname{Ker}[\underset{\sim}{B}_{11} \ \ \underset{\sim}{1}]\right) \end{bmatrix} \tag{82}$$

consists of a set of isolated points. Finally, since $\mathcal{N}_{RG}$ has only isolated operating points, we conclude, by (48) that

$$M = \bigcup_{(\underset{\sim}{v}^*,\underset{\sim}{i}^*)} \left(M_{(\underset{\sim}{v}^*,\underset{\sim}{i}^*)} \cap M^*\right)$$

has only isolated points. ∎

Remark. Note that M^* depends on the initial state, whereas M does not. It is, now, clear geometrically what Ohtsuki and Watanabe [4] have tried to do analytically. They chose a coordinate system on the minimal state space M^* and defined the dynamics on it. The coordinate system, however, is dependent on the initial state.

Note that in the proof of the sufficiency of (1), we did not use condition (iii-b). Hence we have the following.

Corollary 2. Assume Condition A, (i), (ii) and (iii-a) of Theorem 2. Then $\mathcal{N}$ has only isolated equilibria. Moreover, (2), (3), (4) of Theorem 2 are trivially satisfied.

Example 2 of Section 2 can be taken care of by Corollary 2. Note that $\pi_i(\Lambda_L) \cap \{0\} = [-k_1,k_1] \cap \{0\} = \{0\}$. Note also that in terms of the coordinate ϕ_L, we have

$$\Lambda_L = \left\{(\phi_L, k_1 \sin k_2\phi_L) \,|\, \phi_L \in \mathbb{R}\right\}, \ \pi_i(\phi_L) = k_1 \sin k_2\phi_L.$$

The tangent space of Λ_L at 0 satisfies [10,p.4]

$$T_0\Lambda_L = \mathrm{Im}[1 \quad k_1k_2\cos k_2\phi_L]_{\phi_L = \pm\frac{k\pi}{k_2}} = \left\{[1 \quad \pm k_1k_2]x \,|\, x \in \mathbb{R}\right\} = \mathbb{R}\,.$$

Hence

$$\mathrm{Im}(d\pi_i)_{\pm\frac{k\pi}{k_2}} = \left\{(k_1k_2\cos k_2\phi_L)_{\phi_L = \pm\frac{k\pi}{k_2}}\, x \,|\, x \in \mathbb{R}\right\} = \{\pm k_1k_2x \,|\, x \in \mathbb{R}\} = \mathbb{R}\,.$$

and

$$T_0(\{0\}) = \{0\}.$$

Therefore

$$\mathrm{Im}(d\pi_i)_{\pm\frac{k\pi}{k_2}} + T_0(\{0\}) = T_0(\mathbb{R}) \tag{83}$$

which implies

$$\pi_i \pitchfork \{0\}. \tag{84}$$

On the other hand, Λ_L of Fig. 6(c) does not satisfy (84) because for $\phi_L \in [a,b]$, $(d\pi_i)_{\phi_L} = 0$ and hence $\mathrm{Im}(d\pi_i)_{\phi_L}$ is just the point $\{0\}$, thereby violating (83).

Remark. In the special case where all capacitors and inductors are uncoupled 2-terminal elements, condition (iii-a) simply means that at each equilibrium voltage v_k^* for capacitor C_k (resp., current i_k^* for inductor L_k), the tangent to the q_k-v_k curve (resp., ϕ_k-i_k curve) is not parallel to the q_k-axis (resp., ϕ_k-axis).

If we only need necessity part of (1) of Theorem 2, we can relax condition (iii-b). In order to guarantee that the set (48) has a continuum of points, $\pi_q \circ \pi_v^{-1}\left(v^\perp + \mathrm{Ker}[B_{44} \;\; 1]\right)$ and $\pi_\phi \circ \pi_i^{-1}\left(i^\perp + \mathrm{Ker}[1 \;\; -B_{11}^T]\right)$ do not have to be submanifolds globally. If we check the proof, we can generalize the result in the following manner.

Corollary 3. Assume that Condition A is violated and that $\mathcal{N}_{RG}$ has at least one operating point. Suppose that the following holds:

(iii-b') There is a point $(q_C, q_S) \in \mathbb{R}^{n_C+n_S}$ $\left(\text{resp., } (\phi_L, \phi_\Gamma) \in \mathbb{R}^{n_L+n_\Gamma}\right)$ and a neighborhood W_C (resp., W_L) of this point such that $\pi_q \circ \pi_v^{-1}\left(v^\perp + \mathrm{Ker}[B_{44} \;\; 1]\right) \cap W_C$ $\left(\text{resp., } \pi_\phi \circ \pi_i^{-1}\left(i^\perp + \mathrm{Ker}[1 \;\; -B_{11}^T]\right) \cap W_L\right)$ is an $n_{C\mathcal{J}}$ (resp., $n_{L\mathcal{L}}$)-dimensional submanifold of W_C (resp., W_L) and

$$\left\{\pi_q \circ \pi_v^{-1}\left(v^\perp + \mathrm{Ker}[B_{44} \;\; 1]\right) \cap W_C\right\} \pitchfork \left\{\left(q^\perp + \mathrm{Ker}[1 \;\; -B_{44}^T]\right) \cap W_C\right\}$$

$$\left\{\pi_\phi \circ \pi_i^{-1}\left(i^\perp + \mathrm{Ker}[1 \;\; -B_{11}^T]\right) \cap W_L\right\} \pitchfork \left\{\left(\phi^\perp + \mathrm{Ker}[B_{11} \;\; 1]\right) \cap W_L\right\}$$

Then $\mathcal{N}$ has a continuum of equilibria.

Proof. If we restrict the proof of the necessity part of (1) to W_C and W_L, we conclude that the set (48) has a continuum of points. ¤

Remark. Recall that in Theorem 1 we did not need to impose conditions on Λ_C and Λ_L. We only required that $x = (v_C, i_L)$ be chosen as the state variables. The reason we needed the additional condition (iii) of Theorem 2 is that we have to map the equilibria in the $(v_C, v_S, i_L, i_\Gamma)$-space into the (q, ϕ)-space.

Theorem 3. Assume (i), (ii) of Theorem 2 and the following condition:

(iii') Capacitor and inductor constitutive relations are described by

$$(\underset{\sim}{q}_C, \underset{\sim}{q}_S) = \underset{\sim}{f}(\underset{\sim}{v}_C, \underset{\sim}{v}_S) \tag{85}$$

and

$$(\underset{\sim}{\phi}_L, \underset{\sim}{\phi}_\Gamma) = \underset{\sim}{g}(\underset{\sim}{i}_L, \underset{\sim}{i}_\Gamma) \tag{86}$$

where $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are uniformly increasing [6].

Then all the conclusions of Theorem 2 hold. Moreover

(1) $M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)}$ has the following representation:

$$M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)} = \left[\begin{array}{cc|cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^T & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right] \begin{bmatrix} \underset{\sim}{f}\left(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\right) \\ \underset{\sim}{g}\left(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\right) \end{bmatrix} \tag{87}$$

(2) $M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)} \cap M^*$ is a unique point given by

$$M_{(\underset{\sim}{v}^*, \underset{\sim}{i}^*)} \cap M^* = \left[\begin{array}{cc|cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^T & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right] \begin{bmatrix} \underset{\sim}{\mathcal{F}}^{-1}\begin{pmatrix} \underset{\sim}{q}_0 \\ [-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_V^* \\ \underset{\sim}{v}_{G\mathcal{J}}^* \end{bmatrix} \end{pmatrix} \\ \underset{\sim}{\mathcal{G}}^{-1}\begin{pmatrix} \underset{\sim}{\phi}_0 \\ [\underset{\sim}{B}_{21}^T \quad \underset{\sim}{B}_{31}^T] \begin{bmatrix} \underset{\sim}{i}_{R\mathcal{L}}^* \\ \underset{\sim}{i}_I^* \end{bmatrix} \end{pmatrix} \end{bmatrix} \tag{88}$$

where

$$\underset{\sim}{\mathcal{F}}(\underset{\sim}{q}_C, \underset{\sim}{q}_S) \triangleq \begin{bmatrix} [\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^T] \begin{bmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{q}_S \end{bmatrix} \\ [\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\underset{\sim}{f}^{-1}(\underset{\sim}{q}_C, \underset{\sim}{q}_S) \end{bmatrix} \tag{89}$$

$$\underset{\sim}{\mathcal{G}}(\underset{\sim}{\phi}_L, \underset{\sim}{\phi}_\Gamma) \triangleq \begin{bmatrix} [\underset{\sim}{B}_{11} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{\phi}_L \\ \underset{\sim}{\phi}_\Gamma \end{bmatrix} \\ [\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\underset{\sim}{g}^{-1}(\underset{\sim}{\phi}_L \quad \underset{\sim}{\phi}_\Gamma) \end{bmatrix} \tag{90}$$

Proof. We will show that (iii') implies (iii) of Theorem 2. To this end, note first that $(\underset{\sim}{q}_C, \underset{\sim}{q}_S)$ serves as a global coordinate for Λ_C:

$$\Lambda_C = \{(\underset{\sim}{f}^{-1}(\underset{\sim}{q}_C, \underset{\sim}{q}_S), \underset{\sim}{q}_C, \underset{\sim}{q}_S) \mid (\underset{\sim}{q}_C, \underset{\sim}{q}_S) \in \mathbb{R}^{n_C+n_S}\} \tag{91}$$

Hence, in terms of this coordinate, we have

$$\underset{\sim}{\pi}_v = \underset{\sim}{f}^{-1} \tag{92}$$

Since $\underset{\sim}{f}$ is uniformly increasing, it is a global diffeomorphism [6] and so is its inverse $\underset{\sim}{f}^{-1}$. Hence

$$\underset{\sim}{\pi}_v(\Lambda_C) \cap (\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}])$$

is always nonempty. Next, in terms of the above coordinate

$$(\underset{\sim}{d}\underset{\sim}{\pi}_v)(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S) = (\underset{\sim}{D}\underset{\sim}{f}^{-1})(\underset{\sim}{q}_C, \underset{\sim}{q}_S) \tag{93}$$

where the left hand side denotes the derivative map of $\underset{\sim}{\pi}_v$ evaluated at $(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)$ and $\underset{\sim}{D}\underset{\sim}{f}^{-1}$ denotes the Jacobian matrix of $\underset{\sim}{f}^{-1}(\cdot)$. Moreover (91) implies that the <u>tangent space</u> of Λ_C at $(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)$ satisfies the following property [10,p.4]:

$$T_{(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)} \Lambda_C = \mathrm{Im}[(\underset{\sim}{D}\underset{\sim}{f}^{-1})_{(\underset{\sim}{q}_C, \underset{\sim}{q}_S)} \quad \underset{\sim}{1}] = \mathbb{R}^{n_C+n_S} \tag{94}$$

Hence, in terms of the above coordinate

$$(\underset{\sim}{d}\underset{\sim}{\pi}_v)(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)\left(T_{(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)}\Lambda_C\right)$$

$$= (\underset{\sim}{D}\underset{\sim}{f}^{-1})_{(\underset{\sim}{q}_C, \underset{\sim}{q}_S)} \mathrm{Im}[(\underset{\sim}{D}\underset{\sim}{f}^{-1})_{(\underset{\sim}{q}_C, \underset{\sim}{q}_S)} \quad \underset{\sim}{1}] \tag{95}$$

But since $\underset{\sim}{D}\underset{\sim}{f}^{-1}$ is always nonsingular, the right hand side of (95) is $\mathbb{R}^{n_C+n_S}$. Hence

$$(\underset{\sim}{d}\underset{\sim}{\pi}_v)(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)\left(T_{(\underset{\sim}{v}_C, \underset{\sim}{v}_S, \underset{\sim}{q}_C, \underset{\sim}{q}_S)}\Lambda_C\right)$$

$$+ T_{(\underset{\sim}{v}_C, \underset{\sim}{v}_S)}\left(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]\right) = \mathbb{R}^{n_C+n_S} .$$

Since transversality <u>does not depend on a particular choice of coordinate</u>, it follows from (26) and (27) that (64) is satisfied. Similarly, we have (65).

To show (66) we first prove that (79) is nonempty. To this end note first that $\underset{\sim}{\pi}_q$ is the identity map in terms of the coordinate $(\underset{\sim}{q}_C, \underset{\sim}{q}_S)$. It follows from this and (92) that

$$\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1} = \underset{\sim}{f} \quad . \tag{96}$$

Hence, the set (79) assumes the following simplified form:

$$f(v^{\perp} + \mathrm{Ker}[B_{44} \quad 1]) \cap (q^{\perp} + \mathrm{Ker}[1 \quad -B_{44}^{T}]) \tag{97}$$

It follows from (51), (53), (60) and (61) that "non-emptiness of the set defined by (97)" is equivalent to saying that the following simultaneous equations have a solution:

$$[1 \quad -B_{44}^{T}] \begin{bmatrix} q_C \\ q_S \end{bmatrix} = Q_0 \tag{98}$$

$$[B_{44} \quad 1] f^{-1}(q_C, q_S) = [-B_{43} \quad -B_{42}] \begin{bmatrix} v_V^{*} \\ v_{G\mathcal{J}}^{*} \end{bmatrix} \tag{99}$$

To this end, consider the function defined by (89). It follows from Fact D of APPENDIX 1 and the assumption that f is uniformly increasing that the Jacobian matrix

$$(D\mathcal{F})(q_C, q_S) = \begin{bmatrix} [1 \quad -B_{44}^{T}] \\ [B_{44}^{T} \quad 1] \; (Df^{-1})(q_C, q_S) \end{bmatrix} \tag{100}$$

is positive definite uniformly with respect to (q_C, q_S) and hence $\mathcal{F}$ is a global diffeomorphism [6]. This implies that (98) and (99) have a unique solution and hence (97) is nonempty. Since f is a global diffeomorphism, and since $v^{\perp} + \mathrm{Ker}[B_{44} \quad 1]$ is an $n_{C\mathcal{J}}$-dimensional affine submanifold, $f(v^{\perp} + \mathrm{Ker}[B_{44} \quad 1])$ is also an $n_{C\mathcal{J}}$-dimensional submanifold. In order to show the transversality property in (66), note that

$$f(v^{\perp} + \mathrm{Ker}[B_{44} \quad 1]) = \{y \in \mathbb{R}^{n_C + n_S} \mid \hat{f} \circ f^{-1}(y) = 0\} \tag{101}$$

where

$$\hat{f}(x) = [B_{44} \quad 1](x - v^{\perp})$$

By Fact B and (101), (66) holds if

$$\mathrm{rank} \begin{bmatrix} (D\hat{f}) \quad (Df^{-1})(q_C, q_S) \\ [1 \quad -B_{44}^{T}] \end{bmatrix} = \mathrm{rank} \begin{bmatrix} [B_{44} \quad 1] \; (Df^{-1})(q_C, q_S) \\ [1 \quad -B_{44}^{T}] \end{bmatrix} = n_{C\mathcal{J}} + n_{C\mathcal{L}}.$$

But the above matrix is precisely $(D\mathcal{F})(q_C, q_S)$, whose rank has already been shown to be equal to $n_{C\mathcal{J}} + n_{C\mathcal{L}}$. For inductors, we consider (see (63) and (36))

$$[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{\phi}_L \\ \underset{\sim}{\phi}_\Gamma \end{bmatrix} = \underset{\sim}{\Phi}_0 \tag{102}$$

$$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]\, g^{-1}(\underset{\sim}{\phi}_L, \underset{\sim}{\phi}_\Gamma) = [\underset{\sim}{B}_{21}^T \quad \underset{\sim}{B}_{31}^T] \begin{bmatrix} \underset{\sim}{i}_{R\ell}^* \\ \underset{\sim}{i}_I^* \end{bmatrix} \tag{103}$$

and the function defined by (90). A similar argument is valid. Hence all conditions of Theorem 2 are satisfied. Equation (87) follows from (49) and the fact that $\underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v^{-1} = \underset{\sim}{f}$, and $\underset{\sim}{\pi}_\phi \circ \underset{\sim}{\pi}_i^{-1} = g$. Equation (88) follows from (82), the fact that (97) is given by the solution of (98) and (99), and the fact that

$$g(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^T]) \cap (\underset{\sim}{\phi}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{11} \quad \underset{\sim}{1}])$$

is given by the solution of (102) and (103). ¤

Remark. Under the conditions of Theorem 3 one might be tempted to pose the following conjecture: Is the relationship

$$\#(M \cap M^*) = \#(\text{operating points of } \mathcal{N}_{RG})$$

valid, where # denotes the cardinality of a set? This is false, however. For, suppose there are two operating points $(\underset{\sim}{v}_{V_1}^*, \underset{\sim}{v}_{G\mathcal{J}_1}^*)$ and $(\underset{\sim}{v}_{V_2}^*, \underset{\sim}{v}_{G\mathcal{J}_2}^*)$ both of them belonging to $\mathrm{Ker}[-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}]$. Then the above relationship does not hold because

$$[-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_{V_1}^* \\ \underset{\sim}{v}_{G\mathcal{J}_1}^* \end{bmatrix} = \underset{\sim}{0}\ , \quad [-\underset{\sim}{B}_{43} \quad -\underset{\sim}{B}_{42}] \begin{bmatrix} \underset{\sim}{v}_{V_1}^* \\ \underset{\sim}{v}_{G\mathcal{J}_2}^* \end{bmatrix} = \underset{\sim}{0}$$

and hence they give rise to the same equilibrium point.

Corollary 4. Consider the same situation as Theorem 3 except that $\mathcal{N}_{RG}$ has a unique operating point. Then $M \cap M^*$ is a unique point.

Theorem 4. Consider the same situation as Corollary 4 except that the dynamics is linear, i.e., it is described by (17). Then we have:

(i) rank $\underset{\sim}{A} = (n_C - \bar{n}_C) + (n_L - \bar{n}_L)$

(ii) The set of equilibria M is an affine submanifold of dimension $(\bar{n}_C + \bar{n}_L)$.

Proof. Note that the set of equilibria is given by

$$\underset{\sim}{x}^{\perp} + \mathrm{Ker}\, \underset{\sim}{A} \tag{104}$$

for some vector $\underset{\sim}{x}^{\perp}$ in $(\mathrm{Ker}\, \underset{\sim}{A})^{\perp}$. But by (87), this set must be given by

$$M = \left[\begin{array}{cc:cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^{T} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right] \begin{bmatrix} \underset{\sim}{C}(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]) \\ \underset{\sim}{L}(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^{T}]) \end{bmatrix} \tag{105}$$

where $\underset{\sim}{C}$ and $\underset{\sim}{L}$ are the capacitance and inductance matrices, respectively. By Fact B and Fact D, we have

$$\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^{T}] \pitchfork \underset{\sim}{C}(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}])$$

and the intersection is a single point. Let

$$\hat{\underset{\sim}{\pi}} : \mathbb{R}^{n_C+n_S} \to (\mathrm{Ker}\ [\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^{T}])^{\perp}$$

be the orthogonal projection. It follows from the above, that

$$\hat{\underset{\sim}{\pi}}(\underset{\sim}{C}(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}])) \tag{106}$$

is still an $\bar{n}_C$-dimensional affine submanifold. It follows from (74) and the fact that $[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^{T}]$ maps $(\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^{T}])^{\perp}$ onto its image injectively, that

$$[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^{T}]\ \underset{\sim}{C}\ (\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}])$$

is an $\bar{n}_C$-dimensional affine submanifold. A similar argument applies to inductors to give an $\bar{n}_L$-dimensional affine submanifold. This and (104) give (i) and (ii).

¤

Remark. The number $(n_C - \bar{n}_C) + (n_L - \bar{n}_L)$ coincides with the degrees of freedom in the sense of Bers [2].

The set of equilibria M in Theorem 4 is an affine submanifold because the associated state equation is linear. The set M in Corollary 4, however, need not be a submanifold because the submanifold

$$\begin{bmatrix} \underset{\sim}{f}(\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}]) \\ \underset{\sim}{g}(\underset{\sim}{i}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{11}^{T}]) \end{bmatrix}$$

is projected into the $(\underset{\sim}{q},\underset{\sim}{\phi})$-space by the matrix

$$\left[\begin{array}{cc:cc} \underset{\sim}{1} & -\underset{\sim}{B}_{SC}^{T} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{L\Gamma} \end{array}\right].$$

For example, think of a 1-dimensional curve in $\mathbb{R}^3$ whose projection onto $\mathbb{R}^2$ is a curve having self-intersections, which is not a submanifold. However, if $\mathcal{N}$ contains neither capacitor-only loops nor inductor-only cut sets, then the above projection matrix reduces to an identity matrix. In this case we can state:

Corollary 5. Using the same setting as that of Corollary 4 and assuming that $\mathcal{N}$ contains neither capacitor-only loops nor inductor-only cut sets, then

(i) M is an $(\bar{n}_C + \bar{n}_L)$-dimensional submanifold.

(ii) M $\pitchfork$ M*.

The following fact is simple, yet important.

Corollary 6. Let y be any set of variables such that $y = \psi(x)$, where ψ is a global diffeomorphism. Then the state equations can be written in terms of y and Theorems 2-4 and their corollaries are valid in terms of y.

Proof. Conditions required in the results are coordinate-free. Hence everything is preserved under a diffeomorphism. ⌑

As an application of Corollary 6, we note that if all capacitors are C^1, uniformly increasing, and (v_C, q_C) and (v_S, q_S) are not coupled to each other, then the constitutive relations of the capacitors can be written as $q_S = \hat{q}_S(v_S)$ and $v_C = \hat{v}_C(q_C)$, where $\hat{q}_S(\cdot)$ and $\hat{v}_C(\cdot)$ are global diffeomorphisms.[4] It follows from (6) and (46) that

$$q = q_C - B_{SC}^T \hat{q}_S \circ (-B_{SC}\,\hat{v}_C(q_C) - B_{SV}\,v_V) \triangleq h_C(q_C)$$

Now, the Jacobian matrix

$$Dh_C = 1 + B_{SC}^T[D\hat{q}_S \circ D\hat{v}_C(q_C)]B_{SC}$$

is positive definite uniformly with respect to q_C. Hence $h_C(\cdot)$ is uniformly increasing and hence it is a global diffeomorphism. A dual argument applies of course to the inductors. Hence, we conclude that if all capacitors and inductors are uniformly increasing and if (v_C, q_C), (v_S, q_S), (ϕ_L, i_L) and (ϕ_Γ, i_Γ) are not coupled to each other, then we can choose the more usual vector $\hat{x} \triangleq (q_C, \phi_L)$ as the state variables and all results alluded to above remain valid.

Remark. Our results in this paper can also be related to classical techniques from theoretical mechanics where the number of dynamically independent coordinates is minimized through the use of first integrals [11]. In particular, a real-valued function $E(x)$ on the state space is called a first integral for (1) if

$$\frac{dE(x(t))}{dt} = 0 \quad \text{for all } t.$$

[4] All the symbols here pertain to the C-normal tree in Section 2.

In mechanics, quantities such as angular momentum and energy are first integrals. In electrical networks, the energy stored in an L-C network is a first integral. Each such first integral leads to a conservation principle. Now consider

$$\underset{\sim}{G}(\underset{\sim}{x}) \triangleq [\underset{\sim}{1} \quad -\underset{\sim}{B}_{44}^{T}] \begin{bmatrix} \underset{\sim}{q}_{C\mathcal{J}}(\underset{\sim}{x}) \\ \underset{\sim}{q}_{C\mathcal{L}}(\underset{\sim}{x}) \end{bmatrix}$$

$$\underset{\sim}{H}(\underset{\sim}{x}) \triangleq [\underset{\sim}{B}_{11} \quad \underset{\sim}{1}] \begin{bmatrix} \underset{\sim}{\phi}_{L\mathcal{J}}(\underset{\sim}{x}) \\ \underset{\sim}{\phi}_{L\mathcal{L}}(\underset{\sim}{x}) \end{bmatrix}$$

Clearly, then $G_1(\underset{\sim}{x}), \ldots, G_{n_{C\mathcal{J}}}(\underset{\sim}{x})$, $H_1(\underset{\sim}{x}), \ldots, H_{n_{L\mathcal{L}}}(\underset{\sim}{x})$ are all <u>first integrals</u> for (1). Each of these scalar functions has a clear physical meaning: $G_k(\underset{\sim}{x})$ is the <u>net capacitor charge in the k-th capacitor-only cut set</u>, and $H_k(x)$ is the <u>net inductor flux-linkage of the k-th inductor-only loop</u>. Hence our corresponding conservation principle now asserts that each of these quantities is conserved along a trajectory. Observe that <u>each</u> first integral allows us to eliminate one state variable and hence one degree of freedom.

Our next example shows that capacitor-only cut sets and inductor-only loops are not the only situations of practical interest which give rise to an invariant <u>affine</u> submanifold.

<u>Example 3</u>. Consider the circuit of Fig. 9(a), where N_R contains only memoryless elements and the 2-port is an n:1 ratio ideal transformer. Clearly, then

$$i_2 = n\, i_1$$

and hence

$$q_2 - n\, q_1 = \text{constant}.$$

<u>Theorem 5</u>. Let the dynamics of (1) be described in terms of $\underset{\sim}{x} = (\underset{\sim}{q}, \underset{\sim}{\phi})$. Let $\mathcal{N}$ contain elements such that

$$\underset{\sim}{F} \begin{bmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{i}_S \end{bmatrix} = \underset{\sim}{0}$$

$$\underset{\sim}{H} \begin{bmatrix} \underset{\sim}{v}_L \\ \underset{\sim}{v}_T \end{bmatrix} = \underset{\sim}{0}$$

where $\underset{\sim}{F}$ and $\underset{\sim}{H}$ are matrices.

Then each trajectory is constrained to lie on an invariant affine submanifold $\hat{M}$, which depends on the initial state, such that

$$\dim \hat{M} = \dim(\operatorname{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}_{SC}^T])^{\perp} \cap \operatorname{Ker} \underset{\sim}{F}$$

$$+ \dim(\operatorname{Ker}[\underset{\sim}{1} \quad \underset{\sim}{B}_{L\Gamma}])^{\perp} \cap \operatorname{Ker} \underset{\sim}{H} .$$

Remark. The following example shows that one cannot generalize the above situation by constraining capacitor charges and inductor fluxes to nonlinear submanifolds.

Example 4.

Consider the circuit of Fig. 9(b), where N_0 is characterized by

$$i_a = h(i_b)$$

and $h(\cdot)$ is a nonlinear function. Then

$$q_a(t) = q_a(0) + \int_0^t h(i_b(t))dt$$

and there is no obvious way of defining a nonlinear submanifold $f(q_a, q_b) = 0$.

5. Nonautonomous Networks

Some of the results obtained in the previous sections do not depend crucially on the time-invariance of the dynamics. In this section we will show how the results are carried over to nonautonomous networks.

Consider the nonautonomous network $\mathcal{N}$ described by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}(t)) \tag{107}$$

i.e., the time-varying property comes only from independent sources. Recalling the proof of Lemma 2 we see that the affine submanifold M* does not depend on the fact that $\underset{\sim}{u}$ is constant. Since $(\underset{\sim}{x},t) \in \mathbb{R}^{n_C+n_L+1}$, we have the following result:

Theorem 6. Assume the state equation (107) exists with $\underset{\sim}{x} = (\underset{\sim}{q},\underset{\sim}{\phi})$ as the state variables, where $\underset{\sim}{q}$ and ϕ denote the "generalized charge" and "generalized flux" vectors, respectively. Let $\bar{n}_C$ and $\bar{n}_L$ be the number of capacitor-only cut sets and inductor-only loops, respectively. Then

(i) there is an affine submanifold $\hat{M}^*(\underset{\sim}{x}_0,t_0) \subset \mathbb{R}^{n_C+n_L+1}$ which is parametrized by the initial state $(\underset{\sim}{x}_0,t_0)$ such that the trajectory $(\underset{\sim}{x}(t,t_0,\underset{\sim}{x}_0),t) \in \hat{M}^*$ for all t.

(ii) $\hat{M}^*$ is of dimension $(\bar{n}_C+\bar{n}_L+1)$ and is of the form $\hat{M}^* = M^* \times \mathbb{R}$, where M* is an $(\bar{n}_C+\bar{n}_L)$-dimensional affine submanifold of $\mathbb{R}^{n_C+n_L}$. ¤

If we project everything from $\mathbb{R}^{n_C+n_L+1}$ onto $\mathbb{R}^{n_C+n_L}$, then all the arguments of the previous sections are valid except that we need an appropriate concept of equilibrium for (107).

Definition 3. A vector $\hat{\underset{\sim}{x}} \in \mathbb{R}^{n_C+n_L}$ is called an equilibrium for (107) if

$$\underset{\sim}{f}(\hat{\underset{\sim}{x}},\underset{\sim}{u}(t)) = \underset{\sim}{0} \quad \text{for all t.}$$

Definition 4. Let $\mathcal{N}_{RG}$ be obtained as before and let $\underset{\sim}{v}$ and $\underset{\sim}{i}$ be the voltages and the currents associated with $\mathcal{N}_{RG}$. Then a constant vector $(\underset{\sim}{v}^*,\underset{\sim}{i}^*)$ is called an operating point of $\mathcal{N}_{RG}$ if it satisfies the Kirchoff laws and constitutive relations for all t.

Theorem 7. With the new definitions of equilibria and operating points, all the results of the previous sections hold on the projected space $\mathbb{R}^{n_C+n_L}$.

Remark. The class of networks having the properties defined in Definitions 3 and 4, is nonempty. In particular, the variational equation $\dot{\underset{\sim}{y}} = \hat{\underset{\sim}{f}}(\underset{\sim}{y},\underset{\sim}{u}(t))$, associated with (1) in a neighborhood of a particular solution $\hat{\underset{\sim}{x}}(t,t_0,\underset{\sim}{x}_0)$, where $\underset{\sim}{y}(t) \triangleq \underset{\sim}{x}(t) - \hat{\underset{\sim}{x}}(t,t_0,\underset{\sim}{x}_0)$, has an equilibrium point $\underset{\sim}{y} = \underset{\sim}{0}$. In fact, since our nonlinear resistors may be coupled each other, i.e., controlled sources are allowed, it is easy to construct many nontrivial networks belonging to this class.

6. Eventual Passivity on M*

Eventual passivity plays important roles in electrical networks [6], [12-14].

A sufficient condition which can be used to guarantee eventual passivity is given by the Fundamental Topological Hypothesis: There are no cut sets and no loops consisting only of capacitors and inductors.

Observe that this hypothesis excludes, among other things, capacitor-only loops and cut sets, as well as inductor-only loops and cut sets. In this section, we will show that this condition can be relaxed to allow capacitor-only cut sets and inductor-only loops. The following definition is needed to state our relaxed topological hypothesis (Condition B).

Definition 5. A tree containing a minimum number of capacitors is called a C-minimal tree. A cotree containing a minimum number of inductors is called an L-minimal cotree.

Condition B

(i) there is a C-minimal tree containing no inductors.

(ii) there is an L-minimal cotree containing no capacitors.

Remark. Condition B allows capacitor-only cut sets and inductor-only loops although it does not allow L-C cut sets and loops. Condition (i) is equivalent to (115)-(117), while condition (ii) is equivalent to (122)-(124). See APPENDIX 2 for examples of (i) and (ii) above.

In this section we consider nonautonomous network $\mathcal{N}$ described by (107). In order to avoid introducing complicated notations, however, we assume that independent sources are imbedded within the constitutive relations of the nonlinear resistors. Hence Λ_{RG} in (16) depends on $\underset{\sim}{u} = (\underset{\sim}{v}_V, \underset{\sim}{i}_I)$.

Definition 6. The collection of all resistor constitutive relations Λ_{RG} is said to be eventually passive if there is a $k_{RG} > 0$ such that

$$\|(\underset{\sim}{v}_R, \underset{\sim}{v}_G, \underset{\sim}{i}_R, \underset{\sim}{i}_G)\| \geq k_{RG} \tag{108}$$

implies

$$\underset{\sim}{v}_R^T \underset{\sim}{i}_R + \underset{\sim}{v}_G^T \underset{\sim}{i}_G \geq 0 \tag{109}$$

uniformly with respect to $\underset{\sim}{u}$. It is said to be eventually strictly passive if the strict inequality in (109) holds.

Next, let N be the composite n-port (representing the interconnected resistors) seen by the capacitors and inductors [13], and let $\underset{\sim}{v}_p$ and $\underset{\sim}{i}_p$ be its port voltages and currents respectively.

Definition 7. N is said to be eventually passive if there is a $k_p > 0$ such that

$$\|(\underset{\sim}{v}_p, \underset{\sim}{i}_p)\| \geq k_p \tag{110}$$

implies

$$v_p^T i_p \geq 0 \tag{111}$$

uniformly with respect to u. It is said to be eventually strictly passive if the strict inequality in (111) holds.

Although the constitutive relation Λ_{RG} of most practical nonlinear resistors are eventually strictly passive, it does not necessarily imply that the interconnected n-port N is also eventually strictly passive. A sufficient condition for guaranteeing this closure property is given by the Fundamental Topological Hypothesis, which we will now relax.

Theorem 8. Let $\mathcal{N}$ be a nonautonomous network described by (107), where independent sources are imbedded within Λ_{RG}. Let M* be as defined in Section 4. Assume the following:

(i) Λ_{RG} is eventually strictly passive.

(ii) Condition B.

(iii) Λ_C and Λ_L are described by

$$q_C = f(v_C) \text{ and } \phi_L = g(i_L)$$

respectively, where f and g are uniformly increasing.

(iv) State equation exists with $x = (q_C, \phi_L)$ as the state variables.

Then every solution $x(t)$ is eventually uniformly bounded [11] on M*, i.e., there is a bounded set $K \subset M^*$ such that $x(t) \in K$ for all sufficiently large t. (See Fig. 10)

Proof. Let $E(x)$ be the energy stored in the memory elements. Then it follows from Tellegen's theorem that

$$\frac{dE(x(t))}{dt} = -\left(v_R^T(t)\ i_R(t) + v_G^T(t)\ i_G(t)\right). \tag{112}$$

We claim that in order to prove the theorem it suffices to prove that

$$\|(q_C,\ \phi_L)\| \to \infty,\ (q_C,\ \phi_L) \in M^* \tag{113}$$

implies

$$\|(v_R,\ v_G,\ i_R,\ i_G)\| \to \infty \tag{114}$$

In order to see this recall that the eventual strict passivity of Λ_{RG} is uniform with respect to u. Hence, if (113) implies (114), then the right hand side of (112) is strictly negative outside a bounded set and hence the energy must keep decreasing outside some bounded set K. The above condition is equivalent to saying that N is eventually strictly passive on the set $(\underset{\sim}{f}, \underset{\sim}{g})^{-1}(M^*)$, i.e.,

$\|(\underset{\sim}{v}_C, \underset{\sim}{i}_L)\| \to \infty$, $(\underset{\sim}{v}_C, \underset{\sim}{i}_L) \in (\underset{\sim}{f}, \underset{\sim}{g})^{-1}(M^*)$

implies

$\|(\underset{\sim}{v}_R, \underset{\sim}{v}_G, \underset{\sim}{i}_R, \underset{\sim}{i}_G)\| \to \infty$

Pick a <u>C-minimal tree</u> containing no inductors. Then KVL and KCL assume the following special structures:[5]

$$\left[\begin{array}{ccc|cc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \hat{\underset{\sim}{B}}_{RG} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \hat{\underset{\sim}{B}}_{LG} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \hat{\underset{\sim}{B}}_{CG} & \hat{\underset{\sim}{B}}_{CC} \end{array}\right] \left[\begin{array}{c} \hat{\underset{\sim}{v}}_{R\mathcal{L}} \\ \hat{\underset{\sim}{v}}_{L\mathcal{L}} \\ \hat{\underset{\sim}{v}}_{C\mathcal{L}} \\ \hline \hat{\underset{\sim}{v}}_{G\mathfrak{J}} \\ \hat{\underset{\sim}{v}}_{C\mathfrak{J}} \end{array}\right] = \underset{\sim}{0} \qquad \begin{array}{r} (115) \\ (116) \\ (117) \end{array}$$

$$\left[\begin{array}{ccc|cc} -\hat{\underset{\sim}{B}}^T_{RG} & -\hat{\underset{\sim}{B}}^T_{LG} & -\hat{\underset{\sim}{B}}^T_{CG} & \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & -\hat{\underset{\sim}{B}}^T_{CC} & \underset{\sim}{0} & \underset{\sim}{1} \end{array}\right] \left[\begin{array}{c} \hat{\underset{\sim}{i}}_{R\mathcal{L}} \\ \hat{\underset{\sim}{i}}_{L\mathcal{L}} \\ \hat{\underset{\sim}{i}}_{C\mathcal{L}} \\ \hline \hat{\underset{\sim}{i}}_{G\mathfrak{J}} \\ \hat{\underset{\sim}{i}}_{C\mathfrak{J}} \end{array}\right] = \underset{\sim}{0} \qquad \begin{array}{r} (118) \\ (119) \end{array}$$

where R and G denote resistors, C and L denote capacitors and inductors, respectively, $\mathcal{L}$ and $\mathfrak{J}$ denote cotree and tree, respectively.

Pick an <u>L-minimal cotree</u> containing no capacitors. Then KVL and KCL assume the following special structures:

$$\left[\begin{array}{cc|ccc} \underset{\sim}{1} & \underset{\sim}{0} & \bar{\underset{\sim}{B}}_{RL} & \bar{\underset{\sim}{B}}_{RG} & \bar{\underset{\sim}{B}}_{RC} \\ \underset{\sim}{0} & \underset{\sim}{1} & \bar{\underset{\sim}{B}}_{LL} & \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] \left[\begin{array}{c} \bar{\underset{\sim}{v}}_{R\mathcal{L}} \\ \bar{\underset{\sim}{v}}_{L\mathcal{L}} \\ \hline \bar{\underset{\sim}{v}}_{L\mathfrak{J}} \\ \bar{\underset{\sim}{v}}_{G\mathfrak{J}} \\ \bar{\underset{\sim}{v}}_{C\mathfrak{J}} \end{array}\right] = \underset{\sim}{0} \qquad \begin{array}{r} (120) \\ (121) \end{array}$$

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[5] To distinguish the variables associated with the two trees, we attached a "^" to all variables associated with the C-minimal tree, and a "-" to all variables associated with the L-minimal cotree.

$$\begin{bmatrix} -\bar{B}_{RL}^T & -\bar{B}_{LL}^T & 1 & 0 & 0 \\ -\bar{B}_{RG}^T & 0 & 0 & 1 & 0 \\ -\bar{B}_{RC}^T & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_{R\mathcal{L}} \\ \bar{i}_{L\mathcal{L}} \\ \hline \bar{i}_{L\mathcal{J}} \\ \bar{i}_{G\mathcal{J}} \\ \bar{i}_{C\mathcal{J}} \end{bmatrix} = 0 \qquad \begin{matrix} (122) \\ (123) \\ (124) \end{matrix}$$

where the notations have similar meanings as before. It follows from (119) and (121) that

$$[-\hat{B}_{CC}^T \quad 1]\, q_C = \hat{Q}_0 \tag{125}$$

$$[1 \quad \bar{B}_{LL}]\, \phi_L = \bar{\Phi}_0 \tag{126}$$

where $\hat{Q}_0$ and $\bar{\Phi}_0$ depend only on $x(0)$. Since M* <u>does not depend on a particular choice of a tree</u>, M* of (59) must also be given by

$$M^* = \begin{bmatrix} \hat{Q}_0^{\perp} + \mathrm{Ker}[-\hat{B}_{CC}^T \quad 1] \\ \bar{\Phi}_0^{\perp} + \mathrm{Ker}[1 \quad \bar{B}_{LL}] \end{bmatrix} \tag{127}$$

where $\hat{Q}_0^{\perp} \in (\mathrm{Ker}[-\hat{B}_{CC}^T \quad 1])^{\perp}$ and $\bar{\Phi}_0^{\perp} \in (\mathrm{Ker}[1 \quad \bar{B}_{LL}])^{\perp}$.

In order to prove that (113) implies (114), we will first prove that

$$\|q_C\| \to \infty,\ q_C \in \hat{Q}_0^{\perp} + \mathrm{Ker}[-\hat{B}_{CC}^T \quad 1] \tag{128}$$

implies

$$\|\hat{v}_{G\mathcal{J}}\| \to \infty \tag{129}$$

Recall (125) and (117):

$$[-\hat{B}_{CC}^T \quad 1]\, q_C = \hat{Q}_0 \tag{125'}$$

$$[1 \quad \hat{B}_{CC}]\, f^{-1}(q_C) = -\hat{B}_{CG}\, \hat{v}_{G\mathcal{J}} \tag{117'}$$

It follows from a similar argument as that of the previous section that

$$\hat{\mathcal{T}}(q_C) \triangleq \begin{bmatrix} [-\hat{B}_{CC}^T \quad 1]\, q_C \\ [1 \quad \hat{B}_{CC}]\, f^{-1}(q_C) \end{bmatrix} \tag{130}$$

is a global diffeomorphism. Hence $\|q_C\| \to \infty$ implies $\|\hat{\mathcal{T}}(q_C)\| \to \infty$ [15]. Now, if (128) holds, then the first component of $\hat{\mathcal{T}}(q_C)$ is always constant for a fixed $\hat{Q}_0^{\perp}$. Hence

$$\|[1 \quad \hat{B}_{CC}]\, f^{-1}(q_C)\| \to \infty\ . \tag{131}$$

This and (117') imply that

$$\|\hat{\underset{\sim}{v}}_{G\mathcal{J}}\| \to \infty \tag{132}$$

Similarly, the dual analysis shows that

$$\|\underset{\sim}{\phi}_L\| \to \infty \quad \underset{\sim}{\phi}_L \in \bar{\underset{\sim}{\Phi}}_0^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad \bar{\underset{\sim}{B}}_{LL}]$$

implies

$$\|\bar{\underset{\sim}{i}}_{R\mathcal{L}}\| \to \infty \tag{133}$$

Thus (132) and (133) prove that (113) implies (114). ¤

6. Concluding Remarks

Some of the implications of capacitor-only cut sets and inductor-only loops were discussed. Firstly, supposing that the resistive subnetwork $\mathcal{N}_{RG}$ has only isolated operating points, we showed that the network $\mathcal{N}$ has only isolated equilibria if, and only if, there are no capacitor-only cut sets and no inductor-only loops. Hence, if this condition is violated, there are a continuum of equilibria even if the operating point is unique. Secondly, we showed that if there are capacitor-only cut sets and/ or inductor-only loops, then there is an invariant affine submanifold M* on which any trajectory originating from it must remain on it for all times. In this sense, M* can be thought of as the minimal state space for the network. The dimension of M* is the number of the state variables minus the number of linearly-independent capacitor-only cut sets and inductor-only loops. This number turns out, in the linear case, to be the same as the degrees of freedom in the sense of Bers. We also showed that the intersection of M* with the set of equilibria consists only of isolated points. Hence, the network behaves as if it has only isolated equilibria. In Section 6, the result was applied to eventually passive networks. Transversality theory for manifolds and functions has provided us with a powerful tool for the proof of these results.

APPENDIX 1

We will state and prove two facts needed in the proof of the theorems.

Fact D. Let $\underset{\sim}{P}$ be an nxn positive definite matrix and let $\underset{\sim}{B}$ be an (n-m)xm matrix, $n \geq m$. Then the nxn matrix

$$\underset{\sim}{W} \triangleq \begin{bmatrix} [\underset{\sim}{1} & \underset{\sim}{B}] \\ [-\underset{\sim}{B}^T & \underset{\sim}{1}]\underset{\sim}{P} \end{bmatrix}$$

is nonsingular, where $\underset{\sim}{1}$ is the identity matrix of dimension (n-m).

Proof. Let $\underset{\sim}{x} = (\underset{\sim}{x}_1, \underset{\sim}{x}_2)$, $\underset{\sim}{x}_1 \in \mathbb{R}^{n-m}$, $\underset{\sim}{x}_2 \in \mathbb{R}^m$. We will show that $\underset{\sim}{W}\,\underset{\sim}{x} = \underset{\sim}{0}$ implies $\underset{\sim}{x} = \underset{\sim}{0}$. Let

$$\begin{bmatrix} [\underset{\sim}{1} & \underset{\sim}{B}] \\ [-\underset{\sim}{B}^T & \underset{\sim}{1}]\underset{\sim}{P} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{bmatrix} = \underset{\sim}{0}$$

Then

$$[-\underset{\sim}{B}^T \quad \underset{\sim}{1}]\underset{\sim}{P} \begin{bmatrix} -\underset{\sim}{B} \\ \underset{\sim}{1} \end{bmatrix} \underset{\sim}{x}_2 = \underset{\sim}{0}$$

But $\underset{\sim}{P}$ is positive definite implies

$$[-\underset{\sim}{B}^T \quad \underset{\sim}{1}]\underset{\sim}{P} \begin{bmatrix} -\underset{\sim}{B} \\ \underset{\sim}{1} \end{bmatrix}$$

is positive definite. Hence $\underset{\sim}{x}_2 = \underset{\sim}{0}$ and $\underset{\sim}{x}_1 = -\underset{\sim}{B}\underset{\sim}{x}_2 = \underset{\sim}{0}$. ⌑

Fact E. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be linear maps with domain $\mathbb{R}^n$. Then

$\dim \underset{\sim}{A}(\mathrm{Ker}\ \underset{\sim}{B}) = \dim \mathrm{Ker}\ \underset{\sim}{B} - \dim(\mathrm{Ker}\ \underset{\sim}{B} \cap \mathrm{Ker}\ \underset{\sim}{A})$

Proof. For any linear map $\underset{\sim}{T} : X \to Y$, we have the relation [9]

$$\dim(\mathrm{Im}\underset{\sim}{T}) + \dim(\mathrm{Ker}\ \underset{\sim}{T}) = \dim X$$

where X and Y are finite dimensional linear spaces. If we choose $X = \mathrm{Ker}\ \underset{\sim}{B}$, $\underset{\sim}{T} = \underset{\sim}{A}|X$, and $Y = A(\mathrm{Ker}\ B)$, then $\mathrm{Im}\underset{\sim}{T} = \underset{\sim}{A}(\mathrm{Ker}\ \underset{\sim}{B})$ and $\mathrm{Ker}\ \underset{\sim}{T} = (\mathrm{Ker}\ \underset{\sim}{B}) \cap (\mathrm{Ker}\ \underset{\sim}{A})$. The above relationship then follows trivially. ⌑

APPENDIX 2

We will give an example which illustrates the validity of (54) and (58).

Consider the network of Fig. 11(a). Choose the bold branches as the C-normal tree T_C and choose the branches along the dotted path as the L-normal tree T_L. Relabel the capacitor branches such that capacitors belonging to both T_C and T_L are labelled first. Relabel the inductor branches such that inductors belonging to both

T_C^* and T_L^* are labelled first. Under this labelling convention, we have $T_C = \{C_1, C_2, C_3, L_3\}$ and $T_L = \{C_1, R_1, L_2, L_3\}$. Then

$$(\underset{\sim}{v}_C \mid \underset{\sim}{v}_S) = (v_{C1} \; v_{C2}, v_{C3} \mid v_{C4})$$

$$(\underset{\sim}{v}_{C\mathcal{J}} \mid \underset{\sim}{v}_{C\mathcal{L}}) = (v_{C1} \mid v_{C2}, v_{C3}, v_{C4})$$

so that (54) holds. Similarly

$$(\underset{\sim}{i}_L \mid \underset{\sim}{i}_T) = (i_{L1}, i_{L2} \mid i_{L3})$$

$$(\underset{\sim}{i}_{L\mathcal{L}} \mid \underset{\sim}{i}_{L\mathcal{J}}) = (i_{L1} \mid i_{L2}, i_{L3})$$

and hence (58) holds.

We will next give an example illustrating <u>Condition B</u>. Consider the network of Fig. 11(b). $T_{C_{min}} \triangleq \{C_1, R_1, R_2, R_3\}$ is a C-minimal tree containing no inductors and $T^*_{L_{min}} \triangleq \{L_2, R_1, R_2, R_3\}$ is an L-minimal cotree containing no capacitors.

APPENDIX 3

To prove the necessity part of (1) of <u>Theorem 2</u>, suppose there are capacitor-only cut sets. We will show that the set defined by (48) has a continuum of points. To this end define the following orthogonal projection map:

$$\underset{\sim}{\pi} : M_C \to (\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}])^{\perp} \tag{A.1}$$

where

$$M_C = \underset{\sim}{\pi}_q \circ \underset{\sim}{\pi}_v (\underset{\sim}{v}^{\perp} + \mathrm{Ker}[\underset{\sim}{B}_{44} \quad \underset{\sim}{1}])$$

Let

$$\underset{\sim}{y} \in M_C \cap (\underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}]).$$

It follows then from condition (iii-b) and (21) that:

$$T_{\underset{\sim}{y}} M_C + T_{\underset{\sim}{y}} (\underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}]) = \mathbb{R}^{n_C + n_S}. \tag{A.2}$$

This implies that the projection of the tangent space must fill up the orthogonal complement:

$$(\underset{\sim}{d}\underset{\sim}{\pi})_{\underset{\sim}{y}} (T_{\underset{\sim}{y}} M_C) = \left(T_{\underset{\sim}{y}} (\underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}])\right)^{\perp} \tag{A.3}$$

But since the tangent space of a translated linear subspace is identified with the subspace itself, we have:

$$\left(T_{\underset{\sim}{y}} (\underset{\sim}{q}^{\perp} + \mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}])\right)^{\perp} = (\mathrm{Ker}[\underset{\sim}{1} \quad -\underset{\sim}{B}^T_{44}])^{\perp} . \tag{A.4}$$

Substituting (A.3) into (A.4), we obtain

$$(d\pi)_{y} \, (T_{y} M_{C}) = (\mathrm{Ker}[\mathbf{1} \quad -B_{44}^{T}])^{\perp} \tag{A.5}$$

Now note that

$$\dim T_{y} M_{C} = \dim M_{C} = n_{C\mathcal{J}} \tag{A.6}$$

On the other hand

$$\dim \mathrm{Ker}[\mathbf{1} \quad -B_{44}^{T}] = n_{C\mathcal{L}}$$

so that

$$\dim(\mathrm{Ker}[\mathbf{1} \quad -B_{44}^{T}])^{\perp} = n_{C\mathcal{J}} \tag{A.7}$$

By (A.6) and (A.7) we have

$$\dim T_{y} M_{C} = \dim(\mathrm{Ker}[\mathbf{1} \quad -B_{44}^{T}])^{\perp}. \tag{A.8}$$

It follows from (A.5) and (A.8) that $(d\pi)_{y}$ is nonsingular. Hence π is a local diffeomorphism at y. If there is at least one capacitor-only cut set, then $n_{C\mathcal{J}} \geq 1$, and hence $\hat{M}_{C} \triangleq \pi \, M_{C}$ has a continuum of points because a local diffeomorphism maps an open neighborhood onto an open neighborhood. By (74), we see that

$$\pi \, M_{C} \subseteq (\mathrm{Ker}[\mathbf{1} \quad -B_{SC}^{T}])^{\perp}$$

Since the matrix $[\mathbf{1} \quad -B_{SC}^{T}]$ maps $(\mathrm{Ker}[\mathbf{1} \quad -B_{SC}^{T}])^{\perp}$ onto its image space injectively, we see that (49) has a continuum of points and hence (48) has a continuum of points. A similar argument holds for inductor-only loops. ¤

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LIST OF FIGURE CAPTIONS

Fig. 10. A geometrical interpretation of the motion of trajectories along the minimal state space M* which converge toward the compact set $K \subset M^*$.

Fig. 11. An illustration of a C-normal tree T_C, an L-normal tree T_L, a C-minimal tree $T_{C_{min}}$ and an L-minimal cotree $T^*_{L_{min}}$.

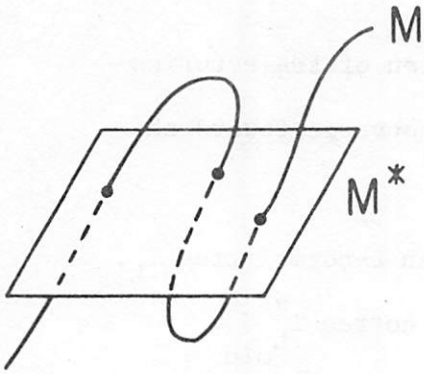
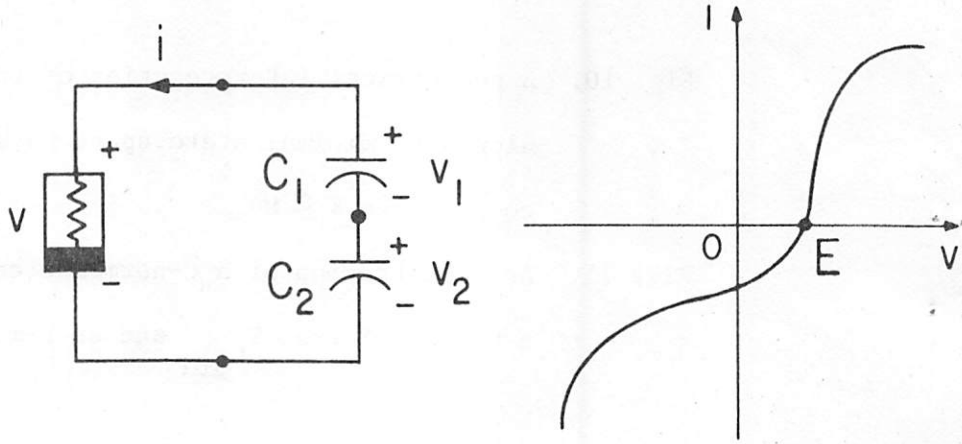


Fig. 1



(a) Fig. 2 (b)

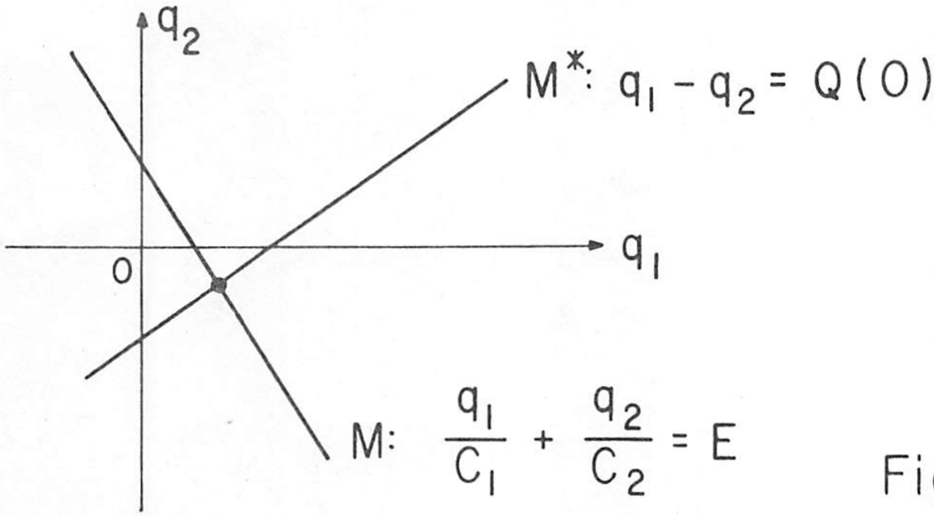
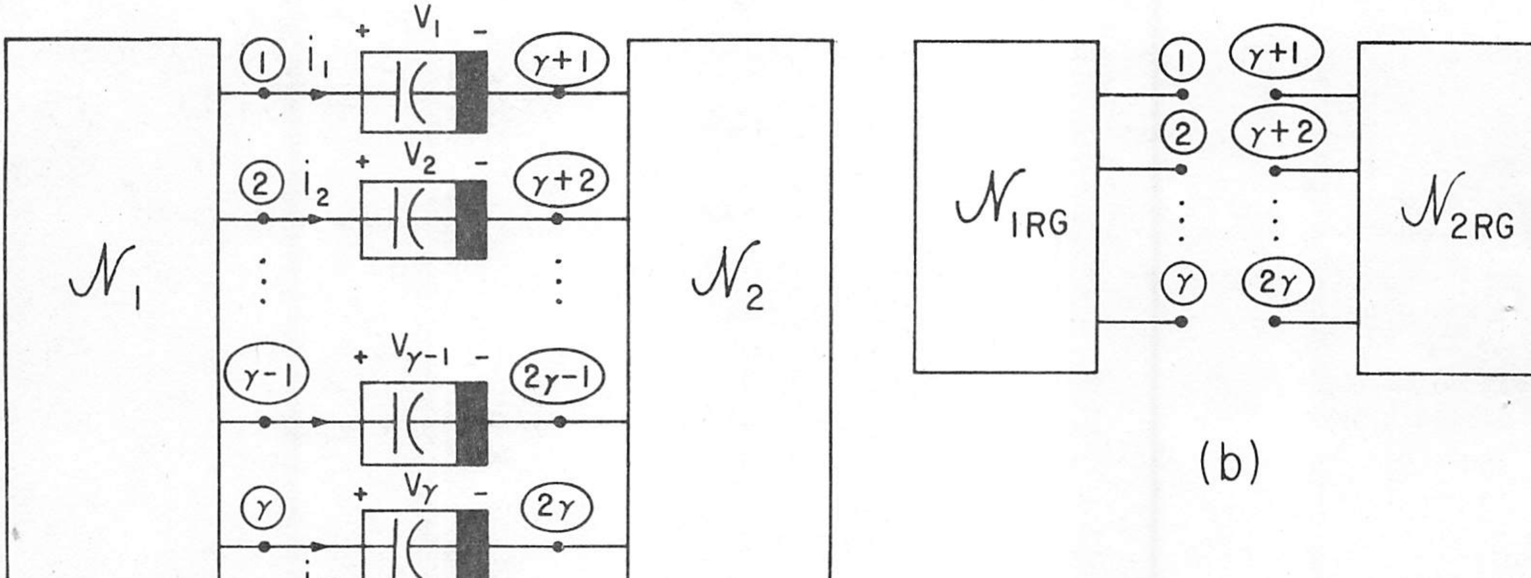
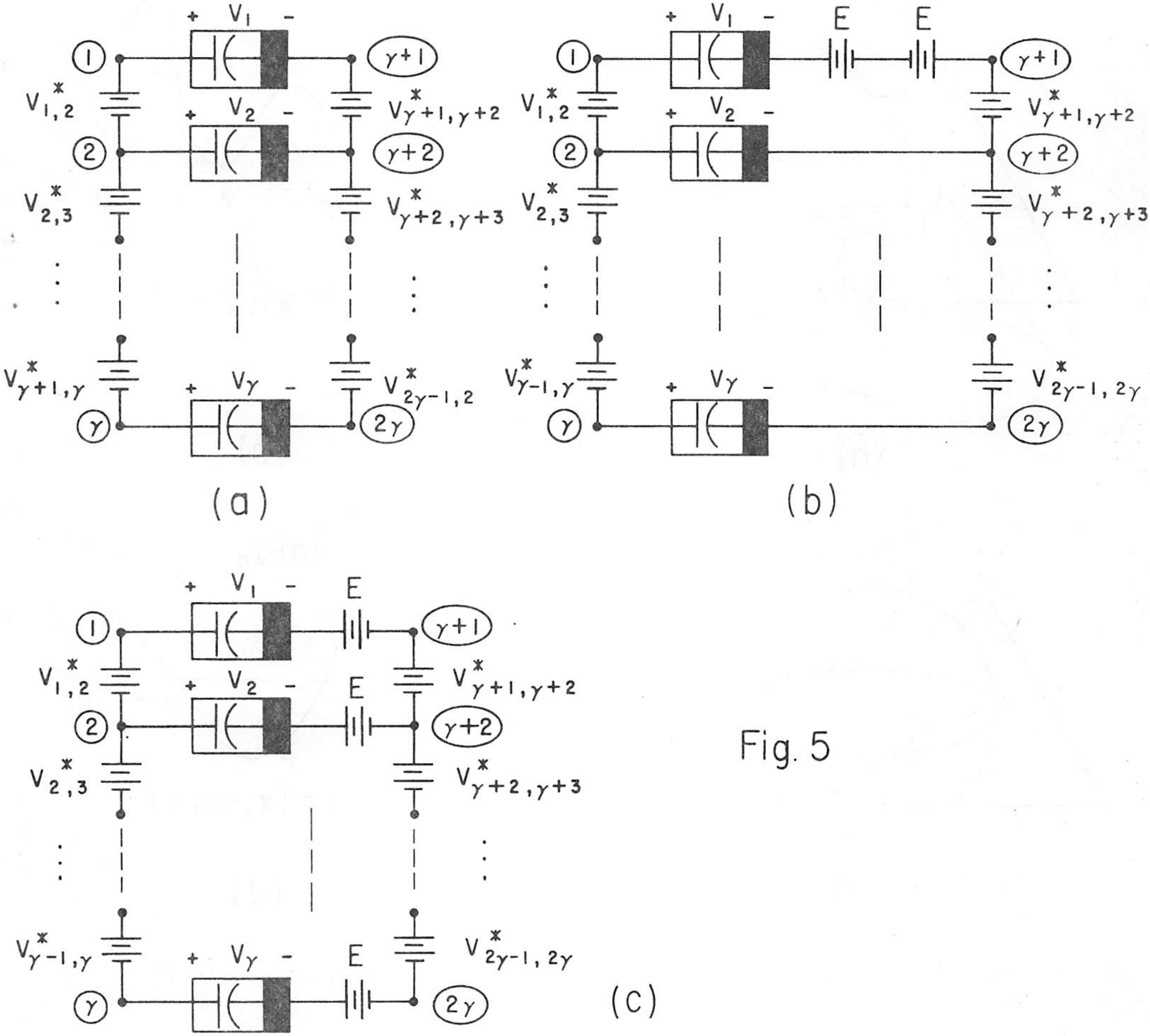


Fig. 3

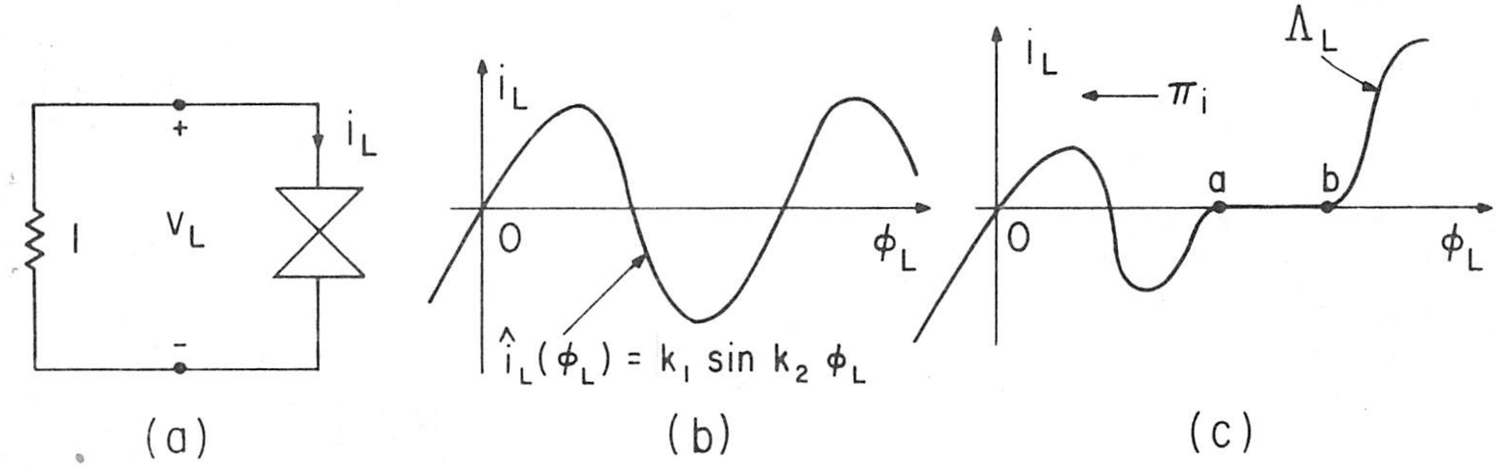


(a) Fig. 4 (b)



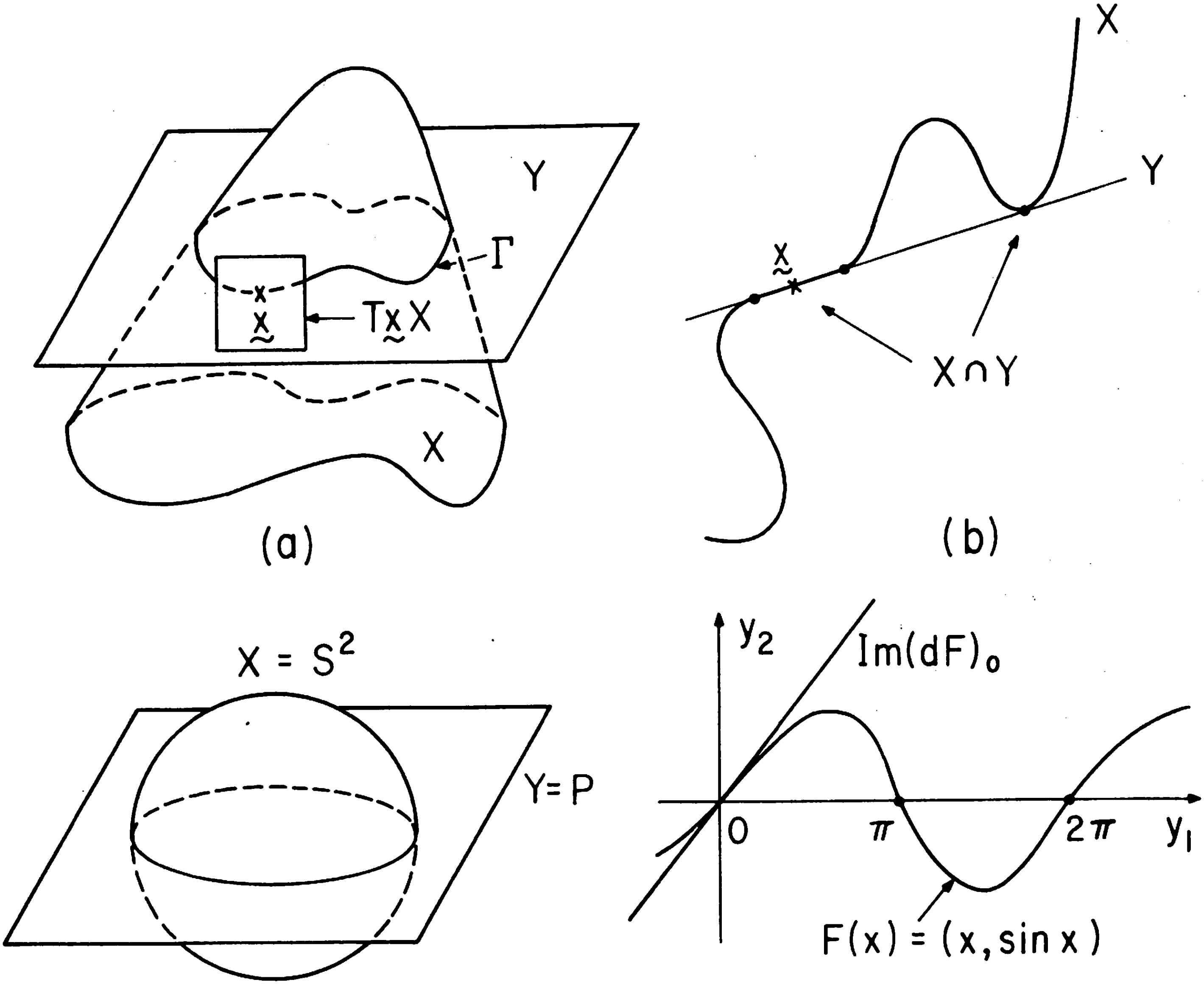
+ V1 -
+ V2 -
+ Vγ -
E
1
2
γ
γ+1
γ+2
2γ
V*1,2
V*2,3
V*γ+1,γ
V*γ-1,γ
V*γ+1,γ+2
V*γ+2,γ+3
V*2γ-1,2
V*2γ-1,2γ
(a)
(b)
(c)

Fig. 5



+
-
I
V_L
i_L
O
φ_L
$\hat{i}_L(\phi_L) = k_1 \sin k_2 \phi_L$
π_i
Λ_L
a
b
(a)
(b)
(c)

Fig. 6


Y
Γ
$T_{\underset{\sim}{x}}X$
X
(a)
X
Y
$\underset{\sim}{x}$
$X \cap Y$
(b)
$X = S^2$
$Y = P$
y_2
$\mathrm{Im}(dF)_0$
0
π
2π
y_1
$F(x) = (x, \sin x)$

(c) Fig. 7 (d)

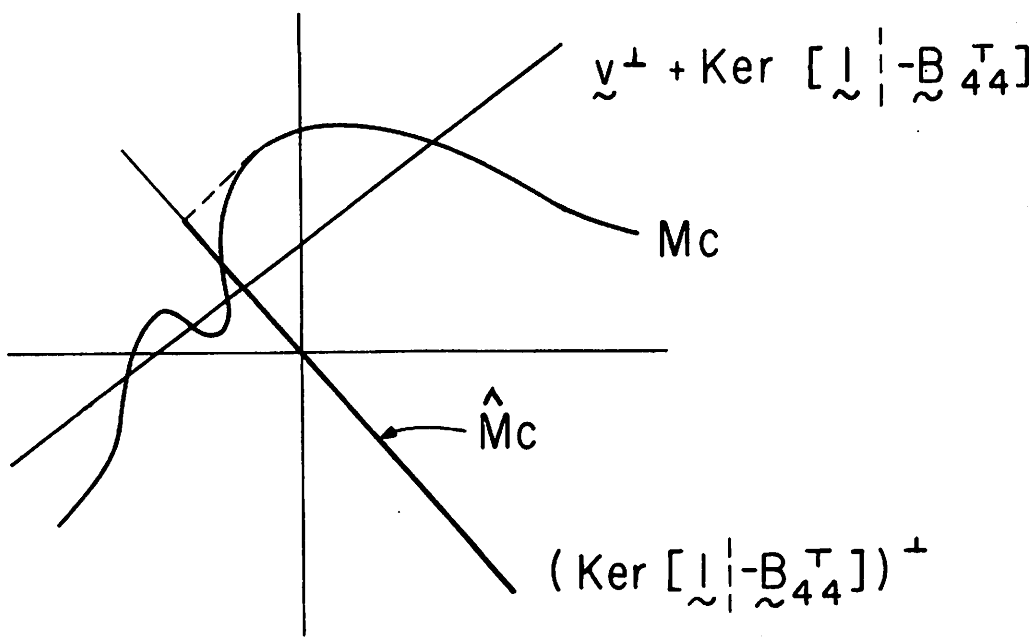
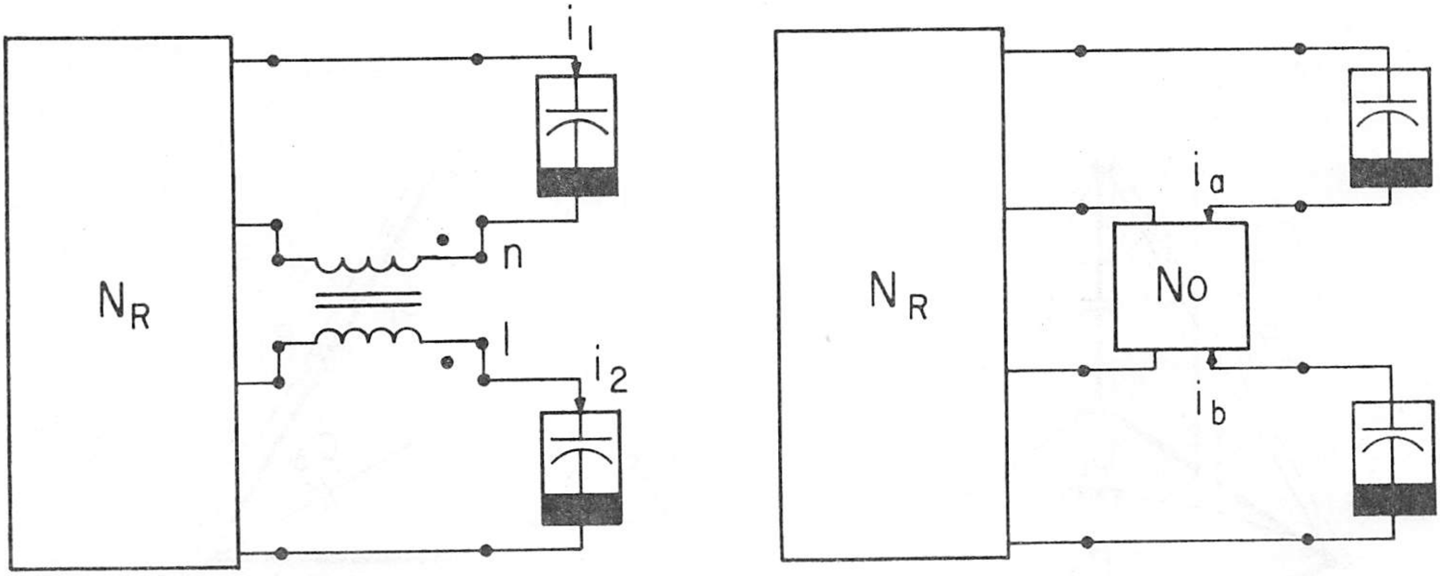

$\underset{\sim}{v}^{\perp} + \mathrm{Ker}\ [\underset{\sim}{I} \mid -\underset{\sim}{B}{}^{T}_{44}]$
Mc
$\hat{M}c$
$(\mathrm{Ker}\ [\underset{\sim}{I} \mid -\underset{\sim}{B}{}^{T}_{44}])^{\perp}$

Fig. 8



(a) (b)

Fig. 9

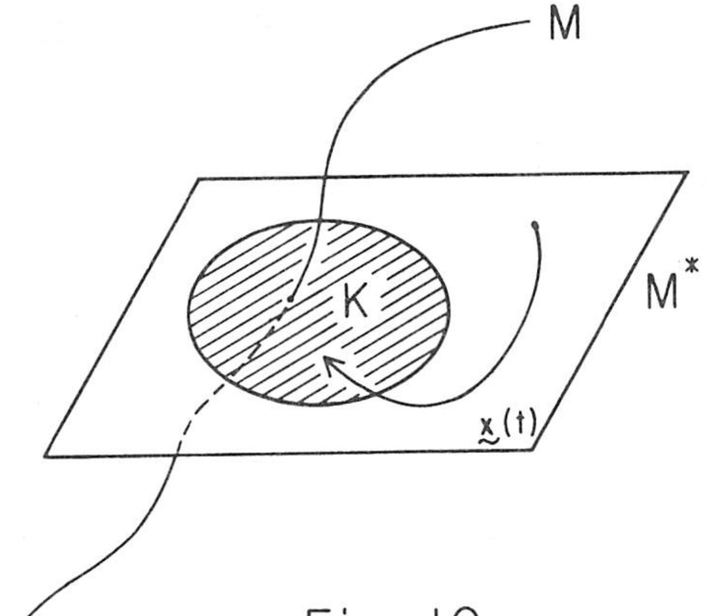


Fig. 10

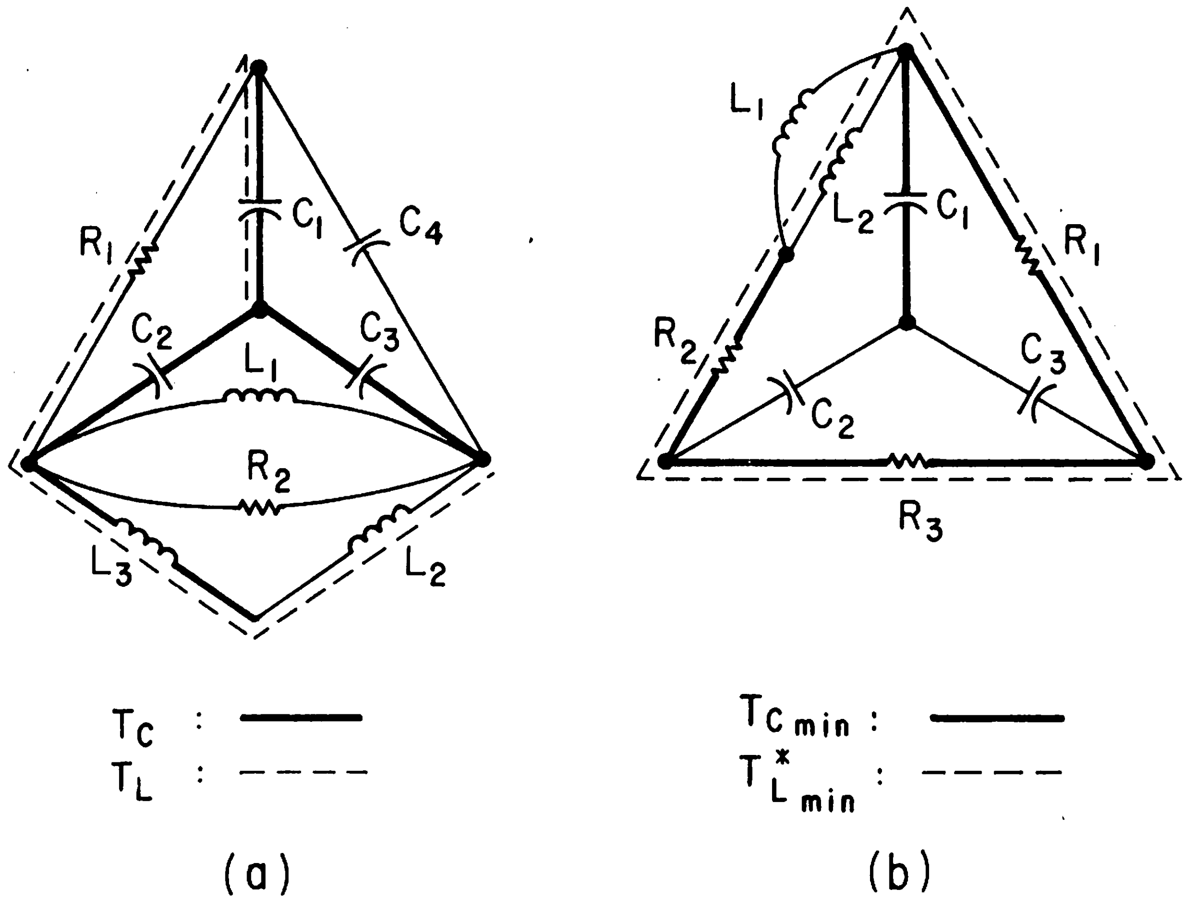

R1
C1
C4
C2
C3
L1
R2
L3
L2
Tc :
TL :
(a)
L1
L2
C1
R1
R2
C3
C2
R3
Tc min :
T*L min :
(b)

Fig. 11