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A UNIFIED THEORY OF SYMMETRY FOR NONLINEAR MULTIPORT AND MULTITERMINAL RESISTORS†

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ABSTRACT

Simple group-theoretic concepts are used to develop a rigorous and comprehensive theory of symmetry for nonlinear multiport and multiterminal resistors which do not rely on geometrical arguments or other ad hoc techniques normally invoked in such studies. This theory unifies all forms of symmetry, including rotation, reflection, and complementary symmetry, into a single framework. It also includes all known nonlinear symmetry principles as special cases. Moreover, a general method for identifying all symmetry characteristics possessed by a nonlinear multiport and multiterminal element is given.

The main results of this paper are:

(1) Several algorithms for synthesizing a nonlinear multiport or multiterminal element having any prescribed form of symmetry are presented. In particular, various examples are given which illustrate how these algorithms can be used to derive well-known symmetrical nonlinear circuit modules such as push-pull amplifiers, complementary-symmetric amplifiers, rectifiers, modulators, etc.

(2) A reduction algorithm is presented which allows a complicated symmetric element to be analyzed by a much simpler reduced element.

(3) A general principle is derived for applying symmetry to achieve frequency separation in nonlinear communication circuits where the even harmonic components are separated from the odd harmonic components.

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I. INTRODUCTION

Many useful results [1-7] have been obtained for symmetric linear circuits. Most of these results rely on the superposition principle and are therefore not valid for nonlinear circuits. The absence of analogous nonlinear results, however, has not prevented engineers from designing symmetric nonlinear circuits -- such as push-pull amplifiers, parametric amplifiers, rectifiers, modulators, detectors, etc., -- using intuition and other ad hoc techniques [8]. In fact, a new form of symmetry having no counterpart in linear circuits has been used extensively in the design of complementary symmetric circuits composed of complementary symmetric elements, such as npn and pnp transistors, n-channel and p-channel FET's, etc. The need for developing a unified theory for nonlinear circuits has been recognized for some time. Two interesting contributions have appeared recently [9-10]. The concept of complementary symmetry for resistive nonlinear networks was formally defined in [9] and shown to be rather useful. Some general results and symmetrical dynamic nonlinear networks have been obtained in [10] using a group-theoretic approach. However, nothing of a unified nature is presently available. Our objective in this paper is to develop such a theory for resistive multiport and multiterminal elements. This theory will be generalized for resistive nonlinear networks in another paper [11].

To motivate the need for a unified theory, let us review some well-known results derived by ad hoc techniques. A multiport or multiterminal resistor R is said to be symmetric with respect to an "axis of symmetry" α if it remains geometrically and electrically invariant upon a rotation of θ degrees about α. If α lies on the plane of R and $\theta = 180°$, then R is said to exhibit reflection symmetry. If α is a line perpendicular to the plane of R, then R is said to exhibit θ-degree rotational symmetry. A multiport or multiterminal resistor R characterized by $\underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$ is said to be complementary symmetric if $\underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{f}(-\underset{\sim}{v},-\underset{\sim}{i}) = \underset{\sim}{0}$. For example, the 2-port resistor R shown in Fig. 1(a) exhibits reflection symmetry, the 3-terminal resistor $\mathcal{R}$ shown in Fig. 1(b) exhibits 120°-rotational symmetry, and the dc circuit model of the OP AMP shown in Fig. 1(c) exhibits complementary symmetry. Observe that the OP AMP can be considered either as a grounded 3-port resistor, or as a 4-terminal resistor. The following three propositions are among the few general symmetry results on nonlinear networks which have been derived by ad hoc techniques:

Proposition 1 [8]. The driving-point plot across two identical nonlinear resistors connected back-to-back in series (see Fig. 2(a)), or back-to-front in parallel (see Fig. 2(b)) is always odd symmetric.

Proposition 2.[1] Consider the basic nonlinear "bridge" 2-port resistor R shown in Fig. 3. Suppose R is voltage-controlled; i.e., each pair of applied voltages v_1 and v_2 gives rise to a unique pair of currents i_1 and i_2. If v_1 has no even harmonics (dc, 2-nd harmonic,..., etc.), and v_2 has no odd harmonics (fundamental, 3-rd harmonic,..., etc.) then also i_1 has no even harmonics and i_2 has no odd harmonics.

Proposition 3 [9]. The driving-point and transfer characteristic plots of any network containing complementary symmetric resistors are odd symmetric.

Propositions 1, 2, and 3 are derived in [8,9,12] by redrawing the network into various geometrical configurations, corresponding to various point transformations, and then showing that certain invariance is achieved. This ad hoc technique is not satisfactory because it depends on being able to draw a network in a particular way so that it "looks" symmetrical to the eyes of the beholder. This "inspection" technique is useful only for simple networks exhibiting simple types of symmetries. However, as we will see shortly, it is possible for an n-port resistor to exhibit up to $2^n n!$ distinct forms of symmetry, many of them are in fact so subtle that they could not be detected by inspection. Moreover, symmetry is an intrinsic property of an element or network and should not depend on how it is drawn geometrically. Our objective therefore is to develop a unified theory of symmetry which depends only on algebraic techniques. Among other things, such a theory should systematically detect, say using a computer, all forms of symmetry possessed by an element or network. It should be completely general in the sense that all known symmetry results and circuits can be derived as special cases. For example, we will show how various symmetrical configurations such as push-pull amplifiers, complementary symmetric amplifiers, nonlinear bridges, etc., can be systematically generated, even though the original circuits must have been discovered with great insights.

The mathematical tools needed to develop this theory are presented in section II. They are not deep and in fact represent only simple extensions of well-known results on permutations, permutation matrices and groups as described in standard textbooks such as [13]. The key notion is the directed permutation of a set of objects having orientations. The fundamental technique used throughout this paper involves the decomposition of a directed permutation into cyclic components.

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[1] This proposition is a rigorous reformulation of some observations given by Penfield [12].

In section III general definitions and properties of π-permuted and π-symmetric multiport resistors are given. They include all known types of symmetry as special cases. An algorithm for detecting all symmetries possessed by a multiport resistor is given in Appendix A. The main result in this section consists of a general algorithm for synthesizing multiports having any prescribed form of symmetry. Many symmetrical circuit configurations can be generated using this approach.

Multiterminal resistors can be handled in almost the same way as multiport resistors. In section IV, we investigate the basic differences and present several useful applications.

In section V two types of applications are obtained concerning the response of symmetric multiport and multiterminal resistors under certain excitations. First, symmetry is used to reduce the computations involved in the analysis of symmetric multiport and multiterminal resistors under symmetric excitations. Next, the concept of a "time-shifted" symmetric excitation is used to derive various interesting symmetry properties of the response waveforms. The use of symmetry for frequency separation turns out to be a natural consequence of these properties.

II. RELEVANT MATHEMATICAL CONCEPTS AND METHODS

Our objective in this section is to present the mathematical tools necessary for developing a unified theory of symmetry for nonlinear circuits, which do not involve any "visual" inspection of the network topology. Most of these results represent straightforward generalizations and applications of well-known concepts from group theory -- which are concerned with "unoriented" objects -- to allow objects having orientations. In particular, each directed object, by definition, can assume one of two distinct orientations. We define the complement "$\bar{x}$" of a directed object "x" to be the same object having the opposite orientation.

Definition 1. A directed permutation[2] π of a set of n (finite) directed objects (also called oriented objects) is a transformation obtained by first permuting these objects and then complementing some of them. (Note the order of these operations.) In particular, suppose object i is transformed into object i' and object j is transformed into the complement of object j', then we will denote this transformation by $\pi = \begin{pmatrix} \cdots & i & \cdots & j & \cdots \\ \cdots & i' & \cdots & \bar{j}' & \cdots \end{pmatrix}$, where the "bar" above j' denotes its complement.

[2]Except for the identity permutation I, we will denote "directed permutations" by lower case Greek letters.

Since π is a one-to-one onto function, we can write $i' = \pi(i)$ and $\bar{j}' = \pi(j)$. Observe that the sequence of the numbers in the upper row need not be consecutive So, for example, the directed permutations $\begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & 3 & \bar{2} \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 & 2 \\ \bar{2} & \bar{1} & 3 \end{pmatrix}$ are identical. The composition of two directed permutations π and σ is the directed permutation obtained by performing first σ and then π, and will be denoted by[3] $\pi \circ \sigma$. The inverse directed permutation, denoted by π^{-1}, of a directed permutation $\pi = \begin{pmatrix} \cdots & i & \cdots & j & \cdots \\ \cdots & i' & \cdots & \bar{j}' & \cdots \end{pmatrix}$ is defined by $\pi^{-1} = \begin{pmatrix} \cdots & i' & \cdots & j' & \cdots \\ \cdots & i & \cdots & \bar{j} & \cdots \end{pmatrix}$. Observe that

$\pi^{-1} \circ \pi = \pi \circ \pi^{-1} = I$, where $I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ is the identity permutation. If a directed permutation π has a smallest positive integer $\ell \neq 0$ such that $\pi^{\ell} \triangleq \underbrace{\pi \circ \pi \cdots \circ \pi}_{\ell \text{ times}} = I$, then ℓ is called the order of π. The complemented permutation, denoted by $\bar{\pi}$, of a directed permutation $\pi = \begin{pmatrix} \cdots & i & \cdots & j & \cdots \\ \cdots & i' & \cdots & \bar{j}' & \cdots \end{pmatrix}$ is defined by $\bar{\pi} = \begin{pmatrix} \cdots & i & \cdots & j & \cdots \\ \cdots & \bar{i}' & \cdots & j' & \cdots \end{pmatrix}$. Observe that $\overline{\pi \circ \sigma} = \bar{\pi} \circ \bar{\sigma}$ and $\bar{\bar{\pi}} = \pi$.

Proposition 4. The set P_n of all directed permutations of n directed objects forms a finite group (with "composition" as binary operation) containing $2^n n!$ elements.

Proof: There are $2^n n!$ directed permutations because we can permute n objects and reverse the orientation of each object by complementation. The composition of any two directed permutations π_1 and π_2 is a directed permutation. The composition is associative and $\pi \circ I = I \circ \pi = \pi$ for all π. Moreover, given any directed permutation π, there exists a directed permutation π^{-1} such that $\pi^{-1} \circ \pi = \pi \circ \pi^{-1} = I$. ⌑

Corollary. Any directed permutation on n symbols has a finite order ℓ.

Proof: This follows from the closure property of P_n. ⌑

Definition 2. Given a directed permutation π on n objects $i_1, i_2, \ldots, i_n$, we call π a cyclic directed permutation if for some object i_j, we have the properties $\pi^k(i_j) \neq i_j$ and $\pi^k(i_j) \neq \bar{i}_j$, whenever $0 < k < n$. Moreover, if $\pi^n(i_j) = i_j$ for some i_j, then π is called a cyclic directed permutation of normal-order, and if $\pi^n(i_j) = \bar{i}_j$ for some i_j, then π is called a cyclic directed permutation of double-

[3] Our notation differs from that adopted in several textbooks on algebra [13,14] where the so-called natural order (which is opposite to ours) is used.

order.

The reason for choosing these terminologies will soon be obvious. Let us consider some examples first: The directed permutation $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \bar{2} & \bar{1} \end{pmatrix}$ is not cyclic, because $\pi^2(1) = \pi(3) = \bar{1}$, $2 < n$, and $\pi(2) = \bar{2}$, $1 < n$. On the other hand the following two directed permutations are cyclic:

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{4} & \bar{5} & \bar{2} & 3 & \bar{1} \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{4} & \bar{5} & 2 & 3 & \bar{1} \end{pmatrix}.$$

The first is of normal-order since $\pi_1^5(1) = 1$, while the second is of double-order since $\pi_2^5(1) = \bar{1}$. Let us rearrange the columns of π_1 and π_2 to emphasize its cyclic nature:

$$\pi_1 = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ \bar{4} & 3 & \bar{2} & \bar{5} & \bar{1} \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ \bar{4} & 3 & 2 & \bar{5} & \bar{1} \end{pmatrix}$$

From this we may infer that in the case of normal-order cycles, a repeated application of π on any object i_j does not return to i_j until all objects have been exhausted. In contrast to this, in the case of double-order cycles, a repeated application of π on i_j always gives $\bar{i}_j$ after n iterations. This observation may be visualized by displaying along a circle the objects, that are obtained by repeated application of π on the object 1, as shown in Figs. 4(a) and (b), respectively. In general we have the following property.

Proposition 5. (a) Given any cyclic directed permutation π on n objects $i_1, i_2, \ldots, i_n$, then for any $m = 1, 2, \ldots, n$, we have the properties $\pi^k(i_m) \neq i_m$ and $\pi^k(i_m) \neq \bar{i}_m$, whenever $0 < k < n$. (b) Moreover, π is either of normal-order or of double-order. If π is of normal-order, then $\pi^n(i_m) = i_m$ for any m, and consequently π is of order n. If π is of double-order, then $\pi^n(i_m) = \bar{i}_m$ for any m, and consequently π is of order 2n.

Proof: (a) In order to prove part (a) we first show that relative to the object i_j inferred to in Def. 2, the elements of the associated sequence $S = \{i_j, \pi(i_j), \ldots, \pi^{n-1}(i_j)\}$ do not duplicate themselves, i.e., for any m, either i_m or $\bar{i}_m$ appears in the sequence S. Since there are precisely n elements in S, it suffices to show that it is impossible for i_m or $\bar{i}_m$ to appear more than once, or for both i_m and $\bar{i}_m$ to appear jointly in S. This can be shown by contradiction. For example, suppose that i_m and $\bar{i}_m$ belong to S, then there exist a k_1 and k_2

such that $i_m = \pi^{k_1}(i_j)$ and $\bar{i}_m = \pi^{k_2}(i_j)$, with $0 < k_1 < n$, $0 < k_2 < n$. It follows that $\pi^{k_1}(i_j) = \pi^{k_2}(\bar{i}_j)$ or $\pi^{|k_1-k_2|}(i_j) = \bar{i}_j$ with $0 < |k_1-k_2| < n$. But this clearly contradicts Def. 2. In order to show (a), let us show that it is impossible to find a k such that $\pi^k(i_m) = i_m$ or $\pi^k(i_m) = \bar{i}_m$ where $0 < k < n$. Consider, for example, $\pi^k(i_m) = i_m$, then since i_m or $\bar{i}_m$ appears once in the sequence S, say $i_m = \pi^{k_1}(i_j)$, then $\pi^{k+k_1}(i_j) = \pi^{k_1}(i_j)$ or $\pi^k(i_j) = i_j$ for $0 < k < n$. This contradicts Def. 2.

(b) Let us show first that the element $\pi^n(i_j)$ is either i_j or $\bar{i}_j$. Suppose the contrary that this element is another object, say, i_m or $\bar{i}_m$ $(m \neq j)$. Since i_m or $\bar{i}_m$ appears in S as $\pi^k(i_j)$ for some k with $0 < k < n$, we have $\pi^n(i_j) = \pi^k(i_j)$ or $\pi^n(i_j) = \pi^k(\bar{i}_j)$. This implies that $\pi^{n-k}(i_j) = i_j$ or $\bar{i}_j$, which contradicts Def. 2. Next, let us show that $\pi^n(i_j) = i_j$ implies $\pi^n(i_m) = i_m$ for all $m = 1,2,\ldots,n$. Since for any i_m there is a k such that either $\pi^k(i_j) = i_m$ or $\bar{i}_m$ where $0 < k < n$, we have $\pi^{-k+n}(i_m) = \pi^{-k}(i_m)$ or $\pi^n(i_m) = i_m$. This implies that the order of π is equal to n. Analogously it can be proven that $\pi^n(i_j) = \bar{i}_j$ implies $\pi^n(i_m) = \bar{i}_m$ for $m = 1,2,\ldots,n$, and that the order of π is 2n. ∎

It follows from Prop. 5 that any cyclic directed permutation of normal-order may be denoted unambiguously by the <u>cyclic notation</u> $(k_1k_2\ldots k_n)$, where $k_{j+1} = \pi(k_j)$, $j = 1,2,\ldots,n-1$, and $k_1 = \pi(k_n)$. Any cyclic permutation of double-order is then denoted by $(k_1k_2\ldots k_n\bar{k}_1\bar{k}_2\ldots\bar{k}_n)$ where $k_{j+1} = \pi(k_j)$, $j = 1,2,\ldots,n-1$ and $k_1 = \pi(\bar{k}_n)$. Observe that there may exist several distinct but equivalent cyclic notations. For example, we can denote our preceding examples π_1 and π_2 as follows:

$$\pi_1 = (1\ \bar{4}\ \bar{3}\ 2\ \bar{5}) = (\bar{4}\ \bar{3}\ 2\ \bar{5}\ 1) = (3\ \bar{2}\ 5\ \bar{1}\ 4) = \ldots \text{ etc.}$$

$$\pi_2 = (1\ \bar{4}\ \bar{3}\ \bar{2}\ 5\ \bar{1}\ 4\ 3\ 2\ \bar{5}) = (4\ 3\ 2\ \bar{5}\ 1\ \bar{4}\ \bar{3}\ \bar{2}\ 5\ \bar{1}) = (3\ 2\ \bar{5}\ 1\ \bar{4}\ \bar{3}\ \bar{2}\ 5\ \bar{1}\ 4) = \ldots \text{ etc.}$$

To simplify our subsequent discussion, we will henceforth assume that the <u>first</u> element in our cyclic notation is <u>not</u> complemented. There is no loss of generality in this assumption since we can always complement every element if necessary. For example, $(\bar{4}\ \bar{3}\ 2\ \bar{5}\ 1) = (4\ 3\ \bar{2}\ 5\ \bar{1})$.

Observe that every cyclic permutation π which involves no complementation of its elements has normal-order. In this case, our definition of cyclic permutation reduces to the conventional definition for non-oriented permutations. The importance of "cyclic directed permutations" in our study of circuit symmetry is due to the following basic theorem, which is a generalization of a classic result from group theory [13,14]:

Theorem 1. Every directed permutation can be decomposed uniquely into a collection of cyclic directed permutations operating on disjoint (mutually exclusive) sets of objects.

Proof: Let π be a directed permutation on a finite number n of objects. Choose any object from 1 to n and call it m_1. Applying the directed permutation π once, we obtain $m_2 = \pi(m_1)$, where m_2 is chosen from the n objects followed possibly by a complementation. By repeating this process we obtain $m_1, m_2, m_3, \ldots$ where $m_{i+1} = \pi(m_i)$. Since there can be at most 2n distinct oriented objects some of the first 2n+1 elements in the sequence $\{m_1, m_2, \ldots, m_{2n+1}, \ldots\}$ must be equal to each other. Let m_i and m_j with $0 < i < j \leq 2n+1$ be two equal elements of this sequence such that $m_k \neq m_i$ for all k such that $i < k < j$. Since π has a unique inverse, $\pi^{-1}(m_i) = \pi^{-1}(m_j)$ and hence $m_{i-1} = m_{j-1}$. Repeating this operation (i-1) times we obtain $m_{j-i+1} = m_1$. Thus the effect of π on $\{m_1, m_2, \ldots, m_{j-i}\}$ constitutes a cyclic directed permutation. Let us delete the objects associated with $\{m_1, m_2, \ldots, m_{j-i}\}$ from the set of n objects. Observe that there are j-i such deleted objects in the case of a normal-order cyclic directed permutation, and (j-i)/2 in the case of a double-order cyclic directed permutation. Repeating the above algorithm on the remaining objects, we obtain another cyclic directed permutation. An iteration of this algorithm must terminate since n is finite. Hence we have succeeded in decomposing the "n" objects into a unique collection of mutually exclusive sets such that π is a cyclic directed permutation on each set. ¤

The resulting cyclic directed permutations are called the cyclic components of π and the collection of these cyclic directed permutations are said to be a decomposition of π into cyclic components. We denote it by

$$\pi = \left(\ldots\right) \ldots \left(i_1^{(j)} \ldots i_{n_j}^{(j)}\right) \ldots \left(i_1^{(\ell)} \ldots i_{n_\ell}^{(\ell)} \bar{i}_1^{(\ell)} \ldots \bar{i}_{n_\ell}^{(\ell)}\right) \ldots \left(\ldots\right), \tag{1}$$

where $\left(i_1^{(j)} \ldots i_{n_j}^{(j)}\right)$ and $\left(i_1^{(\ell)} \ldots i_{n_\ell}^{(\ell)} \bar{i}_1^{(\ell)} \ldots \bar{i}_{n_\ell}^{(\ell)}\right)$ denote a normal-order and a double-order cyclic component, respectively.

Example. Let us decompose $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{4} & \bar{2} & 7 & \bar{5} & 1 & 8 & \bar{6} & 3 & 9 \end{pmatrix}$ into cyclic components:

Starting with 1 we obtain the cycle $(1\ \bar{4}\ 5)$. Choosing next the element 2 we obtain the cycle $(2\ \bar{2})$. Choosing next the element 3, we obtain the cycle $(3\ 7\ \bar{6}\ \bar{8}\ \bar{3}\ \bar{7}\ 6\ 8)$. Choosing next the element 9, we obtain the cycle (9). This exhausts all elements. Hence π is decomposed into 4 cyclic components. The cyclic decomposition of π is then: $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})(3\ 7\ \bar{6}\ \bar{8}\ \bar{3}\ \bar{7}\ 6\ 8)(9)$. This basic technique is simple and will be used frequently in the following sections to derive many symmetry properties in nonlinear circuits.

Corollary. The order ℓ of a directed permutation π is the least common multiple of the orders of its cyclic components.

Proof: Let k_i be the order of the i-th cyclic component, then $\pi^{k_i}(j) = j$ and $\pi^m(j) \neq j$, $0 < m < k_i$ for any object j of the i-th cycle. Thus the least common multiple of the k_i's is the smallest integer k such that $\pi^k(j) = j$ for all objects j. ¤

Under certain operations many aspects of the decomposition (1) of a directed permutation π are preserved. A useful operation is the similarity transformation $\chi^{-1} \circ \pi \circ \chi$ where χ is an arbitrary directed permutation.

Proposition 6. Let (1) be the decomposition of π into cyclic components, then for any arbitrary directed permutation χ, the decomposition of $\chi^{-1} \circ \pi \circ \chi$ into cyclic components is given by:

$$\chi^{-1} \circ \pi \circ \chi = (\ldots) \ldots \left(\chi^{-1}\left(i_1^{(j)}\right) \ldots \chi^{-1}\left(i_{n_j}^{(j)}\right)\right) \ldots$$
$$\left(\chi^{-1}\left(i_1^{(\ell)}\right) \ldots \chi^{-1}\left(i_{n_\ell}^{(\ell)}\right) \chi^{-1}\left(\bar{i}_1^{(\ell)}\right) \ldots \chi^{-1}\left(\bar{i}_{n_\ell}^{(\ell)}\right)\right) \ldots (\ldots) \tag{2}$$

Proof: Consider first the j-th normal-order cyclic component $\left(i_1^{(j)} \ldots i_{n_j}^{(j)}\right)$. Then $i_{k+1}^{(j)} = \pi\left(i_k^{(j)}\right)$, $k < n_j$ and $i_1^{(j)} = \pi\left(i_{n_j}^{(j)}\right)$. This implies that $\chi^{-1}\left(i_{k+1}^{(j)}\right) = \chi^{-1} \circ \pi \circ \chi\left(\chi^{-1}(i_k^{(j)})\right)$, $k < n_j$ and $\chi^{-1}\left(i_1^{(j)}\right) = \chi^{-1} \circ \pi \circ \chi\left(\chi^{-1}(i_{n_j}^{(j)})\right)$. From these relations we obtain the j-th normal-order cyclic component in (2). A similar derivation applies to the double-order cyclic components in (2). ¤

A particular choice of χ yields the following interesting result:

Corollary 1. Let χ_π be the following directed permutation which transforms the set of integers $\{1\ 2\ \dots\ n\}$ into the elements of the cyclic decomposition (1) of π:

$$\chi_\pi = \begin{pmatrix} k_1^{(1)} & k_2^{(1)} & \dots & k_j^{(m)} & \dots \\ i_1^{(1)} & i_2^{(1)} & \dots & i_j^{(m)} & \dots \end{pmatrix} \tag{3}$$

where $k_j^{(m)} = j + \sum_{i=1}^{m} n_i$, then

$$\chi_\pi^{-1} \circ \pi \circ \chi_\pi = \left(1\ 2\ \dots\right) \dots \left(1 + \sum_{m=1}^{j-1} n_m \ \dots \ \sum_{m=1}^{j} n_m\right) \dots$$
$$\left(1 + \sum_{m=1}^{\ell-1} n_m \ \dots \ \sum_{m=1}^{\ell} n_m \quad 1 + \sum_{m=1}^{\ell-1} n_m \ \dots \ \sum_{m=1}^{\ell} n_m\right) \dots \left(\dots\right) \tag{4}$$

Applying this corollary to the preceding directed permutation $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})(3\ 7\ \bar{6}\ \bar{8}\ \bar{3}\ \bar{7}\ 6\ 8)(9)$, we obtain

$$\chi_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & \bar{4} & 5 & 2 & 3 & 7 & \bar{6} & \bar{8} & 9 \end{pmatrix} \text{ and}$$

$$\chi_\pi^{-1} \circ \pi \circ \chi_\pi = (1\ 2\ 3)(4\ \bar{4})(5\ 6\ 7\ 8\ \bar{5}\ \bar{6}\ \bar{7}\ \bar{8})(9) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 1 & \bar{4} & 6 & 7 & 8 & \bar{5} & 9 \end{pmatrix}$$

Observe that the entries in each normal-order cyclic component in (4) are just integers arranged in an increasing consecutive order. A similar observation applies also to double-order cyclic components, except that the entries are followed by their complements to produce twice as many entries. The action of the composition $\chi_\pi^{-1} \circ \pi \circ \chi_\pi$ in (4) is simply to reorder the columns of the permutation π so that the entries in the resulting cyclic components are arranged in a consecutive order.

For reasons that will be obvious soon, two directed permutations π and σ having (1) and (2) as their cyclic decomposition will henceforth be called similar.

Corollary 2. For any two similar directed permutations π and σ, there exists a directed permutation ψ such that $\psi^{-1} \circ \pi \circ \psi = \sigma$.

Proof: Choose χ of (2) for ψ. ¤

Our next objective is to show that there is a one-to-one relationship between the group of directed permutations and a set of associated matrices.

This property will be used to derive a number of useful symmetry results.

Definition 3. A directed permutation matrix is a square matrix whose entries are all zero except one entry in each column and one in each row, which is +1 or −1.

A one-to-one relationship between a "directed permutation" π and a "directed permutation matrix" $\underset{\sim}{P}(\pi)$ can be established as follows:

$$\pi = \begin{pmatrix} \cdots & i & \cdots & j & \cdots \\ \cdots & i' & \cdots & \bar{j}' & \cdots \end{pmatrix} \iff \underset{\sim}{P}(\pi) = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} \leftarrow j' \\ \\ \leftarrow i' \end{matrix} \qquad (5)$$

$$\qquad\qquad\qquad\qquad\qquad \begin{matrix} \uparrow & & \uparrow \\ i & & j \end{matrix}$$

This nxn matrix $\underset{\sim}{P}(\pi)$ transforms an nx1 vector $\underset{\sim}{x}$ into an nx1 vector $\underset{\sim}{P}(\pi)\underset{\sim}{x}$ in accordance with the following rules: The i-th component of $\underset{\sim}{x}$ is the i'-th component of $\underset{\sim}{P}(\pi)\underset{\sim}{x}$, and the j-th component of $\underset{\sim}{x}$ is minus the j'-th component of $\underset{\sim}{P}(\pi)\underset{\sim}{x}$. Observe that if a directed permutation π does not involve any complementation of its objects, the associated matrix $\underset{\sim}{P}(\pi)$ reduces to the conventional permutation matrix with +1 as its only non-zero entries.

Examples. 1. For the directed permutation $\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & 3 & \bar{2} \end{pmatrix}$ we find

$$\underset{\sim}{P}(\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. The "normal-order" cycle $\pi = (1\ 2\ 3)$ has the permutation matrix

$$\underset{\sim}{P}(\pi) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. The "double-order" cycle $\pi = (1\ 2\ 3\ \bar{1}\ \bar{2}\ \bar{3})$ has the directed permutation matrix

$$\underset{\sim}{P}(\pi) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Proposition 7. The collection of all nxn directed permutation matrices form a group (with matrix multiplication as binary operation) isomorphic to the group P_n of directed permutations on "n" objects (with composition as binary operation).

Proof: The map defined in (5) is one-to-one. Furthermore it is easy to check that $\underset{\sim}{P}(\pi_2 \circ \pi_1) = \underset{\sim}{P}(\pi_2)\underset{\sim}{P}(\pi_1)$. ∎

Corollary 1. The collection of all nxn permutation matrices form under matrix multiplication a group isomorphic to the group of all permutations on "n" objects under the composition operation.

Corollary 2. Every directed permutation matrix is orthogonal; namely,

$$\underset{\sim}{P}^T(\pi)\underset{\sim}{P}(\pi) = \underset{\sim}{P}(\pi)\underset{\sim}{P}^T(\pi) = \underset{\sim}{1}_n, \tag{6}$$

where $\underset{\sim}{1}_n$ denotes the unit nxn matrix.

Proof: It follows from Prop. 7 that $\underset{\sim}{P}(\pi^{-1})\underset{\sim}{P}(\pi) = \underset{\sim}{P}(\pi^{-1}\circ\pi) = \underset{\sim}{P}(I) = \underset{\sim}{1}_n$. Moreover the map defined in (5) implies that $\underset{\sim}{P}(\pi^{-1}) = \underset{\sim}{P}^T(\pi)$.

¤

Corollary 3. Given any directed permutation π and its associated directed permutation matrix $\underset{\sim}{P}(\pi)$, there exists a directed permutation χ_π and its associated directed permutation matrix $\underset{\sim}{P}(\chi_\pi)$ such that

$$\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi) = \begin{bmatrix} \begin{bmatrix} 0 & & \pm 1 \\ 1 & \ddots & \\ & \ddots\,1 & 0 \end{bmatrix} & & & \bigcirc \\ & \ddots & & \\ & & \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix} & \\ \bigcirc & & & \ddots \; \begin{bmatrix} \pm 1 \end{bmatrix} \end{bmatrix} \tag{7}$$

is a block-diagonal matrix.

Proof: This is simply the matrix version of Cor. 1 of Prop. 6.

¤

Observe that each cyclic component in π gives rise to one and only one, diagonal block in (7). In particular, a normal-order cyclic component gives rise to a +1 in the upper right corner of the block, and a double-order cyclic component gives rise to a -1. For example, corresponding to π and χ_π given earlier (following Cor. 1 of Prop. 6), we have:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ -1&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{bmatrix} & & & \\ & [-1] & & \\ & & \begin{bmatrix} 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix} & \\ & & & [1] \end{bmatrix} \qquad (8)$$

In section V we will need the eigenvectors of $\underset{\sim}{P}(\pi)$ associated with the eigenvalue +1. Since $\underset{\sim}{P}(\chi_\pi)$ is orthogonal, the matrices $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi)$ and $\underset{\sim}{P}(\pi)$ are similar. Hence, it suffices to investigate the eigenvectors of the similar block-diagonal matrix given in (7).

<u>Proposition 8</u>. The i-th $n_i x n_i$ submatrix of (7) has an eigenvalue +1 iff the upper right-hand entry is +1. The corresponding unique (apart from a scaling constant) eigenvector is given by $[1\ 1\ \dots\ 1]^T$.

<u>Proof</u>: The eigenvectors of the $n_i x n_i$ submatrix $\begin{bmatrix} 0 & & \pm 1\\ 1 & \ddots & \\ & \ddots\ 1 & 0 \end{bmatrix}$ are the solutions of the polynomial equation $\lambda^{n_i} \mp 1 = 0$. Hence $\lambda = 1$ is a solution iff the upper right-hand entry is +1. Consequently, the eigenvector associated with $\lambda = 1$ is given by $\underset{\sim}{x} = [x_1\ x_2\ \dots\ x_{n_i}]^T$, where $x_1 = x_2 = \dots = x_{n_i} = k$. Hence, $\underset{\sim}{x} = k[1\ 1\ \dots\ 1]^T$. ¤

<u>Corollary</u>. The directed permutation matrix $\underset{\sim}{P}(\pi)$ has an eigenvalue +1 iff π

has a cycle of normal-order.

Proof: This corollary follows directly from Cor. 3 of Prop. 7 and Prop. 8. ¤

It follows from this corollary that the eigenvectors associated with the eigenvalue +1 are related to the cyclic components of normal-order. To derive this relationship, we will define first an nxm matrix $\underset{\sim}{S}(\pi)$ associated with the collection of all m cyclic components of normal-order, and then show that the eigenvectors associated with the eigenvalue +1 are linear combinations of columns of $\underset{\sim}{S}(\pi)$.

Definition 4. Let the directed permutation π have in its decomposition "m" cyclic components of normal-order: $\left(i_1^{(1)}\ i_2^{(1)} \ldots\ i_{n_1}^{(1)}\right) \ldots \left(i_1^{(m)}\ i_2^{(m)} \ldots\ i_{n_m}^{(m)}\right)$.

The matrix $\underset{\sim}{S}(\pi)$ is an nxm matrix, whose k-th column is derived from the k-th cyclic component $\left(i_1^{(k)}\ i_2^{(k)} \ldots\ i_{n_k}^{(k)}\right)$ in accordance with the following rule: If $i_j^{(k)}$ is not complemented, place a +1 in the i_j-th row. Otherwise, place a -1. Repeat for $j = 1,2,\ldots,n_k$. Set all other entries in column k equal to zero.

Example. Consider once again the directed permutation $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})\cdot(3\ 7\ \bar{6}\ \bar{8}\ \bar{3}\ \bar{7}\ 6\ 8)(9)$. The normal-order cyclic components are $(1\ \bar{4}\ 5)$ and (9).

Hence $n = 9$, $m = 2$, and $\underset{\sim}{S}(\pi) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Observe that each column of $\underset{\sim}{S}(\pi)$ can be derived from a cyclic component of normal-order by inspection.

Proposition 9. Every eigenvector of $\underset{\sim}{P}(\pi)$ associated with eigenvalue +1 is a unique linear combination of columns of $\underset{\sim}{S}(\pi)$, and vice versa.

Proof. Since $\underset{\sim}{P}(\chi_\pi)$ is orthogonal, $\underset{\sim}{P}(\pi)\underset{\sim}{S}(\pi) = \underset{\sim}{S}(\pi)$ can be rewritten in the form $[\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi)]\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{S}(\pi) = \underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{S}(\pi)$. Hence it suffices to show that the eigenvectors of $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi)$ associated with the eigenvalue +1 are unique linear combinations of the columns of $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{S}(\pi)$. By Cor. 3 of Prop. 7, $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi)$ assumes the block-diagonal form (7). It follows from the definition of $\underset{\sim}{S}(\pi)$ and χ_π that for each submatrix of (7) corresponding to a cycle of normal-order, the block-diagonal matrix $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{P}(\pi)\underset{\sim}{P}(\chi_\pi)$ has a set of 1's in $\underset{\sim}{P}^T(\chi_\pi)\underset{\sim}{S}(\pi)$; namely,

$$
P^T(\chi_\pi)P(\pi)P(\chi_\pi) = \begin{bmatrix} \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{n_1 \times n_1} & 0 \\ 0 & \ddots \\ & & \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{n_m \times n_m} \end{bmatrix}, \quad P^T(\chi_\pi)S(\pi) = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n_1 \times 1} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n_m \times 1} \end{bmatrix} \tag{9}
$$

The column subvector containing all unit entries in $P^T(\chi_\pi)S(\pi)$ is precisely the eigenvector associated with the eigenvalue 1 of the corresponding submatrix (Prop. 8). Since the submatrices on the diagonal of $P^T(\chi_\pi)P(\pi)P(\chi_\pi)$ are disjoint, so are the columns of $P^T(\chi_\pi)S(\pi)$. Hence the columns of $S(\pi)$ are linearly independent eigenvectors of $P(\pi)$ associated with +1. ⌑

__Corollary__. Let n_i be the order of the i-th normal-order cyclic component of π, then

$$
S^T(\pi)S(\pi) = \begin{bmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_m \end{bmatrix}. \tag{10}
$$

__Proof__: From (9) we have

$$
S^T(\pi)S(\pi) = [P^T(\chi_\pi)S(\pi)]^T[P^T(\chi_\pi)S(\pi)] = \begin{bmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_m \end{bmatrix}. \tag{11}
$$

⌑

III. __π-PERMUTED AND π-SYMMETRIC MULTIPORT RESISTORS__

A 2n-terminal element is called an __n-port__ if the terminals can be grouped into n pairs 1 1' 2 2' ... n n' such that the current entering one terminal i of each pair is constrained to leave the other terminal i' -- henceforth called the __port constraint__ --. Each pair i,i' of terminals is called a port i. Let the ports be labelled consecutively from 1 to n and let $v = [v_1 \cdots v_n]^T$ denote the vector of port voltages, and $i = [i_1 \cdots i_n]^T$ the vector of port currents, where the __associated reference convention__ is adopted for each pair (v_j, i_j) of port variables, as shown in Fig. 5. An n-port R is said to be an n-port resistor[4] if it is completely

[4] Throughout this paper, a multiport resistor will be denoted by a __Roman__ R, and a multiterminal resistor will be denoted by a __script__ $\mathcal{R}$.

characterized by a set S of admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$, henceforth called its constitutive relation.

A. Properties of π-permuted and π-symmetric multiport resistors

Definition 5. Given a directed permutation π on n objects and an n-port resistor R, we define the associated π-permuted n-port resistor $\hat{R}$ by the set of admissible pairs $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}})$ such that $\hat{\underset{\sim}{v}} = \underset{\sim}{P}(\pi)\underset{\sim}{v}$, $\hat{\underset{\sim}{i}} = \underset{\sim}{P}(\pi)\underset{\sim}{i}$, where $(\underset{\sim}{v},\underset{\sim}{i})$ is any admissible pair of R.

It follows from Def. 5 that $\hat{R}$ is obtained from R by first relabelling the port numbers in accordance with the rule defined by π, and then interchanging the terminals in each port where a complementation is called for by the directed permutation π. For example, if we choose $\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & 3 & \bar{2} \end{pmatrix}$, then the π-permuted 3-port resistor is shown in Fig. 6. Observe that a π-permuted resistor R is in general distinct from the resistor R and should be treated as a different multiport resistor, when embedded in a circuit. It follows from Prop. 4 that each n-port resistor R induces a total of $2^n n!$ possibly distinct π-permuted n-port resistors. Let us now consider three common special cases:

1) Complementary n-port resistor. This is obtained by choosing $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \bar{1} & \bar{2} & \dots & \bar{n} \end{pmatrix} = \bar{I}$, henceforth referred to as the complementary transformation. In this case, $\hat{R}$ is simply obtained by transposing the terminals of all ports of R.

2) Rotated n-port resistor. This is obtained by choosing the cyclic directed permutation of normal-order $\pi = (1\ 2\ \dots\ n)$. In this case, $\hat{R}$ is simply obtained by relabelling each port j of R by j+1, j = 1,2,...,n-1, and by changing port n into port 1.

3) Involuted n-port resistor. This is obtained by choosing a directed permutation π having the "self-inverse" property $\pi^2 = \pi \circ \pi = I$; i.e. $\pi^{-1} = \pi$. This class of directed permutations includes three common transformations as special cases; namely 1) reflection, 2) 180°-rotation and 3) complementation. By definition, the order of an involution π is 2. It follows then from Cor. of Thm. 1 that the decomposition of an involution π contains only cyclic components of order equal to one or two. This is still a fairly large class, and includes the two port shown in Fig. 3, where $\pi = (1\ \bar{1})(2)$.

Since a π-permuted n-port resistor $\hat{R}$ is obtained from R by merely a port relabelling and/or a complementation operation, it is not surprising to expect that many circuit-theoretic properties of R are inherited by $\hat{R}$. Indeed, we have the following properties:

Proposition 10. If a multiport resistor R is characterized by a constitutive relation $\underset{\sim}{R}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$, then its π-permuted resistor $\hat{R}$ is characterized by

$$\hat{\underset{\sim}{R}}(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}}) \triangleq \underset{\sim}{R}\left(\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{v}},\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{i}}\right) = \underset{\sim}{0}. \tag{12}$$

Proof: Let $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}}) \triangleq (\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i})$ be an admissible pair of $\hat{R}$ (Def. 5). Then it follows from the orthogonality of $\underset{\sim}{P}(\pi)$ that $\underset{\sim}{R}\left(\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{v}},\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{i}}\right) = \underset{\sim}{R}\left(\underset{\sim}{P}^T(\pi)\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}^T(\pi)\underset{\sim}{P}(\pi)\underset{\sim}{i}\right) = \underset{\sim}{R}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$. ¤

Proposition 11. If the constitutive relation of R is C^k-parameterizable [15], in the sense that it is characterized by $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{\rho})$ and $\underset{\sim}{i} = \underset{\sim}{i}(\underset{\sim}{\rho})$, where $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ are C^k-functions, then the π-permuted resistor is also C^k-parameterizable. Furthermore, the dimensions of R and $\hat{R}$ are identical, and R is reciprocal (resp., antireciprocal) iff $\hat{R}$ is reciprocal (resp., antireciprocal).

Proof: Using Def. 5 it easily follows that $\hat{\underset{\sim}{v}} = \underset{\sim}{P}(\pi)\underset{\sim}{v}(\underset{\sim}{\rho})$ and $\hat{\underset{\sim}{i}} = \underset{\sim}{P}(\pi)\underset{\sim}{i}(\underset{\sim}{\rho})$ is a parameterization of $\hat{R}$. This implies that

$$\text{rank}\ \frac{\partial}{\partial \underset{\sim}{\rho}}\ [\underset{\sim}{v}(\underset{\sim}{\rho}),\underset{\sim}{i}(\underset{\sim}{\rho})] = \text{rank}\ \frac{\partial}{\partial \underset{\sim}{\rho}}\ \{\underset{\sim}{P}(\pi)[\underset{\sim}{v}(\underset{\sim}{\rho}),\underset{\sim}{i}(\underset{\sim}{\rho})]\}. \tag{13}$$

Hence, R and $\hat{R}$ have identical dimension [15]. The remaining assertion follows from the definition of reciprocity and antireciprocity [15] and the orthogonality of $\underset{\sim}{P}(\pi)$. ¤

Proposition 12. If R is a non-energic (or energic or passive or active) multiport resistor, then so is the π-permuted resistor $\hat{R}$.

Proof: These invariant properties all follow from the fact that the powers associated with two corresponding admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ of R and $\left(\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i}\right)$ of $\hat{R}$ are equal to each other; namely,

$$\langle \underset{\sim}{v},\underset{\sim}{i} \rangle = \underset{\sim}{v}^T\underset{\sim}{i} = \underset{\sim}{v}^T\underset{\sim}{P}^T(\pi)\underset{\sim}{P}(\pi)\underset{\sim}{i} = \langle \underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i} \rangle. \tag{14}$$

¤

Definition 6. Two multiport resistors R and $\hat{R}$ are said to be identical if every admissible pair $(\underset{\sim}{v},\underset{\sim}{i})$ of R is also an admissible pair of $\hat{R}$, and vice versa. Two multiport resistors R and $\hat{R}$ are said to be isomorphic if there exists a directed permutation π such that for any admissible pair $(\underset{\sim}{v},\underset{\sim}{i})$ of R, $\left(\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i}\right)$ is an admissible pair of $\hat{R}$, and conversely for every admissible pair $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}})$ of $\hat{R}$, $\left(\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{v}},\underset{\sim}{P}^T(\pi)\hat{\underset{\sim}{i}}\right)$ is an admissible pair of R.

It follows from Def. 6 that every π-permuted multiport resistor $\hat{R}$ is isomorphic to R.

Definition 7. A multiport resistor R is said to be π-symmetric iff R is identical to its π-permuted resistor $\hat{R}$; i.e. if $(\underset{\sim}{v},\underset{\sim}{i})$ is an admissible pair of R, then $(\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i})$ is also an admissible pair.

The following properties are useful for checking whether a multiport resistor is π-symmetric or not.

Proposition 13. A multiport resistor R with constitutive relation $R(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$ is π-symmetric iff

$$\underset{\sim}{R}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{R}\left(\underset{\sim}{P}^T(\pi)\underset{\sim}{v},\underset{\sim}{P}^T(\pi)\underset{\sim}{i}\right) = \underset{\sim}{0}. \tag{15}$$

Proof: It follows directly from Prop. 10 and Def. 7. ◻

Example 1. A 1-port nonlinear resistor is bilateral iff $f(v,i) = 0 \Leftrightarrow f(-v,-i) = 0$. It follows from (15) that every bilateral resistor exhibits complementary symmetry with $\pi = (1\ \bar{1})$.

Example 2. Referring back to the dc nonlinear OP AMP circuit model shown in Fig. 1(c), we conclude that it too exhibits complementary symmetry with $\pi = (1\ \bar{1})(2\ \bar{2})(3\ \bar{3})$.

In the common special case where a multiport resistor can be characterized by a hybrid representation [15], we have a unique response to a mixture of current and voltage excitations at the ports. The mixed vectors are defined as follows. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be nxn diagonal matrices satisfying the condition that either $a_{ii} = 1$ and $b_{ii} = 0$, or $b_{ii} = 1$ and $a_{ii} = 0$, where $\underset{\sim}{A} = [a_{ij}]$ and $\underset{\sim}{B} = [b_{ij}]$. If we define the excitation vector $\underset{\sim}{x}$ and the response vector $\underset{\sim}{y}$ by

$$\begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix}, \tag{16}$$

then R is said to be characterized by a hybrid representation if $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are related by

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}), \tag{17}$$

where $\underset{\sim}{h}(\cdot)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. We say that the excitation $\underset{\sim}{x}$ and the response $\underset{\sim}{y}$ are compatible with a directed permutation π if the permuted variables are of the same type; i.e. if $\pi(i) = j$ or $\pi(i) = \bar{j}$, then x_i and x_j are both voltages or both currents. More formally, this compatibility condition is equivalent to the conditions:[5]

$$\underset{\sim}{P}(\pi)\underset{\sim}{A} = \underset{\sim}{A}\,\underset{\sim}{P}(\pi), \qquad \underset{\sim}{P}(\pi)\underset{\sim}{B} = \underset{\sim}{B}\,\underset{\sim}{P}(\pi). \tag{18}$$

Proposition 14. If the excitation $\underset{\sim}{x}$ and the response $\underset{\sim}{y}$ of the hybrid representation $\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x})$ of a multiport resistor R are compatible with a directed permutation π, then R is π-symmetric iff

$$\underset{\sim}{h}(\cdot) = \underset{\sim}{P}^T(\pi)\underset{\sim}{h}\left(\underset{\sim}{P}(\pi)\cdot\right). \tag{19}$$

Proof: It follows from (16) and (18) that

$$\begin{bmatrix} \underset{\sim}{P}(\pi)\underset{\sim}{x} \\ \underset{\sim}{P}(\pi)\underset{\sim}{y} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{P}(\pi)\underset{\sim}{A} & \underset{\sim}{P}(\pi)\underset{\sim}{B} \\ \underset{\sim}{P}(\pi)\underset{\sim}{B} & \underset{\sim}{P}(\pi)\underset{\sim}{A} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} \end{bmatrix} \begin{bmatrix} \underset{\sim}{P}(\pi)\underset{\sim}{v} \\ \underset{\sim}{P}(\pi)\underset{\sim}{i} \end{bmatrix}. \tag{20}$$

Then it follows from Def. 7 that R is π-symmetric iff

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) \Leftrightarrow \underset{\sim}{P}(\pi)\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{P}(\pi)\underset{\sim}{x}). \tag{21}$$

Finally (19) and (21) are equivalent by the orthogonality of $\underset{\sim}{P}(\pi)$. ◻

Observe that our definition of π-symmetry is based on admissible pairs and is independent of any particular hybrid representation. Hence, it suffices to check (19) for any particular representation compatible with π in order to prove or disprove the π-symmetry for a given resistor.

Corollary. A linear multiport resistor described by the hybrid equation $\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{x}$, which is compatible with π, is π-symmetric iff

$$\underset{\sim}{H} = \underset{\sim}{P}^T(\pi)\,\underset{\sim}{H}\,\underset{\sim}{P}(\pi) \tag{22}$$

Equation (22) imposes certain definite structural properties for hybrid matrices exhibiting various forms of π-symmetry. A general algorithm for computing the canonical form of a π-symmetric hybrid matrix (22) is described in Appendix A. Let us consider here some of the more common symmetries of practical interest.

1. Cyclic (rotational) symmetry. The symmetric directed permutation is

[5] Observe that $\underset{\sim}{P}(\pi)\,\underset{\sim}{A}$ permutes the non-zero rows of $\underset{\sim}{A}$ among themselves as well as the zero rows among themselves. Observe also that $\underset{\sim}{P}(\pi)\,\underset{\sim}{A}\,\underset{\sim}{P}^T(\pi)$ permutes the non-zero columns of $\underset{\sim}{P}(\pi)\underset{\sim}{A}$ among themselves, as well as the zero columns among themselves. Since $\underset{\sim}{P}(\pi)\,\underset{\sim}{A}\,\underset{\sim}{P}^T(\pi)$ is a positive semi-definite diagonal matrix, the compatibility property implies that $\underset{\sim}{P}(\pi)\,\underset{\sim}{A}\,\underset{\sim}{P}^T(\pi) = \underset{\sim}{A}$, and hence $\underset{\sim}{P}(\pi)\,\underset{\sim}{A} = \underset{\sim}{A}\,\underset{\sim}{P}(\pi)$. A similar reasoning shows that $\underset{\sim}{P}(\pi)\,\underset{\sim}{B} = \underset{\sim}{B}\,\underset{\sim}{P}(\pi)$.

$\pi = (1\ 2\ \ldots\ n)$. The associated cyclic permutation matrix and the structure of a hybrid matrix are easily found to be:

$$\underset{\sim}{P}(\pi) = \overset{\pi\text{-permutation}}{\begin{bmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & ..1 & 0 \end{bmatrix}} \qquad \underset{\sim}{H} = \overset{\text{structure of } \underset{\sim}{H}}{\begin{bmatrix} A_1 & A_n & A_{n-1} & \cdots & A_2 \\ A_2 & A_1 & A_n & \cdots & A_3 \\ A_3 & A_2 & A_1 & \cdots & A_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 \end{bmatrix}} . \tag{23}$$

It is easy to check that (22) is satisfied. An examination of $\underset{\sim}{H}$ in (23) shows that the "last" entry in any column j is identical to the "first" entry in column j+1. A simple example of a cyclic symmetric hybrid matrix is given by the 3-port <u>circulator</u>:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & R & -R \\ -R & 0 & R \\ R & -R & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{24}$$

2. <u>Block cyclic symmetry</u>. Consider the block-cyclic permutation:

$$\pi = \left(1\ \ k{+}1\ \ 2k{+}1\ \ \ldots\ \ (n{-}1)k{+}1\right)\left(2\ \ k{+}2\ \ 2k{+}2\ \ \ldots\ \ (n{-}1)k{+}2\right)\ \ldots\ \left(k\ \ 2k\ \ 3k\ \ \ldots\ \ nk\right)$$

It is easy to show that the hybrid matrix in this case assumes the same form as (23), except that each A_i in (23) is replaced by a square kxk submatrix $\underset{\sim}{A}_i$ [5].

3. <u>Involution symmetry</u>. Since every involution permutation can be decomposed into a collection of cyclic components of order 1 or 2, the general structure of a hybrid matrix which exhibits involution symmetry can be derived by decomposing $\underset{\sim}{H}$ into various submatrices corresponding to the cyclic components, and then analyzing the general structures of these submatrices. The following is a list of all distinct combinations between two cyclic directed permutations of either order 1 or 2. The structure of $\underset{\sim}{H}$ which exhibits the corresponding symmetry is shown on the right-hand side, where the entries A,B,C,D,E,F,G,K are fixed parameters.

<u>cyclic components</u> <u>hybrid representation</u>

$$\pi_1 = (i\ j)(k\ \ell) \qquad \underset{\sim}{H}_1 = \begin{bmatrix} h_{ii} & h_{ij} & h_{ik} & h_{i\ell} \\ h_{ji} & h_{jj} & h_{jk} & h_{j\ell} \\ h_{ki} & h_{kj} & h_{kk} & h_{k\ell} \\ h_{\ell i} & h_{\ell j} & h_{\ell k} & h_{\ell\ell} \end{bmatrix} = \begin{bmatrix} A & B & G & K \\ B & A & K & G \\ E & F & C & D \\ F & E & D & C \end{bmatrix} \tag{25a}$$

cyclic components		hybrid representation	

$$\pi_2 = (i\ j)(k) \qquad \underset{\sim}{H}_2 = \begin{bmatrix} h_{ii} & h_{ij} & h_{ik} \\ h_{ji} & h_{jj} & h_{jk} \\ h_{ki} & h_{kj} & h_{kk} \end{bmatrix} = \begin{bmatrix} A & B & E \\ B & A & E \\ D & D & C \end{bmatrix} \tag{25b}$$

$$\pi_3 = (i\ j)(k\ \bar{k}) \qquad \underset{\sim}{H}_3 = \begin{bmatrix} h_{ii} & h_{ij} & h_{ik} \\ h_{ji} & h_{jj} & h_{jk} \\ h_{ki} & h_{kj} & h_{kk} \end{bmatrix} = \begin{bmatrix} A & B & E \\ B & A & -E \\ D & -D & C \end{bmatrix} \tag{25c}$$

$$\pi_4 = (i)(j) \qquad \underset{\sim}{H}_4 = \begin{bmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{bmatrix} = \begin{bmatrix} A & D \\ C & B \end{bmatrix} \tag{25d}$$

$$\pi_5 = (i)(j\ \bar{j}) \qquad \underset{\sim}{H}_5 = \begin{bmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{25e}$$

$$\pi_6 = (i\ \bar{i})(j\ \bar{j}) \qquad \underset{\sim}{H}_6 = \begin{bmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{bmatrix} = \begin{bmatrix} A & D \\ C & B \end{bmatrix} \tag{25f}$$

To verify these structures all we have to do is write down the permutation matrix $\underset{\sim}{P}(\pi_j)$ and verify that (22) is satisfied. For example, consider $\pi_1 = (i\ j)(k\ \ell)$. This corresponds to a 4-port resistor which is symmetrical with respect to an interchange of ports i and j and a simultaneous interchange of ports k and ℓ. Observe that (22) is satisfied; namely,

$$\left[\begin{array}{cc|cc} A & B & G & K \\ B & A & K & G \\ \hline E & F & C & D \\ F & E & D & C \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|cc} A & B & G & K \\ B & A & K & G \\ \hline E & F & C & D \\ F & E & D & C \end{array}\right] \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The remaining five cases can be similarly verified.

We can now use the preceding building blocks to derive the general structure of any hybrid matrix $\underset{\sim}{H}$ which exhibits involution symmetry. For example, consider

$\pi = (1\ 2)(3)(4\ 5)(6\ \bar{6})$. Clearly the two cyclic components (1 2)(3) correspond to $\pi_2 = (i\ j)(k)$ and hence the submatrix structure must be as specified in (25b). The two cyclic components $(4\ 5)(6\ \bar{6})$ correspond to $\pi_3 = (i\ j)(k\ \bar{k})$ and hence the submatrix structure must be specified by (25c). On the other hand if we pick the two cycles (1 2)(4 5), then π_1 applies and the corresponding columns and rows of $\underset{\sim}{H}$ must have the structural form specified by (25a). Combining all these observations, we obtain the following structural form for $\underset{\sim}{H}$:

$$\underset{\sim}{H} = \left[\begin{array}{cc|c|cc|c} A & B & V & Q & R & M \\ B & A & V & R & Q & -M \\ \hline X & X & C & S & S & 0 \\ \hline T & U & Z & D & E & L \\ U & T & Z & E & D & -L \\ \hline P & -P & 0 & N & -N & F \end{array}\right] \tag{26}$$

where "0" is a zero.

4. Additional examples of π-permuted symmetry

Example 1. The gyrator $(i_1 = Gv_2, i_2 = -Gv_1)$ exhibits π-symmetry, where $\pi = \begin{pmatrix} 1 & 2 \\ \bar{2} & 1 \end{pmatrix} = (1\ \bar{2}\ \bar{1}\ 2)$. This is easily verified as follows:

$$\begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{27}$$

Observe that this type of symmetry is distinct from the other symmetries already presented.

Example 2. The hybrid coil (Fig. 7) (used in the telephone system) is described by the following hybrid matrix:

$$\begin{bmatrix} i_1 \\ i_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix} \tag{28}$$

It is easy to verify that (22) is satisfied with $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \bar{1} & \bar{4} & 3 \end{pmatrix}$. Hence, this 4-port is π-symmetric.

Example 3. All linear multiports which can be described by a constitutive relation

$$\underset{\sim}{C}\underset{\sim}{v} = \underset{\sim}{D}\underset{\sim}{i} \tag{29}$$

are complementary symmetric. This follows from the choice of $\pi = (1\ \bar{1})(2\ \bar{2})\ldots(n\ \bar{n})$ and the observation that $\underset{\sim}{C}(-\underset{\sim}{v}) = \underset{\sim}{D}(-\underset{\sim}{i})$. As a special case of this general property, we conclude that all ideal n-port transformers, n-winding transformers, n-port circulators, negative impedance converters, negative impedance inverters, gyrators, etc., exhibit complementary symmetry.

The preceding examples show that a multiport resistor may exhibit more than one type of π-permuted symmetries. In the next theorem we prove that all such directed permutations form a group. This result is analogous to the well-known property that all symmetry operations of an object form a group [13].

Proposition 15. The collection S_R of all directed permutations with respect to which a multiport resistor R is symmetric, form a group under the composition operation.

Proof: The collection S_R forms a subset of the group P_n of all $2^n n!$ directed permutations on n objects (Prop. 4). It contains the identity. If $\pi_1 \in S_R$ and $\pi_2 \in S_R$, then so is $\pi_1 \circ \pi_2 \in S_R$, because if $(\underset{\sim}{v},\underset{\sim}{i})$ is an admissible pair, then $\left(\underset{\sim}{P}(\pi_2)\underset{\sim}{v},\underset{\sim}{P}(\pi_2)\underset{\sim}{i}\right)$ is also an admissible pair. Repeating this argument it follows that $\left(\underset{\sim}{P}(\pi_1)\underset{\sim}{P}(\pi_2)\underset{\sim}{v},\underset{\sim}{P}(\pi_1)\underset{\sim}{P}(\pi_2)\underset{\sim}{i}\right) = \left(\underset{\sim}{P}(\pi_1\circ\pi_2)\underset{\sim}{v},\underset{\sim}{P}(\pi_1\circ\pi_2)\underset{\sim}{i}\right)$, is also an admissible pair of R. ◻

Corollary. (a) If a multiport resistor is both π-symmetric and σ-symmetric, then it is also $\pi\circ\sigma$-symmetric. In particular, it is π^ℓ-symmetric, where ℓ is any integer. (b) A multiport is π-symmetric iff it is π^{-1}-symmetric.

Proof: (a) Since $\pi \in S_R$ and $\sigma \in S_R$, also $\pi\circ\sigma$ belongs to this group S_R (closure property). If we let $\sigma = \pi$, then $\pi^2 \in S_R$. Repeating this process, we obtain $\pi^\ell \in S_R$, where ℓ is any integer. (b) Since every element of the group S_R has a finite order k, we have $\pi^k = I$. It follows from $\pi^{k-1} \circ \pi = I$ and the uniqueness of π^{-1} that $\pi^{-1} = \pi^{k-1}$. Hence $\pi^{-1} \in S_R$. Conversely, if $\pi^{-1} \in S_R$, then $(\pi^{-1})^{-1} = \pi \in S_R$. ◻

This corollary indicates that not all $2^n n!$ directed permutations have to be checked exhaustively in order to generate the symmetry group of an n-port resistor. In fact, once a symmetry transformation π is found, many others may be found from π using this corollary. Conversely, when the multiport resistor is not symmetric with respect to a directed permutation π, then many others may be found with respect to which it is also not symmetric. Based on these observations an algorithm is described in Appendix B for generating all symmetry transformations of a multiport resistor using a minimum number of checks.

B. Synthesis of π-symmetric multiport resistors

As will be clear in Section V, there are many communication and power electronic circuits, which use symmetrical nonlinear multiports to achieve various desirable results. Since few intrinsic multiports exhibit symmetry, most symmetrical multiports used in practice are synthesized by a clever interconnection of unsymmetrical components, often arrived at by ad hoc methods. Our objective in this section is to present a unified approach for synthesizing multiport resistors having any prescribed π-symmetry. We will find the concept and properties of π-permuted multiports, presented earlier, to be crucial in this undertaking. Our basic procedure consists of choosing two or more multiports, then permuting their port numbers in an appropriate way, and possibly transposing the terminals of some of the ports (complementation) and finally interconnecting various sets of ports either in series or in parallel with each other. We assume throughout this paper that isolation transformers have been inserted and embedded within the component multiports, whenever necessary to preserve the port constraint property. For example, Fig. 8 shows two 2-port resistors $R^{(1)}$ and $R^{(2)}$ with their first ports connected in series and their second ports connected in parallel. Any isolation transformer is assumed to be embedded already within $R^{(1)}$ and $R^{(2)}$.

Two general synthesis algorithms will be presented for realizing any prescribed $\tilde{\pi}$-symmetric n-port resistor $\tilde{R}$. The algorithms differ from each other in the choice of the building blocks: Algorithm 1 uses two or more identical but unsymmetrical m-port resistors (where $m \leq n$)[6] as building blocks, whereas Algorithm 2 uses one π-symmetric m-port resistor R or two distinct multiport resistors $R^{(1)}$ and $R^{(2)}$, where $R^{(1)}$ has $m^{(1)}$-ports and exhibits $\pi^{(1)}$-symmetry, while $R^{(2)}$ has $m^{(2)}$-ports and exhibits $\pi^{(2)}$-symmetry, where $n \leq m$ or $n \leq m^{(1)} + m^{(2)}$. To demonstrate the generality and utility of these two algorithms, several well-

[6] In Algorithm 1 the number "m" of ports of all building blocks are assumed to be less than or equal to n for the sake of generality. One could alternately assume m = n by introducing trivial "short circuited ports" whenever a series connection is called for, and "open circuited ports" whenever a parallel connection is called for. This artifice is useful from a pedagogical point of view since it is more systematic. However, it involves drawing many unnecessary ports and interconnections, which have to be removed later to obtain the final multiport resistor $\tilde{R}$. Hence at the risk of some loss of clarity in the presentation, we will adopt the more elegant and concise choice of $m \leq n$.

known symmetrical n-ports of practical interest will be systematically synthesized.

Synthesis algorithm 1. Given any directed permutation $\tilde{\pi}$ on n objects, the following procedure will generate a family of n-port resistors having the prescribed $\tilde{\pi}$-symmetry:

Step 1. Decompose $\tilde{\pi}$ into cyclic components (see Theorem 1)

$$\tilde{\pi} = (i_1^{(1)} i_2^{(1)} \ldots i_{\ell_1}^{(1)}) \ldots (i_1^{(k)} i_2^{(k)} \ldots i_{\ell_k}^{(k)}) \ldots (i_1^{(p)} i_2^{(p)} \ldots i_{\ell_p}^{(p)}).$$

Step 2. Determine the order "ℓ" of $\tilde{\pi}$. Recall that $\tilde{\pi}^{\ell} = I$ and that "ℓ" is the least common multiple of the orders of the "p" cyclic components from Step 1. (see Cor. to Thm. 1).

Step 3. Choose "ℓ" identical m-port resistors[7] R ($m \leq n$) and label them $R^{(1)}, R^{(2)} \ldots R^{(\ell)}$. Let $\tilde{\pi}^j(R^{(k)})$ denote the $\tilde{\pi}^j$-permuted m-port resistor associated with $R^{(k)}$, where $\tilde{\pi}^j = \tilde{\pi} \circ \tilde{\pi} \circ \ldots \circ \tilde{\pi}$ (j-times). If $m < n$, there will be more objects to be permuted than there are ports, and $\tilde{\pi}(i)$ is not well-defined if $\tilde{\pi}(i) > m$, or if $\tilde{\pi}(\bar{i}) > m$. Therefore we will henceforth define:

$$\tilde{\pi}^j(i) = \phi, \text{ if } \tilde{\pi}^j(i) > m \text{ or if } \tilde{\pi}^j(\bar{i}) > m,$$

where ϕ denotes the empty set. We will also define the series or parallel connection between port "i" of one m-port resistor $R^{(j)}$ and port "ϕ" of an m-port resistor $\tilde{\pi}^t(R^{(k)})$ to be the original port "i".[8]

Step 4. The $\tilde{\pi}$-symmetric n-port resistor $\tilde{R}$ associated with R is realized by connecting the corresponding ports of the "ℓ" $\tilde{\pi}^j$-permuted m-port resistors $\tilde{\pi}^0(R^{(1)}) = R^{(1)}, \tilde{\pi}^{-1}(R^{(2)}), \ldots, \tilde{\pi}^{-\ell+1}(R^{(\ell)})$ in series, or in parallel with each

---

[7]The m-port resistor R is arbitrary and need not exhibit any form of symmetry. Corresponding to each chosen R, Algorithm 1 generates an associated $\tilde{\pi}$-symmetric n-port resistor $\tilde{R}$. Hence the family of $\tilde{\pi}$-symmetric n-port resistors generated by Algorithm 1 is very large indeed.

[8]This is just a formal way of saying that the series (resp., parallel) connection between port i of $R^{(j)}$ and a short-circuited port (resp., open-circuited port) of $\tilde{\pi}^t(R^{(k)})$ is just port i itself.

other.[9] The choice of series or parallel connection is arbitrary provided that all ports of $\tilde{R}$ which belong to the same cyclic component of π from Step 1 have the same method of connection. For example, if $\tilde{\pi} = (1\ 5\ 3)(4\ 2)$, then ports 1, 3 and 5 of $\tilde{R}$ must be formed by the same method of connection, say all parallel connection. Likewise, ports 2 and 4 must be formed by the same connection method, say all be series connection.

Remark. Step 4 can be replaced by the following more explicit equivalent statements:

Equivalent statement 1. The i-th port of $\tilde{R}$ is obtained by connecting the i-th port of $R^{(1)}$, the $\tilde{\pi}(i)$-th port of $R^{(2)}$... and the $\tilde{\pi}^{\ell-1}(i)$-th port of $R^{(\ell)}$ in series, or in parallel with each other, provided the choice of connection is the same for all ports of $\tilde{R}$ which belong to the same cyclic component of $\tilde{\pi}$.

Equivalent statement 2. Referring to the cyclic decomposition of $\tilde{\pi}$ from Step 1, realize the ports of $\tilde{R}$ in the order listed in the cyclic decomposition as follows: For each cyclic component $k = 1,2,\ldots,p$, choose either series or parallel as the connection method:

1) Port $i_1^{(k)}$ of $\tilde{R}$ is realized by connecting port $i_1^{(k)}$ of $R^{(1)}$, port $i_2^{(k)}$ of $R^{(2)}$,... and port $i_{\ell_k}^{(k)}$ of $R^{(\ell)}$ together.

2) Port $i_2^{(k)}$ of $\tilde{R}$ is realized by connecting port $i_2^{(k)}$ of $R^{(1)}$, port $i_3^{(k)}$ of $R^{(2)}$,... and port $i_{\ell_k}^{(k)}$ of $R^{(\ell-1)}$ and port $i_1^{(k)}$ of $R^{(\ell)}$ together.

q)[10] Port $i_{\ell_k}^{(k)}$ of $\tilde{R}$ is realized by connecting port $i_{\ell_k}^{(k)}$ of $R^{(1)}$, port $i_1^{(k)}$ of $R^{(2)}$,..., and port $i_{\ell_k - 1}^{(k)}$ of $R^{(\ell)}$ together. Again if $m < n$, a port $i_j^{(k)}$ of R is assumed to be ϕ whenever $i_j^{(k)} > m$, or $\bar{i}_j^{(k)} > m$, where ϕ is the empty set.

Before we prove that the n-port resistor $\tilde{R}$ realized by synthesis algorithm 1 is indeed $\tilde{\pi}$-symmetric, let us pause to consider some examples.

---

[9] Observe that alternately, we can interconnect corresponding ports of the "ℓ" $\tilde{\pi}^{(j)}$-permuted m-port resistors $\tilde{\pi}^0(R^{(1)}) = R^{(1)}$, $\tilde{\pi}^1(R^{(2)}),\ldots,\tilde{\pi}^{\ell-1}(R^{(\ell)})$ in series, or in parallel, with each other to obtain a $\tilde{\pi}^{-1}$-symmetric m-port resistor $\tilde{R}'$. However, in view of the corollary of Prop. 15, $\tilde{R}'$ is also $\tilde{\pi}$-symmetric. Moreover, by permuting the resistor numbers of the identical resistors $R^{(1)}\ldots R^{(\ell)}$, it is easy to see that both interconnection schemes in fact gave rise to the same $\tilde{\pi}$-symmetric m-port resistor $\tilde{R}$.

[10] If the k-th cyclic component is a normal-order cycle, then $q = \ell_k$. If the k-th cyclic component is a double-order cycle, then $q = \ell_k/2$.

Example 1. Synthesize a bilateral one-port resistor: $\tilde{\pi} = \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix} = (1\ \bar{1})$. In this case, $\tilde{R}$ is obtained by connecting two identical two terminal resistors $R^{(1)}$ and $R^{(2)}$ "back-to-back" in series or "back-to-front" in parallel. The resulting two-terminal resistor $\tilde{R}$ is bilateral and we obtain Prop. 1 as a trivial application of Algorithm 1.

Example 2. Synthesize a complementary symmetric 2-port resistor: $\tilde{\pi} = \begin{pmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{pmatrix} = (1\ \bar{1})(2\ \bar{2})$. Since ports 1 and 2 belong to distinct cyclic components, Algorithm 1 may be used to generate four distinct 2-port resistors having complementary symmetry; namely, (S_1,S_2), (S_1,P_2), (P_1,S_2), and (P_1,P_2), where S_i (resp., P_i) denote that port i of $R^{(1)}$ is connected in series (resp., in parallel) with the transposed (complemented) port i of $R^{(2)}$. For example, the 2-port $\tilde{R}$ corresponding to the (P_1,P_2) connection is shown in Fig. 9(a). In particular if we choose R to be the "biased" one-transistor 2-port resistor shown in Fig. 9(b), we would obtain the complementary symmetric 2-transistor 2-port $\tilde{R}$ shown in Fig. 9(c), where isolation transformers have been inserted to guarantee that the port constraint property holds. Applying equivalent network transformation techniques to $\tilde{R}$, we obtain the two well-known push-pull transistor amplifier circuits [16] shown in Figs. 9(d) and 9(e), respectively. Hence, we have demonstrated how well-known circuit configurations previously obtained by intuitive or other ad hoc techniques can be systematically generated via Algorithm 1.

Example 3. Synthesize a $\tilde{\pi} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$-symmetric 3-port resistor: $\tilde{\pi} = (1\ 3)(2)$. Since the order ℓ of $\tilde{\pi}$ is equal to 2, we only need two identical m-port resistors $R^{(1)} = R^{(2)} = R$, $m \leq 3$. Let us choose $m = 2$ to illustrate our earlier remark about "empty ports." Suppose we choose "series" connection for the ports belonging to the cyclic component (1 3) and choose "parallel" connection for cyclic component (2). Then port 1 of 3-port $\tilde{R}$ is obtained by connecting port 1 of $R^{(1)}$ in series with port $\tilde{\pi}(1)$ of $R^{(2)}$. But $\tilde{\pi}(1) = 3 > 2$, hence $\tilde{\pi}(1) \stackrel{\Delta}{=} \phi$ and port 1 of $\tilde{R}$ is simply port 1 of $R^{(1)}$. Port 2 of $\tilde{R}$ is obtained by connecting port 2 of $R^{(1)}$ in parallel with port $\tilde{\pi}(2) = 2$ of $R^{(2)}$. Finally, port 3 of $\tilde{R}$ is obtained by connecting the empty port of $R^{(1)}$ in series with port $\tilde{\pi}(3) = 1$ of $R^{(2)}$, which is just port 1 of $R^{(2)}$. The resulting $\tilde{\pi}$-symmetric 3-port $\tilde{R}$ is shown in Fig. 10(a).

Example 4. Synthesize a $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$-symmetric 3-port resistor: $\tilde{\pi} = (1\ 2\ 3)$. Since the order ℓ is equal to 3, we need three identical 3-port resistors $R^{(1)} = R^{(2)} = R^{(3)} = R$, $m \leq 3$. Again let us choose $m = 2$. Since $\tilde{\pi}$ has only one cyclic component, the same method of connection must be chosen for all three ports of $\tilde{R}$. If we choose the series method, we obtain the $\tilde{\pi}$-symmetric 3-port $\tilde{R}$ shown in Fig. 10(b). Observe that port 1 of $\tilde{R}$ is obtained by connecting port 1 of $R^{(1)}$ in series with port $\tilde{\pi}(1) = 2$ of $R^{(2)}$ and port $\tilde{\pi}^2(1) = \phi$ of $R^{(3)}$. Port 2 is obtained by connecting port 2 of $R^{(1)}$ in series with port $\tilde{\pi}(2) = \phi$ of $R^{(2)}$ and port $\tilde{\pi}^2(2) = 1$ of $R^{(3)}$. Finally, port 3 of $\tilde{R}$ is obtained by connecting the empty port of $R^{(1)}$ in series with port $\tilde{\pi}(3) = 1$ of $R^{(2)}$ and port $\tilde{\pi}^2(3) = 2$ of $R^{(3)}$.

Proof of Algorithm 1. To describe the interconnection algebraically we specify how the ports of $R^{(k)}$ are connected relative to $\tilde{R}$ by an $n \times m$ connection matrix $\underset{\sim}{C}^{(k)}$ as follows:

$$
\begin{aligned}
c_{ij}^{(k)} &= 1, \text{ if port } i \text{ of } \tilde{R} \text{ contains port } j \text{ of } R^{(k)} \text{ in the interconnection,} \\
&= -1, \text{ if port } i \text{ of } \tilde{R} \text{ contains the transposed (complemented) port } j \text{ of } R^{(k)}, \\
&= 0, \text{ otherwise.}
\end{aligned}
$$

It follows from Step 4 that

$$
\underset{\sim}{C}^{(1)} = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\ \cdot \\0\end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c}0\\ \cdot \\0\end{array}}\right\} n-m \end{array} \triangleq \underset{\sim}{C},\ \underset{\sim}{C}^{(2)} = \underset{\sim}{P}^T(\tilde{\pi})\underset{\sim}{C},\ \underset{\sim}{C}^{(3)} = \underset{\sim}{P}^T(\tilde{\pi}^2)\underset{\sim}{C}, \ldots,\ \underset{\sim}{C}^{(\ell)} = \underset{\sim}{P}^T(\tilde{\pi}^{\ell-1})\underset{\sim}{C} \tag{30}
$$

Next, we let $\underset{\sim}{x}$, $\underset{\sim}{y}$ denote hybrid n-vectors associated with $\tilde{R}$ and we let $\underset{\sim}{x}^{(k)}$, $\underset{\sim}{y}^{(k)}$ denote the corresponding hybrid m-vectors associated with $R^{(k)}$, $k = 1,2,\ldots,\ell$, such that if port i of $\tilde{R}$ is obtained by a series (resp., parallel) connection, then the i-th component of $\underset{\sim}{x}$ and the j-th component of $\underset{\sim}{x}^{(k)}$ represent voltages (resp., currents) whenever $c_{ij} \neq 0$. Let the corresponding elements in $\underset{\sim}{y}$ and $\underset{\sim}{y}^{(k)}$ denote the opposite variables; namely currents (resp., voltages). Since, by assumption, all ports of $\tilde{R}$ belonging to the same cyclic component are obtained by the same method of connection -- series or parallel --, then corresponding components of $(\underset{\sim}{P}(\tilde{\pi})\underset{\sim}{x},\underset{\sim}{P}(\tilde{\pi})\underset{\sim}{y})$ and $(\underset{\sim}{x},\underset{\sim}{y})$ are of the same type. In terms of the above notations the port variables of $\tilde{R}$ and $R^{(k)}$ are related as follows:

$$\underset{\sim}{x} = \underset{\sim}{C}^{(1)}\underset{\sim}{x}^{(1)} + \underset{\sim}{C}^{(2)}\underset{\sim}{x}^{(2)} + \ldots + \underset{\sim}{C}^{(\ell)}\underset{\sim}{x}^{(\ell)} \tag{31a}$$

$$= \underset{\sim}{C}\underset{\sim}{x}^{(1)} + \underset{\sim}{P}^T(\tilde{\pi})\underset{\sim}{C}\underset{\sim}{x}^{(2)} + \ldots + \underset{\sim}{P}^T(\tilde{\pi}^{\ell-1})\underset{\sim}{C}\underset{\sim}{x}^{(\ell)},$$

$$\underset{\sim}{y}^{(1)} = \underset{\sim}{C}^T\underset{\sim}{y},\ \underset{\sim}{y}^{(2)} = \underset{\sim}{C}^T\underset{\sim}{P}(\tilde{\pi})\underset{\sim}{y},\ \ldots,\ \underset{\sim}{y}^{(\ell)} = \underset{\sim}{C}^T\underset{\sim}{P}(\tilde{\pi}^{\ell-1})\underset{\sim}{y} \tag{31b}$$

If we let $(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}})$ denote the port variables associated with the $\tilde{\pi}$-permuted n-port resistor $\hat{\tilde{R}}$ of $\tilde{R}$, then for each admissible pair $(\underset{\sim}{x},\underset{\sim}{y})$ of $\tilde{R}$ we have $(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}}) = (\underset{\sim}{P}(\tilde{\pi})\underset{\sim}{x},\underset{\sim}{P}(\tilde{\pi})\underset{\sim}{y})$. It follows from (31a) and (31b) that

$$\hat{\underset{\sim}{x}} = \underset{\sim}{P}(\tilde{\pi})\underset{\sim}{C}\underset{\sim}{x}^{(1)} + \underset{\sim}{C}\underset{\sim}{x}^{(2)} + \ldots + \underset{\sim}{P}^T(\tilde{\pi}^{\ell-2})\underset{\sim}{C}\underset{\sim}{x}^{(\ell)} \tag{32a}$$

$$\underset{\sim}{y}^{(1)} = \underset{\sim}{C}^T\underset{\sim}{P}^T(\tilde{\pi})\hat{\underset{\sim}{y}},\ \underset{\sim}{y}^{(2)} = \underset{\sim}{C}^T\hat{\underset{\sim}{y}},\ \ldots,\ \underset{\sim}{y}^{(k)} = \underset{\sim}{C}^T\underset{\sim}{P}(\tilde{\pi}^{\ell-2})\hat{\underset{\sim}{y}} \tag{32b}$$

But since the <u>order</u> of $\tilde{\pi}$ is equal to ℓ, and since $\underset{\sim}{P}(\tilde{\pi})$ is orthogonal, we have $\underset{\sim}{P}(\tilde{\pi}) = \underset{\sim}{P}^T(\tilde{\pi}^{\ell-1})$ and the right-hand sides of (31a) and (32a) (resp., left-hand sides of (31b) and (32b)) are identical (recall that the "ℓ" m-port resistors are identical and hence the labels can be interchanged). Hence $\hat{\tilde{R}}$ and $\tilde{R}$ are identical and $\tilde{R}$ is $\tilde{\pi}$-symmetric (Def. 7). ◻

A multiport resistor may exhibit some π-symmetry as an intrinsic property. If not, a suitable interconnection of the ports of two or more multiport resistors in accordance with Algorithm 1 will give us a new multiport resistor $\tilde{R}$ which exhibits a $\tilde{\pi}$-symmetry. In Algorithm 2, several additional techniques for interconnecting the ports of a π-symmetric m-port to obtain a new n-port which exhibits a different form of symmetry will be presented. For the sake of increased generality, we will first present a simple artifice for combining several distinct but <u>symmetric</u> multiport resistors as one overall multiport resistor: Given a $\pi^{(1)}$-symmetric $m^{(1)}$-port resistor $R^{(1)}$ and a $\pi^{(2)}$-symmetric $m^{(2)}$-port resistor $R^{(2)}$, then the <u>composite</u> m-port resistor R with $m = m^{(1)} + m^{(2)}$ is obtained by retaining the port numbers $1,2,\ldots,m^{(1)}$ of $R^{(1)}$, and by relabelling the port numbers of $R^{(2)}$ by $m^{(1)} + 1$, $m^{(1)} + 2$, ..., $m^{(1)} + m^{(2)} = m$. Clearly, R exhibits a π-symmetry where π acts on $1,2,\ldots,m^{(1)}$ as $\pi^{(1)}$ does, and where π acts on $m^{(1)} + 1$, $m^{(1)} + 2$, ..., m as $\pi^{(2)}$ does on $1,2,\ldots,m^{(2)}$. For example, if $\pi^{(1)} = (1\ 2)$ and $\pi^{(2)} = (1\ 3\ 2)$, then the composite resistor R is a 5-port whose first two ports are identical to those of $R^{(1)}$, and the remaining 3-ports are identical to those of $R^{(2)}$ (after relabelling port numbers 1,2,3 of $R^{(2)}$ by 3,4,5). Clearly, R is π-symmetric, where $\pi = (1\ 2)(3\ 5\ 4)$.

Synthesis algorithm 2. Given a π-symmetric m-port resistor R, then the following port interconnection techniques allow us to generate a large variety of $\tilde{\pi}$-symmetric n-port resistors $\tilde{R}$ with $n \leq m$. Let the cyclic decomposition of π be given by $\pi = c_1 c_2 \ldots$

(A) Direct interconnection of ports belonging to a cyclic component: Let $c_j = (i_1\ i_2\ \ldots\ i_s\ \ldots\ i_\ell)$ be a cyclic component of π of normal order. Let k be any divisor of ℓ i.e. $\ell = ks$. Using consistently a series or parallel interconnection method, form a new n-port $\tilde{R}$ ($n=m-\ell+s$) as follows: Port i_1 of $\tilde{R}$ is realized by interconnecting ports i_1, i_{s+1}, ..., and $i_{(k-1)s+1}$ of R together, where each "complemented" port is transposed before the connection is made. Port i_2 of $\tilde{R}$ is realized by interconnecting ports i_2, i_{s+2}, ..., and $i_{(k-1)s+2}$ of R together. Iterate this procedure until port i_s of $\tilde{R}$ is realized, by interconnecting ports i_s, i_{2s}, ..., i_{ks} of R together. Any other port i of R belonging to the other cycles of π remains unchanged and is simply labelled as port i of $\tilde{R}$. The new n-port resistor $\tilde{R}$ so realized has $n = m - \ell + s$ ports and exhibits π-symmetry, where $\tilde{\pi} = c_1 c_2 \ldots c_{j-1} \tilde{c}_j c_{j+1} \ldots$ and $\tilde{c}_j = (i_1 i_2 \ldots i_s)$ is the new normal-order cycle.

(B) Alternately-transposed interconnection of ports belonging to a cyclic component: Let c_j be a cyclic component of π satisfying one of the following two properties: (1) $c_j = (i_1 i_2 \ldots i_s \ldots i_\ell)$ is a normal-order cyclic component with $\ell = ks$, where k is an even integer. (2) $c_j = (i_1 i_2 \ldots i_s \ldots i_\ell \bar{i}_1 \bar{i}_2 \ldots \bar{i}_s \ldots \bar{i}_\ell)$ is a double-order cyclic component with $\ell = ks$, where k is an odd integer. Using consistently a series or parallel interconnection method, form a new n-port $\tilde{R}$ ($n=m-\ell+s$) as follows: Port i_1 of $\tilde{R}$ is realized by interconnecting ports i_1, $\bar{i}_{s+1}$, i_{2s+1}, $\bar{i}_{3s+1}$, ..., $\bar{i}_{(k-1)s+1}$ (resp., $i_{(k-1)s+1}$ if k is odd) of R together.[11] Port i_2 of $\tilde{R}$ is realized by interconnecting ports i_2, $\bar{i}_{s+2}$, i_{2s+2}, $\bar{i}_{3s+2}$, ..., $\bar{i}_{(k-1)s+2}$ (resp., $i_{(k-1)s+2}$ if k is odd) of R together. Iterate this procedure until port i_s of $\tilde{R}$ is realized by interconnecting ports i_s, $\bar{i}_{2s}$, ..., $\bar{i}_{ks}$ (resp., i_{ks} if k is odd) of R together. The remaining $m-\ell$ ports of R remain unchanged and are simply assigned as ports of $\tilde{R}$. The new n-port resistor $\tilde{R}$ so realized has $n = m - \ell + s$ ports and exhibits $\tilde{\pi}$-symmetry, where $\tilde{\pi} = c_1 c_2 \ldots c_{j-1} \tilde{c}_j c_{j+1} \ldots$ and $\tilde{c}_j = (i_1 i_2, \ldots i_s \bar{i}_1 \bar{i}_2 \ldots \bar{i}_s)$. Hence, the

[11] The only difference from the preceding technique (A) is that every other port index i_k belonging to c_j is complemented before the connection is made.

resulting cyclic component $\tilde{c}_j$ is a double-order cycle.

(C) Interconnection of ports belonging to <u>two compatible cyclic components</u>: Let c_i and c_j be any two cyclic components of π which are compatible in the sense that they have either both normal or both double order, and that they have the same order; i.e., $c_i = (i_1 i_2 \cdots i_\ell)$ and $c_j = (j_1 j_2 \cdots j_\ell)$, or $c_i = (i_1 i_2 \cdots i_\ell \bar{i}_1 \bar{i}_2 \cdots \bar{i}_\ell)$ and $c_j = (j_1 j_2 \cdots j_\ell \bar{j}_1 \bar{j}_2 \cdots \bar{j}_\ell)$. Using consistently a series or parallel interconnection method, form a new n-port $\tilde{R}$ as follows: Port i_1 of $\tilde{R}$ is realized by interconnecting ports i_1 and j_1 of R together. Port 2 of $\tilde{R}$ is realized by interconnecting ports i_2 and j_2 of R together. Iterate this procedure ℓ times until port i_ℓ of $\tilde{R}$ is realized by interconnecting ports i_ℓ and j_ℓ of R. The remaining $m-2\ell$ ports of R remain unchanged and are simply assigned as ports of $\tilde{R}$. The new n-port resistor $\tilde{R}$ so realized has $n = m - \ell$ ports and is $\tilde{\pi}$-symmetric, where $\tilde{\pi} = c_1 c_2 \cdots c_{i-1} c_i c_{i+1} \cdots c_{j-1} c_{j+1} \cdots$.

A cyclic component may have several <u>distinct</u> cyclic notations, for example, $(i_1 i_2, \ldots, i_\ell)$, $(i_2 i_3, \ldots, i_\ell i_1)$, $(\bar{i}_1 \bar{i}_2, \ldots \bar{i}_\ell)$ all describe the same cyclic component. In general, a total of 2ℓ distinct but equivalent cyclic notationsmay be generated by simply shifting the indices consecutively, and by taking their complements. Applying Technique (C) to each such equivalent cyclic notation would result in a distinct $\tilde{\pi}$-symmetric n-port. Hence, Technique (C) alone would allow us to generate a large variety of symmetric n-ports. In general, all three techniques may be combined and an even greater variety of symmetric n-ports can be generated. For example, applying Technique (C) to a <u>composite</u> n-port resistor, we obtain the following useful corollary:

<u>Corollary</u>. Given two distinct π-symmetric n-port resistors $R^{(1)}$ and $R^{(2)}$, a new π-symmetric n-port resistor can be realized by connecting the corresponding ports of $R^{(1)}$ and $R^{(2)}$ in series or in parallel, provided that corresponding ports belonging to the same cyclic component of π are connected in the same way.

The proofs for Techniques (A), (B), and (C) are similar and hence only the proof for Technique (A) will be given. Before we present the proof, it is instructive to consider the following examples.

<u>Example 1</u>. Consider a $\pi^{(1)}$-symmetric 3-port resistor $R^{(1)}$ and a $\pi^{(2)}$-symmetric 3-port resistor $R^{(2)}$, where:

$$\pi^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{3} & \bar{2} \end{pmatrix} = (1\ \bar{1})(2\ \bar{3}), \qquad \pi^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2)(3).$$

The composite 6-port resistor R is π-symmetric, where

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{1} & \bar{3} & \bar{2} & 5 & 4 & 6 \end{pmatrix} = (1\ \bar{1})(2\ \bar{3})(4\ 5)(6).$$ Since the cyclic components $(2\ \bar{3})$ and $(4\ 5)$ have the same order 2 and are both of normal order, we can apply Technique (C): Connect ports 2 and 4 of R in parallel and label it as port 2 of $\tilde{R}$. Next we connect the "transposed" port 3 of R in parallel with port 5 of R and label it as port $\bar{3}$ of $\tilde{R}$. Since the remaining cyclic components $(1\ \bar{1})$ and (6) have different orders, we cannot connect ports 1 and 6. The resulting 4-port resistor $\tilde{R}$, shown in Fig. 11, clearly exhibits $\tilde{\pi}$-symmetry where $\tilde{\pi} = (1\ \bar{1})(2\ \bar{3})(6)$.

Example 2. The 4-port resistor R in Fig. 12(a) exhibits two distinct symmetries, namely,

$$\pi^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{1} & 2 & 4 & 3 \end{pmatrix} = (1\ \bar{1})(2)(3\ 4), \qquad \pi^{(2)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & \bar{2} & \bar{3} & \bar{4} \end{pmatrix} = (1)(2\ \bar{2})(3\ \bar{3})(4\ \bar{4}).$$

Applying Technique (A) to the normal-order cyclic component $(3\ 4)$ of $\pi^{(1)}$, we connect ports 3 and 4 of R in parallel and label the resulting port as port 3 of $\tilde{R}$. Ports 1 and 2 of R remain unchanged and are simply labelled as ports 1 and 2 of $\tilde{R}$, respectively. The resulting 3-port resistor $\tilde{R}$ shown in Fig. 12(b) is $\tilde{\pi}^{(1)}$-symmetric, where $\tilde{\pi}^{(1)} = (1\ \bar{1})(2)(3)$. Let us next check whether the second symmetry $\pi^{(2)}$ is destroyed or preserved by this interconnection. Since ports 3 and 4 belong to different cycles of $\pi^{(2)}$, only Technique (C) can be used. Since both cycles $(3\ \bar{3})$ and $(4\ \bar{4})$ have identical double order, it follows from Technique (C) that $\tilde{R}$ is $\tilde{\pi}^{(2)}$-symmetric with $\tilde{\pi}^{(2)} = (1)(2\ \bar{2})(3\ \bar{3})$.

Example 3. Consider the same 4-port resistor R as in Example 2 (see Fig. 12(a)). Since the cycle $(3\ 4)$ of $\pi^{(1)}$ operates an even number of objects, Technique (B) can be applied. Connect ports 3 and $\bar{4}$ in series and call it port 3 of $\tilde{R}$ as shown in Fig. 12(c) (port $\bar{4}$ is obtained by transposing the two terminals of port 4). Ports 1 and 2 of R remain unchanged and are simply labelled as ports 1 and 2 of $\tilde{R}$. The resulting 3-port resistor $\tilde{R}$ is $\tilde{\pi}^{(1)}$-symmetric, where $\tilde{\pi}^{(1)} = (1\ \bar{1})(2)(3\ \bar{3})$. The effect of this interconnection on $\pi^{(2)}$ can be analyzed by first writing the last cyclic component of $\pi^{(2)}$ in the equivalent cyclic notation $(\bar{4}\ 4)$. It follows then from Technique (C) that $\tilde{R}$ is $\tilde{\pi}^{(2)}$-symmetric, where $\tilde{\pi}^{(2)} = (1)(2\ \bar{2})(3\ \bar{3})$. The 3-port synthesized in this example coincides with the $\tilde{\pi}^{(1)}$ and $\tilde{\pi}^{(2)}$-symmetric network presented by Penfield [12]. Our algorithm demonstrates how this circuit can be systematically synthesized.

Proof of Technique (A) of Algorithm 2. To simplify our algebra, let us relabel the ports of R by introducing the permutation $\rho = \begin{pmatrix} 1 & 2 & \dots & \ell & \dots \\ i_1 & i_2 & \dots & i_\ell & \dots \end{pmatrix}$ to obtain a ρ-permuted resistor $\hat{R}$. Since R is π-symmetric, it follows from Prop. 6 that $\hat{R}$ is $\hat{\pi}$-symmetric, where $\hat{\pi} = \rho^{-1} \circ \pi \circ \rho = \hat{c}_1\hat{c}_2\dots$ and $\hat{c}_1 = (1\ 2\ \dots\ \ell)$ or $\hat{c}_1 = (1\ 2\ \dots\ \ell\ \bar{1}\ \bar{2}\ \dots\ \bar{\ell})$ depending on whether c_j is a normal or a double order cycle. Applying Technique (A) to this $\hat{\pi}$-symmetric resistor $\hat{R}$, we obtain a new n-port $\tilde{\hat{R}}$, where port 1 of $\tilde{\hat{R}}$ is realized by interconnecting ports 1, s+1, ..., (k-1)s+1 of $\hat{R}$ together. Port 2 of $\tilde{\hat{R}}$ is realized by interconnecting ports 2, s+2, ..., (k-1)s+2 of $\hat{R}$ together. Iterating this procedure until port s of $\tilde{\hat{R}}$ is realized by interconnecting ports s, 2s, ..., ks = ℓ of $\hat{R}$ together. All remaining ports $j > \ell$ of $\hat{R}$ remain unchanged and are simply labelled as port j of $\tilde{\hat{R}}$. Observe that ports s+1, s+2, ..., ℓ of $\tilde{\hat{R}}$ are missing after the interconnection and will be treated as empty ports as before. We will prove that the $(m-\ell+s)$-port resistor $\tilde{\hat{R}}$ is $\tilde{\hat{\pi}}$-symmetric, where $\tilde{\hat{\pi}} = \tilde{\hat{c}}_1\hat{c}_2\hat{c}_3\dots$ and $\tilde{\hat{c}}_1 = (1\ 2\ \dots\ s)$ or $\tilde{\hat{c}}_1 = (1\ 2\ \dots\ s\ \bar{1}\ \bar{2}\ \dots\ \bar{s})$ depending on whether c_j is a normal-order or a double-order cycle. Deleting the empty ports, we obtain the following $(m-\ell+s) \times m$ connection matrix $\underset{\sim}{C}$:

$$\underset{\sim}{C} = \left[\begin{array}{c|c|c|c|c} \overbrace{\begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & & \\ 0 & 0 & 0 & \dots & 1 \end{matrix}}^{} & \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{matrix} & \dots & \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{matrix} & 0 \\ \hline & & 0 & & \begin{matrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{matrix} \end{array}\right] \quad \begin{matrix}\text{(first } s \text{ rows; } k\ s{\times}s \text{ blocks)}\end{matrix} \tag{33}$$

By direct computation, it is easy to verify that

$$\underset{\sim}{P}(\tilde{\hat{\pi}})\ \underset{\sim}{C} = \underset{\sim}{C}\ \underset{\sim}{P}(\hat{\pi}), \tag{34}$$

where the rows and columns corresponding to empty ports of $\tilde{\hat{R}}$ have been eliminated in $\underset{\sim}{P}(\tilde{\hat{\pi}})$. Let $(\underset{\sim}{v},\underset{\sim}{i})$ be an admissible pair for $\hat{R}$, and let $(\tilde{\underset{\sim}{v}},\tilde{\underset{\sim}{i}})$ be an admissible pair for $\tilde{R}$, where again the empty ports are omitted in $\tilde{\underset{\sim}{v}}$ and $\tilde{\underset{\sim}{i}}$. Then in terms of the above notations the admissible pairs are related as follows (assuming a series method of interconnection):

$$\tilde{\underset{\sim}{v}} = \underset{\sim}{C}\ \underset{\sim}{v}, \quad \underset{\sim}{i} = \underset{\sim}{C}^T\ \tilde{\underset{\sim}{i}}. \tag{35}$$

In the case of the parallel interconnection the variables $\underset{\sim}{v}$, $\tilde{\underset{\sim}{v}}$ and $\underset{\sim}{i}$, $\tilde{\underset{\sim}{i}}$ are simply interchanged. Now it follows from (35), (34) and the $\hat{\pi}$-symmetry of $\hat{R}$ that $(P(\tilde{\hat{\pi}})\tilde{\underset{\sim}{v}}, P(\tilde{\hat{\pi}})\tilde{\underset{\sim}{i}})$ is also an admissible pair of $\tilde{\hat{R}}$. This proves that $\tilde{\hat{R}}$ is $\tilde{\hat{\pi}}$-symmetric. ◻

IV. π-PERMUTED AND π-SYMMETRIC MULTITERMINAL RESISTOR

An n-terminal resistor $\mathcal{R}$ is an element with n terminals 1 2 ... n, which is completely characterized by a set $\mathcal{S}$ of admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ where $\underset{\sim}{v} = [v_1\ v_2\ \dots\ v_n]^T$ is the vector of the terminal voltages, and $\underset{\sim}{i} = [i_1\ i_2\ \dots\ i_n]^T$ is the vector of the terminal currents, such that for any admissible pair $(\underset{\sim}{v},\underset{\sim}{i})$ of $\mathcal{S}$, the following consistency conditions are satisfied:

1) KCL is satisfied: $\sum_{j=1}^{n} i_j = 0$

2) KVL is satisfied: $([v_1-v_0,\dots,v_n-v_0]^T,\underset{\sim}{i})$ is an admissible pair for any arbitrary reference voltage v_0.

We assume as in the n-port case that the "associated reference convention" is chosen (see Fig. 13(a)). In terms of this convention, condition 1 follows from the fact that the "n" terminals form a cut set, while condition 2 follows from the observation that the datum potential is arbitrary. Consequently, we refer to the collection $\mathcal{S}$ of admissible pairs as an "indefinite representation" (since no one terminal is singled out as datum).

In many practical situations a simpler representation can be derived from a given indefinite representation by choosing one terminal as a datum node (say terminal n)(Fig. 13(b)). In this case, $v_n' = 0$ and the n-terminal resistor $\mathcal{R}$ is completely characterized by the reduced set $\mathcal{S}'$ of admissible pairs $(\underset{\sim}{v}',\underset{\sim}{i}')$ where $\underset{\sim}{v}' = [v_1',v_2',\dots,v_{n-1}']^T$ and $\underset{\sim}{i}' = [i_1',i_2',\dots,i_{n-1}']^T$ are the voltages and the currents at the first n-1 terminals. Since terminal n is now fixed, $\mathcal{S}'$ is called the definite representation.

A multiterminal resistor $\mathcal{R}$ can be transformed from the "indefinite" to the definite representation, and vice versa, by using the following two nx(n-1) matrices:

$$\underset{\sim}{J}_1 = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{J}_2 = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \\ -1 & -1 & \dots & -1 \end{bmatrix} \tag{36}$$

1) The transformation from the definite representation to the indefinite representation is given by:

$$\underset{\sim}{i} = \underset{\sim}{J}_2\underset{\sim}{i}', \quad \underset{\sim}{v} = \underset{\sim}{J}_1\underset{\sim}{v}' \tag{37}$$

2) The transformation from the indefinite representation to the definite representation is given by

$$\underset{\sim}{i}' = \underset{\sim}{J}_1^T\underset{\sim}{i}, \quad \underset{\sim}{v}' = \underset{\sim}{J}_2^T\underset{\sim}{v} \tag{38}$$

Observe that the transformation from the definite (resp., indefinite) representation to the indefinite (resp., definite) representation, and then back again produces the same set $\mathcal{S}'$ (resp., $\mathcal{S}$).

In practice it is useful to list the indefinite (resp., definite) set of admissible pairs by the solutions of a set of algebraic equations $\underset{\sim}{\mathcal{R}}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}$ (resp., $\underset{\sim}{\mathcal{R}}'(\underset{\sim}{v}',\underset{\sim}{i}')=\underset{\sim}{0}$) called the <u>indefinite</u> (resp., <u>definite</u>) <u>constitutive relation</u> of $\mathcal{R}$.

<u>Proposition 16</u>. The transformation from the definite constitutive relation $\mathcal{R}'$ to the indefinite constitutive relation $\mathcal{R}$ is given by:

$$\underset{\sim}{\mathcal{R}}(\underset{\sim}{v},\underset{\sim}{i}) \triangleq \begin{bmatrix} \underset{\sim}{\mathcal{R}}'(\underset{\sim}{J}_2^T\underset{\sim}{v},\underset{\sim}{J}_1^T\underset{\sim}{i}) \\ i_1+i_2+\ldots+i_n \end{bmatrix} = \underset{\sim}{0} \tag{39}$$

The transformation from the indefinite constitutive relation $\mathcal{R}$ to the definite constitutive relation $\mathcal{R}'$ is given by:

$$\underset{\sim}{\mathcal{R}}'(\underset{\sim}{v}',\underset{\sim}{i}') \triangleq \underset{\sim}{\mathcal{R}}(\underset{\sim}{J}_1\underset{\sim}{v}',\underset{\sim}{J}_2\underset{\sim}{i}') = \underset{\sim}{0}. \tag{40}$$

<u>Proof</u>: The proof follows directly from (37) and (38). □

A. <u>Properties of π-permuted and π-symmetric multiterminal resistors</u>

Contrary to many other situations, the indefinite representation is most convenient for studying the symmetry properties of multiterminal resistors, because it treats all terminals alike. Many results of Section III can now be easily rephrased for multiterminal resistors characterized by an indefinite representation. However, not all <u>directed</u> permutations are allowed for n-terminal resistors because consistency condition 1 would clearly be violated if we permute the terminals while changing the sign of some, but not all, terminal currents.

<u>Definition 8</u>. Given an n-terminal resistor $\mathcal{R}$ and a directed permutation π, which <u>complements all n objects, or none</u>, we define the <u>π-permuted n-terminal resistor</u> $\hat{\mathcal{R}}$ by the set of admissible pairs $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}})$ such that $\hat{\underset{\sim}{v}} = \underset{\sim}{P}(\pi)\underset{\sim}{v}$, $\hat{\underset{\sim}{i}} = \underset{\sim}{P}(\pi)\underset{\sim}{i}$,

where $(\underset{\sim}{v},\underset{\sim}{i})$ is any admissible pair of $\mathcal{R}$.

Observe that $\hat{\mathcal{R}}$ is an n-terminal resistor since $\mathcal{R}$ is already an n-terminal resistor and since the two consistency conditions are satisfied for $\hat{\mathcal{R}}$. $\hat{\mathcal{R}}$ can be easily obtained by permuting the terminals of $\mathcal{R}$ in accordance with π, and by using a phase-inverting ideal transformer whenever a terminal is to be complemented. For example, Fig. 14 shows a π-permuted 3-terminal resistor $\hat{\mathcal{R}}$, where $\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{3} & \bar{2} \end{pmatrix}$ is implemented by three phase-inverting ideal transformers.

Proposition 17. Given a multiterminal resistor $\mathcal{R}$ characterized by a definite representation and a directed permutation π, which inverts all terminals or none, then the set of definite admissible pairs of $\hat{\mathcal{R}}$ is given by

$$(\hat{v}',\hat{i}') = \left(\underset{\sim}{J}_2^T \underset{\sim}{P}(\pi)\underset{\sim}{J}_1\underset{\sim}{v}', \underset{\sim}{J}_1^T \underset{\sim}{P}(\pi)\underset{\sim}{J}_2\underset{\sim}{i}'\right), \tag{41}$$

where $(\underset{\sim}{v}',\underset{\sim}{i}')$ is any definite admissible pair of $\mathcal{R}$.

Proof. Let $(\underset{\sim}{v}',\underset{\sim}{i}')$ be an admissible pair of $\mathcal{R}$. Applying transformation (37) and Def. 8, we obtain $(\underset{\sim}{P}(\pi)\underset{\sim}{J}_1\underset{\sim}{v}',\underset{\sim}{P}(\pi)\underset{\sim}{J}_2\underset{\sim}{i}')$ as admissible pair of $\hat{\mathcal{R}}$ (indefinite representation). Equation (41) then follows upon applying the inverse transformation (38).

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Corollary. If π leaves the grounded terminal n invariant; i.e. if $\pi(n) = n$ or $\pi(n) = \bar{n}$, then (41) reduces to

$$(\hat{\underset{\sim}{v}}',\hat{\underset{\sim}{i}}') = \left(\underset{\sim}{P}(\pi_0)\underset{\sim}{v}',\underset{\sim}{P}(\pi_0)\underset{\sim}{i}'\right) \tag{42}$$

where π_0 is simply the directed permutation π restricted to the first n-1 terminals.

In the case where $\mathcal{R}$ is characterized by a definite constitutive relation $\mathcal{R}'$, we have the following more explicit property:

Proposition 18. Given a multiterminal resistor $\mathcal{R}$ characterized by a definite constitutive relation $\mathcal{R}'(\underset{\sim}{v}',\underset{\sim}{i}') = 0$ and a directed permutation π, which complements all terminals or none, then the π-permuted multiterminal resistor $\hat{\mathcal{R}}$ is characterized by the definite constitutive relation

$$\hat{\mathcal{R}}'(\hat{\underset{\sim}{v}}',\hat{\underset{\sim}{i}}') \triangleq \mathcal{R}'\left(\underset{\sim}{J}_2^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_1\hat{\underset{\sim}{v}}', \underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\hat{\underset{\sim}{i}}'\right) = \underset{\sim}{0}. \tag{43}$$

Proof: Similar to that of Prop. 17 but using transformations (39) and (40).

¤

Corollary. If π leaves the grounded terminal invariant, then

$$\hat{\mathcal{R}}'(\hat{\underset{\sim}{v}}',\hat{\underset{\sim}{i}}') \triangleq \mathcal{R}'\left(\underset{\sim}{P}^T(\pi_0)\hat{\underset{\sim}{v}}',\underset{\sim}{P}^T(\pi_0)\hat{\underset{\sim}{i}}'\right) = \underset{\sim}{0} \tag{44}$$

where π_0 is simply the directed permutation π restricted to the first n-1 terminals.

Definition 9. A multiterminal resistor $\mathcal{R}$ is said to be π-symmetric iff $\mathcal{R}$ is identical to its π-permuted resistor $\hat{\mathcal{R}}$; i.e. if $(\underset{\sim}{v},\underset{\sim}{i})$ is an indefinite admissible pair of $\mathcal{R}$, then also $(\underset{\sim}{P}(\pi)\underset{\sim}{v},\underset{\sim}{P}(\pi)\underset{\sim}{i})$ is an indefinite admissible pair.

Proposition 19. A multiterminal resistor $\mathcal{R}$ characterized by a definite constitutive relation $\mathcal{R}'$ is π-symmetric iff

$$\mathcal{R}'(\underset{\sim}{v}',\underset{\sim}{i}') = \underset{\sim}{0} \quad \Leftrightarrow \quad \mathcal{R}'\left(\underset{\sim}{J}_2^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_1\underset{\sim}{v}',\underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\underset{\sim}{i}'\right) = \underset{\sim}{0}. \tag{45}$$

Proof. Follows from Def. 9 and Prop. 18. ¤

Corollary 1. If π leaves the grounded terminal invariant, then (45) reduces to

$$\mathcal{R}'(\underset{\sim}{v}',\underset{\sim}{i}') = \underset{\sim}{0} \quad \Leftrightarrow \quad \mathcal{R}'\left(\underset{\sim}{P}^T(\pi_0)\underset{\sim}{v}',\underset{\sim}{P}^T(\pi_0)\underset{\sim}{i}'\right) = \underset{\sim}{0}. \tag{46}$$

Corollary 2. (a) Let $\mathcal{R}$ be a voltage-controlled resistor characterized by the definite representation $\underset{\sim}{i}' = \underset{\sim}{f}'(\underset{\sim}{v}')$, then $\mathcal{R}$ is π-symmetric iff

$$\underset{\sim}{f}'(\underset{\sim}{v}') = \underset{\sim}{J}_1^T\underset{\sim}{P}(\pi)\underset{\sim}{J}_2\underset{\sim}{f}'\left(\underset{\sim}{J}_2^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_1\underset{\sim}{v}'\right) \qquad \text{for all } \underset{\sim}{v}'. \tag{47}$$

(b) Let $\mathcal{R}$ be a current-controlled resistor characterized by the definite representation $\underset{\sim}{v}' = \underset{\sim}{h}'(\underset{\sim}{i}')$, then $\mathcal{R}$ is π-symmetric iff

$$\underset{\sim}{h}'(\underset{\sim}{i}') = \underset{\sim}{J}_2^T\underset{\sim}{P}(\pi)\underset{\sim}{J}_1\ \underset{\sim}{h}'\left(\underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\underset{\sim}{i}'\right) \qquad \text{for all } i'. \tag{48}$$

Proof: (a) It follows from Prop. 19 that $\mathcal{R}$ is π-symmetric iff

$$\underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\underset{\sim}{f}'(\underset{\sim}{v}') = \underset{\sim}{f}\left(\underset{\sim}{J}_2^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_1\underset{\sim}{v}'\right) \qquad \text{for all } \underset{\sim}{v}'. \tag{49}$$

But

$$\left(\underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\right)\left(\underset{\sim}{J}_1^T\underset{\sim}{P}(\pi)\underset{\sim}{J}_2\right) = \left(\underset{\sim}{J}_1^T\underset{\sim}{P}(\pi)\underset{\sim}{J}_2\right)\left(\underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2\right) = \underset{\sim}{1}_{n-1}, \tag{50}$$

implies

$$\left(\underset{\sim}{J}_1^T\underset{\sim}{P}(\pi)\underset{\sim}{J}_2\right)^{-1} = \underset{\sim}{J}_1^T\underset{\sim}{P}^T(\pi)\underset{\sim}{J}_2.$$

Hence (47) follows from (49).

Assertion (b) follows by duality. ¤

Example 1. Let $\mathcal{R}$ be a 3-terminal resistor characterized by the definite representation:

$$(v_1'-v_2')^2 - v_2'^2 + i_2'^2 = 0 \tag{51a}$$

$$v_1'^2 - v_2'^2 + i_1' + i_2' = 0 \tag{51b}$$

It is difficult to see at first glance that (51) exhibits rotational symmetry i.e. π-symmetry where $\pi = (1\ 3\ 2)$. However, applying Prop. 19 we merely have to check whether (45) is satisfied:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} -v_2' \\ v_1'-v_2' \end{bmatrix} \tag{52a}$$

$$\begin{bmatrix} i_1' \\ i_2' \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} -i_1'-i_2' \\ -i_1' \end{bmatrix} \tag{52b}$$

Substituting each variable in (51) by the transformed variable defined in (52), we obtain

$$v_1'^2 - (v_1'-v_2')^2 + i_1' = 0 \tag{53a}$$

$$v_2'^2 - (v_1'-v_2')^2 - i_2' = 0 \tag{53b}$$

Observe that (53b) is identical to (51a). Moreover, the difference between (53a) and (53b) is identical to (51b). Hence (45) is satisfied and $\mathcal{R}$ is (1 3 2)-symmetric.

Example 2. The dc OP AMP circuit model shown earlier in Fig. 1(c) is a 4-terminal resistor characterized by the following definite representation:

$$i_1' = v_1'/R_d + (v_1'-v_2')/R_c \tag{54a}$$

$$i_2' = v_2'/R_d + (v_2'-v_1')/R_c \tag{54b}$$

$$v_2' = f(v_2'-v_1') + R_0 i_3', \tag{54c}$$

$$\text{where } f(x) = \begin{cases} -E_0 & x < -E_0/A_v \\ A_v x & |x| \le E_0/A_v \\ E_0 & x > E_0/A_v \end{cases} \tag{54d}$$

It is easy to verify that the OP AMP is complementary symmetric i.e. it is π-symmetric with $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix}$. This can be checked using (45), i.e. (54) remains invariant when all voltage and current variables are multiplied by minus 1.

Most of the properties of π-permuted and π-symmetric multiport resistors derived in Section III remain valid, *mutatis mutandis*, for multiterminal resistors. In fact, if one considers an n-terminal resistor as a grounded (n-1)-port resistor, then the properties from Section III become directly applicable. The disadvantage of resorting to this ad hoc technique is that all properties derived for such an (n-1)-port resistor are valid with respect to the same grounded terminal "n," and therefore do not reflect the intrinsic properties of the n-terminal resistor, where all terminals are treated in the same way. In other words, the theory developed in this section is essential for deriving symmetry properties of multiterminal resistors, which are independent of the choice of the datum terminal.

B. Synthesis of π-symmetric multiterminal resistors

The techniques presented in Section III can be used, after appropriate modifications, to synthesize multiterminal resistors $\tilde{\mathcal{R}}$ having any prescribed $\tilde{\pi}$-symmetry, where $\tilde{\pi}$ complements all objects or none. For example, both $\tilde{\pi}^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\tilde{\pi}^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & \bar{1} & \bar{2} \end{pmatrix}$ satisfy this condition. On the other hand, $\tilde{\pi}^{(3)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & 1 & 2 \end{pmatrix}$, $\tilde{\pi}^{(4)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & \bar{1} & 2 \end{pmatrix}$ and $\tilde{\pi}^{(5)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & 1 & \bar{2} \end{pmatrix}$ do not satisfy this condition. The reason for imposing this rule is to guarantee that the consistency conditions defined earlier are satisfied.

Since a separate voltage and current variable are associated with each terminal of a multiterminal resistor, it no longer makes sense to talk about series and parallel connection, as in the case of multiport resistors. Here, two or more terminals are simply connected with each other, resulting in a single terminal. Hence, if we connect the corresponding terminals of "k" n-terminal resistors $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots, \mathcal{R}^{(k)}$, we would obtain a new n-terminal resistor $\mathcal{R}$ whose terminal voltages and currents satisfy:

$$\underset{\sim}{v} = \underset{\sim}{v}^{(1)} = \underset{\sim}{v}^{(2)} = \ldots = \underset{\sim}{v}^{(k)} \tag{55a}$$

$$\underset{\sim}{i} = \sum_{j=1}^{k} \underset{\sim}{i}^{(j)}, \tag{55b}$$

where all voltages are measured with respect to the same datum. If the datum is chosen to be terminal "n" for each element, then $\mathcal{R}$ and $\mathcal{R}^{(j)}$ for $j = 1,2,\ldots,k$ can be described by a definite representation, otherwise we need the indefinite representation.

Two general synthesis algorithms will be presented for realizing a $\tilde{\pi}$-symmetric n-terminal resistor. The algorithms differ from each other in the choice of the building blocks: Algorithm 1' uses two or more identical but unsymmetrical m-terminal resistors (where $m \leq n$) as building blocks, whereas Algorithm 2' uses one π-symmetric m-terminal resistor $\mathcal{R}$, or two distinct multiterminal resistors $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, where $\mathcal{R}^{(1)}$ has $m^{(1)}$ terminals and exhibits $\pi^{(1)}$-symmetry and $\mathcal{R}^{(2)}$ has $m^{(2)}$ terminals and exhibits $\pi^{(2)}$-symmetry, where $n \leq m$ or $n \leq m^{(1)} + m^{(2)}$.

Synthesis algorithm 1'. Let $\tilde{\pi}$ be any prescribed directed permutation of n objects which complements all objects, or none. Determine the order "ℓ" of $\tilde{\pi}$; hence $\tilde{\pi}^{\ell} = I$. Let $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots, \mathcal{R}^{(\ell)}$ denote "ℓ" identical m-terminal resistors and let $\tilde{\pi}^{j}(\mathcal{R}^{(k)})$ denote the $\tilde{\pi}^{j}$-permuted resistor associated with $\mathcal{R}^{(k)}$.[12] Then the n-terminal resistor $\tilde{\mathcal{R}}$ obtained by connecting the corresponding terminals of the resistors $\tilde{\pi}^{-j+1}(\mathcal{R}^{(j)})$, $j = 1,2,\ldots,\ell-1$, is π-symmetric.

The proof that $\tilde{\mathcal{R}}$ exhibits the prescribed $\tilde{\pi}$-symmetry is similar to the proof of Algorithm 1 and will not be given here. We will now illustrate this algorithm by four examples:

Example 1. Synthesize a bilateral 2-terminal resistor; i.e., let $\tilde{\pi} = (1\ \bar{2})$ be prescribed. Since $\ell = 2$ we need two identical two-terminal resistors. Applying Algorithm 1', we obtain the well-known result that the "back-to-front" parallel connection of two identical 2-terminal resistors always leads to a bilateral resistor (see Prop. 1, Fig. 2(b)).

Example 2. Consider the following collection of all directed permutations[13] on three objects which complement all three objects or none:

$$\underline{\text{order } 2} \quad \tilde{\pi}_a = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{2} & \bar{3} \end{pmatrix}, \quad \tilde{\pi}_b = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \tilde{\pi}_c = \begin{pmatrix} 1 & 2 & 3 \\ \bar{2} & \bar{1} & \bar{3} \end{pmatrix},$$

$$\underline{\text{order } 3} \quad \tilde{\pi}_d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \underline{\text{order } 6} \quad \tilde{\pi}_e = \begin{pmatrix} 1 & 2 & 3 \\ \bar{2} & \bar{3} & \bar{1} \end{pmatrix}.$$

The $\tilde{\pi}$-symmetric 3-terminal resistors corresponding to the above five directed permutations are synthesized as shown in Fig. 15(a)-15(e). The symbol $\bar{\mathcal{R}}^{(i)}$ denotes the complement of $\mathcal{R}^{(i)}$, obtained by multiplying all terminal voltages and currents of $\mathcal{R}^{(i)}$ by minus 1. Such an element can be realized with the help

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[12] If $m < n$, there will be more objects to be permuted than there are terminals. Therefore we define $\tilde{\pi}^{j}(i) = \phi$, if $\tilde{\pi}^{j}(i) > m$, or if $\tilde{\pi}^{j}(\bar{I}) > m$, where ϕ denotes the empty set. We also define the connection between terminal "i" of one m-terminal resistor $\mathcal{R}^{(j)}$ and terminal "ϕ" of an m-terminal resistor $\tilde{\pi}^{t}(\mathcal{R}^{(k)})$ to be the original terminal "i."

[13] Up to a relabelling of the three oriented objects.

of <u>phase-inverting</u> ideal transformers (see Fig. 14, for example). However our choice of complementary symmetric elements as building blocks is motivated by the availability of many such physical elements in intrinsic form. For example, the complement of a 2-terminal resistor is obtained by transposing its terminals. The complement of a pnp transistor is an npn transistor having an identical (apart from a negative sign in v_j and i_j) set of characteristics. The complement of a p-channel FET is an n-channel FET, etc. As a concrete application of Algorithm 1', let us choose a "biased" npn transistor for $\mathcal{R}^{(1)}$. Then $\mathcal{R}^{(2)}$ is identical to $\mathcal{R}^{(1)}$, and $\bar{\mathcal{R}}^{(2)}$ is the "complementary biased" pnp transistor in the $\tilde{\pi}_a$-symmetric circuit of Fig. 15(a). The resulting 3-terminal resistor $\tilde{\mathcal{R}}$ shown in Fig. 15(f) coincides with the well-known complementary-symmetric push-pull amplifier circuit.

<u>Example 3</u>. Synthesize a $\tilde{\pi}$-symmetric 4-terminal resistor $\mathcal{R}$, where $\tilde{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. Since $\tilde{\pi}^4 = I$, we need four identical resistors $\mathcal{R}^{(1)}$, $\mathcal{R}^{(2)}$, $\mathcal{R}^{(3)}$ and $\mathcal{R}^{(4)}$. By interconnecting the four identical 2-terminal resistors we obtain the $\tilde{\pi}$-symmetric 4-terminal resistor shown in Fig. 16(a). Observe that terminal 1 of $\tilde{\mathcal{R}}$ is obtained by connecting terminals 1 of $\mathcal{R}^{(1)}$, $\tilde{\pi}(1) = 2$ of $\mathcal{R}^{(2)}$, $\tilde{\pi}^2(1) = \phi$ of $\mathcal{R}^{(3)}$ and $\tilde{\pi}^3(1) = \phi$ of $\mathcal{R}^{(4)}$ together, or in short, by connecting terminals 1 of $\mathcal{R}^{(1)}$ and 2 of $\mathcal{R}^{(2)}$ together. Similarly, terminal 2 of $\tilde{\mathcal{R}}$ is obtained by connecting terminals 2 of $\mathcal{R}^{(1)}$ and 1 of $\mathcal{R}^{(4)}$ together, etc. It follows from an analogous version of Prop. 15 for multiterminal resistors that $\tilde{\mathcal{R}}$ is also $\tilde{\pi}^2$- and $\tilde{\pi}^3$-symmetric.

<u>Example 4</u>. Let $\tilde{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{2} & \bar{3} & \bar{4} & \bar{1} \end{pmatrix}$ be a prescribed symmetry permutation of a 4-terminal resistor $\tilde{\mathcal{R}}$. Again $\ell = 4$ and choosing four identical 2-terminal resistors as in Example 3, we obtain the $\tilde{\pi}$-symmetric 4-terminal resistor shown in Fig. 16(b).

In Algorithm 2' several techniques will be described for making interconnections between terminals of a π-symmetric m-terminal resistor $\mathcal{R}$ in order to obtain a $\tilde{\pi}$-symmetric n-terminal resistor $\tilde{\mathcal{R}}$ with $n \leq m$. This algorithm can also be applied to several symmetric multiterminal resistors by considering them as one overall symmetric multiterminal resistor as follows: Given a $\pi^{(1)}$-symmetric $m^{(1)}$-terminal resistor $\mathcal{R}^{(1)}$ and a $\pi^{(2)}$-symmetric $m^{(2)}$-terminal resistor $\mathcal{R}^{(2)}$, we form the <u>composite</u> m-terminal resistor $\mathcal{R}$ with $m = m^{(1)} + m^{(2)}$ by retaining the terminal numbers $1,2,\ldots,m^{(1)}$ of $\mathcal{R}^{(1)}$, and by relabelling the terminal numbers of $\mathcal{R}^{(2)}$ by $m^{(1)}+1$, $m^{(1)}+2,\ldots,m^{(1)}+m^{(2)} = m$. Clearly, $\mathcal{R}$ exhibits a π-symmetry where π acts on $1,2,\ldots,m^{(1)}$ as $\pi^{(1)}$ does, and where π

acts on $m^{(1)}+1$, $m^{(1)}+2,\ldots,m$ as $\pi^{(2)}$ does on $1,2,\ldots,m^{(2)}$. Since subsequent operations on $\mathcal{R}$ may involve complementations of all or of none of the terminals of $\mathcal{R}$, either one or both of the resistors $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ can be complemented before forming the composite resistor. This results in four cases for the composite resistor; namely, (1) $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}$, (2) $\bar{\mathcal{R}}^{(1)}, \mathcal{R}^{(2)}$, (3) $\mathcal{R}^{(1)}, \bar{\mathcal{R}}^{(2)}$, and (4) $\bar{\mathcal{R}}^{(1)}, \bar{\mathcal{R}}^{(2)}$. Since $\mathcal{R}^{(i)}$ and $\bar{\mathcal{R}}^{(i)}$ have the same symmetry group all four composite resistors exhibit π-symmetry. Observe that since a composite resistor is made up of two or more uncoupled and unconnected resistors, our earlier requirement that π must either complement all or no terminals can be relaxed. For example, $\pi^{(1)}$ may complement all terminals of $\mathcal{R}^{(1)}$, while $\pi^{(2)}$ may not. However, since the cyclic components of π originate either from $\pi^{(1)}$ or $\pi^{(2)}$, π must complement all terminals, or no terminals of <u>each cycle</u> of π. Consequently, the cyclic representation of each cycle of π must have one of three possible general forms: Let $i_1,i_2,\ldots,i_\ell$ denote the original (i.e., uncomplemented) terminals in a cycle of $\pi^{(1)}$ or $\pi^{(2)}$. If this cycle is not complemented in the associated composite permutation π, then this cycle must have the general form $(i_1 i_2 \cdots i_\ell)$ and can clearly only be a normal-order cycle. If π complements all terminals of the cycle, then the cycle is a normal-order cycle $(i_1 \bar{i}_2 i_3 \bar{i}_4 i_5 \cdots \bar{i}_\ell)$ if ℓ is even, or a double-order cycle $(i_1 \bar{i}_2 i_3 \bar{i}_4 \cdots i_\ell \bar{i}_1 i_2 \bar{i}_3 i_4 \cdots \bar{i}_\ell)$ if ℓ is odd. The above observations may be summarized by saying that each cycle of the composite permutation π may assume one of the following three types of cyclic notation: Type 1: $(i_1 i_2 \cdots i_\ell)$, Type 2: $(i_1 \bar{i}_2 i_3 \bar{i}_4 i_5 \cdots \bar{i}_\ell)$, and Type 3: $(i_1 \bar{i}_2 i_3 \bar{i}_4 \cdots i_\ell \bar{i}_1 i_2 \bar{i}_3 i_4 \cdots \bar{i}_\ell)$. We are now ready to present Algorithm 2', which parallels that presented in the preceding section.

<u>Synthesis Algorithm 2'</u>. Given a π-symmetric m-terminal resistor $\mathcal{R}$, then the following terminal interconnection techniques allow us to generate a large variety of $\tilde{\pi}$-symmetric n-terminal resistors $\tilde{\mathcal{R}}$ with $n \leq m$. Let the cyclic decomposition of π be given by $\pi = c_1 c_2 \cdots$,

(A) Interconnection of terminals belonging to a cyclic component of <u>normal-order</u> (type 1 or 2). Let c_j be a cyclic-component of π of normal-order with cyclic notation: (1) $c_j = (i_1 i_2 \cdots i_s \cdots i_\ell)$ where $\ell = ks$ and (2) $c_j = (i_1 \bar{i}_2 \cdots \bar{i}_s i_{s+1} \cdots \bar{i}_\ell)$, where $\ell = ks$ is an even integer with s even. We form a $\tilde{\pi}$-symmetric resistor $\tilde{\mathcal{R}}$ ($n=m-\ell+s$) as follows: Terminal i_1 of $\tilde{\mathcal{R}}$ is realized by connecting terminals $i_1, i_{s+1}, \ldots, i_{(k-1)s+1}$ of $\mathcal{R}$ together. Terminal i_2

of $\tilde{\mathcal{R}}$ is realized by connecting terminal $i_2, i_{s+2}, \ldots, i_{(k-1)s+2}$ of $\mathcal{R}$ together. Iterate this procedure until terminal i_s of $\tilde{\mathcal{R}}$ is realized by connecting terminals $i_s, i_{2s}, \ldots, i_{ks}$ of $\mathcal{R}$ together. Any other terminal i of $\mathcal{R}$ belonging to the other cycles of π remains unchanged and is simply labelled as terminal i of $\tilde{\mathcal{R}}$. The new resistor $\tilde{\mathcal{R}}$ so realized has $n = m - \ell + s$ terminals and exhibits $\tilde{\pi} = c_1 c_2 \ldots c_{j-1} \tilde{c}_j c_{j+1} \ldots$ and, $\tilde{c}_j = (i_1 i_2 \ldots i_s)$ if $c_j = (i_1 i_2 \ldots i_s \ldots i_\ell)$, or $\tilde{c}_j = (i_1 \bar{i}_2 \ldots \bar{i}_s)$ if $c_j = (i_1 \bar{i}_2 \ldots \bar{i}_s i_{s+1} \ldots \bar{i}_\ell)$.

(B) Interconnection of terminals belonging to a cyclic component of Type 2 or Type 3: Let c_j be a cyclic component of π satisfying one of the following two properties: (1) c_j has a type 2 cyclic-notation $c_j = (i_1 \bar{i}_2 \ldots i_s \bar{i}_{s+1} \ldots \bar{i}_\ell)$, where $\ell = ks$ is an even integer, with k even and s odd. (2) c_j has a type 3 cyclic notation $c_j = (i_1 \bar{i}_2 \ldots i_s \bar{i}_{s+1} \ldots i_\ell \bar{i}_1 i_2 \ldots \bar{i}_s i_{s+1} \ldots \bar{i}_\ell)$, where $\ell = ks$ is an odd integer with k odd and s odd. Under the above assumptions, we can form a $\tilde{\pi}$-symmetric n-terminal resistor $\tilde{\mathcal{R}}(n=m-\ell+s)$ as follows: Terminal i_1 of $\tilde{\mathcal{R}}$ is realized by connecting terminals $i_1, i_{s+1}, \ldots, i_{(k-1)s+1}$ of $\mathcal{R}$ together. Terminal i_2 of $\tilde{\mathcal{R}}$ is realized by connecting terminals $i_2, i_{s+2}, \ldots, i_{(k-1)s+2}$ of $\mathcal{R}$ together. Iterate this procedure until terminal i_s of $\tilde{\mathcal{R}}$ is realized by connecting terminals $i_s, i_{2s}, \ldots, i_{ks}$ of $\mathcal{R}$ together. Any other terminal i of $\mathcal{R}$ belonging to the other cycles of π remains unchanged and is simply labelled as terminal i of $\tilde{\mathcal{R}}$. The new resistor $\tilde{\mathcal{R}}$ so realized has $n = m - \ell + s$ terminals and exhibits $\tilde{\pi}$-symmetry, where $\tilde{\pi} = c_1 c_2 \ldots c_{j-1} \tilde{c}_j c_{j+1} \ldots$, and $\tilde{c}_j = (i_1 \bar{i}_2 \ldots i_s \bar{i}_1 i_2 \ldots \bar{i}_s)$. Hence, the resulting cyclic component $\tilde{c}_j$ has a double order.

(C) Interconnection of terminals belonging to two compatible cyclic components. Let c_i and c_j be any two cyclic components of π which are compatible in the sense that they have the same order, and have the same type of cyclic notation; namely, (1) type 1 notation $c_i = (i_1 i_2 \ldots i_\ell)$ and $c_j = (j_1 j_2 \ldots j_\ell)$, (2) type 2 notation $c_i = (i_1 \bar{i}_2 i_3 \bar{i}_4 \ldots i_\ell)$ and $c_j = (j_1 \bar{j}_2 j_3 \bar{j}_4 \ldots j_\ell)$, where ℓ is even,

(3) type 3 notation $c_i = (i_1 \bar{i}_2 \ldots i_\ell \bar{i}_1 i_2 \ldots \bar{i}_\ell)$ and $c_j = (j_1 \bar{j}_2 \ldots j_\ell \bar{j}_1 j_2 \ldots \bar{j}_\ell)$ where ℓ is odd. Under the above assumptions, we can form a $\tilde{\pi}$-symmetric n-terminal resistor $\tilde{\mathcal{R}}$ $(n=m-\ell)$ as follows: Terminal i_1 of $\tilde{\mathcal{R}}$ is realized by connecting terminals i_1 and j_1 of $\mathcal{R}$ together. Terminal i_2 of $\tilde{\mathcal{R}}$ is realized by connecting terminals i_2 and j_2 of $\mathcal{R}$ together. Iterate this procedure ℓ times until terminal i_ℓ of $\tilde{\mathcal{R}}$ is realized by connecting terminals i_ℓ and j_ℓ of $\mathcal{R}$ together. Any other terminal i of $\mathcal{R}$ belonging to the other cycles of π remains unchanged and is simply labelled as terminal i of $\tilde{\mathcal{R}}$. The new resistor $\tilde{\mathcal{R}}$ so realized has $n = m - \ell$ terminals and is $\tilde{\pi}$-symmetric, where $\tilde{\pi} = c_1 c_2 \ldots c_{i-1} c_i c_{i+1} \ldots c_{j-1} c_{j+1} \ldots$.

Just as in the n-port case, a large variety of distinct π-symmetric n-terminal resistors may be generated from the same cyclic components having distinct but equivalent cyclic notation. However since the complements of the terminals are not available in the multiterminal case only "ℓ" distinct interconnection schemes are generated. A combination of Techniques (A), (B) and (C) would give rise to an even larger variety of $\tilde{\pi}$-symmetric n-terminal resistors. For example, the following well-known corollaries follow directly from Technique (C):

Corollary 1. Given two distinct π-symmetric n-terminal resistors $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, then the n-terminal resistor $\tilde{\mathcal{R}}$ obtained by interconnecting corresponding terminals of $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ is also π-symmetric.

Corollary 2. Let $\mathcal{R}^{(j)}$ denote a complementary symmetric $m^{(j)}$-terminal resistor, $j = 1,2,\ldots,k$. Let $\tilde{\mathcal{R}}$ be an n-terminal resistor obtained by interconnecting terminals of $\mathcal{R}^{(1)}$, $\mathcal{R}^{(2)}$,... $\mathcal{R}^{(k)}$ with each other. Then $\tilde{\mathcal{R}}$ is also complementary symmetric.

Remark: Proposition 3 is a special case of Cor. 2.

The proof of the validity of Techniques (A), (B), and (C) of Algorithm 2' is analogous to the proof of Algorithm 2 and will not be given. Instead, we will illustrate the application of Algorithm 2' with an example:

Example. Let $\mathcal{R}$ be a π-symmetric m-terminal resistor, where $\pi = (1\ \bar{2}\ 3\ \bar{4})(5\ \bar{6})(7\ \bar{8}\ 9\ \overline{10}\ 11\ \overline{12})$ and $m = 12$. We will apply Techniques (A), (B), and (C) to synthesize a $\tilde{\pi}$-symmetric n-terminal resistor $\tilde{\mathcal{R}}$ where $\tilde{\pi} = (1\ \bar{2})(3\ \bar{4}\ 5\ \bar{3}\ 4\ \bar{5})$ and $n = 5$: Applying first Technique (A) to the first normal-order cyclic component $(1\ \bar{2}\ 3\ \bar{4})$, we connect terminals 1 and 3 together and label it as terminal 1 of $\tilde{\mathcal{R}}$. Similarly, we connect terminals 2 and 4 together and label it as terminal 2 of $\tilde{\mathcal{R}}$. The resulting cycle is $(1\ \bar{2})$. Next, let us apply Technique (B) to the third cyclic component $(7\ \bar{8}\ 9\ \overline{10}\ 11\ \overline{12})$ which has a Type 2 cyclic representation. Let us choose $s = 3$ and obtain a new double-order cycle $(7\ \bar{8}\ 9\ \bar{7}\ 8\ \bar{9})$ by connecting terminals 7 with 10, 8 with 11, and 9 with 12, and relabelling them as terminals 7, 8, and 9, respectively. Applying Technique (C) to the second cyclic component $(5\ \bar{6})$ of π, and the reduced first component $(1\ \bar{2})$, we connect terminals 1 with 5, and 2 with 6, and relabel them as terminals 1 and 2 of $\tilde{\mathcal{R}}$, respectively. Finally, we relabel terminals 7, 8, and 9 by 3, 4, and 5, respectively to obtain the 5-terminal resistor $\tilde{\mathcal{R}}$. Clearly, $\tilde{\mathcal{R}}$ is $\tilde{\pi}$-symmetric, where $\tilde{\pi} = (1\ \bar{2})(3\ \bar{4}\ 5\ \bar{3}\ 4\ \bar{5})$.

V. SOME APPLICATIONS OF SYMMETRY IN NONLINEAR CIRCUITS

When a π-symmetric multiport or multiterminal resistor is driven by an arbitrary excitation $\underset{\sim}{x}$, then in general the response $\underset{\sim}{y}$ has no special properties. However if the excitation $\underset{\sim}{x}(t)$ is periodic of period T, and π-symmetric in the sense that $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$ for all t, or time-shifted π-symmetric in the sense that $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t+kT/\ell)$ for all t, where k and ℓ are integers, then we will show that under rather mild conditions, the response $\underset{\sim}{y}(t)$ also exhibits the same symmetry property. This property can be exploited to simplify the analysis, or to derive some frequency separation properties. In fact, many communication circuits make implicit use of such frequency separation properties.

A. Application 1: Simplifications under π-symmetric excitations

Under the additional standing assumption that the response is unique, we will show that a symmetric excitation $\underset{\sim}{x}(t)$ $\left(\underset{\sim}{P}(\pi)\underset{\sim}{x}(t)=\underset{\sim}{x}(t)\right)$ implies a symmetric response (Prop. 20), namely $\underset{\sim}{P}(\pi)\underset{\sim}{y}(t) = \underset{\sim}{y}(t)$. This latter constraint imposes additional relationships that must be satisfied by the components of $\underset{\sim}{y}$. Consequently, only a subset of components of $\underset{\sim}{y}$ need be determined by circuit analysis. The remaining components are then obtained by direct substitution. This approach greatly reduces the computational complexity of the analysis problem. Since the vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are eigenvectors of $\underset{\sim}{P}(\pi)$ associated with the eigenvalue 1, the matrix $\underset{\sim}{S}(\pi)$ of all linearly independent eigenvectors associated with the eigenvalue 1 (Def. 4), comes into play in a natural way. In particular, a symmetrically reduced resistor having fewer ports or terminals can be defined with the help of $\underset{\sim}{S}(\pi)$. We will show that the analysis of the original n-terminal resistor under the π-symmetric excitation $\underset{\sim}{x}(t)$ can be reduced to the analysis of the reduced m-terminal resistor ($m<n$) driven by a reduced set of excitations.

Some of the results in the following discussion have been derived earlier by Desoer and Lo [10] using group representation theory. Our approach here is more direct and simpler in the sense that only elementary properties of the directed permutation π derived earlier in Section II are used. Moreover our computational method is much easier.

Proposition 20. If a π-symmetric multiport (resp., multiterminal) resistor R (resp., $\mathcal{R}$) has a unique response $\underset{\sim}{y}(t)$ to a π-symmetric excitation $\underset{\sim}{x}(t)$, namely $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$, where the hybrid vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are compatible with the directed permutation π, then the response $\underset{\sim}{y}(t)$ is also π-symmetric in the sense that $\underset{\sim}{P}(\pi)\underset{\sim}{y}(t) = \underset{\sim}{y}(t)$ for all t.

Proof: It suffices to consider one instant of time. Let $(\underset{\sim}{x},\underset{\sim}{y})$ be an admissible pair of R. Then R is π-symmetric implies that $\left(\underset{\sim}{P}(\pi)\underset{\sim}{x},\underset{\sim}{P}(\pi)\underset{\sim}{y}\right)$ is also an admissible pair. It follows from the π-symmetry that $(\underset{\sim}{x},\underset{\sim}{P}(\pi)\underset{\sim}{y})$ is an admissible pair. But the response is unique by hypothesis, hence we must have $\underset{\sim}{P}(\pi)\underset{\sim}{y} = \underset{\sim}{y}$.

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Definition 10. Given a π-symmetric n-port resistor R (resp., n-terminal resistor $\mathcal{R}$), let $(\underset{\sim}{x},\underset{\sim}{y})$ denote any hybrid admissible pair compatible with π. We define the associated π-symmetrically reduced m-port resistor R_o (resp., m-terminal resistor $\mathcal{R}_o$), $m < n$, to be characterized by a corresponding collection of admissible pairs such that $(\underset{\sim}{x}_o,\underset{\sim}{y}_o)$ is an admissible pair of R_o (resp., $\mathcal{R}_o$) iff $\left(\underset{\sim}{S}(\pi)\underset{\sim}{x}_o,\underset{\sim}{S}(\pi)(\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi))^{-1}\underset{\sim}{y}_o\right)$ is an admissible pair of R (resp., $\mathcal{R}$).[14]

Proposition 21. If a π-symmetric multiport resistor R (resp., multiterminal resistor $\mathcal{R}$) is characterized by a constitutive relation $\underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{0}$, where $(\underset{\sim}{x},\underset{\sim}{y})$ are mixed variables compatible with π, then the associated π-symmetrically reduced resistor R_o (resp., $\mathcal{R}_o$) is characterized by the constitutive relation

$$\underset{\sim}{F}_o(\underset{\sim}{x}_o,\underset{\sim}{y}_o) \triangleq \underset{\sim}{F}(\underset{\sim}{S}(\pi)\underset{\sim}{x}_o,\underset{\sim}{S}(\pi)(\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi))^{-1}\underset{\sim}{y}_o) = \underset{\sim}{0} \tag{56}$$

Proof: Follows directly from Def. 10. ¤

Observe that the power input to both R and R_o (resp., $\mathcal{R}$ and $\mathcal{R}_o$) are the same, because $\underset{\sim}{x}^T\underset{\sim}{y} = \underset{\sim}{x}_o^T\underset{\sim}{y}_o$ in view of Def. 10.

Proposition 22. If a π-symmetric multiport resistor R (resp., multiterminal resistor $\mathcal{R}$), has a unique response $\underset{\sim}{y}$ to a π-symmetric excitation $\underset{\sim}{x}(t)$ and $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ are compatible with π, then the response $\underset{\sim}{y}(t)$ is also π-symmetric and is given explicitly by:

$$\underset{\sim}{y}(t) = \underset{\sim}{S}(\pi)(\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi))^{-1}\underset{\sim}{y}_o(t), \tag{57}$$

where $\underset{\sim}{y}_o(t)$ is the response of the π-symmetrically reduced resistor R_o (resp., $\mathcal{R}_o$) to the excitation $\underset{\sim}{x}_o(t)$, where $\underset{\sim}{x}_o(t)$ is related to $\underset{\sim}{x}(t)$ by $\underset{\sim}{x}(t) = \underset{\sim}{S}(\pi)\underset{\sim}{x}_o(t)$.

Proof: It suffices to consider one instant of time. Since $\underset{\sim}{x}$ is an eigenvector of $\underset{\sim}{P}(\pi)$ associated with the eigenvalue 1, it follows from Prop. 9 that $\underset{\sim}{x}$ is a unique linear combination of the columns of $\underset{\sim}{S}(\pi)$. Hence, there exists an $\underset{\sim}{x}_o$ such that $\underset{\sim}{x} = \underset{\sim}{S}(\pi)\underset{\sim}{x}_o$. By Prop. 20, the unique response $\underset{\sim}{y}$ of R (resp., $\mathcal{R}$) to the excitation $\underset{\sim}{x}$ is also π-symmetric and hence $\underset{\sim}{y}$ is also an eigenvector of $P(\pi)$ with

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[14]Recall from (10) that $\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi)$ is an mxm nonsingular diagonal matrix. Hence, its inverse exists. The reason for introducing this scaling matrix will soon become clear.

eigenvalue 1. It follows from Prop. 9 again that $\underset{\sim}{y}$ is a linear combination of the columns of $\underset{\sim}{S}(\pi)$; namely $\underset{\sim}{y} = \underset{\sim}{S}(\pi)\underset{\sim}{w}$, where $\underset{\sim}{w}$ is a unique m-vector. Now since $\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi)$ is a nonsingular diagonal constant matrix we can choose

$$\underset{\sim}{w} \triangleq [\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi)]^{-1}\underset{\sim}{y}_o, \tag{58}$$

where y_o is an m-vector. Observe that y_o is uniquely determined because $\underset{\sim}{w}$ is unique and $\underset{\sim}{S}(\pi)$ depends only on π. Substituting (58) into $\underset{\sim}{y} = \underset{\sim}{S}(\pi)\underset{\sim}{w}$, we obtain (57). Hence, we have shown that

$$(\underset{\sim}{x},\underset{\sim}{y}) \triangleq \left(\underset{\sim}{S}(\pi)\underset{\sim}{x}_o, \underset{\sim}{S}(\pi)(\underset{\sim}{S}^T(\pi)\underset{\sim}{S}(\pi))^{-1}\underset{\sim}{y}_o\right) \tag{59}$$

is an admissible pair of R (resp., $\mathcal{R}$). It follows from Def. 10 that $(\underset{\sim}{x}_o,\underset{\sim}{y}_o)$ is an admissible pair of the π-symmetrically reduced resistor R_o (resp., $\mathcal{R}_o$).

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Proposition 22 is very useful because it allows us to solve a higher dimensional problem involving n variables by solving an equivalent lower dimensional problem involving only m < n variables, where m is the number of distinct cyclic components of normal order of the directed permutation π. The algorithm for implementing Prop. 22 is as follows:

<u>Symmetry reduction algorithm</u>. Given a π-symmetric n-port resistor R (resp., n-terminal resistor $\mathcal{R}$), driven by a π-symmetric excitation $\underset{\sim}{x}(t)$. Given also that there exists a <u>unique</u> response $\underset{\sim}{y}(t)$, where $(\underset{\sim}{x},\underset{\sim}{y})$ is a mixed pair of variables compatible with π.

1. Decompose π into cyclic components. Identify the normal-order cyclic components and label them consecutively: $c_1, c_2, \ldots c_m$.
2. Form the nxm matrix $\underset{\sim}{S}(\pi)$ by inspection (using Def. 4).
3. Find the constitutive relation of the π-symmetrically-reduced resistor R_o (resp., $\mathcal{R}_o$) using (56).
4. Find the unique m-vector excitation $\underset{\sim}{x}_o(t)$ by inspection from $\underset{\sim}{x}(t) = \underset{\sim}{S}(\pi)\underset{\sim}{x}_o(t)$.
5. Apply the excitation $\underset{\sim}{x}_o(t)$ to R_o (resp., $\mathcal{R}_o$) and compute the m-vector response $\underset{\sim}{y}_o(t)$.
6. Substitute $\underset{\sim}{y}_o(t)$ into (57) to obtain the n-vector response $\underset{\sim}{y}(t)$ of R (resp., $\mathcal{R}$).

An example illustrating this algorithm is described in Appendix C.

It follows from the preceding algorithm, that the less normal-order cyclic components there are in π, the less computations are involved. In the special case where π is block-cyclic; i.e., $\pi = \left(1\ m{+}1\ \ldots\ (k{-}1)m{+}1\right)\ldots\left(m\ 2m\ 3m\ \ldots\ km\right)$

the analysis of the n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) where $n = km$ reduces to the analysis of a m-port resistor R_o (resp., m-terminal resistor $\mathcal{R}_o$).

The preceding algorithm is strictly algebraic. Much insight concerning symmetry can be gained by deriving a corresponding physical interpretation. Let us first consider Steps 1-5 for a π-symmetric n-port resistor. In order to satisfy $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$, all ports of R belonging to the same cyclic component c_j of π must be excited by identical sources with appropriate polarity (i.e., the polarity must be reversed for a complemented port). Remember that the mixed variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are assumed to be compatible with π, which implies that all variables of one cycle are either all voltages or all currents. Hence if c_j is driven by voltage sources (resp., current sources), then all ports belonging to c_j may be connected in parallel (resp., in series), provided the terminals in each complemented port are transposed, thereby reducing the n_j ports belonging to c_j into a single port. How about the ports belonging to double-order cycles? Suppose π contains a double-order cycle $\sigma = (i_1 i_2 \ldots i_k \bar{i}_1 \bar{i}_2 \ldots \bar{i}_k)$, then it follows from Prop. 4 that R is also π^k-symmetric. But π^k contains the double-order cyclic component σ^k, which is easily seen to be decomposable into $\sigma^k = (i_1 \bar{i}_1)(i_2 \bar{i}_2) \ldots (i_k \bar{i}_k)$. In other words, each port belonging to a double-order cyclic component of π also exhibits complementary symmetry. Since the complement of a voltage source (resp., current source) is only identical to the original source if it is a 0-volt voltage source (resp., 0-ampere current source) it follows from $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$ and $\underset{\sim}{P}(\pi)\underset{\sim}{y}(t) = \underset{\sim}{y}(t)$, that under the conditions of Prop. 22, each port of R belonging to a double-order cyclic component of π must have zero port voltage and zero port current and hence must be terminated by a nullator in order to obtain R_o.

For example, suppose the 7-port resistor R shown in Fig. 17(a) is π-symmetric with $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})(3\ \bar{6})(7)$. To derive R_o, we identify 3 normal-order cyclic components $(1\ \bar{4}\ 5)(3\ \bar{6})$ and (7) and a double-order component $(2\ \bar{2})$. Suppose ports 1,4,5,7 are driven by π-symmetric current sources, and ports 2, 3 and 6 are driven by π-symmetric voltage sources. Our preceding analysis shows that this excitation is π-symmetric only if the voltage source across port 2 has zero voltage. Hence R_o is a 3-port resistor as shown in Fig. 17(a). Observe that port 1 of R_o is a current-driven port obtained by connecting ports 1, $\bar{4}$ (transpose of port 4), and 5 in series; port 2 is a voltage-driven port obtained by connecting ports 3 and $\bar{6}$ (transpose of port 6) in parallel; and port 3 is a current-driven port consisting of port 7 by itself. Observe that port 2 of R is terminated in a nullator and embedded within R_o.

A similar interpretation can be given for $\mathcal{R}_o$ associated with an n-terminal resistor $\mathcal{R}$. Since there is only one possible connection for two terminals, in contrast with the two possibilities (series or parallel) for ports, the reduced resistor $\mathcal{R}_o$ given in Def. 10 has an even simpler physical interpretation if it is voltage driven ($\underset{\sim}{x}=\underset{\sim}{v}$). Here the terminals belonging to the same normal-order cyclic component of π are connected together, assuming a <u>phase-inverting transformer</u> has been inserted for each complemented terminal. Each terminal belonging to a double-order cyclic component is likewise connected to one terminal of a nullator (the second terminal is connected to the common datum).

For example, suppose that the voltage-driven 5-terminal resistor $\mathcal{R}$ shown in Fig. 17(b) is π-symmetric, where $\pi = (1\ \bar{2})(3\ \bar{3})(4\ \bar{5})$. Since terminal 3 belongs to a double-order cycle, its terminal voltage and current are identically zero and it is therefore terminated by a nullator. Since there are only 2 normal-order cyclic components, $\mathcal{R}_o$ is a 2-terminal resistor shown in Fig. 17(b).

Consider Step 6 of the symmetry reduction algorithm. In view of the π-symmetry of R (resp., $\mathcal{R}$) and our preceding interpretation of R_o (resp., $\mathcal{R}_o$), it follows that the response voltage (resp., current) of each port (resp., terminal) belonging to each normal order cyclic component c_j of order n_j is equal to $1/n_j$ times the voltage (resp., current) response of the corresponding port in R_o (resp., terminal in $\mathcal{R}_o$). But this is precisely the response obtained from (57), using Cor. of Prop. 9.

For example, referring again to Fig. 17(a), it follows that the voltages across ports 1, $\bar{4}$ and 5 are equal to 1/3 of the voltage across port 1 of R_o. Similarly, the currents in ports 3 and $\bar{6}$ of R are equal to 1/2 of the current of port 2 of R_o. For the 5-terminal resistor $\mathcal{R}$ shown in Fig. 17(b), we found the response current of terminals 1 and $\bar{2}$ of $\mathcal{R}$ are identical to 1/2 of the current of terminal 1 of $\mathcal{R}_o$. Similarly, the response currents of terminals 4 and $\bar{5}$ are equal to 1/2 of the current of terminal 2 of $\mathcal{R}_o$.

The usefulness of Prop. 22 motivates our search for analogous reduction techniques involving an excitation symmetry other than $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$. Since the preceding algorithm depends <u>crucially</u> on the property that $\underset{\sim}{x}(t)$ is an eigenvector of $\underset{\sim}{P}(\pi)$, and since for <u>nonlinear</u> elements, only real excitations make sense, it follows that any analogous technique would require at the very least, that the excitation vector $\underset{\sim}{x}(t)$ be a <u>real</u> eigenvector of $\underset{\sim}{P}(\pi)$. But since $\underset{\sim}{P}(\pi)$ is a directed permutation matrix, by (7) all the eigenvalues of $\underset{\sim}{P}(\pi)$ are roots of unity and thus the only real eigenvectors are those associated with an eigenvalue +1 or -1. Hence, the only possible analogous case requires an anti-π-symmetric excitation vector $\underset{\sim}{x}(t)$; i.e., $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = -\underset{\sim}{x}(t)$. Under this condition, a careful analysis of our preceding derivation would show that the response $\underset{\sim}{y}(t)$ is also anti-π-symmetric in the sense that $\underset{\sim}{P}(\pi)\underset{\sim}{y}(t) = -\underset{\sim}{y}(t)$, provided that in addition to being π-symmetric, the n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) is also complementary symmetric; i.e., symmetric with respect to $\bar{I} = (1\ \bar{1})(2\ \bar{2})\ldots(n\ \bar{n})$. It follows from Prop. 4 that R (resp., $\mathcal{R}$) is then also $\hat{\pi}$-symmetric, where $\hat{\pi} = \pi \circ \bar{I} = \bar{\pi}$. But the directed permutation matrix $\underset{\sim}{P}(\hat{\pi}) = \underset{\sim}{P}(\bar{\pi}) = -\underset{\sim}{P}(\pi)$ and hence $\underset{\sim}{P}(\hat{\pi})\underset{\sim}{x}(t) = \underset{\sim}{x}(t)$. We will summarize the above observations as follows:

Proposition 23. If R (resp., $\mathcal{R}$) is both π-symmetric and complementary symmetric (i.e., $\bar{I}$-symmetric), and if the excitation $\underset{\sim}{x}(t)$ is anti-π-symmetric, then both R (resp., $\mathcal{R}$) and the excitation $\underset{\sim}{x}(t)$ are $\bar{\pi}$-symmetric.

Proposition 23 guarantees that our earlier symmetry reduction algorithm is also applicable for anti-π-symmetric excitations. Hence, there is no need to develop a separate algorithm for handling anti-π-symmetric cases. Proposition 23 has the following physical interpretation: If an n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) is excited by anti-π-symmetric sources, then by transposing the terminals of some of the sources, the excitations can always be made to exhibit $\bar{\pi}$-symmetry, provided R (resp., $\mathcal{R}$) exhibits π- and $\bar{I}$-symmetry. It follows from the preceding observations that our symmetry reduction algorithm is in fact the most general method that can be derived for nonlinear multiport and multiterminal resistors.[15]

B. Application 2: Frequency separation under time-shifted π-symmetric excitations

Many communication circuits make use of symmetry configurations so that the odd and even harmonic components of various waveforms are separated and extracted at separated ports. Although the network may contain inductors and capacitors [12], the frequency separation is achieved solely by symmetry properties and not

[15] For linear R,L,C elements, phasors may be used in the frequency domain and hence complex eigenvectors of $\underset{\sim}{P}(\pi)$ are allowed. In this case, additional symmetry reduction techniques may be derived. The well-known method of symmetrical components widely used in analyzing linear power circuits is a case in point [1,3].

by any filtering operation. For this reason, to uncover the mechanisms which led to this frequency separation phenomenon, it is convenient to eliminate all irrelevant inductors and capacitors, or to replace them by resistors.

Now suppose the n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) is π-symmetric, where π has order ℓ. Suppose the excitation satisfies the property $\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t + m'/m)$ for all t, where m and m' are integers. Then we have

$$\underset{\sim}{P}^{\ell}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t + m'\ell/m) = \underset{\sim}{x}(t), \text{ for all } t \tag{60}$$

since $\underset{\sim}{P}^{\ell}(\pi) = \underset{\sim}{1}_n$. Hence $\underset{\sim}{x}(t)$ must be periodic and $m'\ell/m$ must be a multiple of the period T of $\underset{\sim}{x}(t)$ or $k'T = m'\ell/m$. Therefore the most general form of <u>time-shifted symmetry</u> for the excitation vector $\underset{\sim}{x}(t)$ is given by

$$\underset{\sim}{P}(\pi)\underset{\sim}{x}(t) = \underset{\sim}{x}(t + kT/\ell), \text{ for all } t \tag{61}$$

where $0 < k < \ell$ is satisfied by a suitable choice of m'' in $k' - m''\ell = k$.

<u>Theorem 2</u>. Let R (resp., $\mathcal{R}$) be a π-symmetric time-invariant n-port (resp., n-terminal) resistor. Let the mixed pair of hybrid variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be compatible with π, and let the response $\underset{\sim}{y}(t)$ be unique for each excitation $\underset{\sim}{x}(t)$. If the excitation $\underset{\sim}{x}(t)$ is T-periodic and satisfies (61), where ℓ is the order of π, then the response $\underset{\sim}{y}(t)$ exhibits the same time-shifted symmetry; namely,

$$\underset{\sim}{P}(\pi)\underset{\sim}{y}(t) = \underset{\sim}{y}(t + kT/\ell), \text{ for all } t. \tag{62}$$

<u>Proof</u>: Let $\left(\underset{\sim}{x}(t),\underset{\sim}{y}(t)\right)$ be an admissible pair, then $\left(\underset{\sim}{x}(t + kT/\ell),\underset{\sim}{y}(t + kT/\ell)\right)$ is an admissible pair since R (resp., $\mathcal{R}$) is time invariant. Since R (resp., $\mathcal{R}$) is π-symmetric, $\left(\underset{\sim}{P}(\pi)\underset{\sim}{x}(t),\underset{\sim}{P}(\pi)\underset{\sim}{y}(t)\right)$ is also an admissible pair. Now, if $\underset{\sim}{x}(t)$ satisfies (61), then $\left(\underset{\sim}{x}(t + kT/\ell),\underset{\sim}{P}(\pi)\underset{\sim}{y}(t)\right)$ is also an admissible pair. It follows from the uniqueness of the response that $\underset{\sim}{y}(t)$ must satisfy (62).

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<u>Corollary 1</u>. If a bilateral voltage (resp., current) controlled one-port resistor is driven by an odd-symmetric voltage source $v(t + T/2) = -v(t)$ $\left(\text{resp., current source } i(t + T(2) = -i(t)\right)$, then the current (resp., voltage) response is also odd symmetric.

<u>Proof</u>: Choose $\underset{\sim}{P}(\pi) = -1$, $k = 1$, and $\ell = 2$ in Thm. 2. ¤

<u>Corollary 2</u>. Let R be a voltage-controlled π-symmetric 2-port resistor, where $\pi = \begin{pmatrix} 1 & 2 \\ \bar{1} & 2 \end{pmatrix}$: Let the excitation voltages satisfy

$$v_1(t + T/2) = -v_1(t), \text{ for all } t \tag{63a}$$

$$v_2(t + T/2) = -v_2(t), \text{ for all } t. \tag{63b}$$

Then the current response must satisfy the same property:

$$i_1(t + T/2) = -i_1(t), \text{ for all } t \tag{64a}$$

$$i_2(t + T/2) = -i_2(t), \text{ for all } t. \tag{64b}$$

Proof: Choose $\underset{\sim}{P}(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $k = 1$ and $\ell = 2$ in Thm. 2. ⌑

Observe that (63a) and (64a) imply that $v_1(t)$ and $i_1(t)$ contain only odd harmonics, whereas (63a) and (64b) imply that $v_2(t)$ and $i_2(t)$ contain only even harmonics. Hence ports 1 and 2 separate the frequencies into odd and even harmonic components, respectively. An example of a 2-port resistor having this property is shown earlier in Fig. 3. In fact, Prop. 2 in Section I is a direct consequence of Cor. 2. Observe that this frequency separation property is taken advantage of implicitly in the full-wave rectifier circuit obtained by replacing each resistor in Fig. 3 by a diode. If we excite port 1 with an ac voltage waveform and port 2 with a dc voltage, then the current in port 1 will not contain any even harmonic components and the current in port 2 will not contain any odd harmonic components.

Another 2-port resistor R that exhibits the same π-symmetry where $\pi = \begin{pmatrix} 1 & 2 \\ \bar{1} & 2 \end{pmatrix}$ is characterized by:

$$v_1^2 + i_1^2 = 1 \tag{65a}$$

$$v_2 + i_1^2 = 1 \tag{65b}$$

If we excite ports 1 and 2 by $v_1(t) = \sin t$ and $v_2(t) = \sin 2t$ then $i_1(t) = \sin t - \cos t$ and $i_2(t) = \cos t$ constitute a possible response. Observe that $v_1(t)$ and $v_2(t)$ satisfy (63) and $i_1(t)$ and $i_2(t)$ do not satisfy (64). This paradox can be resolved by observing that (65) does not <u>not</u> give a <u>unique</u> solution to the excitation. This example clearly shows that our standing uniqueness assumption is in fact necessary for the various symmetry properties to hold.

<u>Corollary 3</u>. Let R be a <u>complementary</u> symmetric (i.e., a $(1\ \bar{1})(2\ \bar{2})$-symmetric) two-port resistor characterized by a hybrid representation $i_1 = h_1(v_1, i_2)$ and $i_2 = h_2(v_1, i_2)$. If the excitation $\underset{\sim}{x} = [v_1\ i_2]^T$ satisfies

$$v_1(t + T/2) = -v_1(t) \tag{66a}$$

$$i_2(t + T/2) = -i_2(t), \tag{66b}$$

then the response $\underset{\sim}{y} = [i_1 \; v_2]^T$ must satisfy the same symmetry property:

$$i_1(t + T/2) = -i_1(t) \tag{67a}$$

$$v_2(t + T/2) = -v_2(t). \tag{67b}$$

Proof: Choose $\underset{\sim}{P}(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $k = 1$ and $\ell = 2$ in Thm. 2. ¤

As an application of Cor. 3, let $i_2(t) \equiv 0$ for all t. Then Cor. 3 implies that if $v_1(t)$ contains only odd harmonics, so do $i_1(t)$ and $v_2(t)$. For example, the push-pull amplifier circuits shown in Figs. 9(c), (d) and (e) and Fig. 15(f) are all designed using this arrangement. The absence of even-order harmonics at both input and output ports make these amplifier configurations superior to other nonsymmetric configurations in terms of both harmonic distortion and dynamic range.

VI. CONCLUDING REMARKS

The concepts of "directed permutation" and its decomposition into cyclic components play a dominant role in this paper. While these concepts are new, they are really simple generalizations of classical concepts from group theory. These results provide us with the natural tools for studying symmetry in nonlinear circuits, and should be equally useful in future works in this area [11]. Although this paper considers only multiterminal and multiport resistors, all results also hold, mutatis mutandis, for inductors, capacitors, and memristors.

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APPENDIX

APPENDIX A: GENERAL FORM FOR THE HYBRID MATRIX OF A LINEAR π-SYMMETRIC RESISTOR

Our objective in this section is to identify the general structure of the hybrid matrix H associated with a π-symmetric n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) which is compatible with the hybrid vectors x, and y. It follows from (22) that our problem reduces to that of finding a general nxn matrix solution H to the equation

$$P(\pi)H = HP(\pi). \tag{A-1}$$

An algorithm for finding the general nxn matrix solution X to the matrix equation $AX = XA$ for an arbitrary nxn matrix A is described by Gantmacher [17] and is therefore applicable to our problem. However, by taking advantage of the special properties of the directed permutation matrix $P(\pi)$, we will derive a more practical algorithm.

Using the directed permutation χ_π defined in (3) and defining $H' \triangleq P(\chi_\pi^{-1})HP(\chi_\pi)$, we obtain the following expression upon substituting $H = P^T(\pi)HP(\pi)$ from (A-1):

$$H' = [P^T(\chi_\pi)P^T(\pi)P(\chi_\pi)][P(\chi_\pi^{-1})HP(\chi_\pi)][P^T(\chi_\pi)P(\pi)P(\chi_\pi)]$$

Now defining $P(\sigma) \triangleq P^T(\chi_\pi)P(\pi)P(\chi_\pi)$ and making use of (7), we obtain

$$H' = P^T(\sigma)H'P(\sigma) = \begin{bmatrix} \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \pm 1 & & & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix} \\ & & & \ddots \end{bmatrix} H' \begin{bmatrix} \begin{bmatrix} 0 & & & \pm 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix} \\ & & & \ddots \end{bmatrix} \tag{A-2}$$

where the cyclic decomposition of $\sigma \triangleq \chi_\pi^{-1} \circ \tilde{\pi} \circ \chi_\pi$ is given by $\sigma = c_1c_2\cdots c_t$, where $c_j = (s{+}1\ s{+}2\ \ldots\ s{+}\ell_j)$ or $c_j = (s{+}1\ s{+}2\ \ldots\ s{+}\ell_j\ \overline{s{+}1}\ \overline{s{+}2}\ \ldots\ \overline{s{+}\ell_j})$ and $s = \sum_{i=1}^{j-1} \ell_i$. In terms of H' (A-1) now assumes the form $P(\sigma)H' = H'P(\sigma)$, a much simpler equation to solve in view of the special structure of $\mathbf{P}(\sigma)$. Once $\mathbf{H}'$ is determined, we can recover $\mathbf{H} = \mathbf{P}(\chi_\pi)\mathbf{H}'\mathbf{P}^T(\chi_\pi)$. To find $\mathbf{H}'$, let us partition the matrix $\mathbf{H}'$ into t^2 blocks, corresponding to the "t" diagonal blocks of $\mathbf{P}(\sigma)$, or to the "t" cycles of σ. Since the matrix $\mathbf{P}(\sigma)$ is block diagonal, each block $\mathbf{H}'_{ij}$ in (A-2) is obtained by premultiplying $\mathbf{H}'_{ij}$ by the j-th diagonal block of $\mathbf{P}(\sigma)$ and postmultiplying by the transposed of the i-th diagonal block of $\mathbf{P}(\sigma)$. Hence the structure of $\mathbf{H}'$ can be determined by identifying the structure of <u>each</u> block $\mathbf{H}_{ij}$ separately. An analysis of (A-2) will reveal that only 5 cases need be considered and we will denote these 5 "prototype" blocks by $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$ and $\mathbf{H}_5$, respectively.

<u>Case 1</u>. $\mathbf{H}_1$ is a diagonal mxm block made up of columns s+1...s+m and rows s+1...s+m of $\mathbf{H}'$, and σ has a normal-order cycle (s+1 s+2 ... s+m). In this case, the structure of $\mathbf{H}_1$ in (A-2) must satisfy the condition:

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} \mathbf{H}_1 \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} . \tag{A-3}$$

<u>Case 2</u>. $\mathbf{H}_2$ is a diagonal mxm block made up of columns s+1 ... s+m and rows s+1 ... s+m of $\mathbf{H}'$, and σ has a double-order cycle (s+1 s+2 ... s+m $\overline{s+1}$ $\overline{s+2}$ ... $\overline{s+m}$). In this case, the structure $\mathbf{H}_2$ in (A-2) must satisfy the condition:

$$\mathbf{H}_2 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -1 & & & 0 \end{bmatrix} \mathbf{H}_2 \begin{bmatrix} 0 & & & -1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{A-4}$$

<u>Case 3</u>. $\mathbf{H}_3$ is an off-diagonal kxm block made up of columns s+1 ... s+m and rows q+1 ... q+k of $\mathbf{H}'$, and σ has two normal-order cycles (s+1 ... s+m) and (q+1 ... q+k). In this case the structure of $\mathbf{H}_3$ in (A-2) must satisfy the condition:

$$\mathbf{H}_3 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} \mathbf{H}_3 \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{A-5}$$

where the left permutation matrix has dimension kxk and the right permutation matrix has dimension mxm.

<u>Case 4</u>. $\mathbf{H}_4$ is an off-diagonal kxm block made up of columns s+1 ... s+m and rows q+1 ... q+k of $\mathbf{H}'$, and σ has a normal-order cycle (s+1 ... s+m) and a

double-order cycle $(q+1 \ldots q+k\ \overline{q+1} \ldots \overline{q+k})$. In this case, the structure of $\underset{\sim}{H}_4$ in (A-2) must satisfy the condition:

$$\underset{\sim}{H}_4 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -1 & & & 0 \end{bmatrix} \underset{\sim}{H}_4 \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \quad \text{(A-6)}$$

Case 5. $\underset{\sim}{H}_5$ is an off-diagonal kxm block made up of columns $s+1 \ldots s+m$ and rows $q+1 \ldots q+k$ of $\underset{\sim}{H}'$, and σ has two double-order cycles $(s+1 \ldots s+m\ \overline{s+1} \ldots \overline{s+m})$ and $(q+1 \ldots q+k\ \overline{q+1} \ldots \overline{q+k})$. In this case the structure of $\underset{\sim}{H}_5$ in (A-2) must satisfy the condition:

$$\underset{\sim}{H}_5 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -1 & & & 0 \end{bmatrix} \underset{\sim}{H}_5 \begin{bmatrix} 0 & & & -1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \quad \text{(A-7)}$$

Solutions to equations((A-3)-(A-7))can be easily found by analyzing the effect of a right multiplication of $\underset{\sim}{H}_i$ with the matrix $\underset{\sim}{P} = \begin{bmatrix} 0 & \ddots & \underline{+1} \\ 1 & \ddots & \\ & \ddots 1 & 0 \end{bmatrix}$. If the upper right entry is + 1, then a right multiplication of matrix $\underset{\sim}{H}_i$ with $\underset{\sim}{P}$ implies a cyclic rotation of the columns of $\underset{\sim}{H}_i$, i.e. the first column is shifted into the second position, the second column is shifted into the third position, ..., and the last column is shifted into the first position. If the upper right entry of $\underset{\sim}{P}$ is -1, then a right multiplication of $\underset{\sim}{H}_i$ with $\underset{\sim}{P}$ performs the same cyclic rotation except that the last column of $\underset{\sim}{H}_i$ is multiplied by -1 before being shifted into the first position. A left multiplication of $\underset{\sim}{H}_i$ with $\underset{\sim}{P}^T$ operates analogously on the <u>rows</u>. Let us now derive the solutions of (A-3)-(A-7):

<u>Case 1</u>: The general solution to (A-3) is easily found to have <u>m free parameters</u> $a_1 \ldots a_m$, and is in fact just the <u>cyclic matrix</u> obtained earlier in (23):

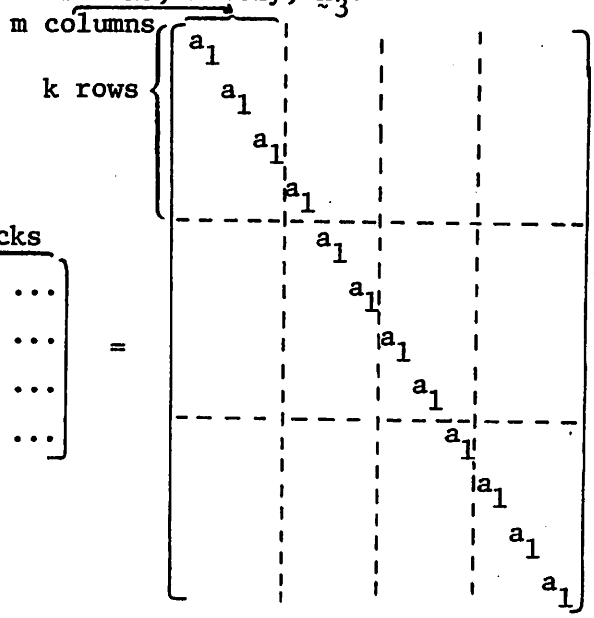
$$\underset{\sim}{H}_1 = \begin{bmatrix} a_1 & a_m & a_{m-1} & \cdots & a_2 \\ a_2 & a_1 & a_m & \cdots & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots & & \cdots \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{bmatrix} \quad \text{(A-8)}$$

<u>Case 2</u>: The general solution to (A-4) is easily found to have m free parameters $a_1\ a_2 \ldots a_m$, and assumes the following form:

$$
\mathbf{H}_2 = \begin{bmatrix} a_1 & -a_m & -a_{m-1} & \cdots & -a_2 \\ a_2 & a_1 & -a_m & \cdots & -a_3 \\ a_3 & a_2 & a_1 & \cdots & -a_4 \\ \cdots & \cdots & \cdots & & \cdots \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{bmatrix}. \qquad \text{(A-9)}
$$

Case 3: The general solution to (A-5) is not as simple as in cases 1 and 2 since the cyclic rotation of the k rows and the m columns can generate several distinct patterns depending on the greatest common divider of k and m, henceforth denoted by gcd (k,m). The effect of a cyclic shift of the rows and the columns on $\mathbf{H}_3$ can be more easily visualized by forming the following mk x mk square matrix $\mathbf{C}_3$ made up of mk identical blocks, namely, $\mathbf{H}_3$.

$$
\mathbf{C}_3 = \underbrace{\overbrace{\begin{bmatrix} \mathbf{H}_3 & \mathbf{H}_3 & \mathbf{H}_3 & \cdots \\ \mathbf{H}_3 & \mathbf{H}_3 & \mathbf{H}_3 & \cdots \\ \mathbf{H}_3 & \mathbf{H}_3 & \mathbf{H}_3 & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}}^{k \text{ blocks}}}_{m \text{ blocks}} = \qquad \text{(A-10)}
$$



Now let us investigate the effect of shifting all columns of $\mathbf{C}_3$ to the right by one position, and then shifting all rows of the resulting matrix from top to bottom by one position. Since $\mathbf{C}_3$ is a square matrix, the structure of $\mathbf{C}_3$ must be cyclic as in $\mathbf{H}_1$. This implies that all diagonal elements of $\mathbf{C}_3$ must be identical and that all elements located on a line parallel to the diagonal line of $\mathbf{C}_3$ must also be identical. Moreover, since $\mathbf{C}_3$ contains mk identical blocks, (A-5) requires that after the above shifting transformations, the corresponding mk blocks remain identical to each other.

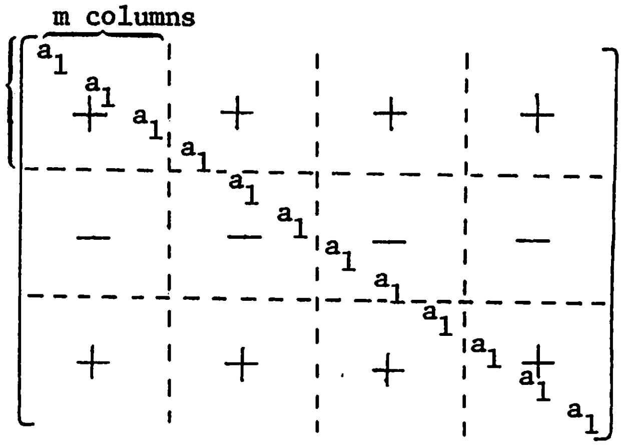
We first consider the special case that k and m are coprime or gcd(k,m) = 1. In this case, a careful analysis of (A-10) will reveal that the diagonal entries of $\mathbf{C}_3$ actually pass through all entries of $\mathbf{H}_3$ in the sense that if we cut out all entries of $\mathbf{H}_3$ located along each -45°-line (parallel to the diagonal line) and line them up, we would obtain precisely the diagonal entries of $\mathbf{C}_3$. In other words, the diagonal entries of $\mathbf{C}_3$ are made up of a union of the entries of $\mathbf{H}_3$ located along each of the (m+k−1) lines parallel to the diagonal line of $\mathbf{H}_3$. It follows from the above observations that <u>if gcd(k,m) = 1, then all entries of $\mathbf{H}_3$ must be identical</u>.

In the general case, an analogous analysis of (A-10) will reveal that the diagonal entries of $\mathbf{C}_3$ pass through only km/gcd(k,m) entries of $\mathbf{H}_3$, where each entry is passed over gcd(k,m) times. If we repeat the above analysis on the entries along each −45°-line parallel to the diagonal line of $\mathbf{C}_3$, we will find that they too pass through only km/gcd(k,m) entries of $\mathbf{H}_3$, where each entry is passed over gcd(k,m) times. It follows from the above observations that in general <u>$\mathbf{H}_3$ contains gcd(k,m) free parameters located in such a way that (A-5) is satisfied</u>. For example, let k = 6, m = 4, and let the corresponding normal-order cycles be (1 2 3 4 5 6) and (7 8 9 10). In this case, $\mathbf{H}_3$ is a 6x4 matrix and gcd(6,4) = 2. Hence $\mathbf{C}_3$ is a 24x24 square matrix whose diagonal entries are made up of a union of 24/2 = 12 entries of $\mathbf{H}_3$. An analysis of this matrix shows that $\mathbf{H}_3$ must be given as follows:

$$\mathbf{H}_3 = \begin{bmatrix} a_1 & a_2 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_1 \end{bmatrix} \qquad \text{(A-11)}$$

<u>Case 4</u>: The general solution $\mathbf{H}_4$ of (A-6) can be determined as before by analyzing the effect of the column and row shifting transformations imposed by (A-6), which differs from that of (A-5) only in the additional operation where the last row of $\mathbf{H}_4$ is multiplied by minus 1 before it is shifted into the first row of $\mathbf{H}_4$. To analyze the structure of $\mathbf{H}_4$, it is convenient to form the following mk x mk square matrix $\mathbf{C}_4$:

$$\mathbf{C}_4 = \underbrace{\overbrace{\begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 & \mathbf{H}_4 \cdots \\ -\mathbf{H}_4 & -\mathbf{H}_4 & -\mathbf{H}_4 \cdots \\ \mathbf{H}_4 & \mathbf{H}_4 & \mathbf{H}_4 \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}^{\text{k blocks}}}_{\text{m blocks (rows)}}$$



(A-12)

Observe that $\underset{\sim}{C}_4$ contains k identical "block" columns made up of an alternation of $\underset{\sim}{H}_4$ and $-\underset{\sim}{H}_4$ blocks. To emphasize this sign alternation property, each block in (A-12) is indicated by a plus or a minus sign. It follows from (A-5) and the above construction that all diagonal entries of $\underset{\sim}{C}_4$ must be identical to each other; say a_1. This condition may conflict with the condition that the (1,1) entry of all $\underset{\sim}{H}_4$'s of $\underset{\sim}{C}_4$, are equal to a_1. Depending on the least common multiple of k and m, henceforth denoted by $\ell cm(k,m)$ there are two possibilities to consider: (1) $\ell cm(k,m)/k$ = odd integer. In this case, a careful analysis of $\underset{\sim}{C}_4$ will reveal that the (1,1) element of $\underset{\sim}{C}_4$ is identical to the (1,1) element of $\underset{\sim}{H}_4$, whereas the (p,p) element $\underset{\sim}{C}_4$ is identical to the (1,1) element of $-\underset{\sim}{H}_4$, where $p = \ell cm(k,m) + 1$. But since the diagonal elements are identical, the (1,1) element of $\underset{\sim}{H}_4$ and $-\underset{\sim}{H}_4$ must be zero. A similar analysis on the other elements of $\underset{\sim}{C}_4$ shows that <u>if $\ell cm(k,m)/k$ is odd, then all entries of $\underset{\sim}{H}_4$ are identically zero</u>. (2) $\ell cm(k,m)/k$ = even integer. In this case, a similar analysis of $\underset{\sim}{C}_4$ shows that $\underset{\sim}{H}_4$ contains <u>gcd(k,m) free parameters</u>. For example, let k = 6, m = 4 and let the corresponding cycles be (1 2 3 4 5 6 $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$ $\bar{6}$) and (7 8 9 10). Since $\ell cm(6,4)/6 = 2$ and $gcd(6,4) = 2$, $\underset{\sim}{H}_4$ contains two free parameters. The structure of $\underset{\sim}{H}_4$ can be determined by imposing the cyclic conditions on $\underset{\sim}{C}_4$ and is found to be as follows:

$$\underset{\sim}{H}_4 = \begin{bmatrix} a_1 & -a_2 & -a_1 & a_2 \\ a_2 & a_1 & -a_2 & -a_1 \\ -a_1 & a_2 & a_1 & -a_2 \\ -a_2 & -a_1 & a_2 & a_1 \\ a_1 & -a_2 & -a_1 & a_2 \\ a_2 & a_1 & -a_2 & -a_1 \end{bmatrix} \qquad \text{(A-14)}$$

<u>Case 5</u>: The general solution $\underset{\sim}{H}_5$ of (A-7) can be determined by the same technique as before. In this case, we form the following mk x mk square matrix $\underset{\sim}{C}_5$:

$$\underset{\sim}{C}_5 = \begin{bmatrix} \underset{\sim}{H}_5 & -\underset{\sim}{H}_5 & \underset{\sim}{H}_5 & \cdots \\ -\underset{\sim}{H}_5 & \underset{\sim}{H}_5 & -\underset{\sim}{H}_5 & \cdots \\ \underset{\sim}{H}_5 & -\underset{\sim}{H}_5 & \underset{\sim}{H}_5 & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix} \qquad \text{(A-15)}$$

It follows from (A-7) and our construction of $\underset{\sim}{C}_5$ that the diagonal entries of $\underset{\sim}{C}_5$ are all equal to each other. Again, to determine the structure of $\underset{\sim}{H}_5$, two possibilities involving the following condition must be considered:

$$\frac{\ell cm(k,m)}{k} = \text{even} \qquad \text{and} \qquad \frac{\ell cm(k,m)}{m} = \text{even}, \tag{A-16a}$$

$$\frac{\ell cm(k,m)}{k} = \text{odd} \qquad \text{and} \qquad \frac{\ell cm(k,m)}{m} = \text{odd}. \tag{A-16b}$$

(1) Neither (A-16a) nor (A-16b) is satisfied. In this case, all entries of $\underset{\sim}{H}_5$ must be identically zero. For example, let $k = 6$, $m = 4$, and let the corresponding cycles be given by $(1\ 2\ 3\ 4\ 5\ 6\ \bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{5}\ \bar{6})$ and $(7\ 8\ 9\ 10\ \bar{7}\ \bar{8}\ \bar{9}\ \overline{10})$. Then $\frac{\ell cm(6,4)}{6} = 2$ and $\frac{\ell cm(6,4)}{4} = 3$ imply that neither (A-16a) nor (A-16b) is satisfied and we have $\underset{\sim}{H}_5 = \underset{\sim}{0}$.

(2) Either (A-16a) or (A-16b) is satisfied. In this case, $\underset{\sim}{H}_5$ contains <u>gcd(k,m) free parameters</u>.

We will close this section by presenting two examples for illustrating the application of the preceding five cases:

<u>Example 1</u>. Determine the general structure of the hybrid matrix $\underset{\sim}{H}$ of a π-symmetric 7-port linear resistor R, where $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \bar{4} & \bar{2} & \bar{6} & \bar{5} & 1 & \bar{3} & 7 \end{pmatrix}$. The step-by-step procedure is as follows:

1) Find the cyclic decomposition of π: $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})(3\ \bar{6})(7)$.

2) Choose the directed permutation $\chi_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & \bar{4} & 5 & 2 & 3 & \bar{6} & 7 \end{pmatrix}$ to transform π into σ, where $\sigma = (1\ 2\ 3)(4\ \bar{4})(5\ 6)(7)$.

3) Since σ has 4 cycles, there are 16 blocks in $\underset{\sim}{H}'$. The diagonal blocks from the top left to bottom right location are easily found as follows:

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}, \quad \begin{bmatrix} d \end{bmatrix}, \quad \begin{bmatrix} e & f \\ f & e \end{bmatrix}, \quad \begin{bmatrix} g \end{bmatrix}.$$

4) Determine the structure of the off-diagonal blocks below the diagonal line[16]

a) The cycles (1 2 3) and $(4\ \bar{4})$ give rise to a 1x3 matrix of type $\underset{\sim}{H}_4 = [0\ 0\ 0]$ since $[\ell cm(3,1)]/3 = 1 = \text{odd}$.

b) The cycles (1 2 3) and (5 6) give rise to a 2x3 matrix $\underset{\sim}{H}_3 = \begin{bmatrix} h & h & h \\ h & h & h \end{bmatrix}$ containing one free parameter h since $\gcd(3,2) = 1$.

c) The cycles (1 2 3) and (7) give rise to a 1x3 matrix $\underset{\sim}{H}_3 = [t\ t\ t]$.

d) The cycles $(4\ \bar{4})$ and (5 6) give rise to a 2x1 matrix of type $\underset{\sim}{H}_4 = [a\ -a]^T$ since $[\ell cm(2,1)]/1 = 2 = \text{even}$.

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[16] Those above the diagonal line are found by analogous methods.

e) The cycles $(4\ \bar{4})$ and (7) give rise to a 1x1 matrix of $\underset{\sim}{H}_4 = [0]$ since $[\ell cm(1,1)]/1 = 1 = \text{odd}$.

f) The cycles (5 6) and (7) give rise to a 1x2 matrix $\underset{\sim}{H}_3 = [w\ w]$.

5) Collecting the preceding submatrices together, we obtain the following general structure for $\underset{\sim}{H}'$.

$$\underset{\sim}{H}' = \left[\begin{array}{ccc|c|cc|c} a & c & b & 0 & p & p & s \\ b & a & c & 0 & p & p & s \\ c & b & a & 0 & p & p & s \\ \hline 0 & 0 & 0 & d & r & -r & 0 \\ \hline h & h & h & q & e & f & u \\ h & h & h & -q & f & e & u \\ \hline t & t & t & 0 & w & w & g \end{array}\right] \tag{A-17}$$

6) Finally, we obtain the general structure of the desired hybrid matrix $\underset{\sim}{H}$ by using the following inverse transformation on $\underset{\sim}{H}'$:

$$\underset{\sim}{H} = \underset{\sim}{P}(\chi_\pi)\underset{\sim}{H}'\underset{\sim}{P}^T(\chi_\pi) =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c|cc|c} a & c & b & 0 & p & p & s \\ b & a & c & 0 & p & p & s \\ c & b & a & 0 & p & p & s \\ \hline 0 & 0 & 0 & d & r & -r & 0 \\ \hline h & h & h & q & e & f & u \\ h & h & h & -q & f & e & u \\ \hline t & t & t & 0 & w & w & g \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & p & -c & b & -p & s \\ 0 & d & r & 0 & 0 & r & a \\ h & q & e & -h & h & -f & u \\ -b & 0 & -p & a & -c & p & -s \\ c & 0 & p & -b & a & -p & s \\ -h & q & -f & h & -h & e & -u \\ t & 0 & w & -t & t & -w & g \end{bmatrix} \tag{A-18}$$

Example 2. To demonstrate the complete generality of our results, we will rederive the structure of the hybrid matrix of a 6-port resistor R first derived by Dorfman [18] which is symmetrical with respect to the following three unoriented permutation matrices[17]:

[17]It follows from the closure property of the symmetry group that R is also symmetric with respect to the group of 12 permutations generated by π_1, π_2, and π_3 via the composition operation.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} = (1\ 2\ 3)(4\ 5\ 6), \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (1\ 4)(2\ 5)(3\ 6),$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix} = (1\ 2)(3)(4\ 5)(6).$$

1) It is easy to see that the π_1-symmetry of $\underset{\sim}{H}$ requires that $\underset{\sim}{H}$ assumes the following structure:

$$\underset{\sim}{H} = \left[\begin{array}{ccc|ccc} a & c & b & \ell & n & m \\ b & a & c & m & \ell & n \\ c & b & a & n & m & \ell \\ \hline g & k & h & d & f & e \\ h & g & k & e & d & f \\ k & h & g & f & e & d \end{array}\right] \tag{A-19}$$

2) For the π_1-symmetric hybrid matrix $\underset{\sim}{H}$ in (A-19) to be also π_2-symmetric, we must have $d = a$, $e = b$, $f = c$, $\ell = g$, $m = h$, and $n = k$; namely

$$\underset{\sim}{H} = \begin{bmatrix} a & c & b & g & k & h \\ b & a & c & h & g & k \\ c & b & a & k & h & g \\ g & k & h & a & c & b \\ h & g & k & b & a & c \\ k & h & g & c & b & a \end{bmatrix} \tag{A-20}$$

3) Finally for π_1 <u>and</u> π_2-symmetric hybrid matrix $\underset{\sim}{H}$ in (A-20) to be also π_3-symmetric, we must further require $b = c$ and $k = h$; namely,

$$\underset{\sim}{H} = \begin{bmatrix} a & b & b & g & h & h \\ b & a & b & h & g & h \\ b & b & a & h & h & g \\ g & h & h & a & b & b \\ h & g & h & b & a & b \\ h & h & g & b & b & a \end{bmatrix} \tag{A-21}$$

This hybrid matrix is precisely the one derived by Dorfman via a much more involved method.

APPENDIX B: ALGORITHM FOR FINDING THE SYMMETRY GROUP OF AN n-PORT RESISTOR

Let R be an n-port resistor. Let G denote the collection of all directed π-permutations such that R is π-symmetric. Then Prop. 4 shows that G forms a <u>subgroup</u> of the group P_n of all $2^n n!$ directed permutations. Let B denote the complement of G, hence $\pi \in B \Leftrightarrow \pi \notin G$. Clearly G can be generated by exhaustively checking each directed permutation $\pi \in P_n$. The following algorithm makes use of simple group-theoretic concepts to generate G in a much more efficient way. Our algorithm is based on the following properties of composition between two directed permutations:[18]

(1) If $\pi \in G$ and $\sigma \in G$,
then $\pi^{-1} \in G$, $(\pi \circ \sigma) \in G$, $(\sigma \circ \pi) \in G$ and $\rho \in G$ whenever $\rho = \pi^i$ $(i \in Z)$.

(2) If $\pi \in B$ and $\sigma \in G$,
then $\pi^{-1} \in B$, $(\pi \circ \sigma) \in B$, $(\sigma \circ \pi) \in B$ and $\rho \in B$, whenever $\rho^i = \pi (i \in Z)$.[19]

These properties can be easily verified using the "closure" property of a group. For example, to prove that $(\pi \circ \sigma) \in B$ in (2), suppose the contrary, then $(\pi \circ \sigma) \in G$. This implies $((\pi \circ \sigma) \circ \sigma^{-1}) \in G$ and hence $\pi \in G$, a contradiction. It follows from properties (1) and (2) that G is a group, and B is a union of some cosets of G[13-14].

The flow chart for our algorithm is shown in Fig. 18. We start with G containing the identity permutation I, and B being equal to the empty set ϕ. For each $\pi \in (B \cup G)$, we check whether $\pi \in G$, or $\pi \in B$. In the former case, we use property (1) to generate the following family

$$\{\pi^i \circ \sigma | \sigma \in G,\ i \in N\} \cup \{\sigma \circ \pi^i | \sigma \in G,\ i \in N\}$$

(where N denotes the set of nonnegative integers) and place them in G. Simultaneously, we use property (2) to generate the following family

$$\{(\rho \circ \sigma)^j | \rho \in B,\ \sigma \in G,\ j = \pm 1\}$$

and place them in B. On the other hand, if $\pi \in B$, then we use property (2) to generate the following family

---

[18] As a mnemonic tool, we choose G and B to stand for <u>G</u>ood or <u>B</u>ad subsets of P_n. Property (1) has been used in [19] for finding partial symmetries of Boolean functions.

[19] If $\rho^i = \pi$ where $i \in Z$ (Z denotes the set of all natural numbers), we say ρ is an i-th order <u>root</u> of π.

$$\{\rho\circ\sigma | \sigma \in G,\ \rho^i = \pi,\ i \in Z\} \cup \{\sigma\circ\rho | \sigma \in G,\ \rho^i = \pi,\ i \in Z\} \cup B$$

and call this the updated set B. The next step is to count the total number of elements present in G and B. If $\#G + \#B = 2^n n!$, where $\#G$ and $\#B$ denote the number of elements in G and B, respectively, we are done. Otherwise, return to step (2) and choose another directed permutation belonging neither to G nor B, and repeat steps (3) and (4).

Clearly, the choice of the new element in step (2) affects the efficiency of this algorithm in a significant way. If $\pi \in G$ and is of a high order, then many new elements of both G and B will be efficiently generated. If $\pi \in B$ has a high-order root ρ such that $\rho^i = \pi$, then many new elements in B will be efficiently generated. Hence, a good strategy is to choose a new element π which has either a high order ($\ell >> 1$, $\pi^\ell = I$), or has many roots ($\ell >> 1$, $\rho^\ell = \pi$). Unfortunately, these two criteria often represent conflicting requirements. Let us now consider a specific example:

<u>Example</u>. Let R be a 2-port resistor characterized by:

$$v_1 = i_2^3$$

$$v_2 = -i_1^3$$

Since $n = 2$, there are eight possible directed permutations. (1) Initialization: $G = \left\{\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\right\}$, $B = \{\phi\}$. (2) Choose a directed permutation $\pi_1 = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$. (3) We check and find $\pi_1 \notin G$. Hence $\pi_1 \in B$ and $B = \left\{\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\right\}$. We return to (2) and choose $\pi_2 = \begin{pmatrix}1 & 2\\ \bar{1} & \bar{2}\end{pmatrix}$. We check and find $\pi_2 \in G$. Hence $G = \left\{\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{1} & \bar{2}\end{pmatrix}\right\}$ and $B = \left\{\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{2} & \bar{1}\end{pmatrix}\right\}$. Since $\#G + \#B = 4$, we return to (2) and choose $\pi_3 = \begin{pmatrix}1 & 2\\ 1 & \bar{2}\end{pmatrix}$. We check and find $\pi_3 \notin G$. Hence $B = \left\{\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{2} & \bar{1}\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & \bar{2}\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{1} & 2\end{pmatrix}\right\}$. Since $\#G + \#B = 6$, we return to (2) and choose $\pi_4 = \begin{pmatrix}1 & 2\\ \bar{2} & 1\end{pmatrix}$. We check and find $\pi_4 \in G$. Hence $G = \left\{\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{2} & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & \bar{1}\end{pmatrix}, \begin{pmatrix}1 & 2\\ \bar{1} & \bar{2}\end{pmatrix}\right\}$. Since $\#B + \#G = 8$, G is the symmetry group of the 2-port resistor R.

APPENDIX C: EXAMPLE ILLUSTRATING THE SYMMETRY REDUCTION ALGORITHM

To illustrate the application of the symmetry reduction algorithm, consider the following hybrid matrix

$$\mathbf{H} = \begin{bmatrix} a & 0 & p & -c & b & -p & s \\ 0 & d & r & 0 & 0 & r & 0 \\ h & q & e & -h & h & -f & u \\ -b & 0 & -p & a & -c & p & -s \\ c & 0 & p & -b & a & -p & s \\ -h & q & -f & h & -h & e & -u \\ t & 0 & w & -t & t & -w & g \end{bmatrix} \qquad \text{(C-1)}$$

The structure of $\mathbf{H}$ in (C-1) has been derived earlier in (A-18) for a π-symmetric 7-port resistor R, where $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \bar{4} & \bar{2} & \bar{6} & \bar{5} & 1 & \bar{3} & 7 \end{pmatrix}$. If we drive R with the π-symmetric excitation $\mathbf{x} = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7]^T = [1\ 0\ 2\ -1\ 1\ -2\ 3]^T = \mathbf{P}(\pi)\mathbf{x}$, we would obtain by direct substitution the response $\mathbf{y} = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7]^T$, where $y_1 = -y_4 = y_5 = a + b + c + 4p + 3s$, $y_2 = 0$, $y_3 = -y_6 = 2(e+f) + 3(h+u)$, and $y_7 = 3(t+g) + 4w$. To obtain $\mathbf{y}$ using the reduction algorithm, we proceed as follows:

1) Decompose π into cyclic components: $\pi = (1\ \bar{4}\ 5)(2\ \bar{2})(3\ \bar{6})(7)$.

2) Form the 7x3 matrix:

$$\mathbf{S}(\pi) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T .$$

3) Form the 3x3 reduced hybrid matrix:

$$\mathbf{H}_0 = \mathbf{S}^T(\pi)\mathbf{H}\mathbf{S}(\pi) = \begin{bmatrix} 3(a+b+c) & 6p & 3s \\ 6h & 2(e+f) & 2u \\ 3t & 2w & g \end{bmatrix}$$

4) Form the 3x1 π-reduced excitation $\mathbf{x}_0$, by inspection, such that

$\mathbf{x} = \mathbf{S}(\pi)\mathbf{x}_0$; namely, $\mathbf{x}_0 = [1\ 2\ 3]^T$.

5) Calculate the 3x1 π-reduced response:

$$\mathbf{y}_0 = \mathbf{H}_0\mathbf{x}_0 = \begin{bmatrix} 3(a+b+c) + 12p + 9s \\ 6h + 4(e+f) + 6u \\ 3(t+g) + 4w \end{bmatrix}$$

6) Recover the original response $\mathbf{y}$ by using (57):

$$\underset{\sim}{y} = \underset{\sim}{S}(\pi) \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{y}_0 = \underset{\sim}{S}(\pi) \begin{bmatrix} a + b + c + 4p + 3s \\ 6h + 4(e+f) + 6u \\ 3(t+g) + 4w \end{bmatrix} .$$

Hence indeed the same response is obtained. Observe the savings in computation could be quite significant if the dimension of the hybrid matrix $\underset{\sim}{H}$ is large and if the response due to several different π-symmetric excitations are to be computed. Observe also that in many cases, $\underset{\sim}{H}_0$ is either known or can be easily obtained by inspection of the reduced resistor.

Now suppose the response $\underset{\sim}{y}$ of the above resistor R due to an anti-π-symmetric excitation $\underset{\sim}{x} = -\underset{\sim}{P}(\pi)\underset{\sim}{x}$ is desired. Since R is linear, $\underset{\sim}{H}$ is <u>complementary</u> symmetric and it follows from <u>Prop. 23</u> that R is $\bar{\pi}$-symmetric, where $\bar{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 6 & 5 & \bar{1} & 3 & \bar{7} \end{pmatrix}$, and the response of R to an anti-π-symmetric excitation is identical to the response of R to a π-symmetric excitation. To verify this, consider the 7x1 $\bar{\pi}$-symmetric excitation $\underset{\sim}{x} = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7] = [0\ 1\ 2\ 0\ 0\ 2\ 0]^T$. By direct substitution, we calculate $\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{x} = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7]^T$, where $y_1 = y_4 = y_5 = y_1 = 0$, $y_2 = 4r + d$, and $y_3 = y_6 = g + 2(e-f)$. Now using the reduction algorithm, we proceed as follows:

1) $\bar{\pi} = (1\ 4\ 5\ \bar{1}\ \bar{4}\ \bar{5})(2)(3\ 6)(7\ \bar{7})$

2) $\underset{\sim}{S}(\bar{\pi}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^T$

3) $\underset{\sim}{H}_0 = \underset{\sim}{S}^T(\bar{\pi})\underset{\sim}{H}\underset{\sim}{S}(\bar{\pi}) = \begin{bmatrix} d & 2r \\ 2g & 2(e-f) \end{bmatrix}$

4) $\underset{\sim}{x}_0 = [1\ 2]^T$

5) $\underset{\sim}{y}_0 = [d+4r\quad 2g+4(e-f)]^T$

6) $\underset{\sim}{y} = \underset{\sim}{S}(\bar{\pi}) \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \underset{\sim}{y}_0 = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7]^T$.

Again, the same response is obtained, as it should be.

APPENDIX D: SYMMETRY CONDITIONS IN TERMS OF GENERALIZED COORDINATES

In many cases the obvious port variables (resp., terminal variables) voltage $\underset{\sim}{v}$ and current $\underset{\sim}{i}$ are often not the most convenient choices. It is sometimes desirable to introduce a pair of generalized coordinates $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ which include as special cases the hybrid variables consisting of a mixture of both currents and voltages, as well as the scattering variables consisting of the incident and reflected voltages. Our objective in this section is to derive the conditions which guarantee that the results obtained in the preceding sections in terms of the variables $\underset{\sim}{v}$ and $\underset{\sim}{i}$ can be translated in terms of $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ by simply substituting $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ for $(\underset{\sim}{v},\underset{\sim}{i})$.

Definition D1. Given the voltage n-vector $\underset{\sim}{v}$ and the current n-vector $\underset{\sim}{i}$, the generalized coordinate variables $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are n-vectors defined by the linear transformation

$$\begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} \triangleq \underset{\sim}{\Omega} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix}, \tag{D-1}$$

where $\underset{\sim}{\Omega}$ is a $2n \times 2n$ non-singular real constant matrix.[20]

Since $\underset{\sim}{\Omega}$ is invertible the inverse transformation is given by

$$\begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\alpha} & \underset{\sim}{\beta} \\ \underset{\sim}{\gamma} & \underset{\sim}{\delta} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} = \underset{\sim}{\Omega}^{-1} \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} \tag{D-2}$$

This is a very general transformation which includes many interesting special cases. For example, the previously defined mixed variables (16) form an interesting special case of generalized coordinates. From (16) we see that in this case

$$\underset{\sim}{\Omega} = \underset{\sim}{\Omega}^{-1} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} \end{bmatrix}. \tag{D-3}$$

Another interesting case applicable for linear n-port or n-terminal resistors is the complex coordinate transformation

$$\underset{\sim}{\Omega} = \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{r}^{1/2} + \frac{1}{2}\underset{\sim}{r}^{-1/2}(\underset{\sim}{z}-\underset{\sim}{z}^*) & \underset{\sim}{r}^{1/2} - \frac{1}{2}\underset{\sim}{r}^{-1/2}(\underset{\sim}{z}-\underset{\sim}{z}^*) \\ -\underset{\sim}{r}^{-1/2} & \underset{\sim}{r}^{-1/2} \end{bmatrix} \tag{D-4}$$

where

---

[20] In the case of linear resistors even a complex $\underset{\sim}{\Omega}$ is allowed.

$$\underset{\sim}{z} = \begin{bmatrix} z_1 & & 0 \\ & z_2 & \\ & & \ddots \\ 0 & & & z_n \end{bmatrix} = \begin{bmatrix} r_1+j\chi_1 & & & 0 \\ & r_2+j\chi_2 & & \\ & & \ddots & \\ 0 & & & r_n+j\chi_n \end{bmatrix} \tag{D-5}$$

and where $(r+j\chi)^* = r - j\chi$ and $r_i \geq 0$, for $1 \leq i \leq n$. The complex number z_i is called the <u>normalization port number</u> (resp., <u>terminal number</u>). The inverse coordinate transformation is

$$\underset{\sim}{\Omega}^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\underset{\sim}{r}^{-1/2} & -\frac{1}{2}\underset{\sim}{z}^*\underset{\sim}{r}^{-1/2} \\ \frac{1}{2}\underset{\sim}{r}^{-1/2} & \frac{1}{2}\underset{\sim}{z}\underset{\sim}{r}^{-1/2} \end{bmatrix} \tag{D-6}$$

The corresponding generalized coordinate variables defined by (D-1) are called the <u>incident voltage vector</u> $\underset{\sim}{\eta}$, and the <u>reflected voltage vector</u> $\underset{\sim}{\xi}$, respectively.

Since there is in general a one-to-one relationship between the vectors $(\underset{\sim}{v},\underset{\sim}{i})$ and the vectors $(\underset{\sim}{\xi},\underset{\sim}{\eta})$, any multiport resistor R (resp., multiterminal resistor $\mathcal{R}$) is characterized by a set S' of <u>generalized admissible pairs</u> $(\underset{\sim}{\xi},\underset{\sim}{\eta})$, obtained by substituting the admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ of (D-1) into (D-2).

Using equations (D-1), (D-2) and Def. 7 it is possible to check whether a resistor characterized by a set S' of generalized admissible pairs is π-symmetric. However, all algebraic results (which have been expressed in terms of $\underset{\sim}{v}$ and $\underset{\sim}{i}$) of this paper can be translated in terms of $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ if the π-symmetry can be verified by simply checking

$$(\underset{\sim}{\xi},\underset{\sim}{\eta}) \in S' \Longleftrightarrow \left(\underset{\sim}{P}(\pi)\underset{\sim}{\xi},\underset{\sim}{P}(\pi)\underset{\sim}{\eta}\right) \in S'. \tag{D-7}$$

Observe that (D-7) consists of a simple substitution of the variables $\underset{\sim}{v}$, $\underset{\sim}{i}$ of Def. 7 by $\underset{\sim}{\xi}$, $\underset{\sim}{\eta}$ and is therefore much easier to check. Conditions on $\underset{\sim}{\Omega}$ guaranteeing (D-7) are given in Prop. D1.

<u>Proposition D1</u>. Let π be a directed permutation and let $\underset{\sim}{\Omega}$ be a coordinate transformation matrix such that

$$\begin{bmatrix} \underset{\sim}{P}(\pi) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}(\pi) \end{bmatrix} \underset{\sim}{\Omega} = \underset{\sim}{\Omega} \begin{bmatrix} \underset{\sim}{P}(\pi) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}(\pi) \end{bmatrix}. \tag{D-8}$$

Then an n-port resistor R (resp., n-terminal resistor $\mathcal{R}$) is π-symmetric iff (D-7) is satisfied. In other words, $(\underset{\sim}{\xi},\underset{\sim}{\eta})$ is a generalized admissible pair of R iff $\left(\underset{\sim}{P}(\pi)\underset{\sim}{\xi},\underset{\sim}{P}(\pi)\underset{\sim}{\eta}\right)$ is a generalized admissible pair of R.

Proof: From (D-1), (D-2) and Def. 7 we see that R (resp., $\mathcal{R}$) is π-symmetric iff

$$(\underset{\sim}{\xi},\underset{\sim}{\eta}) \in S' \Leftrightarrow (\hat{\underset{\sim}{\xi}},\hat{\underset{\sim}{\eta}}) \in S' \tag{D-9}$$

where

$$\begin{bmatrix} \hat{\underset{\sim}{\xi}} \\ \hat{\underset{\sim}{\eta}} \end{bmatrix} = \underset{\sim}{\Omega}^{-1} \begin{bmatrix} \underset{\sim}{P}(\pi) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}(\pi) \end{bmatrix} \underset{\sim}{\Omega} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix} . \tag{D-10}$$

Using (D-8) we can reduce (D-10) to $\hat{\underset{\sim}{\xi}} = \underset{\sim}{P}(\pi)\underset{\sim}{\xi}$ and $\hat{\underset{\sim}{\eta}} = \underset{\sim}{P}(\pi)\underset{\sim}{\eta}$. ¤

In the "mixed variable" case, i.e., when $\underset{\sim}{\Omega}$ is given by (D-3), it is easily checked that (D-8) is equivalent to

$$\underset{\sim}{P}(\pi)\underset{\sim}{A} = \underset{\sim}{A}\underset{\sim}{P}(\pi) \tag{D-11a}$$

$$\underset{\sim}{P}(\pi)\underset{\sim}{B} = \underset{\sim}{B}\underset{\sim}{P}(\pi) . \tag{D-11b}$$

Observe that (D-11) is precisely the condition derived earlier in (18) which requires that the mixed variables be <u>compatible with π</u>, as it should be.

In the "scattering variable" case defined by ((D-4)-(D-6)), we can simplify (D-8) by first recasting it into the following equivalent form:

$$\underset{\sim}{\Omega}^{-1} \begin{bmatrix} \underset{\sim}{P}(\pi) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}(\pi) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{P}(\pi) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}(\pi) \end{bmatrix} \underset{\sim}{\Omega}^{-1} . \tag{D-8'}$$

Since $\underset{\sim}{P}(\pi)$ and $\underset{\sim}{r}$ are real matrices, (D-8') is equivalent to the following:

$$\frac{1}{2}\,\underset{\sim}{r}^{-1/2}\underset{\sim}{P}(\pi) = \frac{1}{2}\,\underset{\sim}{P}(\pi)\underset{\sim}{r}^{-1/2} \tag{D-12a}$$

$$\frac{1}{2}\,\underset{\sim}{z}\underset{\sim}{r}^{-1/2}\underset{\sim}{P}(\pi) = \frac{1}{2}\,\underset{\sim}{P}(\pi)\underset{\sim}{z}\underset{\sim}{r}^{-1/2} , \tag{D-12b}$$

Substituting (D-12a) into (D-12b) we obtain

$$\underset{\sim}{z}\underset{\sim}{P}(\pi) = \underset{\sim}{P}(\pi)\underset{\sim}{z} . \tag{D-13}$$

It is easy to check that (D-12a) and (D-12b) are equivalent to (D-13). Using the cyclic decomposition of a directed permutation, (Section II) we can interpret (D-13) as follows: The <u>normalization port numbers</u> (resp., <u>terminal numbers</u>) associated with the ports (resp., terminals) belonging to the same cyclic component of π must be the same.

It follows from Prop. D1 that if (D-8) is satisfied, then Prop. 13, 14 and the Cor. of Prop. 14 can be immediately rephrased in terms of generalized coordinates.

<u>Corollary</u>. Given a pair of generalized coordinate variables $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$, a generalized coordinate transformation matrix $\underset{\sim}{\Omega}$, and a directed permutation π satisfying (D-8). Then we have the following equivalent symmetry conditions:

1) A multiport resistor R with constitutive relation $\underset{\sim}{R}(\underset{\sim}{\xi},\underset{\sim}{\eta}) = \underset{\sim}{0}$ is π-symmetric iff

$$\underset{\sim}{R}(\underset{\sim}{\xi},\underset{\sim}{\eta}) = \underset{\sim}{0} \Longleftrightarrow \underset{\sim}{R}(\underset{\sim}{P}(\pi)\underset{\sim}{\xi},\underset{\sim}{P}(\pi)\underset{\sim}{\eta}) = \underset{\sim}{0}. \tag{D-14}$$

2) A mutliport resistor R with constitutive relation $\underset{\sim}{\xi} = \underset{\sim}{h}(\underset{\sim}{\eta})$ is π-symmetric iff

$$\underset{\sim}{h}(\cdot) = \underset{\sim}{P}^T(\pi)\underset{\sim}{h}(\underset{\sim}{P}(\pi)\cdot). \tag{D-15}$$

3) A linear multiport resistor R with constitutive relation $\underset{\sim}{\xi} = \underset{\sim}{\Lambda}\underset{\sim}{\eta}$ (Λ is called the <u>constitutive matrix</u>) is π-symmetric iff

$$\underset{\sim}{\Lambda}\underset{\sim}{P}(\pi) = \underset{\sim}{P}(\pi)\underset{\sim}{\Lambda}. \tag{D-16}$$

To illustrate this Corollary, let us apply (D-16) to the scattering matrix of a <u>circulator</u>. To derive the scattering matrix from the current controlled representation $\underset{\sim}{v} = \underset{\sim}{Z}\underset{\sim}{i}$ given in (24), we choose the normalization port numbers $z_1 = z_2 = z_3 = R$ in ((D-4)-(D-6)). The generalized coordinate transformation matrix is

$$\underset{\sim}{\Omega} = R^{1/2}\begin{bmatrix} \underset{\sim}{1}_3 & \underset{\sim}{1}_3 \\ -R^{-1}\underset{\sim}{1}_3 & R^{-1}\underset{\sim}{1}_3 \end{bmatrix}.$$

The incident and reflected voltages $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are given by:

$$\underset{\sim}{\xi} = \frac{1}{2}R^{-1/2}(\underset{\sim}{v}-R\underset{\sim}{i}) \tag{D-17a}$$

$$\underset{\sim}{\eta} = \frac{1}{2}R^{-1/2}(\underset{\sim}{v}+R\underset{\sim}{i}) \tag{D-17b}$$

The scattering representation is then given by

$$\underset{\sim}{\xi} = [\underset{\sim}{Z}-R\underset{\sim}{1}_3][\underset{\sim}{Z}+R\underset{\sim}{1}_3]^{-1}\underset{\sim}{\eta}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\underset{\sim}{\eta} = \underset{\sim}{S}\underset{\sim}{\eta} \tag{D-18}$$

where $\underset{\sim}{S}$ is the scattering matrix of the circulator, whose current controlled representation has been shown in Section III to be π-symmetric, where $\pi = (1\ 2\ 3)$. Substituting $\underset{\sim}{P}(\pi)$ and $\underset{\sim}{\Omega}$ into (D-8), we can verify that this equation is satisfied; namely,

$$
\left[\begin{array}{ccc|ccc}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\left[\begin{array}{ccc|ccc}
R^{1/2} & 0 & 0 & R^{1/2} & 0 & 0 \\
0 & R^{1/2} & 0 & 0 & R^{1/2} & 0 \\
0 & 0 & R^{1/2} & 0 & 0 & R^{1/2} \\
\hline
-R^{-1/2} & 0 & 0 & R^{-1/2} & 0 & 0 \\
0 & -R^{-1/2} & 0 & 0 & R^{-1/2} & 0 \\
0 & 0 & -R^{-1/2} & 0 & 0 & R^{-1/2}
\end{array}\right]
=
$$

$$
\left[\begin{array}{ccc|ccc}
R^{1/2} & 0 & 0 & R^{1/2} & 0 & 0 \\
0 & R^{1/2} & 0 & 0 & R^{1/2} & 0 \\
0 & 0 & R^{1/2} & 0 & 0 & R^{1/2} \\
\hline
-R^{-1/2} & 0 & 0 & R^{-1/2} & 0 & 0 \\
0 & -R^{-1/2} & 0 & 0 & R^{-1/2} & 0 \\
0 & 0 & -R^{-1/2} & 0 & 0 & R^{-1/2}
\end{array}\right]
\left[\begin{array}{ccc|ccc}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] . \qquad \text{(D-19)}
$$

The π-symmetry of the circulator can be proved by substituting $\underset{\sim}{S}$ for $\underset{\sim}{\Lambda}$ in (D-16); namely,

$$
\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \qquad \text{(D-20)}
$$

FIGURE CAPTIONS

Fig. 1. (a) A reflection-symmetric 2-port resistor R, (b) a 120°-rotational symmetric 3-terminal resistor $\mathcal{R}$ and (c) a 4-terminal or grounded 3-port OP AMP, whose dc circuit model exhibits complementary symmetry.

Fig. 2. (a) Back-to-back series connection and (b) back-to-front parallel connection.

Fig. 3. A nonlinear bridge circuit as a symmetric 2-port.

Fig. 4. Geometric interpretation of the two types of cyclic directed permutations (a) normal-order cycle and (b) double-order cycle.

Fig. 5. An n-port resistor R with port variables satisfying the associated reference convention.

Fig. 6. The $\begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & 3 & \bar{2} \end{pmatrix}$-permuted 3-port resistor $\hat{R}$ associated with R.

Fig. 7. A hybrid coil.

Fig. 8. Example of interconnection of two 2-port resistors: port 1 of $R^{(1)}$ is connected in series with port 1 of $R^{(2)}$, and port 2 of $R^{(1)}$ is connected in parallel with port 2 of $R^{(2)}$.

Fig. 9. Synthesis of a $\begin{pmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{pmatrix}$-symmetric 2-port resistor which gives rise to a push-pull amplifier.

Fig. 10. Synthesis of a π-symmetric 3-port resistor $\tilde{R}$ by interconnecting identical 2-port resistors $R^{(1)}$, $R^{(2)}$ and/or $R^{(3)}$ using Algorithm 1.

Fig. 11. A $\begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & \bar{3} & \bar{2} & 6 \end{pmatrix}$-symmetric 4-port resistor $\tilde{R}$, obtained by interconnecting a $\begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{3} & \bar{2} \end{pmatrix}$-symmetric 3-port resistor $R^{(1)}$ and a $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$-symmetric 3-port resistor $R^{(2)}$ using Algorithm 2.

Fig. 12. (a) A 4-port resistor which exhibits $(1\ \bar{1})(2)(3\ 4)$- and $(1)(2\ \bar{2})(3\ \bar{3})(4\ \bar{4})$-symmetry. (b) and (c), two possible interconnections of ports 3 and 4 of (a) using Algorithm 2.

Fig. 13. (a) An n-terminal resistor $\mathcal{R}$ and (b) a grounded n-terminal resistor $\mathcal{R}$, both with terminal variables satisfying the associated reference convention.

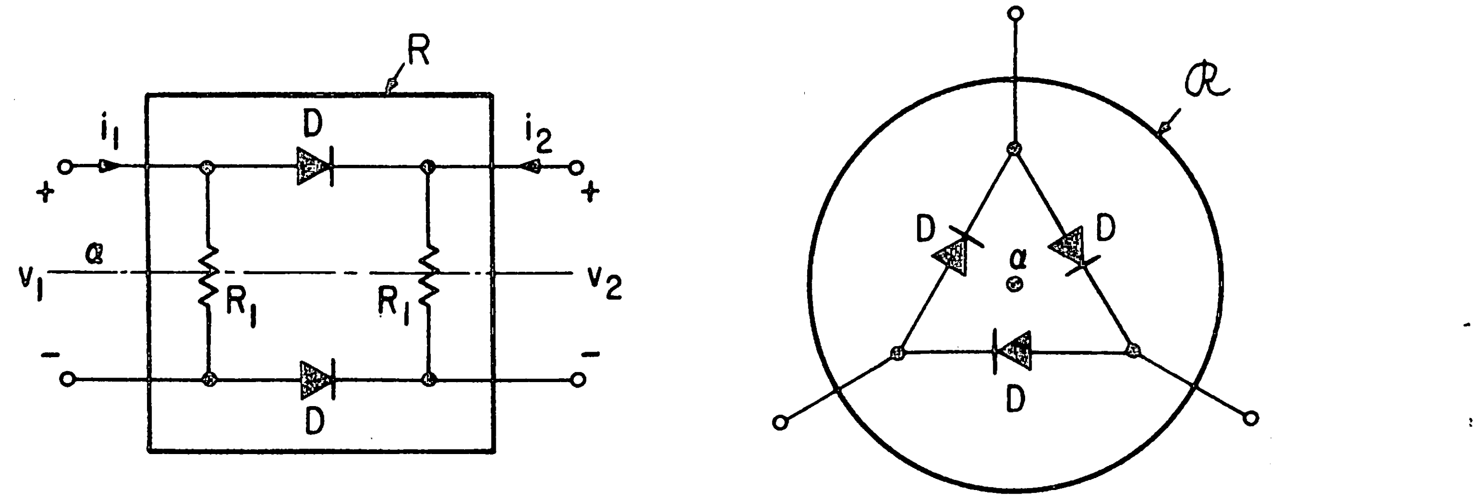
Fig. 14. A π-permuted resistor $\hat{\mathcal{R}}$ associated with a 3-terminal resistor $\mathcal{R}$, where $\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{3} & \bar{2} \end{pmatrix}$.

Fig. 15. Synthesis of a π-symmetric 3-terminal resistors by interconnection of identical 3-terminal resistors using Algorithm 1'.

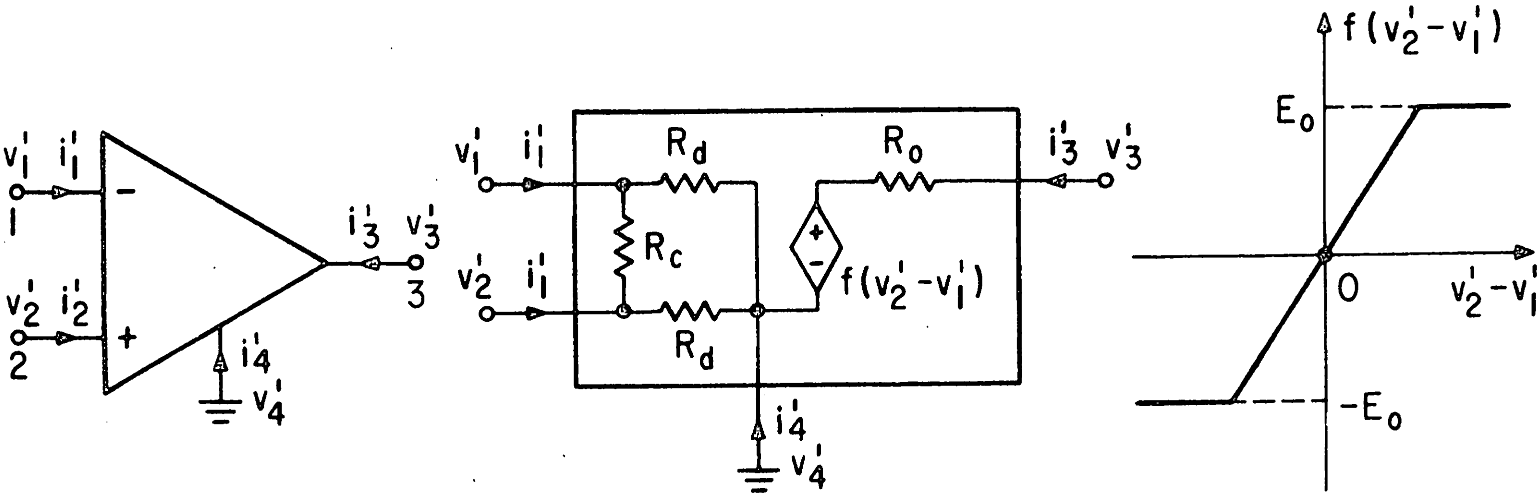
Fig. 16. Synthesis of symmetric 4-terminal resistors from identical 2-terminal resistors using Algorithm 1'.

Fig. 17. Example of a π-symmetrically reduced 3-port resistor R_o of a 7-port resistor R and (b) an example of a π-symmetrically reduced 2-terminal resistor $\mathcal{R}_o$ of a 5-terminal resistor $\mathcal{R}$.

Fig. 18. Algorithm for finding the symmetry group of an n-port resistor R.

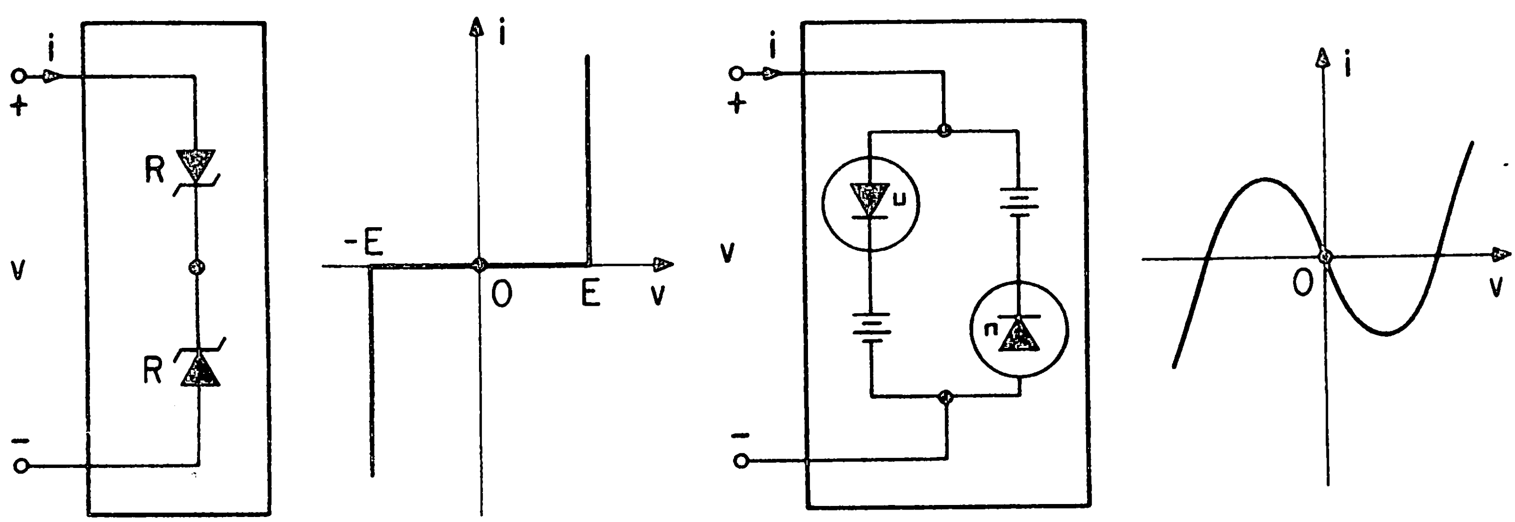

R
i1
i2
D
D
+
+
a
v1
v2
R1
R1
-
-
ℛ
α

(a) (b)


v'1
i'1
1
v'2
i'2
2
i'3
v'3
3
i'4
v'4
Rd
Ro
Rc
Rd
f(v'2 - v'1)
Eo
-Eo
O
v'2 - v'1

(c)

Fig. 1


i
+
R
v
R
-
i
-E
O
E
v
i
+
u
v
n
-
i
O
v

(a) (b)

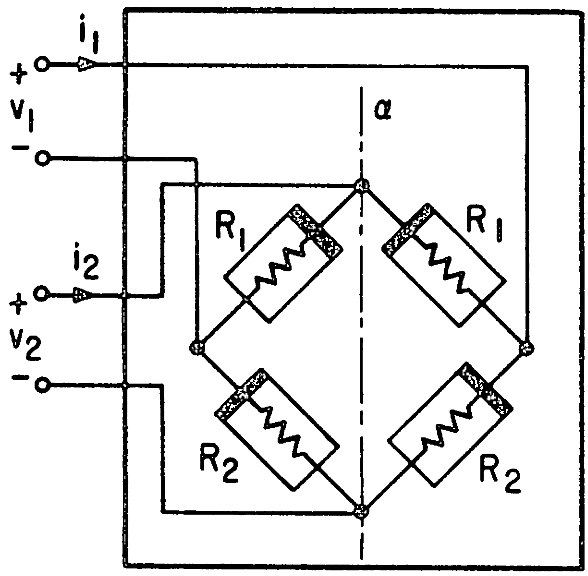


Fig. 3

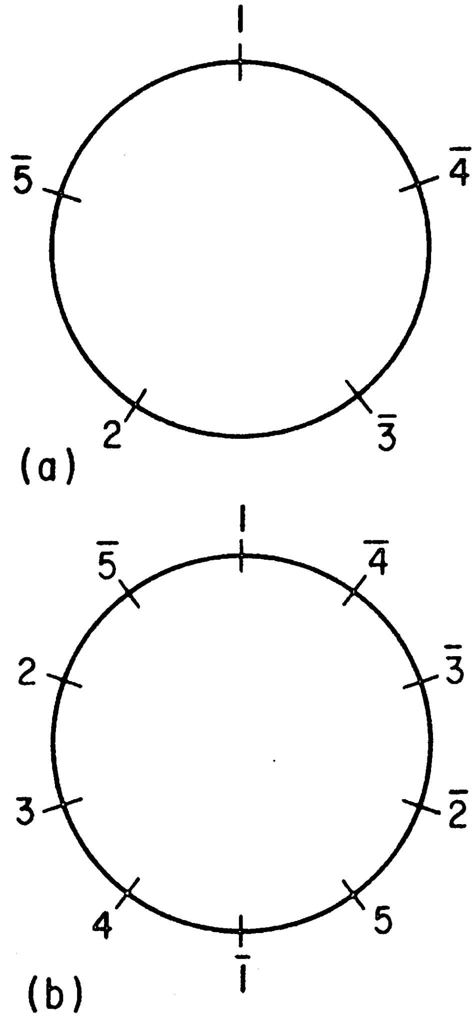


Fig. 4

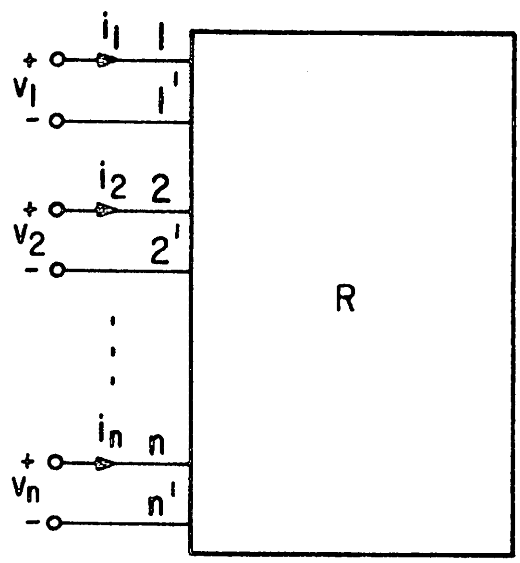


Fig. 5

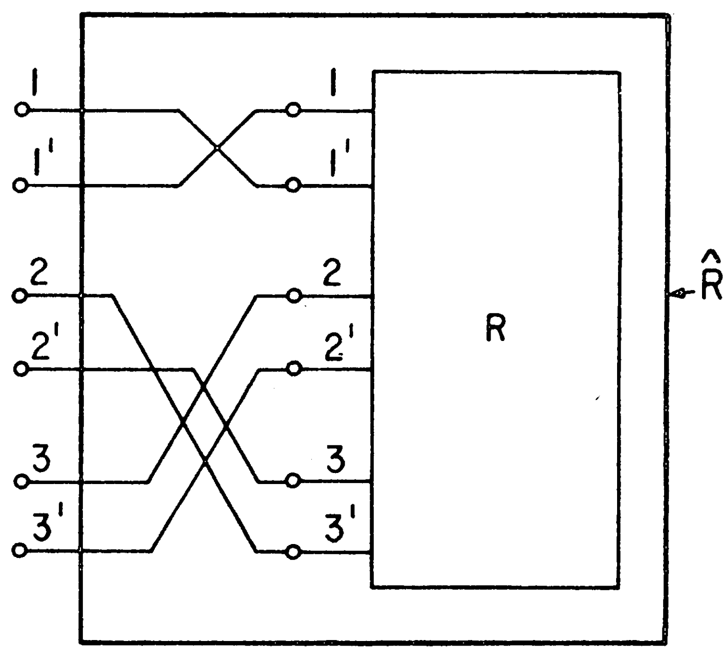


Fig. 6

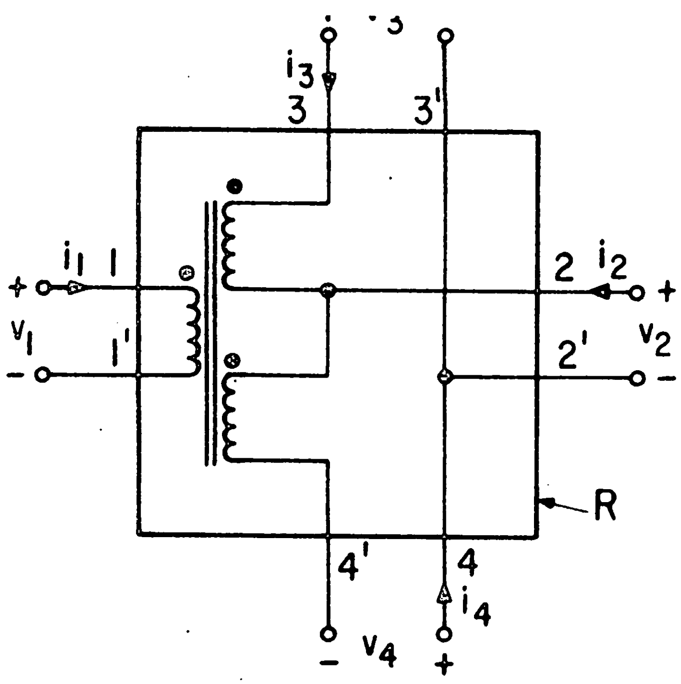


Fig. 7

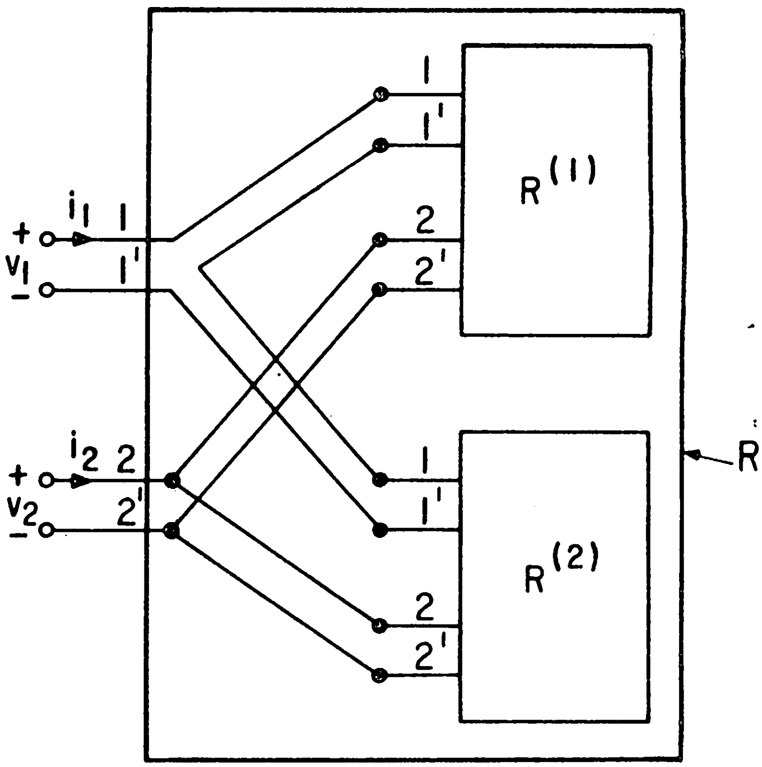


Fig. 8

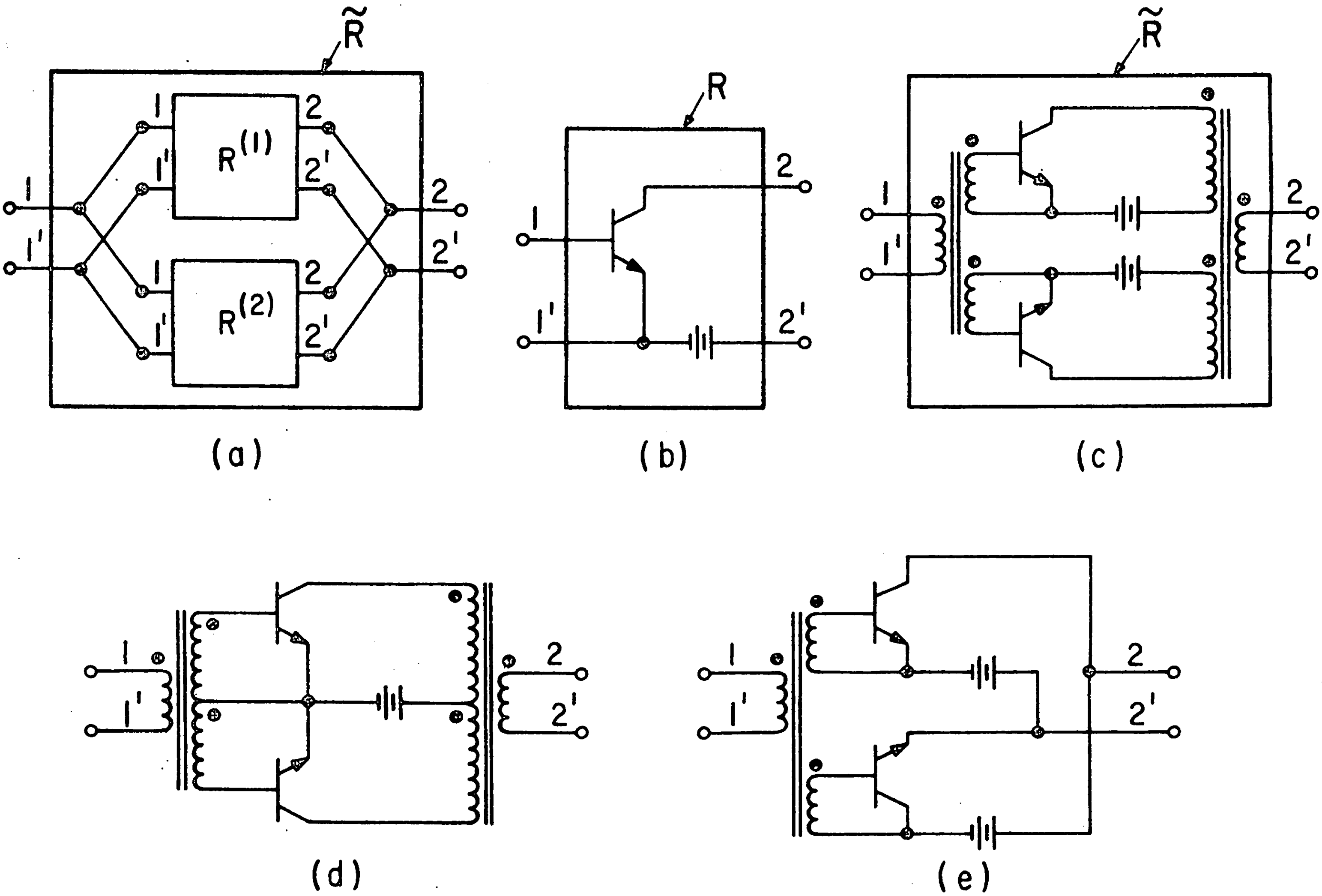


Fig.9

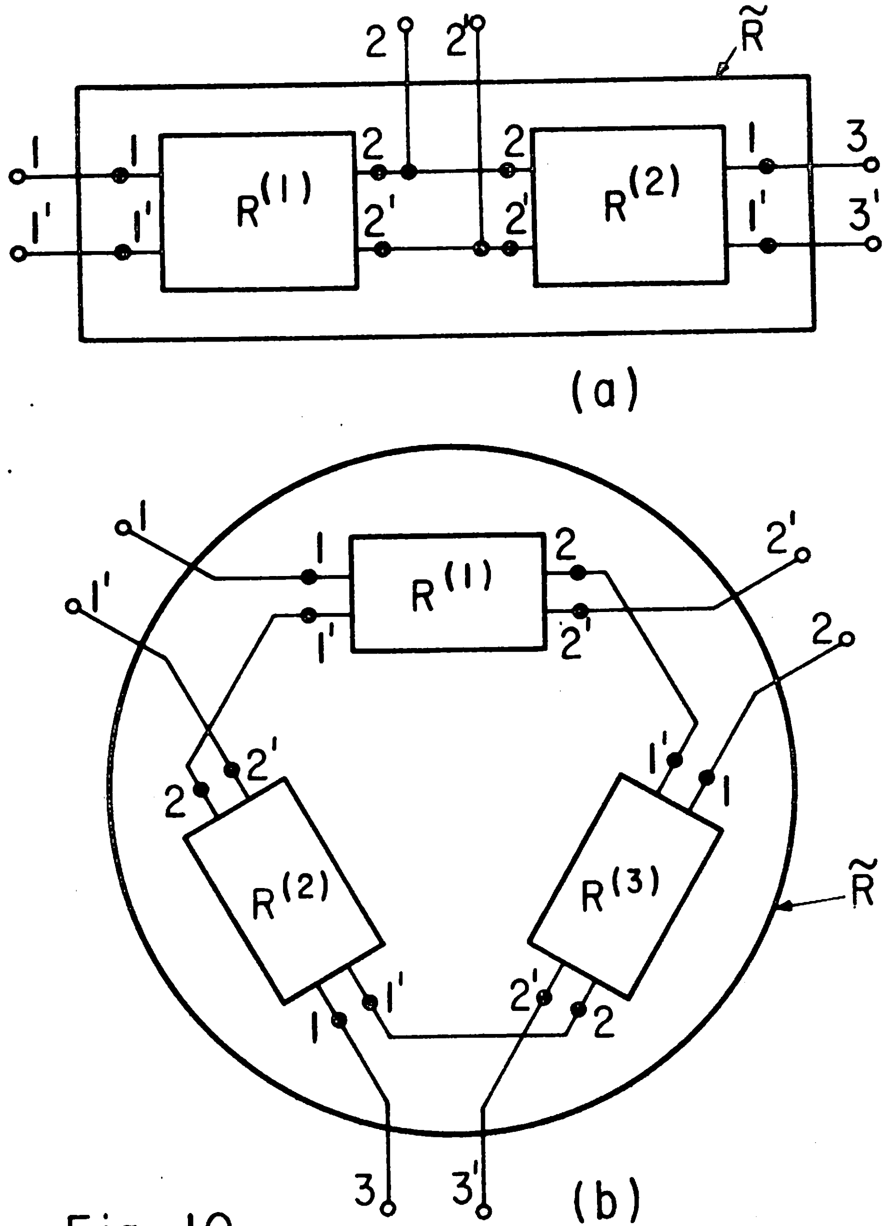


Fig. 10

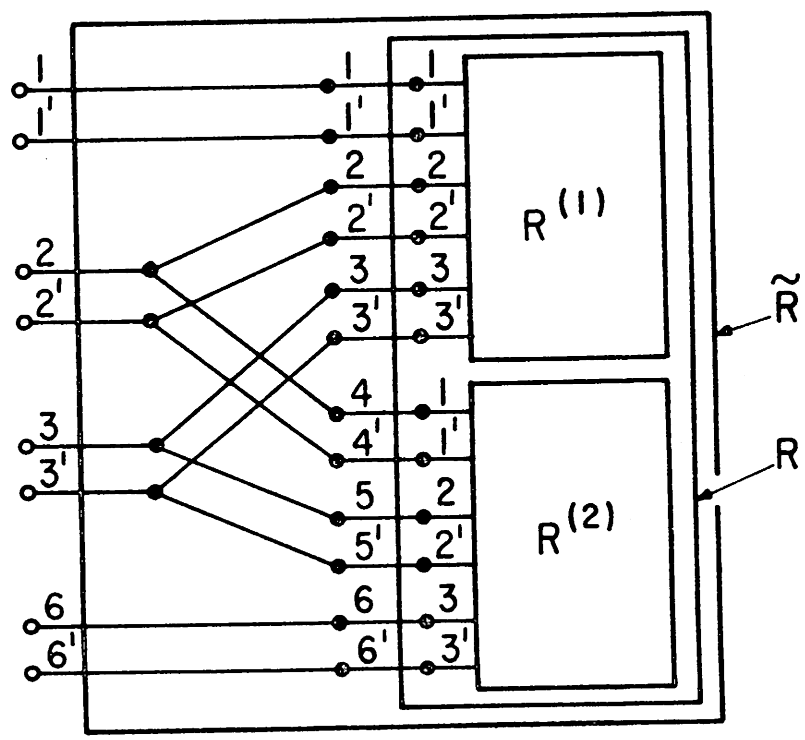


Fig. 11

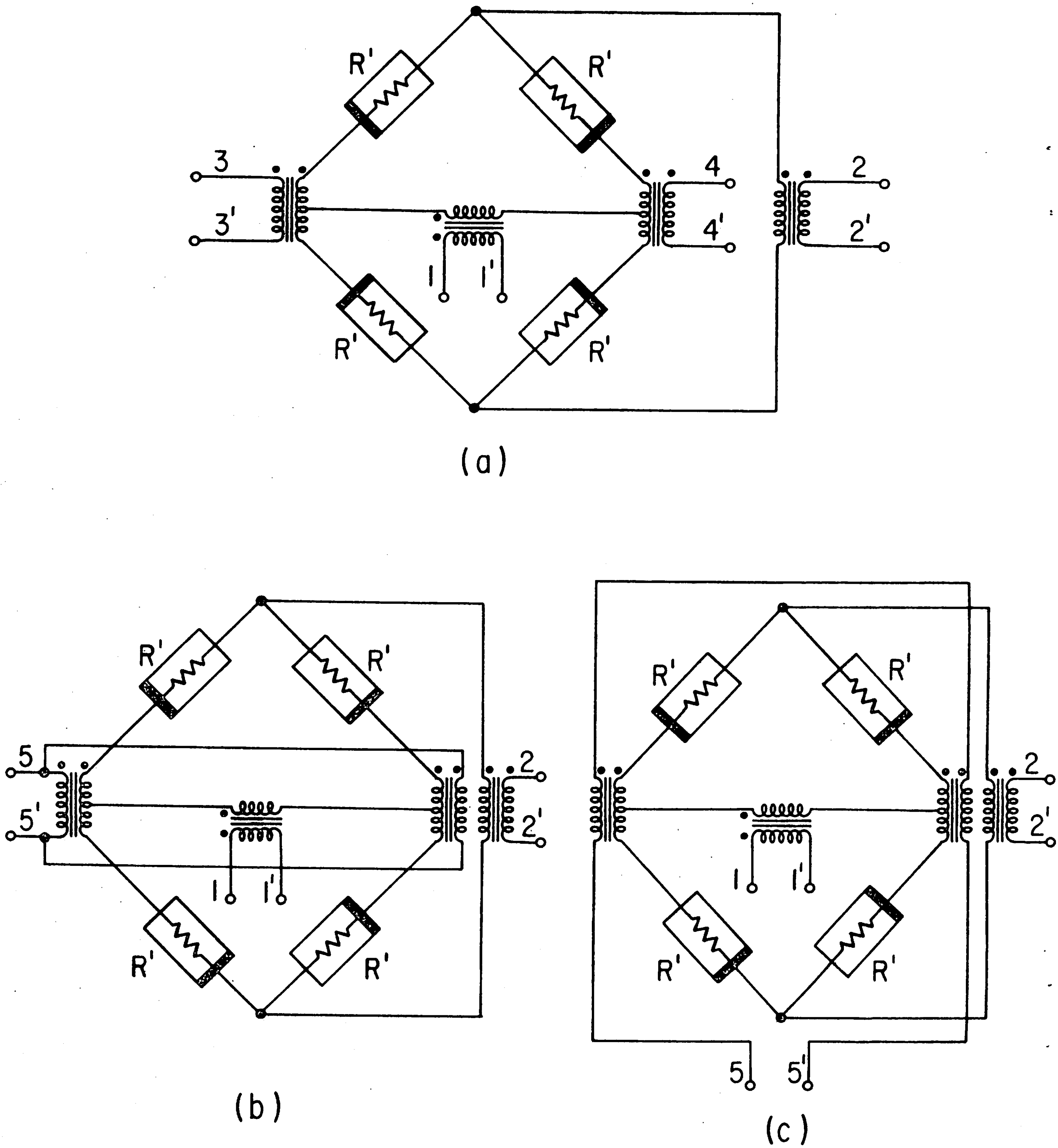


Fig. 12

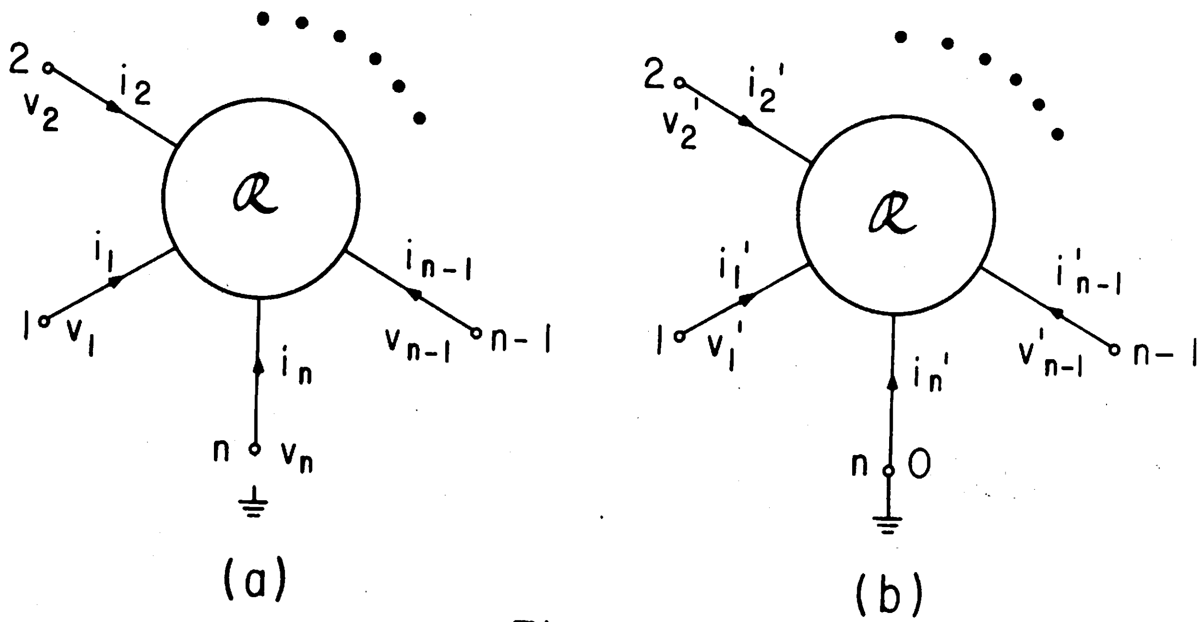

2
v_2
i_2
$\mathcal{R}$
i_1
1
v_1
i_{n-1}
v_{n-1}
n-1
i_n
n
v_n
(a)
2
v_2'
i_2'
$\mathcal{R}$
i_1'
1
v_1'
i'_{n-1}
v'_{n-1}
n-1
i_n'
n
0
(b)

Fig. 13

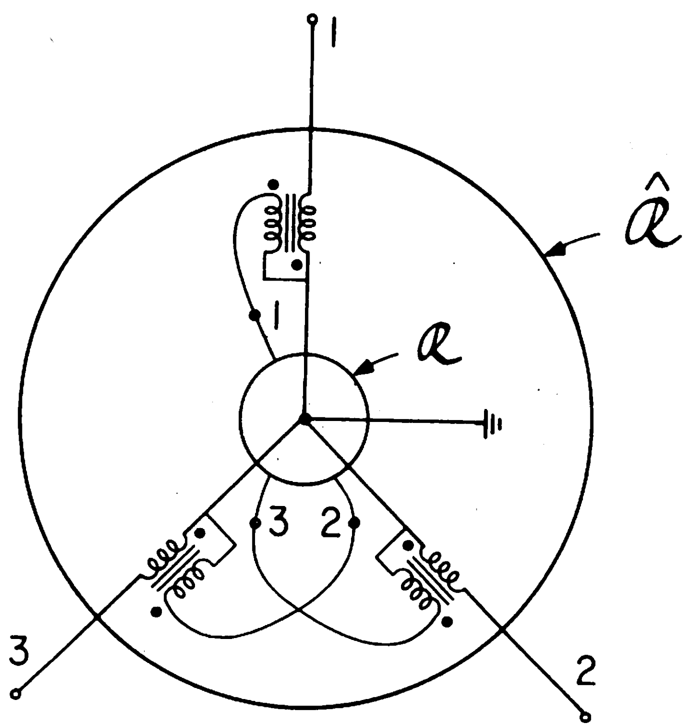

1
$\hat{\mathcal{R}}$
1
$\mathcal{R}$
3
2
3
2

Fig. 14

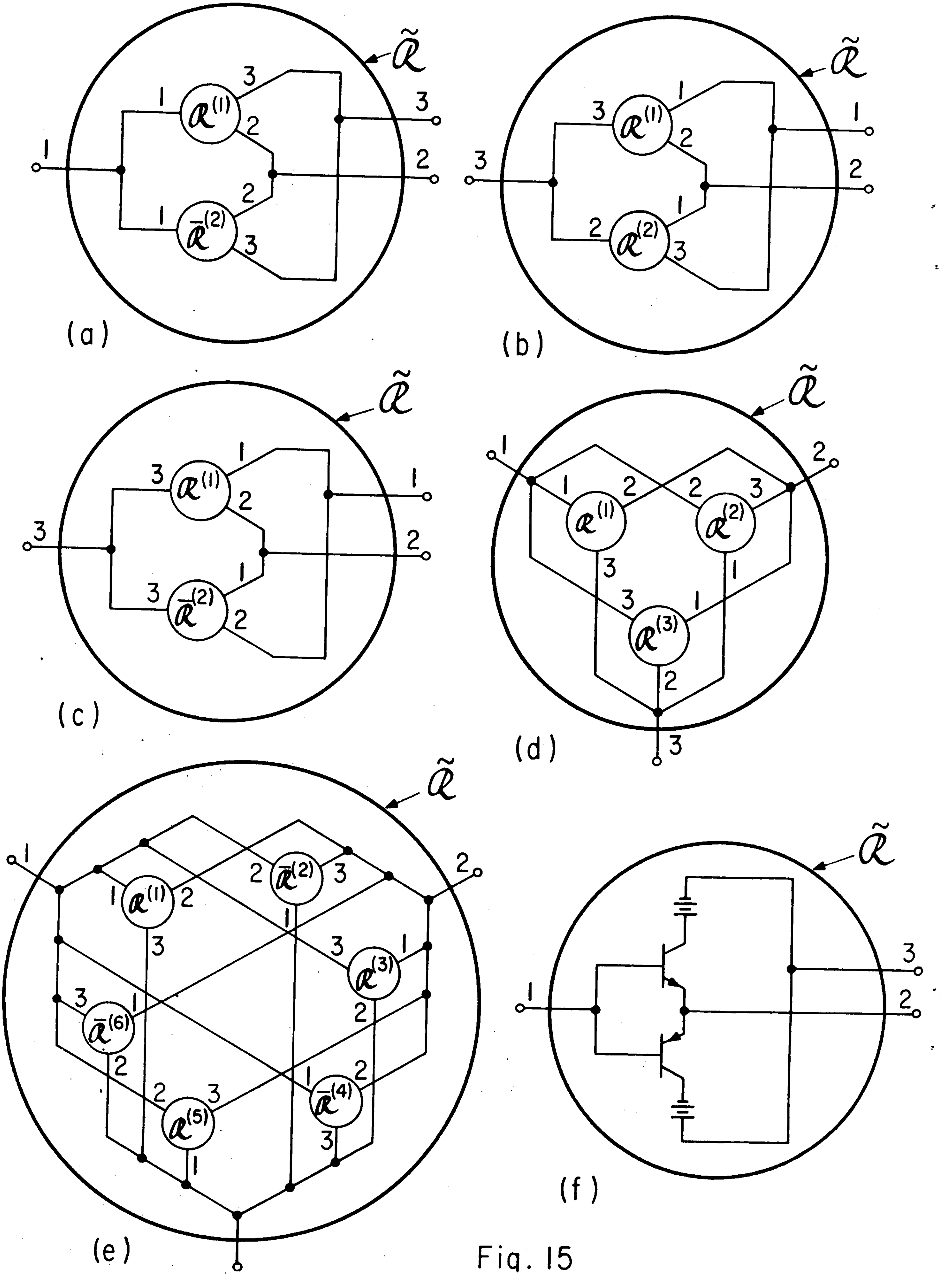
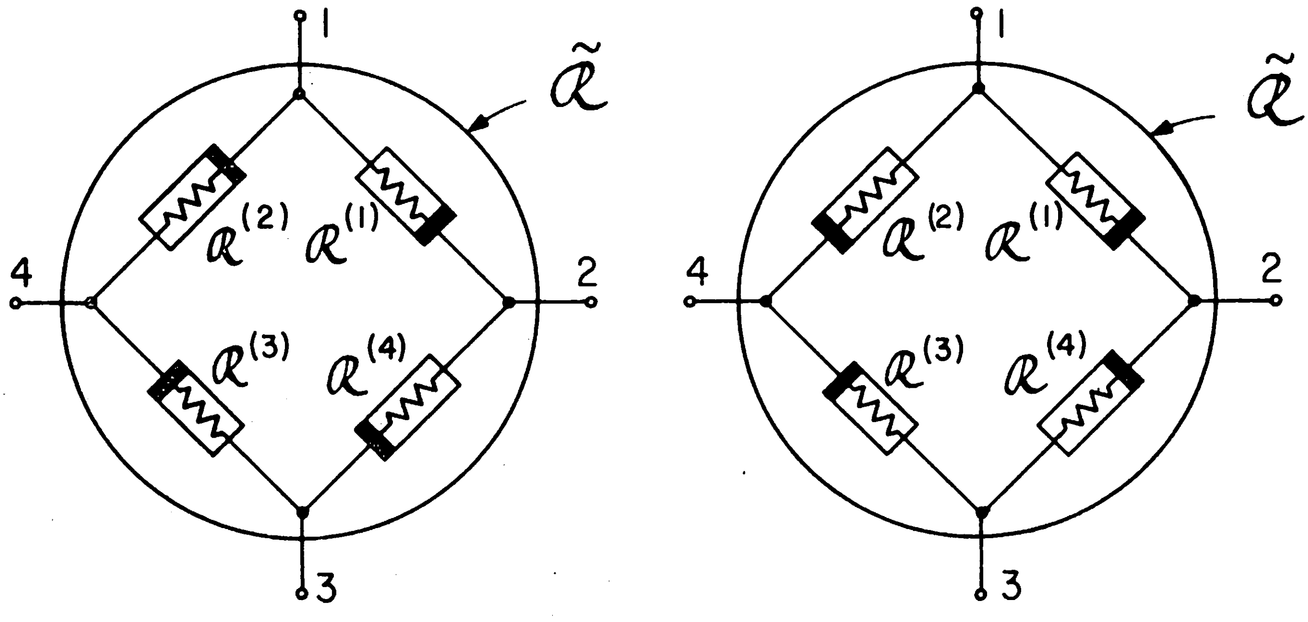
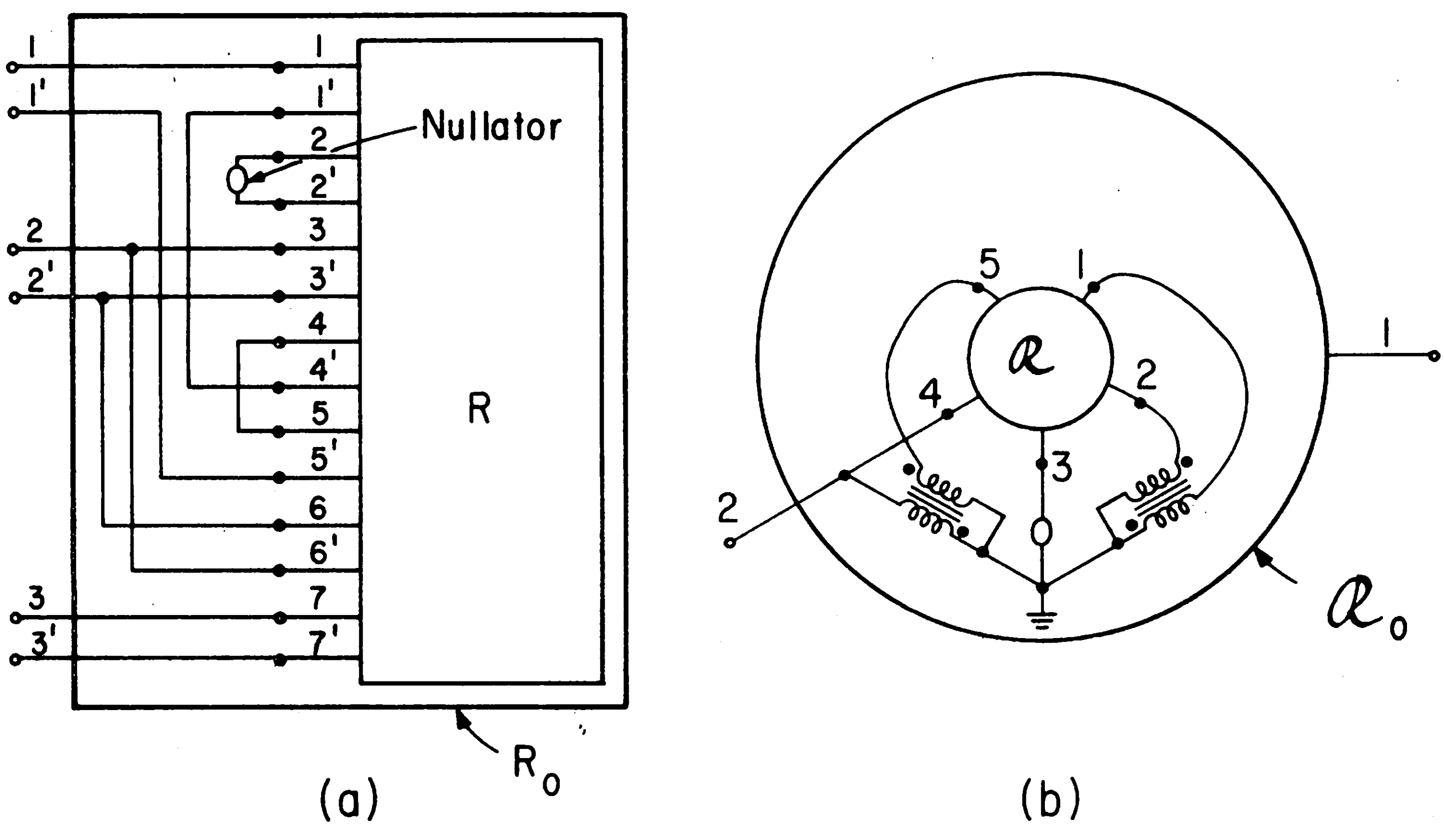


Fig. 15



(a) (b)

Fig. 16



(a) (b)

Fig. 17

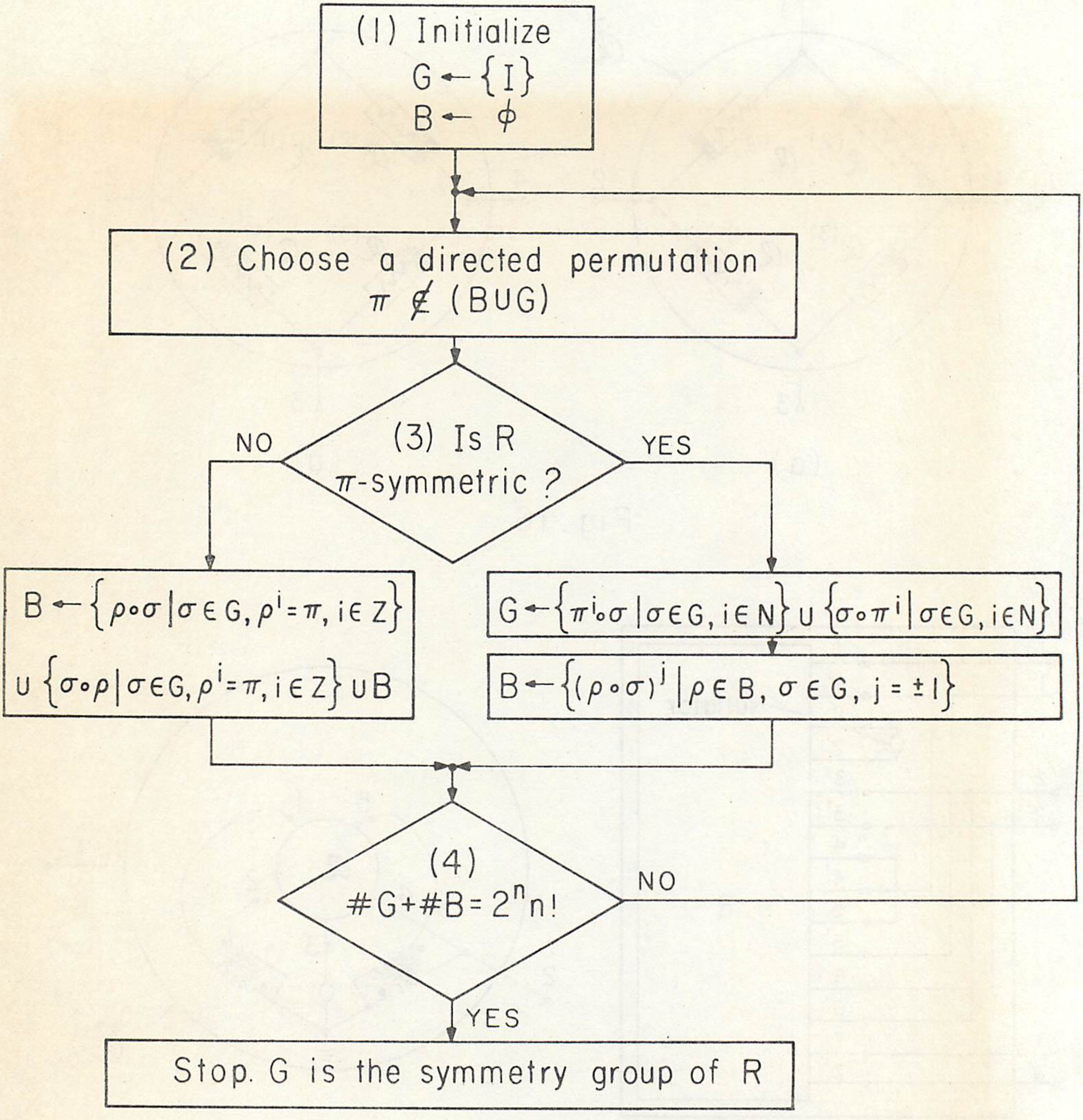

(1) Initialize
G ← {I}
B ← φ
(2) Choose a directed permutation
π ∉ (BUG)
(3) Is R
π-symmetric ?
NO
YES
B ← {ρ∘σ | σ ∈ G, ρ^i = π, i ∈ Z}
∪ {σ∘ρ | σ ∈ G, ρ^i = π, i ∈ Z} ∪ B
G ← {π^i∘σ | σ ∈ G, i ∈ N} ∪ {σ∘π^i | σ ∈ G, i ∈ N}
B ← {(ρ∘σ)^j | ρ ∈ B, σ ∈ G, j = ±1}
(4)
#G+#B = 2^n n!
NO
YES
Stop. G is the symmetry group of R

Fig. 18