A UNIFIED THEORY OF SYMMETRY FOR NONLINEAR RESISTIVE NETWORKS

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ABSTRACT

Based on the interconnections and constitutive relations of multiterminal and/or multiport resistors, a general and purely algebraic definition of symmetry of a nonlinear network is given. Examples show that the network geometry, although frequently useful in detecting simple symmetries, can conceal or destroy more subtle forms of symmetries.

The main results of this paper are based on group theory and on the decomposition of a directed permutation introduced in [11]. These results generalize many existing *ad hoc* techniques used for special circuits having special symmetries:

(1) An algorithm is presented for checking whether a network possesses any form of symmetry.

(2) It is shown that for a suitable choice of the reference nodes, a symmetric network has a symmetric solution, provided the network has a *unique* solution.

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(3) Techniques are described for creating terminal (resp., port) entries in a symmetric network, in order to obtain a symmetric multiport (resp., multiterminal) resistor.

(4) A reduction technique for symmetric networks is described, which generalizes the well-known bisection technique and unifies various algebraic and graphical reduction methods.

I. INTRODUCTION

In linear circuit theory many powerful results [1-7] have been obtained for symmetric circuits. All of these rely on the superposition principle and are therefore not applicable for nonlinear networks. This, however, has not prevented engineers from designing symmetric nonlinear circuits by ad hoc methods [8] (push-pull and parametric amplifiers, rectifiers, modulators and detectors). Also the evolution of solid state technology and especially the production of complementary elements pushed toward an essentially new type of symmetry, called complementary symmetry. In short there is a lack of general results and systematic approaches dealing with symmetry in nonlinear circuits. In this area a few papers [9,10] have appeared recently. A unification of mirror and complementary symmetry has been obtained in [9]. Using group representations, Desoer and Lo [10] showed how to reduce the network equations and how to simplify the stability conditions for periodic oscillations in nonlinear dynamic networks. Our paper is the second of two papers, dealing with symmetry in nonlinear elements and networks described by algebraic equations. The first paper [11] is entitled, "A unified theory of symmetry for nonlinear multiport and multiterminal resistors." Our approach is unifying in the sense that it includes all types of symmetry and that it generalizes many ad hoc results to arbitrary symmetries. This approach also consolidates these ad hoc results by giving precise and general definitions and rigorous proofs.

In order to give a feeling for the need for such an approach in dealing with symmetry in nonlinear resistive networks, we review some ad hoc results. Usually two kinds of symmetry are defined: geometric symmetry and complementary symmetry [8].

A network N is said to be symmetric with respect to a transformation $T(\cdot)$ such as a rotation, a reflection or a translation if it can be drawn such that after making the geometric transformation $T(\cdot)$, we obtain a new network $T(N)$ which is identical to N, except possibly for some labellings, i.e., N and $T(N)$ have identical topology and the corresponding elements have identical constitutive relations. A network N is said to be complementary symmetric if it is identical with its complementary network $\bar{N}$ which is obtained by complementing all elements [8], i.e., multiplying all port voltages and currents in each element's constitutive relation by minus 1. A network is said to be symmetric with respect to both a complementation and a transformation if it is identical with $T(\bar{N})$. In Fig. 1(a) a network is given which is reflection symmetric with respect to an axis[1] drawn through nodes ① ⑥ ⑦ . The network in Fig. 1(b) exhibits complementary symmetry if the operational amplifier is complementary symmetric [10-11] and if $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$ are bilateral. A two-terminal or one-port resistor is said to be bilateral if it is identical with its complement. In Fig. 1(c) a complementary reflection symmetric network is given, i.e., a network which is symmetric with respect to a complementation followed by a reflection about an axis through nodes ① ② ⑤ , provided that $R^{(1)}$ and $R^{(2)}$ are bilateral resistors.

[1] Since this network is not planar a rigorous analysis of the symmetry of this network should be done in the 3-dimensional space. This would also solve the apparent difficulty in Fig. 1(a) where nodes ② and ③ are not precisely reflected into each other. This observation demonstrates one difficulty that could arise when checking the symmetry of a network by inspection of its circuit diagram.

If a port entry is made in a symmetric network between two suitably chosen nodes of N, then the driving-point (DP) plot exhibits odd symmetry. The case of a mirror symmetric, complementary, and complementary mirror symmetric network has been dealt with in [11].

Proposition 1. DP plots of symmetric networks.

(a) The DP plot across a pair of driving-point terminals connected to any pair of symmetrically located nodes of a reflection symmetric network is odd symmetric.

(b) The DP plot across any pair of driving-point terminals connected to any pair of nodes of a complementary symmetric network is odd symmetric.

(c) The DP plot across any pair of driving-point terminals connected to any pair of nodes located along the symmetry axis of a complementary reflection symmetric network is odd symmetric.

Examples of DP plots having properties (a), (b), and (c) can be obtained by making a "soldering-iron" entry across nodes 4 and 5 in Fig. 1(a), across any pair of nodes in Fig. 1(b), and across nodes 2 and 5 in Fig. 1(c), respectively.

Proposition 2. v_o-versus-v_{in} TC plots of symmetric networks:

(a) Let N be a reflection symmetric network having an input port across two symmetrically located nodes. Then the v_o-vs.-v_{in} transfer characteristic (TC) plot is odd symmetric if v_o is also measured across a pair of symmetrically located nodes and is even symmetric if v_o is measured across a pair of nodes located along the symmetry axis.

(b) Let N be a complementary symmetric network. Then any v_o-vs.-v_{in} TC plot is odd symmetric.

(c) Let N be a complementary reflection symmetric network having an input port across two nodes located along the symmetry axis. Then the v_o-vs.-v_{in} TC plot is <u>odd symmetric</u> if v_o is also measured across a pair of nodes located along the symmetry axis.

Examples of TC plots having properties (a), (b), and (c) can also be constructed using the networks in Fig. 1. The TC plot of a symmetrically-driven differential amplifier (Fig. 1(a)) whose output terminals are also symmetrically situated is odd symmetric. (b) Any TC plot associated with Fig. 1(b) is odd symmetric. (c) The TC plot of the push-pull amplifier driven across nodes ① - ⑤ and whose output is measured across nodes ② - ⑤ is odd symmetric.

The next proposition shows that the symmetry of a network can be exploited to reduce the complexity of the analysis [8].

<u>Proposition 3</u>. If the reflection symmetric network of Fig. 2(a) has a unique solution, then this solution can also be derived from the bisected network in Fig. 2(b). An analogous result applies for the 180° rotational symmetric[2] network of Fig. 2(c) and its bisected network in Fig. 2(d).

For the symmetric lattice a special reduction technique has been developed called the symmetric lattice property [8].

<u>Proposition 4</u>. If the symmetric lattice network (Fig. 3) with $R^{(2)} = R^{(3)}$ and $R^{(4)} = R^{(5)}$ has a unique solution, then the currents and voltages associated with the identical resistors $R^{(2)}$ and $R^{(3)}$ are identical. Similarly, those associated with resistors $R^{(4)}$ and $R^{(5)}$ are also identical.

[2]The network is invariant upon rotating it by 180° about an axis through node ②, perpendicular to the paper.

The preceding definitions and results are not satisfactory in view of the following observations:

1) The preceding definition of symmetry requires that the network be drawn in a specific form. This definition is undesirable because symmetry is an inherent property of a network and should not depend on how it is drawn. Furthermore, since a symmetry transformation is one which leaves all structural relations undisturbed [12, p.144], a network symmetry transformation should rely only upon the constitutive relations and the interconnection of these elements. Since we deal with networks with a finite number of nodes and resistors, these transformations can be described by finite permutations. These ideas will lead to a more general definition of symmetry in Section II.

2) No general results are available relating the properties of a solution of a network to the symmetry properties of the network. Using the algebraic definition of symmetry, we derive in Section III symmetry properties of the unique solution of a symmetric network. This result includes the symmetric lattice property (Prop. 4) as a special case.

3) The results of Prop. 1 and 2 are restricted to mirror symmetric networks and to even and odd symmetries. In extending the line of results of Prop. 2 one would expect in case (c) that an even symmetric TC plot would be obtained by measuring v_o across a pair of nodes symmetric with respect to the symmetry axis. This is indeed true and will be shown in Section IV where we will deal with the general problem of making a π-symmetric multiport or multiterminal resistor from a symmetric network by making suitable port (or terminal) entries in the network.

4) The preceding bisection technique (Prop. 3) is restricted to reflection symmetric or 180° rotational symmetric networks where no

resistors are located on the symmetry axis or on the rotation axis. We will generalize this reduction technique in Section V to all symmetries.

In this paper, we consider resistive networks obtained by interconnecting multiterminal and multiport resistors. These resistors can be described by a set of admissible pairs, by a constitutive relation, or by a hybrid representation. We will denote the multiport resistors by R and the multiterminal resistors by $\mathcal{R}$. Multiport resistors are always assumed to be intrinsic (with internal isolation transformers already imbedded if necessary) so that no problems arise in the interconnection. All multiterminal resistors are assumed to be given via their indefinite representation since we want to be able to find all possible symmetries. If necessary it can always be recast into this form using equations (39-40) given in [11]. Unless otherwise mentioned, we always assume that associated reference directions are chosen (Fig. 4). In general a rectangular box (resp., circular box) represents a multiport (resp., two-terminal) resistors we use more in the case of one-port (two-terminal) resistors we use more frequently the conventional symbol (as in Fig. 3). This avoids the use of too many terminal labels, because the terminal at the darkened edge is then the primed terminal 1' of the one-port or the terminal 2 of the two-terminal element. If there is no darkened edge, then the one-port (resp., two-terminal element) is bilateral.

The mathematical tools are mainly combinatorial and are described in Section II of [11]. As before, the key notions are the directed permutation and its decomposition into cyclic components [11]. To save space, we refer all notations not defined explicitly in this paper to [11]. For the relevant mathematical techniques, we refer the reader to [13].

II. TRANSFORMED AND SYMMETRIC RESISTIVE NETWORKS

A resistive network is obtained by interconnecting multiterminal and/or multiport resistors. In circuit theory this interconnection is usually given by a circuit diagram. However the particular configuration in which the network is drawn is irrelevant as long as the resistor terminals or ports are connected in the same way. In other words, no other geometric aspects involved in the drawing of a network than the network interconnection are relevant. Consequently, in this paper, we give a rigorous and general definition of symmetry, which is only based on the combinatorial aspects of the interconnection and the algebraic aspects of the constitutive relations of the resistors. Our definition is mainly graph-theoretic in nature and is in fact inspired by the definition of isomorphic hypergraphs in [14, p.411].

A time invariant resistive network N is completely characterized by the following three sets of information:

1) the time-invariant multiport resistors $R^{(j)}$, $j = 1,\dots,m$, with ports $[1,j]$, $[2,j],\dots,[n_j,j]$ and with terminals $(1,j)(1',j)\dots(n_j,j)(n_j',j)$ and their associated sets of admissible pairs $(\underset{\sim}{v}^{(j)},\underset{\sim}{i}^{(j)})$,

2) the time-invariant multiterminal resistors $\mathcal{R}^{(j)}$, $j = m_1+1,\dots,m$, with terminals $(1,j),\dots,(n_j,j)$ and their associated sets of indefinite admissible pairs $(\underset{\sim}{v}^{(j)},\underset{\sim}{i}^{(j)})$, and

3) a set of nodes $\mathcal{N} = \{ ①, ②, \ldots; ⓝ \}$ which is a partition of the set of all resistor terminals into subclasses such that all terminals belonging to a given subclass are connected together. For example, the notation ⓚ denoted by

$$ⓚ = \{(i_1,j_1),i_2,j_2)\ldots\} \tag{1}$$

implies that terminal i_1 of resistor j_1, terminal i_2 of resistor j_2, etc. are connected together and this terminal is called node ⓚ . In short, a network N is completely specified by $N = (R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$.

The advantage of this characterization is that it is independent of the drawings and thus ideally suited for computer analysis. It also allows a general definition of a network permutation and a network symmetry to be given devoid of irrelevant details. Observe that the node interconnection equation (1) is just a formal way of specifying the wiring instructions of a network; namely, join together terminal i_1 of resistor j_1, terminal i_2 of resistor j_2, etc., and call it node ⓚ (Fig. 5).

To illustrate the preceding method for describing a network, consider the network characterized by $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $\mathcal{R}^{(4)}$, $R^{(5)}$ with constitutive relations $R^{(1)}(\cdot,\cdot)$, $R^{(2)}(\cdot,\cdot),\ldots,R^{(5)}(\cdot,\cdot)$, and the following node interconnections:

$$\begin{array}{ll} ① = \{(1,1),(1,2)\} & ④ = \{(1',3),(1,4)\} \\ ② = \{(1',1),(1',2)\} & ⑤ = \{(2,4),(1,5)\} \\ ③ = \{(2,2),(1,3)\} & ⑥ = \{(2',2),(3,4)(1',5)\} \end{array} \tag{2}$$

The circuit diagram corresponding to these specifications is shown in Fig. 6. We say that terminal i_1 of resistor j_1 is connected to node ⓚ if $(i_1,j_1) \in$ ⓚ .

Observe that multiterminal and multiport resistors are denoted and treated differently in this paper. The reason for this is that the standard procedure of considering an n+1 terminal resistor as a grounded n-port resistor may destroy some element symmetries as observed in [11].

The interconnection can also be uniquely characterized by a generalized incidence matrix $\underset{\sim}{A}$. This is a "port or terminal-node incidence" matrix and not the common branch-node incidence matrix. The generalized incidence matrix $\underset{\sim}{A}$ has as many rows as there are nodes (ⓝ), and as many columns, as there are ports of $R^{(j)}$ or terminals of $\mathcal{R}^{(j)}$, namely $r = \sum_{j=1}^{m} n_j$. Order the ports and terminal labels (i,j) and [i,j] lexicographically, i.e. $(i_1,j_1) < (i_2,j_2)$ (resp., $(\bar{i}_1,j_1) < (\bar{i}_2,j_2)$, $[i_1,j_1] < [i_2,j_2]$ or $[i_1,j_1] < [i_2,j_2]$) if $i_1 < i_2$ and $j_1 \leq j_2$. If terminal (i,j) of a multiterminal resistor is connected to node ⓚ then a +1 placed in the ⓚ-th row of the column corresponding to (i,j). Repeat for all terminals of the multiterminal resistors. In the column corresponding to the port [i,j] we set a +1 in the ⓚ-th row if terminal (i,j) is connected to node ⓚ and a -1 in the ⓚ-th row if terminal (i'j) is connected to node ⓚ. Repeat for all ports. Let the remaining entries of A be zero. Conversely any interconnection of the resistors is completely characterized by any ⓝ x r matrix $\underset{\sim}{A}$ such that each column corresponding to a port has only two nonzero elements +1, -1, and each column corresponding to a terminal has only one nonzero element +1.

For example, in Fig. 6, the generalized incidence matrix $\underset{\sim}{A}$ is given by:

$$\underset{\sim}{A} = \begin{array}{cc} \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \\ \begin{array}{cccccccc} [1,1] & [1,2] & [2,2] & [1,3] & (1,4) & (2,4) & (3,4) & [1,5] \end{array} & \end{array} \qquad (3)$$

Observe that there are 6 rows and 8 columns since there are 6 nodes, 3 terminals (corresponding to $\mathcal{R}^{(4)}$) and 5 ports (corresponding to $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ and $R^{(5)}$. Observe also that even though a one-port resistor can also be considered as a two-terminal resistor, the former is preferred since it is represented by only one column, compared to two in the latter case, in the generalized incidence matrix $\underset{\sim}{A}$. Hence, the total number of columns of $\underset{\sim}{A}$ is therefore smaller.

Since our network may contain both multiport and multiterminal resistors, and may have several connected components, it may be necessary to specify more than one <u>reference node</u> with respect to which the port voltages are measured. Clearly, <u>every connected component of the network needs at least one reference node</u>. However, several reference nodes may be needed in one connected component since the reference node for one port of a multiport resistor need not have any relationship with the reference node for another port. For example, the network of Fig. 6 needs two reference nodes, say nodes 2 and 6 . The following algorithm shows how to assign reference nodes:

1) Replace all n-port resistors by n uncoupled one-port resistors.

2) Identify the connected components of this new network. The nodes of each connected component i in this new network form sets $\mathcal{N}_i$, $i = 1,2,\ldots,\ell$, which we call connected sets of nodes. These ℓ connected sets of nodes form a partition of the nodes of N; namely, $\mathcal{N}_i \cap \mathcal{N}_j = \phi$ if $i \neq j$ and $\bigcup_{i=1}^{\ell} \mathcal{N}_i = \mathcal{N}$. Clearly, any multiterminal resistor has all its terminals connected to nodes belonging to one particular $\mathcal{N}_i$.

3) For each connected set $\mathcal{N}_i$ of nodes, choose one node (arbitrarily) and call it the reference node for $\mathcal{N}_i$. The voltage at a node is always measured with respect to the reference node associated with the connected set of nodes. The terminal voltages of multiterminal resistors connected to this same set of nodes are all measured with respect to this reference node.

Definition 1. For a given set of references nodes, a solution of a network N is any set of vectors $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$, called the voltage and current distribution $\underset{\sim}{v}$, $\underset{\sim}{i}$, and the node voltage $\underset{\sim}{u}$ with

$$\underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}^{(1)} \\ \vdots \\ \underset{\sim}{v}^{(m)} \end{bmatrix}, \quad \underset{\sim}{i} = \begin{bmatrix} \underset{\sim}{i}^{(1)} \\ \vdots \\ \underset{\sim}{i}^{(m)} \end{bmatrix}, \quad \text{and } \underset{\sim}{u} \begin{bmatrix} u_{1} \\ \vdots \\ u_{n} \end{bmatrix} \tag{4}$$

such that the following conditions are satisfied:

1) $(\underset{\sim}{v},\underset{\sim}{i})$ is an admissible pair of resistors of N, i.e. the $(\underset{\sim}{v}^{(j)},\underset{\sim}{i}^{(j)})$ is an admissible pair of resistor j of N,

2) KCL is satisfied, i.e., the sum of the currents leaving any node is zero,

3) KVL is satisfied, i.e., any port voltage of a multiport resistor is equal to the difference between the node voltages at its terminals and

any terminal voltage of a multiterminal resistor is equal to the node voltage at this terminal.

A network may have no solution, one solution, or many solutions. Using the generalized incidence matrix, KCL and KVL can easily be formulated as follows

$$\underset{\sim}{A}\underset{\sim}{i} = \underset{\sim}{0} \tag{5a}$$

$$\underset{\sim}{A}^T\underset{\sim}{u} = \underset{\sim}{v} \tag{5b}$$

This can easily be proved using the definitions of $\underset{\sim}{A}$, $\underset{\sim}{v}$, $\underset{\sim}{i}$, $\underset{\sim}{u}$ and noting that associated reference directions are chosen. It is also easy to show that the number of solutions is independent from the choice of reference nodes.

A. <u>Permuted network</u>

Given a circuit diagram, let N and $\hat{N}$ be two networks associated with the same circuit diagram, i.e., two networks N and $\hat{N}$ whose elements, nodes, and port (resp., terminal) labels of each internal multiport resistor are assigned in two different ways. Since the same circuit diagram is involved, the two sets of solutions are clearly related by a one-to-one transformation. Although these two networks are isomorphic to the eyes of the beholder, it is far from a trivial task to establish their isomorphic nature if N and $\hat{N}$ were described <u>not</u> by a circuit diagram, but by the <u>algebraic</u> methods described above. In fact, the only way a computer could claim that N and $\hat{N}$ are isomorphic is to produce a one-to-one transformation between the relevant data describing N and $\hat{N}$; namely, the port (resp., terminal) labels, the resistor number and the node number. When such a transformation exists in the sense to be defined shortly, we say $\hat{N}$ is the <u>permuted network</u> associated with N. It turns out that a completely rigorous and unambiguous study of symmetry in nonlinear networks requires that we define the notion of permuted network in precise terms.

Our definition of a network permutation will be based upon the notion of a π-permuted multiport resistor and a π-permuted multiterminal resistor [11]. To simplify our discussion, let us first introduce the phase-inverting ideal transformer (Fig. 7(a)) which is defined by $v_1 = -v_2$ and $i_1 = -i_2$ (note the unconventional reference current direction for i_2). In order to simplify this symbol, we will henceforth drop the ground terminal and represent it by a small square box (Fig. 7(b)). This element is a useful artifice for complementing the voltage $v_i^{(j)}$ and the current $i_i^{(j)}$ of a terminal i of a resistor j (Fig. 7(c)). The free terminal $(\bar{i},j)$ is then at a voltage $-v_i^{(j)}$ and carries a current $-i_i^{(j)}$. Consequently, it makes sense to call $(\bar{i},j)$ the complemented terminal of (i,j). Observe that our choice of the unconventional current reference for i_2 in Fig. 7(a) is motivated by this complementation operation. Clearly, two complementations applied in tandem results in the original voltage and current i.e., $(\bar{\bar{i}},j) = (i,j)$. Observe also that complementation of two terminals of a port which is followed by the interchange of the two terminals (Fig. 7(d)) also results in the original port. This equivalence will be useful in many instances. Algebraically this implies that in the description of a network the following two sets of ordered pairs are equivalent

$$\{(i,j),(i',j)\} \Leftrightarrow \{(\bar{i}',j),(\bar{i},j)\} \tag{6}$$

Since we deal with many objects like currents, voltages, ports, terminals, which have two possible orientations, the following notion of a "directed permutation" is indispensable in this paper. A directed permutation π of "n" oriented objects is a transformation obtained by first permuting some of these objects and then changing the orientation (complementing) of some of them. The directed permutation $\pi = \begin{pmatrix} \ldots i \ldots j \ldots \\ \ldots i_1 \ldots \bar{j}_1 \ldots \end{pmatrix}$ transforms object i into object i_1 and object j into the complement of object j_1 where i, j, i_1, j_1 represent objects with the normal orientation. We write $i_1 = \pi(i)$ and $\bar{j}_1 = \pi(j)$. Corresponding to the directed permutation π we define the directed permutation matrix $\underset{\sim}{P}(\pi)$ by

$$\underset{\sim}{P}(\pi) = \begin{array}{c} \begin{bmatrix} & \vdots & & \vdots & \\ \ldots & 0 & \ldots & -1 & \ldots \\ & \vdots & & \vdots & \\ \ldots & 1 & \ldots & 0 & \ldots \\ & \vdots & & \vdots & \end{bmatrix} \begin{array}{l} \\ \leftarrow j_1 \\ \\ \leftarrow i_1 \\ \\ \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ i & j \end{array} \end{array} \tag{7}$$

Definition 2. Given an n-port resistor R characterized by a set S of admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$, and a directed permutation π of n objects, we define the associated π-permuted n-port resistor $\hat{R}$ by the set of admissible pairs $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}})$ such that $\hat{\underset{\sim}{v}} = \underset{\sim}{P}(\pi)\underset{\sim}{v}$, $\hat{\underset{\sim}{i}} = \underset{\sim}{P}(\pi)\underset{\sim}{i}$, where $(\underset{\sim}{v},\underset{\sim}{i})$ is an admissible pair of R. We often denote $\hat{R}$ by $\pi(R)$. Given an n-terminal resistor $\mathcal{R}$ characterized by a set $\mathcal{S}$ of indefinite admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ and a directed permutation π which complements all n objects or none, we define the π-permuted n-terminal resistor $\hat{\mathcal{R}}$ by the set of indefinite admissible pairs $(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}})$ such that $\hat{\underset{\sim}{v}} = \underset{\sim}{P}(\pi)\underset{\sim}{v}$ $\hat{\underset{\sim}{i}} = \underset{\sim}{P}(\pi)\underset{\sim}{i}$, where $(\underset{\sim}{v},\underset{\sim}{i})$ is an admissible pair $\mathcal{R}$. We often denote $\hat{\mathcal{R}}$ by $\pi(\mathcal{R})$.

Permuted resistors can be easily synthesized from the original resistors by permuting ports or terminals and introducing some complementations. Observe that the complementation of a port, which implies the complementation of the port voltage and current, can be achieved in two different ways: by interchanging the port terminals, or by complementing both terminals. If π transforms port i into port i_1 and port j into port $\bar{j}_1$, we denote this operation on the corresponding pair of terminals i, i', and j, j' by

$$\pi(i) = i_1,\ \pi(i') = i_1' \quad \text{or} \quad \pi(i) = \bar{i}_1',\ \pi(i') = \bar{i}_1 \tag{8a}$$

$$\pi(j) = \bar{j}_1,\ \pi(j') = \bar{j}_1' \quad \text{or} \quad \pi(j) = j_1',\ \pi(j') = j_1. \tag{8b}$$

Note that we have abused our notation slightly by using the same symbol π to denote the transformation of a port, as well as a terminal. Recall that a terminal can be complemented by connecting a "complementation element" to this terminal. For example, the $\begin{pmatrix} 1 & 2 & 3 \\ \bar{2} & 3 & \bar{1} \end{pmatrix}$ - permuted 3-port resistor $\hat{R}$ associated with a given resistor R can be synthesized as shown in Fig. 7(e). Similarly, the $\begin{pmatrix} 1 & 2 & 3 \\ \bar{2} & \bar{3} & \bar{1} \end{pmatrix}$ - permuted 3-terminal resistor $\hat{\mathcal{R}}$ associated with a given resistor $\mathcal{R}$ is synthesized in Fig. 7(f). Observe that in the special case where $\pi = \begin{pmatrix} 1 & 2 \ldots n \\ \bar{1} & \bar{2} \ldots \bar{n} \end{pmatrix}$, the π-permuted multiterminal resistor, henceforth called the <u>complementary multiterminal resistor $\bar{\mathcal{R}}$</u>, is obtained by simply complementing all terminals. Such complementary elements, however, may also be available in intrinsic form, as in the case of complementary transistors and FET's.

Since a resistive network N is characterized by a set of resistors and a set of nodes, before introducing the notion of a permuted network $\hat{N}$, let us first define two permutations, one involving the set of terminals and ports of the resistors, and the other involving the set of nodes.

<u>Definition 3</u>. The couple $(\underset{\sim}{\pi},\sigma)$ is said to be a "<u>port-terminal permutation</u>" of a set of m multiport or multiterminal resistors if

1) $\underset{\sim}{\pi} = [\pi^{(1)} \ldots \pi^{(m)}]^T$ where $\pi^{(j)}$ is a directed permutation of the terminals or ports of the j-th resistor,

2) σ is a permutation of the resistors.

A directed permutation "ρ" which permutes and/or complements the nodes of a network N is said to be a "<u>node permutation</u>". The complement of a node ⓚ $= \{(i_1,j_1), i_2,j_2) \ldots (i_n,j_n)\}$ (Fig. 7(g)) is defined by

$$\bar{ⓚ} = \{(\bar{i}_1,j_1),(\bar{i}_2,j_2)\ldots(\bar{i}_n,j_n)\} \tag{9}$$

and is obtained by the complementation operation shown in Fig. 7(h). Observe that $\bar{\bar{ⓚ}}$ = ⓚ

We see that as a result of the terminal and port permutation $(\underset{\sim}{\pi},\sigma)$, terminal (or port) i of resistor j is transformed into terminal (or port) $\pi^{(j)}(i)$, of resistor $\sigma(j)$, and is denoted by

$$(\underset{\sim}{\pi},\sigma)(i,j) = (\pi^{(j)}(i),\sigma(j))$$

or

$$(\underset{\sim}{\pi},\sigma)[i,j] = [\pi^{(j)}(i),\sigma(j)].$$

The port-terminal permutation $(\underset{\sim}{\pi},\sigma)$ can also be written as

$$(\underset{\sim}{\pi},\sigma) = \begin{pmatrix} (1,1) & \ldots[i,j] & \ldots \\ (\pi^{(1)}(1),\sigma^{(1)}) & \ldots[\pi^{(j)}(i),\sigma^{(j)}] & \ldots \end{pmatrix} \tag{10}$$

where the upper row is a list of all resistor terminals and ports.

Given two port-terminal permutations $(\underset{\sim}{\pi}_1,\sigma_1)$ and $(\underset{\sim}{\pi}_2,\sigma_2)$ such that we can operate first with $(\underset{\sim}{\pi}_1,\sigma_1)$ on a set of resistors and then with $(\underset{\sim}{\pi}_2,\sigma_2)$ on the resulting set of resistors, we define the <u>composition</u> $(\underset{\sim}{\pi}_3,\sigma_3) = (\underset{\sim}{\pi}_2,\sigma_2)\circ(\underset{\sim}{\pi}_1,\sigma_1)$ to be the port-terminal permutation obtained by first applying $(\underset{\sim}{\pi}_1,\sigma_1)$ and then applying $(\underset{\sim}{\pi}_2,\sigma_2)$. This product is given by

$$\pi_3^{(j)} = \pi_2^{(\sigma_1(j))} \circ \pi_1^{(j)}, \quad \sigma_3 = \sigma_2 \circ \sigma_1 \tag{11a}$$

where

$$\underset{\sim}{\pi}_k = [\pi_k^{(1)} \ldots \pi_k^{(m)}]^T \quad k = 1,2,3 \tag{11b}$$

and exists if $\pi_2^{(\sigma_1(j))}$ and $\pi_1^{(j)}$ operate on the same number of objects for all j. It follows from (11a) that the inverse of $(\underset{\sim}{\pi},\sigma)$ exists and is unique. It is given by

$$(\underset{\sim}{\pi},\sigma)^{-1} = (\underset{\sim}{\pi}_1,\sigma^{-1}) \tag{12a}$$

with

$$\pi_1^{(\sigma(j))} = (\pi^{(j)})^{-1}. \tag{12b}$$

We order the terminal and port labels (i,j) and [i,j] lexicographically. According to this order they are mapped one-to-one into the integers 1,2,...,r. This allows us to define the r×r matrix $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)$ as follows

$$\underset{\sim}{P}(\underset{\sim}{\pi},\sigma) = \begin{array}{c} \left[\begin{array}{c|c|c} & \bigcirc & \\ \hline \bigcirc & \underset{\sim}{P}(\pi^{(j)}) & \bigcirc \\ \hline & \bigcirc & \end{array}\right] \\ \uparrow \\ j\text{-th block} \end{array} \leftarrow \sigma(j)\text{-th block} \tag{13}$$

The effect of the multiplication of a vector $\underset{\sim}{x}$ by $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)$ is to rearrange variables of the ports and terminals of the old lexicographic order to that of the new lexicographic order and then possibly complementing some of them.

Before giving the formal definition of a permuted network, we will first introduce this notion on a nontrivial network N (Fig. 6 and equation (2)). Suppose we choose a "port-terminal permutation" $(\underset{\sim}{\pi},\sigma)$ with

$\pi^{(1)} = \pi^{(5)} = \begin{pmatrix}1\\1\end{pmatrix}$, $\pi^{(2)} = \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $\pi^{(3)} = \begin{pmatrix}1\\\bar{1}\end{pmatrix}$, $\pi^{(4)} = \begin{pmatrix}1 & 2 & 3\\\bar{2} & \bar{3} & \bar{1}\end{pmatrix}$,

$\sigma = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\2 & 3 & 4 & 5 & 1\end{pmatrix}$ and a "node permutation" $\rho = \begin{pmatrix}① & ② & ③ & ④ & ⑤ & ⑥\\② & ③ & \bar{④} & \bar{⑤} & \bar{⑥} & \bar{①}\end{pmatrix}$. The above choice for $(\underset{\sim}{\pi},\sigma)$ and ρ completely specifies the $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network $\hat{N}$. The recipe for constructing $\hat{N}$ from N consists of 4 steps:

(1) Replace each resistor j of N (Fig. 6) by an equivalent resistor in Fig. 8(a) as follows. Let $\hat{R}^{(\sigma(j))}$ (resp. $\hat{\mathcal{R}}^{(\sigma,j)}$) denote the $\pi^{(j)}$-permuted resistor of the resistor $R^{(j)}$ (resp. $\mathcal{R}^{(j)}$). Then resistor $R^{(j)}$ (resp. $\mathcal{R}^{(j)}$) is the $(\pi^{(j)})^{-1}$-permuted resistor of $\hat{R}^{(\sigma(j))}$ (resp. $\mathcal{R}^{(\sigma(j))}$). In Fig. 8(a) we synthesize resistor $R^{(j)}$ (resp. $\mathcal{R}^{(j)}$) starting from $\hat{R}^{(\sigma(j))}$ (resp. $\mathcal{R}^{(\sigma(j))}$) using the above described procedure for making a permuted resistor. Since N and this new network are equivalent, the currents and voltages of Fig. 8(a) can be derived immediately from those of Fig. 6. Observe also that this implies that terminal i of resistor $R^{(j)}$ (resp. $\mathcal{R}^{(j)}$) becomes terminal $\pi^{(j)}(i)$ of resistor $R^{(\sigma(j))}$ (resp. $\mathcal{R}^{(\sigma(j))}$). This map corresponds to the port terminal permutation $(\underset{\sim}{\pi},\sigma)$.

(2) Relabel each node ⓚ by ρ(ⓚ) (Fig. 8(a)). The only effect of this operation on the solution is a directed permutation of the node voltages.

(3) Looking carefully at the network of Fig. 8(a) and discarding all lables related to the original network N we see that the new network is composed of some phase-inverting ideal transformers and of the resistors $\hat{R}^{(1)}$, $\hat{R}^{(2)}$, $\hat{R}^{(3)}$, $\hat{R}^{(4)}$, $\hat{\mathcal{R}}^{(5)}$ with $\hat{R}^{(1)} = R^{(5)}$, $\hat{R}^{(2)} = R^{(1)}$, $\hat{R}^{(3)} = R^{(2)}$, $\hat{R}^{(4)} = \pi^{(3)}(R^{(3)}) = \bar{R}^{(3)}$, $\hat{\mathcal{R}}^{(5)} = \pi^{(4)}(\mathcal{R}^{(4)})$. Observe also that some nodes in Fig. 8(a) are complemented: $\bar{④}\ \bar{⑤}\ \bar{⑥}\ \bar{①}$. These

complemented nodes can be eliminated as shown in Fig. 8(b) by using the node complementation technique shown in Figs. 7(g) and 7(h).

(4) Using the fact that two complementations in tandem are equivalent to a short circuit and the fact that the complementation of a port can be accomplished by complementing both terminals or by interchanging two terminals, we can eliminate all complementation elements and obtain the network of Fig. 8(c) which is the $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network $\hat{N}$.

Observe that an arbitrary choice of the port-terminal permutation $(\underset{\sim}{\pi},\sigma)$ and node permutation ρ may not necessarily result in a network $\hat{N}$ without complementation elements. In order to have this property for a permuted network we will have to impose some consistency conditions on the triple $(\underset{\sim}{\pi},\sigma,\rho)$.

Observe also that the left part of this network $\hat{N}$ in Fig. 8(a) is not complemented by $(\underset{\sim}{\pi},\sigma,\rho)$ although the right part is. Hence we have here a transformation which is more general than the ones given in the introduction. With this motivation we are ready to give algebraic definitions of a network permutation and of a permuted network.

Definition 4. Given a resistive network $N = (R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$ where each node belonging to $\mathcal{N} = \{①\ldots ⓝ\}$ is as defined in (1). The triple (π,σ,ρ) is said to be a network permutation if:

1) $(\underset{\sim}{\pi},\rho)$ is a port-terminal permutation of the ports and terminals of the resistors of N.

2) ρ is a directed permutation of the nodes of N.

3) the following consistency conditions are satisfied: (a) All terminals of multiterminal resistors connected to any node ⓚ are either all complemented by (π,σ) if ⓚ is complemented by ρ or all are uncomplemented by ρ if ⓚ is not complemented by ρ. (b) The two

nodes associated with any port of a multiport resistor are either both complemented by ρ, or both are uncomplemented. The $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network $\hat{N} = (\hat{R}^{(\sigma(j))}, \hat{\mathcal{R}}^{(\sigma(j))}, \hat{\mathcal{N}})$ is then defined by

1) $\hat{R}^{(\sigma(j))}$ (or $\hat{\mathcal{R}}^{(\sigma(j))}$) is the $\pi^{(j)}$-permuted resistor of $R^{(j)}$ (or $\mathcal{R}^{(j)}$)

2) $\hat{\mathcal{N}} = \{①\ \dots\ ⓝ\}$ is the set of nodes resulting from

$$\rho(ⓚ) = \{\left(\pi^{(j_1)}(i_1),\sigma(j_1)\right), \left(\pi^{(j_2)}(i_2),\sigma(j_2)\right),\dots\} \tag{14}$$

where eventually some complementations have to be removed by complementing both sides of (12) using (9) and by using (6) at the ports.

Observe that the elimination of all complements in all nodes (14) is always possible because of the consistency condition. Indeed condition (3a) guarantees that an eventual complementation of (14) brings all terminals of multiterminal resistors and $\rho(ⓚ)$ into the uncomplemented form. Similarly, condition (3b) guarantees that any pair of terminals (i,j) and (i',j) or port i of a multiport resistor $\hat{R}^{(j)}$ appear as (i,j) and (i',j), or as $(\bar{i},j)$ and $(\bar{i}',j)$ in nodes (k_1) (k_2), or in nodes $\overline{(k_1)}$ $\overline{(k_2)}$ and thus complemented port terminals can be eliminated using (6)

In terms of the incidence matrix, condition 3 of a network permutation implies that $(\underset{\sim}{\pi},\sigma,\rho)$ are such that $\underset{\sim}{P}(\rho)\underset{\sim}{A}\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)^T$ is again an incidence matrix. This is precisely the incidence matrix $\hat{A}$ of $\hat{N}$ or

$$\hat{\underset{\sim}{A}} = \underset{\sim}{P}(\rho)\underset{\sim}{A}\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)^T \tag{15}$$

Let us now derive the $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network of N (Fig. 6), (2) using definition 4 directly. Conditions 1) and 2) are obviously satisfied for the above $(\underset{\sim}{\pi},\sigma,\rho)$. The set of nodes from (14) is given by

$$
\begin{array}{lll}
2 = \{(1,2),(1,3)\} & \bar{5} = \{(1,4)(\bar{2},5)\} & \\
3 = \{(1',2),(1',3)\} & \bar{6} = \{(\bar{3},5),(1,1)\} & (16) \\
\bar{4} = \{(2,3),(1',4)\} & \bar{1} = \{(2',3),(\bar{1},5),(1',1)\} &
\end{array}
$$

Consistency condition 3(a) is satisfied since only the complements $\bar{5}$, $\bar{6}$, $\bar{1}$, $(\bar{1},5)$, $(\bar{2},5)$, and $(\bar{3},5)$ appear. Similarly, consistency condition 3(b) is satisfied since (1,1), (1',1) appear in $\bar{6}$, $\bar{1}$; (1,2), (1',2) appear in 2 , 3 ; (2,3), (2',3) appear in $\bar{4}$, $\bar{1}$; and (1',4)(1,4) appear in $\bar{4}$, $\bar{5}$. Using (6) and (9) to eliminate all complements in (16) we obtain

$$
\begin{array}{lll}
1 = \{(2,3),(1,5),(1,1)\} & 4 = \{(2',3),(1,4)\} & \\
2 = \{(1,2),(1,3)\} & 5 = \{(1',4),(2,5)\} & (17) \\
3 = \{(1',2),(1',3)\} & 6 = \{(3,5),(1',1)\} &
\end{array}
$$

This node set together with the resistors $\hat{R}^{(1)} = R^{(5)}$, $\hat{R}^{(2)} = R^{(1)}$, $\hat{R}^{(3)} = R^{(2)}$, $\hat{R}^{(4)} = \pi^{(3)}(R^{(3)}) = \bar{R}^{(3)}$ and $\hat{\mathcal{R}}^{(5)} = \pi^{(4)}(\mathcal{R}^{(4)})$ define precisely the network shown in Fig. 8(c).

The incidence matrix $\hat{\underset{\sim}{A}}$ of this new network $\hat{N}$ can be derived from $\underset{\sim}{A}$ using (15) and is

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \qquad (18)
$$

[1,1][1,2][1,3][2,3][1,4](1,5)(2,5)(3,5)

$$
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}
\left[\begin{array}{c|c|cc|c|ccc}
0 & 1 & \multicolumn{2}{c|}{0} & 0 & \multicolumn{3}{c}{0} \\ \hline
0 & 0 & 1 & 0 & 0 & \multicolumn{3}{c}{0} \\
 & & 0 & 1 & & & & \\ \hline
0 & 0 & \multicolumn{2}{c|}{0} & -1 & \multicolumn{3}{c}{0} \\ \hline
 & & & & & 0 & -1 & 0 \\
0 & 0 & \multicolumn{2}{c|}{0} & 0 & 0 & 0 & -1 \\
 & & & & & -1 & 0 & 0 \\ \hline
1 & 0 & \multicolumn{2}{c|}{0} & 0 & \multicolumn{3}{c}{0}
\end{array}\right]
$$

Our definition of a network permutation is very general and contains many special cases.

1) Choose $\pi^{(i)} = I$, $\sigma = I$, then the network permutation performs a relabeling of the nodes.

2) Choose $\sigma = I$, $\rho = I$, then the network permutation performs a relabeling of the terminals or ports.

3) Choose $\pi^{(i)} = \bar{I}$, $\sigma = I$, $\rho = \bar{I}$ then the network permutation performs a complementation. $\hat{N}$ is the complemented network denoted by $\bar{N}$.

4) If $(\underset{\sim}{\pi},\sigma)$ is a port-terminal permutation of a network N and ρ is a node permutation and if both do not involve any complementations then condition 3 is automatically satisfied and $(\underset{\sim}{\pi},\sigma,\rho)$ is a network permutation.

It is clear that condition 3) of Def. 4 requires a major subset of the ports (terminals) and nodes of a network to be all or not complemented. What is the smallest subset that is subject to this constraint? Our next proposition shows that each such set of nodes is precisely a <u>connected set of nodes</u>.

<u>Proposition 5</u>. In any network permutation $(\underset{\sim}{\pi},\sigma,\rho)$ of a network N the node permutation ρ either complements all or none of the nodes belonging to a connected set of nodes. Conversely there exists a network permutation, such that the node permutation ρ complements all nodes of some connected sets of nodes and no other nodes.

<u>Proof:</u> From condition 3) it follows that terminals of multiterminal resistors connected to one node have to be either all or not complemented and that the two terminals of a port have to be either both complemented or not complemented. From condition 1) it follows that terminals of the same multiterminal resistor have to be either all or not complemented. These facts imply that nodes belonging to a connected set of nodes have to be either all or not complemented. Since there are no other conditions there exist network permutations complementing only the nodes of some prescribed connected sets of nodes. ◻

<u>Proposition 6</u>. (a) Given a network permutation $(\underset{\sim}{\pi}_1,\sigma_1,\rho_1)$ transforming network N into network $\hat{N}$ and network permutation $(\underset{\sim}{\pi}_2,\sigma_2,\rho_2)$ transforming $\hat{N}$ into $\tilde{N}$ then the composition $(\underset{\sim}{\pi}_2,\sigma_2,\rho_2) \circ (\underset{\sim}{\pi}_1,\sigma_1,\rho_1)$ is a network permutation transforming N into $\tilde{N}$.

(b) Let N be a network containing m_1 multiport and m_2 multiterminal resistors. Assume N has n nodes and these nodes are made up of a union of ℓ connected sets of nodes. Let the j-th resistor be an n_j-port resistor for $j = 1,\ldots,m_1$ and an n_j-terminal resistor for $j = m_1+1,\ldots,m$, where $m = m_1+m_2$. Then the collection of <u>all</u> distinct <u>network permutations</u> contains exactly M elements, where

$$M = \left(\prod_{j=1}^{m_1} (nj!2^{n_j})\right)\left(\prod_{j=m_1+1}^{m} nj!\right) (m!)(n!)_2^{\ell} \tag{19}$$

Proof: The composition of the two network permutations is

$$(\underset{\sim}{\pi}_3,\sigma_3,\rho_3) = (\underset{\sim}{\pi}_2,\sigma_2,\rho_2) \circ (\underset{\sim}{\pi}_1,\sigma_1,\rho_1) \tag{20a}$$

with

$$\pi_3^{(j)} = \pi_2^{(\sigma_1(j))} \circ \pi_1^{(j)} \tag{20b}$$

$$\sigma = \sigma_2 \circ \sigma_1,\ \rho_3 = \rho_2 \circ \rho_1 \tag{20c}$$

and is easily seen to satisfy all conditions. The total number of distinct network permutationsis the product of the following items: 1) the number of permutations of the m_1 multiport resistors, 2) the number of permutations of the $m_2 = m-m_1$ multiterminal resistors not involving complementations 3) the number of the permutations of the resistors, 4) the number of the permutations of the nodes not involving complementations, 5) the integer 2^{ℓ} where ℓ is the number of connected sets of nodes. ⌑

The most important property of a permuted network $\hat{N}$ is that its solution can be easily derived from that of the original network N. Indeed, if we let $\underset{\sim}{v}$ and $\underset{\sim}{i}$ denote the voltages and currents of all resistor ports and terminals, and let $\underset{\sim}{u}$ denote the set of all node-to-datum voltages, then we have the following:

Theorem 1. For any solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ of N, the associated $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network $\hat{N}$ has a solution

$$(\hat{\underset{\sim}{v}},\hat{\underset{\sim}{i}},\hat{\underset{\sim}{u}}) = (\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{v},\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{i},\underset{\sim}{P}(\rho)\underset{\sim}{u}) \tag{21}$$

Proof. Since $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ is a solution of N

1) $(\underset{\sim}{v}^{(j)},\underset{\sim}{i}^{(j)})$ is an admissible pair for the j-th resistor of N,

2) $\underset{\sim}{A}\underset{\sim}{i} = \underset{\sim}{0}$ and

3) $\underset{\sim}{A}^T\underset{\sim}{u} = \underset{\sim}{v}$.

Using the orthogonality of the directed permutation matrix, the definition of the resistors of $\hat{N}$, and (15), we obtain by applying the appropriate directed permutations the following relations:

1) $\left(\hat{\underset{\sim}{v}}^{(\sigma(j))},\hat{\underset{\sim}{i}}^{(\sigma(j))}\right) = \left(\underset{\sim}{P}(\pi^{(j)})\underset{\sim}{v}^{(j)},\underset{\sim}{P}(\pi^{(j)})\underset{\sim}{i}^{(j)}\right)$ is an admissible pair of the $\sigma(j)$-th resistor of $\hat{N}$,

2) $\hat{\underset{\sim}{A}}\hat{\underset{\sim}{i}} = \left(\underset{\sim}{P}(\rho)\underset{\sim}{A}\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)^T\right)\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{i} = \underset{\sim}{0}$, and

3) $\hat{\underset{\sim}{A}}^T\hat{\underset{\sim}{u}} = \left(\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{A}^T\underset{\sim}{P}(\rho)^T\right)\underset{\sim}{P}(\rho)\underset{\sim}{u} = \underset{\sim}{P}(\underset{\sim}{\pi},\rho)\underset{\sim}{v} = \hat{\underset{\sim}{v}}$.

This implies that (21) is a solution of $\hat{N}$ where ρ relates the reference node(s) of the solution of N to the reference nodes of the solution of $\hat{N}$. ⌑

It should not be too surprising that the solution of permuted networks can be derived from one another since a network permutation involves only operations which permute and/or complement variables of the solution. No operation destroys solutions or introduces new ones. This implies that N and $\hat{N}$ have the same number of solutions.

Corollary: If $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ is a solution of a network N, then the complementary network $\bar{N}$ has a solution $(-\underset{\sim}{v},-\underset{\sim}{i},-\underset{\sim}{u})$.

This corollary [8,9] is used extensively in logic circuits. It is a common practice to obtain the solution of a circuit in negative logic by complementing all currents and voltages of the solution of the corresponding circuit in positive logic.

As a result of this theorem it is easy to derive the solutions of a permuted network $\hat{N}$ from those of N. So it makes sense to define identical and isomorphic networks as follows.

Definition 5. Two networks $N = (R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$ and $\hat{N} = (\hat{R}^{(j)}, \hat{\mathcal{R}}^{(j)}, \hat{\mathcal{N}})$ are said to be identical if

1) $R^{(j)}$ (resp. $\mathcal{R}^{(j)}$) and $\hat{R}^{(j)}$ (resp. $\hat{\mathcal{R}}^{(j)}$) are identical for all j.

2) the nodes of N and $\hat{N}$ are identical

Two networks N and $\hat{N}$ are said to be isomorphic if there is a network permutation, so that $\hat{N}$ is identical with the permuted network of N.

In terms of network drawings two networks are identical if their circuit diagrams can be made coincident such that corresponding nodes, ports, terminals, and resistors have the same labels, and such that the corresponding resistors are identical. That is, they are exact duplicates in all aspects. On the other hand, if two networks differ only by the labels assigned to the respective nodes, ports, terminals, and resistors, then they are isomorphic to each other.

B. Symmetric network

Definition 6. A network N is $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric if N is identical to its $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network.

Examples.

1) Our first example demonstrates that there exist symmetric networks whose associated symmetry permutations $(\underset{\sim}{\pi},\sigma,\rho)$ has $\rho = I$. In other words, the nodes are invariants of the transformations. Consider the network N shown in Fig. 9(a), where we have considered the two identical 1-Ω resistors as two-terminal resistors (rather than one-ports) in order to show the generality of the definition. This network is described by

$$\mathcal{R}^{(1)}: v_1^{(1)} - v_2^{(1)} = 1\text{V}, \text{ and } \quad \mathcal{R}^{(i)}: v_1^{(i)} - v_2^{(i)} = i_1^{(i)} = -i_2^{(i)}, \; i = 2,3;$$

$$① = \{(1,1),(1,2),(1,3)\} \qquad ② = \{(2,1),(2,2),(2,3)\} \tag{22}$$

It is easy to verify that N is invariant under an interchange of the resistors $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(3)}$, or more precisely, under the network permutation $(\underset{\sim}{\pi},\sigma,\rho)$ with $\pi^{(j)} = \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$ for i = 1,2,3 and $\sigma = \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}$ and $\rho = \begin{pmatrix}① & ②\\① & ②\end{pmatrix}$.

2) Our next example demonstrates that there exist symmetries which are of the form $(\underset{\sim}{I},I,\rho)$. Consider the network N in Fig. 9(b) where the two distinct nonlinear resistors $R^{(1)}$ and $R^{(2)}$ are considered as <u>one-ports</u>, rather than two-terminal resistors. The two nodes are ① = {(1,1),(1,2)}, ② = {(1',1),(1',2)}. If we apply the network permutation $(\underset{\sim}{\pi},\sigma,\rho)$ with $\pi^{(1)} = \pi^{(2)} = I = \begin{pmatrix}1\\1\end{pmatrix}$, $\sigma = I = \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$ and $\rho = \begin{pmatrix}① & ②\\② & ①\end{pmatrix}$, we would obtain a new network $\hat{N}$ which is identical to the original. In fact this form of symmetry always exists whenever the ports of two multiport resistors are connected in parallel with each other.

3) Consider next the complementary mirror-symmetric "push-pull amplifier" circuit shown in Fig. 10(a).

This network is described by

$$\begin{aligned}
\mathcal{R}^{(1)} : i_2^{(1)} &= \alpha_F I_{ES}(\exp((v_1^{(1)}-v_3^{(1)})/v_T-1)) - I_{CS}(\exp((v_1^{(1)}-v_2^{(1)})/v_T-1)) \\
&= f_2(v_1^{(1)},v_2^{(1)},v_3^{(1)}) \\
i_3^{(1)} &= -I_{ES}(\exp((v_1^{(1)}-v_3^{(1)})/v_T-1)) + \alpha_R I_{CS}(\exp((v_1^{(1)}-v_2^{(1)})/v_T-1)) \\
&= f_3(v_1^{(1)},v_2^{(1)},v_3^{(1)}) \\
i_1^{(1)} &+ i_2^{(1)} + i_3^{(1)} = 0
\end{aligned}$$

$$\mathcal{R}^{(2)} : i_2^{(2)} = -f_2(-v_1^{(2)},-v_2^{(2)},-v_3^{(2)}),\; i_3^{(2)} = -f_3(-v_1^{(2)},-u_2^{(2)},-v_3^{(2)}),$$

$$i_1^{(2)} + i_2^{(2)} + i_3^{(2)} = 0$$

$$\mathcal{R}^{(3)} : v_1^{(3)} - v_2^{(3)} = 5V,\ i_1^{(3)} = -i_2^{(3)}$$

$$\mathcal{R}^{(4)} : v_1^{(4)} - v_2^{(4)} = -5V,\ i_1^{(4)} = -i_2^{(4)}$$

$$\mathcal{R}^{(5)} : v_1^{(5)} - v_2^{(5)} = i_1^{(5)} = -i_2^{(5)} \tag{23}$$

and

$$① = \{(1,1),(1,2)\},\quad ② = \{(3,1),(1,5),(3,2)\},\quad ③ = \{(2,1),(1,3)\},$$

$$④ = \{(2,2),(1,4)\},\quad ⑤ = \{(2,3),(2,4),(2,5)\}$$

The push-pull amplifier is symmetric with respect to the following symmetry transformation $(\underset{\sim}{\pi},\sigma,\rho)$:

$$\pi^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{2} & \bar{3} \end{pmatrix},\ \pi^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{2} & \bar{3} \end{pmatrix},\ \pi^{(3)} = \begin{pmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{pmatrix},\ \pi^{(4)} = \begin{pmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{pmatrix},$$

$$\pi^{(5)} = \begin{pmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{pmatrix},\ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix},\ \rho = \begin{pmatrix} ① & ② & ③ & ④ & ⑤ \\ \bar{①} & \bar{②} & \bar{④} & \bar{③} & \bar{⑤} \end{pmatrix}.$$

As an illustration of the algebraic nature of the definition of symmetry, we check this symmetry without using the circuit diagram. First it has to be shown that $(\underset{\sim}{\pi},\sigma,\rho)$ is a network permutation. The first two conditions are obviously satisfied and the consistency condition requires in this case that all terminals and all nodes are complemented. The $(\underset{\sim}{\pi},\sigma,\rho)$ transformed network $\hat{N}$ is given then as follows:

The resistors are

$$\hat{\mathcal{R}}^{(1)} = \pi^{(2)}(\mathcal{R}^{(2)}) = \bar{\mathcal{R}}^{(2)},\ \hat{\mathcal{R}}^{(2)} = \pi^{(1)}(\mathcal{R}^{(1)}) = \bar{\mathcal{R}}^{(1)},$$

$$\hat{\mathcal{R}}^{(3)} = \pi^{(4)}(\mathcal{R}^{(4)}) = \bar{\mathcal{R}}^{(4)},\ \hat{\mathcal{R}}^{(4)} = \pi^{(3)}(\mathcal{R}^{(3)}) = \bar{\mathcal{R}}^{(3)},$$

$$\hat{\mathcal{R}}^{(5)} = \pi^{(5)}(\mathcal{R}^{(5)}) = \bar{\mathcal{R}}^{(5)},$$

and the nodes are

$$\bar{\textcircled{1}} = \rho(\textcircled{1}) = \{(\pi^{(1)}(1),\sigma(1)),(\pi^{(2)}(1),\sigma(2))\} = \{(\bar{1},2),(\bar{1},1)\}$$

$$\bar{\textcircled{2}} = \rho(\textcircled{2}) = \{(\pi^{(1)}(3),\sigma(1),(\pi^{(5)}(1),\sigma(5)),(\pi^{(2)}(3),\sigma(2))\}$$

$$= \{(\bar{3},2),(\bar{1},5),(\bar{3},1)\}$$

$$\bar{\textcircled{3}} = \rho(\textcircled{4}) = \{(\pi^{(2)}(2),\sigma(2)),(\pi^{(4)}(1),\sigma(4))\} = \{(\bar{2},1),(\bar{1},3)\}$$

$$\bar{\textcircled{4}} = \rho(\textcircled{3}) = \{(\pi^{(1)}(2),\sigma(1)),(\pi^{(3)}(1),\sigma(3))\} = \{(\bar{2},2),(\bar{1},4)\}$$

$$\bar{\textcircled{5}} = \rho(\textcircled{5}) = \{(\pi^{(3)}(2),\sigma(3)),(\pi^{(4)}(2),\sigma(4)),(\pi^{(5)}(2),\sigma(5))\}$$

$$= \{(\bar{2},4),(\bar{2},3),(\bar{2},5)\}$$

The complements of the nodes and the complemented terminals can be eliminated by using (9):

$$\textcircled{1} = \{(1,2),(1,1)\},\ \textcircled{2} = \{(3,2)(1,5),(3,1)\},\ \textcircled{3} = \{(2,1),(1,3)\}$$

$$\textcircled{4} = \{(2,2),(1,4)\},\ \textcircled{5} = \{(2,4),(2,3),(2,5)\}$$

It is now easy to check that N and $\hat{N}$ are identical since $\hat{\mathcal{R}}^{(1)} = \mathcal{R}^{(1)} = \bar{\mathcal{R}}^{(2)}$, $\hat{\mathcal{R}}^{(2)} = \mathcal{R}^{(2)} = \bar{\mathcal{R}}^{(1)}$, $\hat{\mathcal{R}}^{(3)} = \mathcal{R}^{(3)} = \bar{\mathcal{R}}^{(4)}$, $\hat{\mathcal{R}}^{(4)} = \mathcal{R}^{(4)} = \bar{\mathcal{R}}^{(3)}$, $\hat{\mathcal{R}}^{(5)} = \mathcal{R}^{(5)} = \bar{\mathcal{R}}^{(5)}$ and since the nodes are the same.

It is important to note that apart from the interconnection the presence of the above symmetry in the network depends critically on three properties of the constitutive relations: (1) The resistor $\mathcal{R}^{(2)}$ is the complemented resistor of $\mathcal{R}^{(1)}$, i.e. $\mathcal{R}^{(2)} = \bar{\mathcal{R}}^{(1)}$. This implies that $\mathcal{R}^{(2)}$ is a pnp transistor which is the complement of the npn transistor $\mathcal{R}^{(1)}$, (2) The voltage sources $\mathcal{R}^{(3)}$ and $\mathcal{R}^{(4)}$ are the complement of each other, (3) The resistor $\mathcal{R}^{(5)}$ is bilateral, i.e., $\mathcal{R}^{(5)} = \bar{\mathcal{R}}^{(5)}$, which follows from the linearity of $\mathcal{R}^{(5)}$.

To provide additional insight we present a graphical verification of this symmetry. This involves the determination of the $(\underset{\sim}{\pi},\sigma,\rho)$ permuted network $\hat{N}$ of Fig. 10(a). A similar analysis as in Fig. 8

produces via Fig. 10(b) the permuted network $\hat{N}$ of Fig. 10(c). Observe that $\hat{\mathcal{R}}^{(1)}$ (resp. $\hat{\mathcal{R}}^{(2)}$) is a an npn (resp. pnp) transistor. Again using the above three properties of the constitutive relations it is easily established that the networks of Fig. 10(a) and 10(c) are identical.

4) If a network is symmetric with respect to $(\bar{I},I,\bar{I})$, we call it complementary symmetric. This corresponds to the definition given in Sec. 1. It is easy to verify that a network is complementary symmetric iff every component is complementary symmetric. In particular this shows that any linear network is complementary symmetric.

Observe that our definition of symmetry is extremely general and contains as a special case the three ad hoc symmetries mentioned in the introduction. It will be shown later that the definition of geometric symmetry given in the introduction provides a useful technique for detecting the presence of some form of symmetries in most instances.

Let $(\underset{\sim}{\pi},\sigma,\rho)$ be a network permutation. Observe that when σ permutes two resistors which have a different number of terminals or ports, or when ρ permutes nodes which are associated with a different number of incident terminals, the resulting permuted network cannot be identical to the original network.

Now consider a network N with n nodes ① ... ⓝ and m resistors. Let the nodes be made up of ℓ connected set of nodes, and let the j-th resistor be an n_j-port resistor $R^{(j)}$ for $j = 1,\ldots,m_1$, or an n_j-terminal resistor $\mathcal{R}^{(j)}$ for $j = m_1+1,\ldots,m$. Let $\mathcal{P}$ be the set of all network permutations $(\underset{\sim}{\pi},\sigma,\rho)$ of N which satisfy the following conditions:

1) for $j = 1,\ldots,m$ the resistors j and $\sigma(j)$ of N are either both multiport resistors with the <u>same</u> number of ports, or both multiterminal resistors with the <u>same</u> number of terminals,

2) for ⓚ = ①,...,ⓝ, the nodes ⓚ and ρ(ⓚ) of N both have the <u>same</u> number of incident terminals.

<u>Proposition 7</u>. Given a network N, the set $\mathcal{P}$ forms a finite group (with "composition" as binary operation) containing

$$\left(\prod_{j=1}^{m_1} (n_j!2^{n_j})\right)\left(\prod_{j=m_1+1}^{m} n_j!\right) \quad s_1 s_2 2^{\ell} \text{ elements, where}$$

$$s_1 = (\prod_i \text{number of i-port resistors})(\prod_i \text{number of i-terminal resistors})$$

$$s_2 = \prod_i \text{number of nodes having n incident terminals.}$$

<u>Proof</u>: It is easy to check that $\mathcal{P}$ has indeed the predicted number of elements. The composition (20) of two elements of $\mathcal{P}$ always exists and is again an element of $\mathcal{P}$ (closure law). The network permutation $(\underset{\sim}{I},I,I)$ where I is the unit permutation and where $\underset{\sim}{I} = [I,\ldots,I]^T$, is called the unit network permutation. The composition of two network permutations belonging to $\mathcal{P}$ is associative since the composition of permutations in (20b,c)) is associative. By (12) and (20) every element of $\mathcal{P}$ has an inverse. This implies that $\mathcal{P}$ is a group. ¤

Now with regard to the network permutations of $\mathcal{P}$, we can apply all results obtained in Section II of [11]. Among them the cyclic decomposition of a directed permutation is the most important. Since $\mathcal{P}$ is a finite group we call the smallest integer d such that $\nu^d = \nu \circ \nu \circ \ldots \circ \nu = (\underset{\sim}{I},I,I)$ the order of the network permutation $\nu \in \mathcal{P}$. Analogously we call the order e of a port-terminal permutation $(\underset{\sim}{\pi},\sigma)$ of $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$ the smallest integer e such that $(\pi,\sigma)^e = (\underset{\sim}{I},I)$.

Proposition 8. Given a network permutation $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$, then we have:

1) $(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ can be written uniquely as a product of disjoint cycles either of ports or of terminals,

$$(\underset{\sim}{\pi},\sigma) = ((i_o,j_o),(i_1,j_1)\ldots(i_{q-1},j_{q-1}))\ldots(\;). \tag{24a}$$

2) σ can be uniquely written as a product of cycles either of multiport resistors where all resistors have the same number of ports, or of multiterminal resistors where all resistors have the same number of terminals, i.e.,

$$\sigma = (j_o,j_1,\ldots,j_{s-1})\ldots(\cdot), \tag{24b}$$

where s is a divider of q.

3) ρ can be uniquely written as a product of disjoint cycles

$$\rho = (k_o\, k_1 \cdots k_{p-1})\ldots(\;\;), \tag{24c}$$

such that all nodes of a cycle have the same number of incident terminals.

4) The order e of $(\underset{\sim}{\pi},\sigma)$ is a multiple of the order of σ, and the order d of $(\underset{\sim}{\pi},\sigma,\rho)$ is the least common multiple (ℓcm) of e and the order of ρ.

Proof:

1) Since $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$ the composition $(\underset{\sim}{\pi},\sigma)$ o $(\underset{\sim}{\pi},\sigma)$ exists and performs a directed permutation of the set of ports and terminals. Hence, (24a) follows from Theorem 1 of [11].

2) In (24a) we have from the definition of a port-terminal permutation that $(j_o,j_1,\ldots,j_{q-1})$ is obtained by repeatedly applying σ to j_o. In view of Theorem 1 of [11], the corresponding cycle of σ must accordingly be $(j_o,j_1,\ldots,j_{s-1})$, where s is a divider of q.

3) (24c) can be obtained immediately by applying Theorem 1 of [11].

4) The order e of $(\underset{\sim}{\pi},\sigma)$ is the smallest integer such that $(\underset{\sim}{\pi},\sigma)^e = (\underset{\sim}{I},I)$. This implies $\sigma^e = I$ and thus e is a multiple of the order of σ. Analogously it is easy to prove that the order d of $(\underset{\sim}{\pi},\sigma,\rho)$ is a multiple of e and of the order of ρ. Call $g = \ell cm(e, \text{order of } \rho)$, then $(\underset{\sim}{\pi},\sigma,\rho)^g = (I,I,I)$ and thus $g = d$. ¤

We say that a terminal i of resistor j is "incident at 'or' connected to" node (k) if $(i,j) \in$ (k) or equivalently if $(\bar{i},j) \in \bar{(k)}$. Using the previous proposition, we obtain the following interesting property.

Proposition 9. Let $\hat{N}$ be the $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network of N with $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$. If the i-th terminal of resistor j is connected to node (k) in N, then the $\pi^{(j)}(i)$-th terminal of resistor $\sigma(j)$ is connected to node $\rho($(k)$)$ in $\hat{N}$. In other words, corresponding to the cycles ((k_0) (k_1) ... (k_{p-1})) of ρ, $((i_0,j_0)(i_1,j_1)\ldots(i_{q-1},j_{q-1}))$ $(\underset{\sim}{\pi},\sigma)$, and $(j_0 j_1,\ldots,j_{s-1})$ of σ, if terminal i_0 of resistor j_0 is connected to node (k_0) , then s is a divider of q. Moreover, for any integer d, terminal i_{d_1} of resistor j_{d_2} is connected in $\hat{N}$ to node (k_{d_3}) , where d_1 (resp. d_2,d_3) is the remainder of the division of d by q (resp. by s,p) i.e. $d = qq_1+d_1$ with d_1,q_1 integers and $0 \leq d_1 < q$.

Proof: Apply the definition of a network permutation to N and use Prop. 8. ¤

A network may exhibit many distinct forms of symmetry. Again all these symmetry permutations form a group, henceforth called the symmetry group of a network.

Proposition 10: The set $\mathcal{S}$ of all network permutations with respect to which a given network N is symmetric forms under the composition operation a group, which is a subgroup of $\mathcal{P}$.

Proof: It is easy to see that $\mathcal{S} \subset \mathcal{P}$. By an analogous proof as in Prop. 14 in [11], it can be shown that $\mathcal{S}$ satisfies the closure property and hence is a group. $\square$

Corollary 1: a) Let ν_1 and ν_2 be two symmetry network permutations of a network N. Then $\nu_1 \circ \nu_2$ and ν_1^{ℓ}, with ℓ an integer, are symmetry network permutations.

b) A network is ν-symmetric iff it is ν^{-1}-symmetric.

Corollary 2: Let N be a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network. If the i-th terminal of resistor j is connected to node (k) in N, then also the $\pi^{(j)}(i)$-th terminal of resistor $\sigma(j)$ in N is connected to node $\rho((k))$. In other words, corresponding to the cycles $((k_0)(k_1)\cdots(k_{p-1}))$ of ρ, $((i_0,j_0)(i_1,j_1),\ldots,(i_{q-1},j_{q-1}))$ of $(\underset{\sim}{\pi},\sigma)$, and $(j_0 j_1,\ldots,j_{s-1})$ of σ, if terminal i_0 of resistor j_0 is connected to node (k_0), then s is a divider of q and for any integer d, terminal i_{d_1} of resistor j_{d_2} is connected to node (k_{d_3}), where d_1 (resp. d_2,d_3) is the remainder of the division of d by q (resp. s,p).

Let us now apply this corollary and the cyclic decomposition technique to the symmetry permutation of the network of Fig. 10(a). The cyclic decompositions are:

$$(\underset{\sim}{\pi},\sigma) = \big((1,1)(\bar{1},2)\big)\big((2,1)(\bar{2},2)\big)\big((3,1)(\bar{3},2)\big)\big((1,3)(\bar{1},4)\big)\big((2,3)(\bar{2},4)\big)$$
$$\big((1,5)(\bar{1},5)\big)\big((2,5)(\bar{2},5)\big)$$

$$\rho = ((1)(\bar{1}))((2)(\bar{2}))((3)(\bar{4}))((5)(\bar{5})), \quad \sigma = (1\ 2)(3\ 4)(5).$$

The order of $(\underset{\sim}{\pi},\sigma,\rho)$ is equal to 2, since the order of $(\underset{\sim}{\pi},\sigma)$ and of ρ

are both 2. The symmetry group consists of the unit permutation and the above permutation. The predictions of Cor. 2 are that if terminal (1,1) is connected to node 1, then terminal $(\bar{1},2)$ is connected to node $\bar{1}$, or terminal (1,2) is connected to 1. Analogous observations can be made at other terminals. The cyclic decompositions of $(\underset{\sim}{\pi},\sigma)$ σ and ρ will also be useful in describing the symmetry properties of the solution (Sec. 3) and in reducing a symmetric network (Sec. 5).

C. Useful techniques for detecting symmetries in a network

Often one is interested in the symmetry group of a network. This problem can be attacked in essentially two different ways. The first is the most common and is based on geometrical constructions: the network is redrawn (on a plane or in the 3-dimensional space) such that it completely coincides with itself (the graphs coincide and the resistors are identical) after some geometric transformations (such as a reflection or a rotation and/or some complementation. This is a useful technique for detecting symmetries with pencil and paper for a simple network. It is not suited for large networks where a computer must be used. This geometric method gives also a justification for the ad hoc definition of symmetry given in the introduction. However, this method is not general enough. We will demonstrate later by an example that some symmetries cannot be detected by this method. This is because the 3-dimensional space, and certainly the plane, does not possess enough "rooms" for identifying certain symmetries. The second technique is more algebraic since it relies directly on the algebraic definitions of a network and of symmetry. Instead of checking all possible network permutations, we first select all network permutations (a subgroup of $\mathcal{P}$) which leave the graph or the incidence matrix invariant, and then eliminate those resistors which are not identical and which are thus electrically different. This technique is completely general and more suited for computer detection of symmetries.

The geometric detection technique is based on the following result.

Proposition 11. Given a network N and its network diagram D in the plane or in the 3-dimensional space. Apply the following operations: 1) Select some connected sets $\mathcal{N}_i$ of nodes and complement all multiterminal resistors and all ports of multiport resistors incident at these nodes. 2) Apply an isometric transformation (such as a rotation or reflection or both) on the resulting network diagram. If the original network diagram D and the new network diagram $\hat{D}$ coincide with each other, and if corresponding resistors have identical constitutive relations, then the network is symmetric with respect to $(\underset{\sim}{\pi},\sigma,\rho)$, where $\pi^{(j)}$, σ and ρ are the directed permutations resulting from the complementation and the matching of the coincident resistors (all nodes belonging to complemented connected sets $\mathcal{N}_i$ of nodes have undergone an implicit complementation since all the incident terminals are complemented).

Proof: The set of directed permutations $\underset{\sim}{\pi},\sigma,\rho$ form a network permutation for N because $\underset{\sim}{\pi},\sigma$ and ρ represent terminal (resp. port), resistor and node (directed) permutations respectively, and since the consistency condition is guaranteed by the fact that all resistors connected to nodes of $\mathcal{N}_i$ are complemented. It follows then that the $(\underset{\sim}{\pi},\sigma,\rho)$-permuted network $\hat{N}$ coincides with N. ◻

We illustrate this technique first with the network diagram D of Fig. 10(a). In our search for a symmetry we can either choose everything to be complemented or nothing since there is only one connected set of nodes. In complementing everything we obtain the new network diagram of Fig. 11(a). Observe that 1) the complementation of an npn transistor $\mathcal{R}^{(1)}$ is a pnp transistor identical with $\mathcal{R}^{(2)}$, 2) the complementation of a voltage source $\mathcal{R}^{(3)}$ is obtained by interchanging its terminals and thus $\bar{\mathcal{R}}^{(3)}$ is identical with $\mathcal{R}^{(4)}$ and 3) the linear resistor $\mathcal{R}^{(5)}$ is identical with its complement $\bar{\mathcal{R}}^{(5)}$. We reflect this network diagram with respect to the axis ① ② ⑤ while preserving the labellings and obtain the diagram $\hat{D}$ of Fig. 11(b). Using the above three facts we see that the network diagrams D and $\hat{D}$ of Fig. 10(a) and 11(b) coincide and that the corresponding resistors have identical constitutive equations and thus the network is symmetric. From the matching of the coincident elements we obtain the symmetry network permutation $(\underset{\sim}{\pi},\sigma,\rho)$. The top row contains the labels of D and the lower row contains the labels of $\hat{D}$ including the eventual complementations of terminals, ports, or nodes:

$$\rho = \begin{pmatrix} ① & ② & ③ & ④ & ⑤ \\ \bar{①} & \bar{②} & \bar{④} & \bar{③} & \bar{⑤} \end{pmatrix}$$

$$(\underset{\sim}{\pi},\rho) = \begin{pmatrix} (1,1)(2,1)(3,1)(1,2)(2,2)(3,2)(1,3)(2,3)(1,4)(2,4)(1,5)(2,5) \\ (\bar{1},2)(\bar{2},2)(\bar{3},2)(\bar{1},1)(\bar{2},1)(\bar{3},1)(\bar{1},4)(\bar{2},4)(\bar{1},3)(\bar{2},3)(\bar{1},5)(\bar{2},5) \end{pmatrix},$$

which is the same symmetry as found before.

Remarks: 1) Since we only consider networks with a finite number of nodes and resistors, the classification theorem of isometries in space [15] shows that we only have to consider reflections and rotations and a combination of both.

2) Observe that it may be necessary to draw the network in the 3-dimensional space in order to identify certain forms of symmetry. For example, consider the network N of Fig. 11(c) as drawn in the 3-dimensional space. Let us apply Prop. 11 by complementing first

all resistors connected to nodes of the sets $\mathcal{N}_1$ and $\mathcal{N}_2$ and then rotating it about the x-axis by 180°. Since the two drawings coincide with each other, and since the corresponding resistors are identical N is symmetric. The associated node permutation is given by $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{2} & \bar{1} & \bar{4} & \bar{3} & \bar{6} & \bar{5} \end{pmatrix}$. The σ and $\pi^{(i)}$ permutations can also be determined as soon as all resistors and terminals are labeled. Another symmetry of the same network can be found as follows. First complement everything that is connected to nodes 1 and 2 i.e. to nodes of $\mathcal{N}_1$ Next make a mirror reflection with respect to the plane x-y. The two drawings are identical. The corresponding node permutations is

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{2} & \bar{1} & 4 & 3 & 5 & 6 \end{pmatrix}.$$

3) Observe also that it may be necessary to redraw a network several times before uncovering any form of symmetry. For example, the circuit shown in Fig. 11(d) with $R^{(3)}$ linear does not seem to exhibit any form of symmetry at all. However, after interchanging the location of nodes 3 and 4 (Fig. 11(e)), the line connecting nodes 1 and 4 clearly forms an axis of symmetry.

A major drawback of this technique for detecting symmetries is that it <u>does not guarantee that all symmetries can be found</u>. To show this, consider the network shown in Fig. 12, where the one-port resistors $\{R^{(1)}R^{(2)}R^{(3)}\}$ (resp., $\{R^{(4)},R^{(5)},R^{(6)},R^{(7)}\}$, $\{R^{(8)},R^{(9)},\dots,R^{(19)}\}$) are assumed to be identical, but need not be bilateral. The node set is given by:

$$\textcircled{1} = \{(1',1),(1,2),(1',10),(1',11),(1',12),(1',13)\}$$

$$\textcircled{2} = \{(1',2),(1,3),(1',14),(1',15),(1',16),(1',17)\}$$

$$\textcircled{3} = \{(1,1),(1',3),(1',8),(1',9),(1',18),(1',19)\}$$

$$\textcircled{4} = \{(1',4),(1,7),(1,9),(1,11),(1,14)\} \tag{25}$$

$$\textcircled{5} = \{(1,4),(1',5),(1,8),(1,10),(1,15)\}$$

$$\textcircled{6} = \{(1,5),(1',6),(1,12),(1,17),(1,19)\}$$

$$\textcircled{7} = \{(1,6),(1',7),(1,13),(1,16),(1,18)\}$$

It can be easily verified that this network is $(\underset{\sim}{\pi},\sigma,\rho)$=symmetric with

$\pi^{(j)} = I,\ j = 1,\ldots,19,$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 & 12 & 10 & 17 & 15 & 16 & 14 & 8 & 19 & 9 & 18 & 11 & 13 \end{pmatrix},$$

and $\rho = \begin{pmatrix} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} \\ \textcircled{2} & \textcircled{3} & \textcircled{1} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{4} \end{pmatrix} = (\textcircled{1}\ \textcircled{2}\ \textcircled{3})(\textcircled{4}\ \textcircled{5}\ \textcircled{6}\ \textcircled{7})$. The difficulty in drawing this network can be seen more clearly from the cyclic decomposition of ρ. The two cycles imply that such a network diagram should have two axes, one through the center of the triangle ① ② ③ and the other through the center of the square ④ ⑤ ⑥ ⑦, and the network should be invariant under a 120° rotation about the first axis followed by a 90°-rotation about the second axis. A similar problem arises when one wants to draw in the two or three-dimensional space a symmetric multiterminal or multiport resistor such that its symmetry can be derived from an isometric operation. A simple example where this is impossible is a (1 2 3)(4 5 6 7)-symmetric resistor.

The other symmetry detection technique is directly based on the definition of symmetry: A network permutation $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$ is a symmetry network permutation if and only if

1) $\underset{\sim}{P}(\rho)\underset{\sim}{A}\underset{\sim}{P}^T(\underset{\sim}{\pi},\rho) = \underset{\sim}{A}$ (26)

and

2) the resistor $\sigma(j)$ is the $\pi^{(j)}$-permuted resistor of the resistor j.

The problem of finding the group of all symmetry permutations can be solved sequentially as follows.

1) First solve the combinatorial problem of finding the group of all network permutations such that (26) is satisfied.

2) Then check the condition on the resistors. Not all cases need be verified exhaustively since we can use the algorithm described in Appendix B of [11]. The first problem is equivalent to finding the group of symmetry operations of a directed hypergraph. In the case of a planar graph Weinberg has described algorithms for finding the symmetry group [16-17]. His algorithms are based on canonical codes for the planar graph. In the general case more exhaustive techniques are unavoidable, although the structure of the problem and the use of the algorithm of Appendix B in [11] allow great savings. We present here such an algorithm. But first observe that (26) implies

$$\underset{\sim}{P}(\rho)\underset{\sim}{A}\underset{\sim}{A}^T = \underset{\sim}{A}\underset{\sim}{A}^T P(\rho) \tag{27}$$

<u>Algorithm for finding the symmetry group $\mathcal{S}$ of a network N</u>

Given a network $N = (R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$ find the group $\mathcal{S}$ of all $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$ such that the network is $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric.

1) Find set M_k of nodes which have the same number k of terminals for $k = 1,2,\ldots$. Find the set L_k (resp. L_k') of multiport (resp. multiterminal) resistors which have the same number k of ports (resp. terminals) for $k = 1,2,\ldots$. Find also the ℓ connected sets of nodes $\mathcal{N}_k$, $k = 1,\ldots,\ell$. Then from the definition of $\mathcal{P}$, $(\underset{\sim}{\pi},\sigma,\rho)$ has to be such that ρ does not interchange elements of the M_k with different k, and σ does not interchange elements of the L_k (resp. L_k') with different k.

2) Find the group G_ρ of all permutations ρ such that (27) is satisfied. Mowshowitz [18] has described an algorithm to find all solutions in two steps: first solve $\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{A}^T = \underset{\sim}{A}\underset{\sim}{A}^T X$ for all complex $\underset{\sim}{X}$ [19] and then identify those solutions which are permutation matrices. Since the second step is by nature exhaustive we prefer a direct exhaustive procedure. Consider $\underset{\sim}{A}\underset{\sim}{A}^T$ as the hybrid matrix of a linear multiport resistor and find the group G_ρ of all symmetry permutations of this resistor using the algorithm of Appendix B of [11]. Observe that only the permutations which do not interchange elements of the sets M_k are valid candidates as elements of G_ρ, since this condition implies that the diagonal of AA^T is invariant under ρ.

3) Find the group G of all permutations $\underset{\sim}{\pi},\sigma,\rho$ such that $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$, $\rho \in G_\rho$ and $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{A} = \underset{\sim}{A}\underset{\sim}{P}(\rho)$. We use the Algorithm of Appendix B of [11] to find for a <u>given</u> $\rho \in G_\rho$ all solutions of $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma) = \underset{\sim}{A}\underset{\sim}{P}(\rho)$ for $(\underset{\sim}{\pi},\sigma,\rho) \in \mathcal{P}$. Two important group theoretic observations can greatly reduce the number ρ's to be considered. If for a given ρ there is no solution then also for any ρ_0 such that $\rho_0^m = \rho$ for some integer m there is no solution. Second if the solutions for any ρ_1 and ρ_2 are found then any solution for $\rho_1\rho_2$ can be found by making the composition of the solutions for ρ_1 and ρ_2.

4) As a result of the previous steps we have a group G of permutations which leave all graph theoretic aspects of the network invariant. Since the symmetry operations can complement any of the ℓ connected parts of the network the group G has to be extended to a group G' of $2^{\ell}(\#G)$ directed permutations. In other words, any permutation contained in G gives rise to 2^{ℓ} directed permutations in G'.

5) Find the symmetry group $\mathcal{S} \subset G'$. For any directed permutation $(\underset{\sim}{\pi},\sigma,\rho) \in G'$ check if the $\pi^{(j)}$-permuted resistor of the j-th resistor and the $\sigma(j)$-th resistor of N are identical, for all j. Again the number of elements of G' to be checked can be reduced significantly by the following two rules: First, if $\rho \in G'$, $\rho \notin \mathcal{S}$ then for all $\rho_0 \in G'$ and $\rho_0^m = \rho$ for some integer $m, \rho_0 \notin \mathcal{S}$. Second, if $\rho_1 \in \mathcal{S}$ and $\rho_2 \in \mathcal{S}$, then $\rho_1 \circ \rho_2 \in \mathcal{S}$.

III. PROPERTIES OF SOLUTIONS OF A SYMMETRIC NETWORK

It will be proved in general in this section that if the solution of a symmetric network is unique then this <u>solution is symmetric</u> if it is measured with respect to suitable reference nodes. If we have reference nodes which are invariant under the symmetry node permutation ρ, this property follows almost immediately from Theorem 1 and is stated in Theorem 2. We call a node ⓚ invariant under ρ if $\rho(ⓚ) = ⓚ$ or $\rho(ⓚ) = \overline{ⓚ}$. Since it may not be possible in general to find invariant nodes the remainder of this section is devoted to a systematic procedure for introducing reference nodes and/or selecting reference nodes such that the solution is symmetric.

<u>Theorem 2</u>. If the solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ of a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N is <u>unique</u> and if the reference node(s) is (are) invariant under ρ, then

$$\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{v} = \underset{\sim}{v} \tag{28a}$$

$$\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{i} = \underset{\sim}{i} \tag{28b}$$

$$\underset{\sim}{P}(\rho)\underset{\sim}{u} = \underset{\sim}{u} \tag{28c}$$

Proof: From Thm. 1, the symmetry of N, and the invariance of the reference node(s), we see that

$$(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u}) \text{ and } (\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\,\underset{\sim}{v},\underset{\sim}{P}(\underset{\sim}{\pi},\rho)\underset{\sim}{i},\underset{\sim}{P}(\rho)\underset{\sim}{u})$$

are solutions of N with respect to the same set of reference node(s). The theorem now follows from the uniqueness of the solution. ¤

Corollary 1. Let N be a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network having a unique solution and reference nodes which are invariant under ρ.

1) If ρ has a cyclic decomposition $\rho = (\ ) \ldots (n_1\ n_2\ \ldots\ n_\ell) \ldots (\ )$, then the voltages at all nodes belonging to each cyclic component and having the same orientation in the cycle, say $n_{i_1}, n_{i_2}, \ldots, n_{i_k}$ are equal to each other. Similarly, the voltages of the remaining nodes, say $\overline{n_{j_1}}, \overline{n_{j_2}}, \ldots, \overline{n_{j_k}}$, in the same cyclic component (which must necessarily have the opposite orientation) are equal to the negative of the voltage of the first set of nodes.

2) Analogously for the cyclic decomposition

$$(\underset{\sim}{\pi},\sigma) = \Big(\quad\Big)\ldots\ \Big((i_1,j_1)(i_2,j_2)\ \ldots\ (i_k,j_k)\Big)\ \ldots\ \Big(\quad\Big),$$

the voltages at all resistor terminals belonging to each cyclic component and having the same orientation are equal to each other. Similarly, the voltages of the remaining resistor terminals in the same cyclic component (which must necessarily have the opposite orientation) are equal to the negative of the corresponding voltages of the first set of terminals.

3) Analogous statements also apply to the resistor terminal currents.

The symmetric lattice property (Prop. 4) in Sec. 1 can now be proved with the help of Thm. 2 and the above Corollary. The one-port resistors $R^{(2)}$ and $R^{(3)}$ (resp., $R^{(4)}$ and $R^{(5)}$) in Fig. 3 are identical. The node set is:

$$①=\{(1,1),(1,2),(1,5)\},\qquad ③=\{(1',2),(1,4),(1,6)\}$$

$$②=\{(1',1),(1',3),(1',4)\},\qquad ④=\{(1,3),(1',5),(1',6)\}\ .$$

The symmetry transformation is $\pi^{(j)}=\begin{pmatrix}1\\1\end{pmatrix}$ for $j=1,\ldots,6$, and

$\sigma=\begin{pmatrix}1&2&3&4&5&6\\1&3&2&5&4&6\end{pmatrix}$ and $\rho=\begin{pmatrix}①&②&③&④\\\bar{②}&\bar{①}&\bar{④}&\bar{③}\end{pmatrix}$. Observe that no node is invariant under ρ. In order to apply Thm. 2 and Cor. 1, let us introduce a phase-inverting ideal transformer (Fig. 7a), which is a 3-terminal resistor described by $v_1-v_3=v_3-v_2$, $i_1=i_2$, $i_1-i_2+i_3=0$. By connecting this 3-terminal resistor to the symmetric lattice as shown in Fig. 13, the symmetry is preserved and the currents and voltages are not modified since terminal (3,7) does not carry any current. Nodes ③ ④ and ⑤ become

$$③=\{(1',2),(1,4),(1,6),(1,7)\}$$

$$④=\{(1,3),(1',5),(1',6),(2,7)\}$$

$$⑤=\{(3,7)\}$$

and the node symmetry operation is $\rho'=\begin{pmatrix}①&②&③&④&⑤\\\bar{②}&\bar{①}&\bar{④}&\bar{③}&\bar{⑤}\end{pmatrix}$. Observe that node ⑤ is now an invariant node and can be chosen as our reference node. Since $\rho'=(①\ \bar{②})(③\ \bar{④})(⑤\ \bar{⑤})$, it follows from Cor. 1 that the voltages at nodes ① and ② {resp., nodes ③ and ④} are the negative of each other. Since $(\underset{\sim}{\pi}',\sigma') = \Big([1,1]\Big)\Big([1,2]\,[1,3]\Big)\Big([1,4]\,[1,5]\Big)$ $\Big([1,6]\Big)\Big((1,7)(\bar{2},7)\Big)\Big((3,7)\,(\bar{3},7)\Big)$, this implies, among other things, that the voltages and currents in resistors $R^{(2)}$ and $R^{(3)}$ are equal to each other. This proves Prop. 4. As a result of this corollary we see that the voltages of many nodes in the network are equal to each other, or differ by a negative sign. The great advantage of the cyclic decomposition is now obvious: it allows us to select these nodes immediately. This observation will enable us to construct in Sec. V a practical algorithm for simplifying a symmetric network.

<u>Corollary 2</u>. If a complementary symmetric network has a unique solution, then all current and voltage solutions are <u>zero</u>.

<u>Proof</u>: Since $\pi^{(j)} = \bar{I}$, $\sigma = I$, and $\rho = \bar{I}$, we have $\underset{\sim}{P}(\rho) = -\underset{\sim}{1}_n$ and $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma) = -\underset{\sim}{1}_r$, it follows from Thm. 2 that $\underset{\sim}{v} = -\underset{\sim}{v}$, $\underset{\sim}{i} = -\underset{\sim}{i}$, $\underset{\sim}{u} = -\underset{\sim}{u}$. ¤

A general remark concerning Thm. 2 and its corollaries is that the <u>uniqueness of the solution</u> is <u>essential</u> to guarantee the symmetry of the solution. Counterexamples such as the Eccles-Jordan multivibrator (Fig. 14) show that this condition cannot be relaxed.

We have already seen that a symmetric network (such as Fig. 3) may not have an invariant node. An artifice was then introduced as in Fig. 13 which allows such a node to be generated without affecting the solutions of the original network. Another common situation of a symmetric network having no invariant nodes is given by the class of rotational symmetric networks. The network in Fig. 15 is a case in point. We will now describe a general method for selecting reference nodes such that the unique solution is symmetric. In order to do this, let us first investigate the relationship between the connected sets of nodes

$\mathcal{N}_1, \mathcal{N}_2, \ldots,$ and the symmetry node operation ρ. Let $\tilde{\rho}$ be called the <u>directed permutation of the connected sets of nodes</u> and be defined by

$$\tilde{\rho}(\mathcal{N}_i) = \{\rho(\text{k}) \mid \text{k} \in \mathcal{N}_i\}. \tag{29}$$

The following proposition then shows that $\tilde{\rho}$ is indeed a directed permutation and is the directed permutation induced by ρ on the connected sets of nodes.

<u>Proposition 12</u>. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network $N = (R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$ with connected sets of nodes $\mathcal{N}_1, \mathcal{N}_2, \ldots,$ then for any i there exists a j such that

$$\tilde{\rho}(\mathcal{N}_i) = \mathcal{N}_j \text{ or } \bar{\mathcal{N}}_j \tag{30}$$

<u>Proof</u>: Any two nodes n_1 n_2 of $\mathcal{N}_i$ are interconnected via some ports of multiport resistors and/or via some multiterminal resistors. Since the network permutation preserves this interconnection, $\rho(n_1)$ and $\rho(n_2)$ are in the same way interconnected in the permuted network and thus belong to the same consistent set of nodes. Since the nodes of $\mathcal{N}_i$ are either all or not complemented, we have

$$\tilde{\rho}(\mathcal{N}_i) \subset \mathcal{N}_j \text{ or } \bar{\mathcal{N}}_j.$$

Applying the inverse permutation $(\underset{\sim}{\pi},\sigma,\rho)^{-1}$ which is also a symmetry permutation, and repeating the previous argument, we obtain

$$\tilde{\rho}^{-1}(\mathcal{N}_j) \subset \mathcal{N}_i \text{ or } \bar{\mathcal{N}}_i$$

because ρ^{-1} is the node symmetry permutation of $(\underset{\sim}{\pi},\sigma,\rho)^{-1}$. ¤

This proposition implies that the connected set of nodes determine an equivalence relation in the set of nodes, which is invariant under ρ [13;p.166].

Corollary 1. $\tilde{\rho}$ is a directed permutation of the connected sets of nodes and can thus be written as

$$\tilde{\rho} = \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 & \cdots \\ \mathcal{N}_i & \mathcal{N}_j & \cdots \end{pmatrix} \tag{31}$$

or by its cyclic decomposition

$$\tilde{\rho} = (\mathcal{N}_1 \mathcal{N}_i \cdots)(\quad)\cdots(\quad). \tag{32}$$

Corollary 2. Given a connected set of nodes $\mathcal{N}_{i_0}$ with cycle

$$(\mathcal{N}_{i_0}\ \mathcal{N}_{i_1}\ \cdots \mathcal{N}_{i_{m-1}}), \tag{33}$$

then any node $n_{i_0} \in \mathcal{N}_{i_0}$ has a cycle

$$(n_{i_0}\, n_{i_1} \cdots\, n_{i_{k-1}}) \tag{34}$$

whose order k is a multiple of m, and $n_{i_j} \in \mathcal{N}_{i_{j_1}}$, where j_1 is the remainder of the division of j by m.

Definition 7. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N we call a set of reference nodes of N compatible with the symmetry permutation $(\underset{\sim}{\pi},\sigma,\rho)$ if

1) for all connected sets of nodes belonging to the same cyclic component of $\tilde{\rho}$, the reference nodes also belong to the same cyclic component of ρ, and

2) for any connected set of nodes $\mathcal{N}_i$, the cycle of ρ, which contains the reference node of $\mathcal{N}_i$, only contains nodes which have all the same orientation or which have all both orientations.

Constructively, these conditions imply the following. Choose some cyclic components of ρ such that each connected set of nodes $\mathcal{N}_i$ has one or more node(s) appearing each in just one selected cyclic component of ρ. Then condition 1 is automatically satisfied, if we choose the reference nodes among the nodes of the selected components of ρ. Condition 2 requires additionally that for each selected component γ_j and for each connected set of nodes $\mathcal{N}_i$, any two different nodes $\textcircled{n'}$ and $\textcircled{n''}$ of $\mathcal{N}_i$ appearing in γ_j can only have the following orientations in this cycle $\textcircled{n'}$, $\textcircled{n''}$, or $\overline{\textcircled{n'}}$, $\overline{\textcircled{n''}}$ or $\textcircled{n'}$, $\textcircled{n''}$, $\overline{\textcircled{n'}}$, $\overline{\textcircled{n''}}$. Thus the mixed cases $\textcircled{n'}$, $\overline{\textcircled{n''}}$ and $\overline{\textcircled{n'}}$, $\textcircled{n''}$ are ruled out by condition 2. So for example if $\textcircled{1}$, $\textcircled{4}$, $\textcircled{7} \in \mathcal{N}_1$, $\textcircled{2}$, $\textcircled{5}$, $\textcircled{8} \in \mathcal{N}_2$ and $\textcircled{3}$, $\textcircled{6}$, $\textcircled{9} \in \mathcal{N}_3$ and if $\rho = (\textcircled{1}\ \overline{\textcircled{2}}\ \textcircled{3}\ \overline{\textcircled{4}}\ \textcircled{5}\ \overline{\textcircled{6}})(\textcircled{7}\ \overline{\textcircled{8}}\ \textcircled{9}\ \overline{\textcircled{7}}\ \textcircled{8}\ \overline{\textcircled{9}})$, the component $(\textcircled{1}\ \overline{\textcircled{2}}\ \textcircled{3}\ \overline{\textcircled{4}}\ \textcircled{5}\ \overline{\textcircled{6}})$ cannot contain reference nodes satisfying condition 2, while the component $(\textcircled{7}\ \overline{\textcircled{8}}\ \textcircled{9}\ \overline{\textcircled{7}}\ \textcircled{8}\ \overline{\textcircled{9}})$ can. In fact the nodes $\textcircled{7}$, $\textcircled{8}$, $\textcircled{9}$ form a compatible set of references nodes.

Before showing that Thm. 2 can be extended to the case of a network N with a set of reference nodes compatible with the symmetry, we answer two important questions. Can a compatible set of reference nodes always be found for any symmetry permutation and any network N? And how can such a set of reference nodes be found? It is easy to see that any set of reference nodes which is invariant under ρ is compatible with the symmetry permutation and thus we are dealing with a larger class than that of Thm. 2. However there exist networks which have no set of reference nodes compatible with the symmetry permutation, the network of Fig. 3 being a case in point. This difficulty has been solved in Fig. 13 by adding a phase-inverting ideal transformer whose terminal 3 is free. This does not modify the solution of the network and preserves the symmetry. We now present a simple algorithm which extends this technique and shows that by adding some phase-inverting ideal transformers a compatible set of reference nodes can always be found.

Algorithm. Construction of a compatible set of reference nodes. Given a (π,σ,ρ)-symmetric network N introduce phase-inverting ideal transformers in order to obtain an equivalent symmetric network and find a set of reference nodes compatible with the symmetry.

1) Find the cyclic decomposition of ρ, the directed permutation $\tilde{\rho}$ of the connected set of nodes and its cyclic decomposition. The assignment of reference nodes in each connected set of nodes and the eventual introduction of phase-inverting ideal transformers are applied to one cyclic component of $\tilde{\rho}$ at a time. So step 2 has to be repeated for all cycles of $\tilde{\rho}$.

2) Consider the cycle $\beta = (\mathcal{N}_{i_0} \mathcal{N}_{i_1} \cdots \mathcal{N}_{i_{m-1}})$ of $\tilde{\rho}$. One can choose arbitrarily any cyclic component $\gamma = (n_{i_0} n_{i_1} \cdots n_{i_{k-1}})$ of ρ such that $n_{i_0} \in \mathcal{N}_{i_0}$[3]. Then by Cor. 2 of Thm. 2 k is a multiple of m or k = ms and $n_{i_0}, n_{i_m}, n_{i_{2m}} \cdots n_{(s-1)m} \in \mathcal{N}_{i_0}$ and $n_{i_1}, n_{i_{m+1}}, n_{i_{2m+1}} \cdots n_{i_{(s-1)m+1}} \in \mathcal{N}_{i_1}$ and so on. If we choose the reference nodes of $\mathcal{N}_{i_0}\ \mathcal{N}_{i_1} \cdots \mathcal{N}_{i_{m-1}}$ among the nodes of γ, condition 1 of Def. 7 is automatically satisfied. The condition 2 of Def. 7, however, requires some further analysis of the cycle β. It is clear that β can be of normal or of double order. If β is a normal-

[3]Observe that $\mathcal{N}_{i_j}$ (resp. n_{i_j}) may as well stand for a complemented connected set of nodes (resp. node) as for an uncomplemented set.

order cycle then condition 2 is satisfied if we choose any node among n_{i_j} $n_{i_{m+j}}$ $\cdots$ $n_{i_{(s-1)m+1}}$ as reference nodes for $\mathcal{N}_{i_j}$ and repeat this for $j = 0,1,\ldots,m-1$. If β is a double-order cycle then m is even and $\mathcal{N}_{i_j} = \overline{\mathcal{N}}_{i_{m/2+j}}$ for $j = 0,1,\ldots,m/2-1$. Two cases have to be considered according to the fact that n_{i_0} and $\overline{n_{i_{m/2}}}$ are the same or not. a) If $n_{i_0} = \overline{n_{i_{m/2}}}$ then $n_{i_j} = \overline{n_{i_{m/2+j}}}$ for $j = 1,\ldots,m/2-1$, and condition 2 is satisfied if we choose the reference node for $\mathcal{N}_{i_j}$ arbitrarily among n_{i_j} $n_{i_{m+j}}$ $\cdots$ $n_{i_{(s-1)m+j}}$ and repeat this for $j = 0,1,\ldots,m-1$. b) In the case $n_{i_0} \neq \overline{n_{i_{m/2}}}$ we have then $n_{i_j} \neq \overline{n_{i_{m/2+j}}}$ for all $j = 1,\ldots,m/2-j$ and condition 2 of Def. 7 cannot be satisfied. Therefore we introduce k/2 phase-inverting ideal transformers T_i, $i = 0,1,\ldots,k/2$ as follows. Select two nodes k_1 and k_2 of γ such that $k_1 \in \mathcal{N}_{i_0}$ and $\overline{k_2} \in \mathcal{N}_{i_{m/2}}$. Connect terminals 1 and 2 of T_o to nodes k_1 and k_2 and leave terminal 3 of T_o open. Connect terminals 1 and 2 of T_1 to nodes $\rho(k_1)$ and $\rho(k_2)$ and leave terminal 3 open, and so on until terminals 1 and 2 of $T_{k/2}$ are connected to nodes $\rho^{k/2-1}(k_1)$ and $\rho^{k/2-1}(k_2)$ and terminal 3 is left open. The new network has k/2 new nodes at the third terminals of the T_i, $i = 0,\ldots,k/2$. Since no current is flowing in the T_i the solution of the old network can be immediately recovered from the solution of the corresponding part of the new network. The new network has also a symmetry permutation which acts on the old part of the network as the old symmetry permutation did on the old network. The node permutation contains a new double-order cycle of of the new k/2 nodes. Thus condition 2 is satisfied if we choose the m reference nodes among these k/2 nodes.

Example. We apply the algorithm to the network of Fig. 16(a) with the following resistors: the 5-port resistor $R^{(1)}$ which is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{1} & 2 & 4 & 5 & 3 \end{pmatrix}$-symmetric, the complementary 3-terminal resistors (transistors) $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(3)}$, the identical 2-terminal resistors $\mathcal{R}^{(4)}$, $\mathcal{R}^{(5)}$ and $\mathcal{R}^{(6)}$. The node set is given by

① = {(1,1),(3,2),(3,3)}	⑥ = {(4,1),(1,5)}
② = {(1',1),(2,2),(2,3)}	⑦ = {(5,1),(1,6)}
③ = {(1,2),(2,1)}	⑧ = {(3',1),(2,4)}
④ = {(1,3),(2',1)}	⑨ = {(4',1),(2,5)}
⑤ = {(3,1),(1,4)}	⑩ = {(5',1),(2,6)}.

This network is $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric with

$$\pi^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{1} & 2 & 4 & 5 & 3 \end{pmatrix}, \quad \pi^{(j)} = \begin{pmatrix} 1 & 2 & 3 \\ \bar{1} & \bar{2} & \bar{3} \end{pmatrix}, \quad j = 2,3 \text{ and } \pi^{(j)} = I, \; j = 4,5,6$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}, \quad \rho = \begin{pmatrix} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ & ⑨ & ⑩ \\ \bar{①} & \bar{②} & \bar{④} & \bar{③} & ⑥ & ⑦ & ⑤ & ⑨ & ⑩ & ⑧ \end{pmatrix}.$$

In step 1 of the algorithm we find the consistent sets of nodes

$$\mathcal{N}_1 = \{①, ②, ③, ④\}, \; \mathcal{N}_2 = \{⑤, ⑧\}, \; \mathcal{N}_3 = \{⑥, ⑨\}, \; \mathcal{N}_4 = \{⑦, ⑩\}.$$

The directed permutation of the consistent sets is

$$\tilde{\rho} = \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 & \mathcal{N}_3 & \mathcal{N}_4 \\ \bar{\mathcal{N}}_1 & \mathcal{N}_3 & \mathcal{N}_4 & \mathcal{N}_2 \end{pmatrix}.$$

The decompositions in cycles are

$$\rho = (①\,\bar{①})(②\,\bar{②})(③\,\bar{④})(⑤\,⑥\,⑦)(⑧\,⑨\,⑩),$$

$$\tilde{\rho} = (\mathcal{N}_1\bar{\mathcal{N}}_1)(\mathcal{N}_2\mathcal{N}_3\mathcal{N}_4).$$

In <u>step 2</u> we try to find a reference node for $\mathcal{N}_1$ with cycle $(\mathcal{N}_1\bar{\mathcal{N}}_1)$. We choose the cyclic component $(③\,\bar{④})$ of ρ. Since $(\mathcal{N}_1\bar{\mathcal{N}}_1)$ is a double-order cycle and since $③ \neq \bar{④}$ we have to introduce a phase-inverting ideal transformer between nodes ③ and ④ (Fig. 16b). The reference node in $\mathcal{N}_1$ is then the new node ⑪. Observe that this voltage divider need not be introduced if we had chosen cyclic component $(①\,\bar{①})$. In <u>step 2</u> for the cycle $(\mathcal{N}_2\mathcal{N}_3\mathcal{N}_4)$ we search for reference nodes for the other consistent sets. We choose a cyclic component $(⑧\,⑨\,⑩)$ of ρ containing nodes in $\mathcal{N}_2\mathcal{N}_3\mathcal{N}_4$. Since $(\mathcal{N}_2\mathcal{N}_3\mathcal{N}_4)$ is a normal-order cycle, condition 2 is automatically satisfied if we choose ⑧ (resp. ⑨, ⑩) as reference node for $\mathcal{N}_2$ (resp. $\mathcal{N}_3, \mathcal{N}_4$).

It should be clear from this example that the number of phase-inverting ideal transformers can be minimized by choosing as often as possible the reference nodes in double-order cycles of ρ for connected sets of nodes which appear in a double-order cycle of $\tilde{\rho}$.

Observe also that if no nodes are complemented by ρ no phase-inverting transformers have to be introduced, since the cycles of ρ and $\tilde{\rho}$ contain no complements.

<u>Theorem 3</u>. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N with a set of reference nodes compatible with the symmetry and with a unique solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ then

$$\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{v} = \underset{\sim}{v} \tag{35a}$$

$$\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)\underset{\sim}{i} = \underset{\sim}{i} \tag{35b}$$

$$\underset{\sim}{P}(\rho)\underset{\sim}{u} = \underset{\sim}{u} \tag{35c}$$

Proof. The main problem is to remove the asymmetry in the location of the reference nodes. Consider any connected set of nodes $\mathcal{N}_i$ and its cyclic component in $\tilde{\rho}$. Call r the number of connected sets of nodes in this cycle. Then the order of the cycle is equal to r (resp. 2r) if it is a normal-order cycle (resp. double-order cycle). Consider the cyclic component β of ρ which contains the reference node of $\mathcal{N}_i$. Now ρ^r has a cyclic component γ which is composed precisely of the nodes of β contained in $\mathcal{N}_i$. Consider now the symmetry permutation $(\underset{\sim}{\pi},\sigma,\rho)^r$. Since any cycle of ρ^r is composed of every r-th node of a cycle of ρ the chosen set of reference nodes is also compatible with $(\underset{\sim}{\pi},\sigma,\rho)^r$. After a relabelling of the nodes, node ① is the reference node and this cyclic component is $\gamma = (① \bar{②} \ldots \bar{ⓝ})$ or $\gamma = (① \bar{②} \ldots ⓝ \bar{①} ② \ldots \bar{n})$ with n odd. Observe that $(① ② \ldots ⓝ)$ cannot appear since the reference nodes are compatible with $(\underset{\sim}{\pi},\sigma,\rho)^r$. In general, the node vector contains the following part $[0\ u_{②} u_{③} \ldots u_{ⓝ}]^T$. It follows from Thm. 1 and the symmetry of the network with the relabelled nodes that $\underset{\sim}{P}(\gamma)[0\ u_{②} \ldots u_{ⓝ}]^T$ is also a valid part of the node vector. But this solution has reference node ② . By making node ① again the reference node and using the uniqueness of the solution we obtain

$$\begin{bmatrix} 0 & & & \\ -1 & 1 & & \\ -1 & & 1 & \\ \vdots & & & \ddots \\ -1 & & & & 1 \end{bmatrix} \underset{\sim}{P}(\gamma) \begin{bmatrix} 0 \\ u_{②} \\ \vdots \\ u_{ⓝ} \end{bmatrix} = \begin{bmatrix} 0 \\ u_{②} \\ \cdots \\ u_{ⓝ} \end{bmatrix} \tag{36}$$

In the case $\gamma = (1\ 2 \ldots n)$ this reduces to

$$\begin{bmatrix} -1 & & & & -1 \\ 1 & & & & -1 \\ & \ddots & & & \\ & & \ddots & -1 & -1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_{2} \\ \vdots \\ \\ u_{n} \end{bmatrix} = \underset{\sim}{0} \tag{37a}$$

and in the case $\gamma(1\ \bar{2} \ldots n\ \bar{1}\ 2 \ldots \bar{n})$ with n odd we have

$$\begin{bmatrix} -1 & & & & 1 \\ -1 & & & & 1 \\ & \ddots & & & \\ & & \ddots & -1 & 1 \\ & & & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{2} \\ \vdots \\ \\ u_{n} \end{bmatrix} = \underset{\sim}{0}. \tag{37b}$$

Since matrix in (37) is of full rank, the only solution is

$$u_{2} = u_{3} \cdots = u_{n} = 0$$

This implies that nodes $1\ 2 \ldots n$ can be connected to each other and so the asymmetry in the reference node 1 is removed. After repeating this process for all connected sets we obtain a network with invariant reference nodes. Hence the theorem follows from (28). ¤

Cor. 1 of Thm. 2 can also be extended to this case.

Corollary 1. If a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N has a unique solution, and if the reference nodes are compatible with the symmetry permutation, then we have the following properties:

1) In any cycle γ of the decomposition of ρ into cyclic components, all nodes of γ with the same orientation have the same voltage; nodes with opposite orientations have opposite voltages.

2) In any cycle δ of the decomposition of $(\underset{\sim}{\pi},\sigma)$ into cyclic components, all the terminals of δ with the same orientation have the same voltage and carry the same current; terminals with opposite orientations have opposite voltages and currents.

In the case of cycles of double-order we arrive at the following conclusion.

Corollary 2. If a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N has a unique solution, and if the reference nodes are compatible with the symmetry permutation, then we have the following properties:

1) In any cycle of ρ of double-order, all nodes have zero voltage.

2) In any cycle of $(\underset{\sim}{\pi},\sigma)$ of double-order all terminals have zero current and zero voltage.

IV. SYNTHESIS OF A SYMMETRIC MULTIPORT OR MULTITERMINAL RESISTOR FROM A RESISTIVE NETWORK.

In many practical situations, such as in signal processing, port entries are made in a network and loaded with one-port resistors. The transfer characteristic from one entry to another, or the voltage-current driving-point characteristic at one entry, are analyzed [8,pp.228-236]. Interesting frequency separation properties can be obtained if there is some symmetry in the transfer characteristic [11].

Here we consider in general the problem of synthesizing a symmetrical multiport or multiterminal resistor from a resistive network. For the case where the network is already symmetric, or nearly symmetric, an easy method is described.

Let us describe first how port entries and terminal entries can be made in a resistive network.

Definition 8. Let N be a resistive network, and let $\mathcal{B}$ be an operator which selects n nodes of N where external terminals are attached (Fig. 17). The set of admissible terminal voltages and currents at these n terminals determine an equivalent n-terminal resistor $\mathcal{R}$, henceforth called the n-terminal resistor $\mathcal{R}$ derived from the network N by the operator $\mathcal{B}$.

Observe that the "equivalence" applies only externally since there need not be any relationship between the currents and voltages inside two networks which give rise to the same multiterminal resistor. Corresponding to the operator $\mathcal{B}$ we can construct a selection matrix $\underset{\sim}{B}$. $\underset{\sim}{B}$ has n rows and a number of columns equal to the number of nodes of N and has a 1 at entry i,j if the node ⓙ of N is connected to terminal i of $\mathcal{R}$, and zero otherwise. With this selection matrix $\underset{\sim}{B}$ we can describe $\mathcal{R}$ algebraically as follows: $(\underset{\sim}{v}^*,\underset{\sim}{i}^*)$ is an admissible pair of $\mathcal{R}$ if there exist vectors $\underset{\sim}{v} = [\underset{\sim}{v}^{(1)T}\ldots\underset{\sim}{v}^{(m)T}]^T$ and $\underset{\sim}{i} = [\underset{\sim}{i}^{(1)T}\ldots\underset{\sim}{i}^{(m)T}]^T$ such that

$$\begin{aligned}
&1)\ (\underset{\sim}{v},\underset{\sim}{i}) \text{ is an admissible pair of resistors of N}\\
&2)\ \underset{\sim}{A}\underset{\sim}{i} = \underset{\sim}{B}^T\underset{\sim}{i}^*\\
&3)\ \underset{\sim}{A}^T\underset{\sim}{u} = \underset{\sim}{v}\\
&4)\ \underset{\sim}{v}^* = \underset{\sim}{B}\underset{\sim}{u}
\end{aligned} \tag{38}$$

where $\underset{\sim}{A}$ is the incidence matrix of N.

Port entries can be made in a network by two essentially different methods: 1) The two terminals of the port can be soldered to an arbitrarily chosen pair of nodes of the network. This entry is henceforth called a soldering-iron port entry. 2) On the other hand a port can be inserted in series with an arbitrarily chosen terminal of a resistor. This entry is henceforth called a pliers-type port entry.

Definition 9. Let N be a resistive network and let $\mathcal{C}$ be an operator which maps a subset of the union of all "node pairs" and of all resistor "terminals" in a one-to-one manner onto "n" external ports. The selected node pairs correspond to the external ports created by soldering-iron entries, and the selected terminals correspond to the ports created by pliers-type entries (Fig. 18). The set of all admissible port voltages and currents at these n-ports determine an equivalent n-port resistor $\mathcal{R}$, called the n-port resistor derived from the network N by the operator $\mathcal{C}$. The port current constraint can be satisfied by connecting an isolation transformer, or by terminating the port externally by a 2-terminal element.

Observe from the operator $\mathcal{C}$ we can construct selection matrices $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ as follows:

1) The matrix $\underset{\sim}{C}_1$ has n rows and a number of columns equal to the number of nodes of N, and has a +1 (resp. -1) at the entry i,j if node j is connected to terminal i (resp. i') of R, and a zero otherwise.

2) The matrix $\underset{\sim}{C}_2$ has n rows and a number of columns equal to the sum $r = \sum_{i=1}^{m} n_i$ of the number of terminals of multiterminal resistors and ports of multiport resistors of N. As before we order these terminals and ports lexicographically. a) If port k is made by a pliers-type entry in terminal (i,j) i.e. the i-th terminal of the j-th multiterminal resistor then a +1 or -1 is placed in the k-th row of the column corresponding to (i,j) of $\underset{\sim}{C}_2$. The sign ± depends on the polarity of the port k. If terminal k' is nearer to the multiterminal resistor than

terminal k we have a +1, and -1 otherwise. b) Analogously if port k is made by a pliers-type entry in either terminal of the port [i,j], i.e., the i-th port of the j-th <u>multiport resistor</u>, then a + 1 or -1 is placed in the k-th row of the column corresponding to [i,j] of $\underset{\sim}{C}_2$. The sign ± depends on the polarity of the port k and on the terminal of port [i,j] selected for port entry. If the new port k and the port [i,j] are connected to each other with terminals k and (i',j), or with terminals k' and (i,j), we have a +1, or a -1 respectively.

In the case where all port entries are of the soldering-iron type, the admissible pairs $(\underset{\sim}{v}^*,\underset{\sim}{i}^*)$ of R are described by the following equations:

$$\begin{aligned} &1)\quad (\underset{\sim}{v},\underset{\sim}{i}) \text{ are admissible pairs for the resistors of N}\\ &2)\quad \underset{\sim}{A}\underset{\sim}{i} = \underset{\sim}{C}_1^T\underset{\sim}{i}^*\\ &3)\quad \underset{\sim}{A}^T\underset{\sim}{u} = \underset{\sim}{v}\\ &4)\quad \underset{\sim}{v}^* = \underset{\sim}{C}_1\underset{\sim}{u} \end{aligned} \tag{39}$$

In case there are only pliers-type entries, the admissible pairs $(\underset{\sim}{v}^*,\underset{\sim}{i}^*)$ of R are described by the following equations:

$$\begin{aligned} &1)\quad (\underset{\sim}{v},\underset{\sim}{i}) \text{ are admissible pairs for the resistors of N}\\ &2)\quad \underset{\sim}{A}\underset{\sim}{i} = \underset{\sim}{0}\\ &3)\quad \underset{\sim}{A}^T\underset{\sim}{u} = \underset{\sim}{v} + \underset{\sim}{C}_2^T\underset{\sim}{v}^*\\ &4)\quad \underset{\sim}{i}^* = \underset{\sim}{C}_2\underset{\sim}{i} \end{aligned} \tag{40}$$

In the most general case where both types of entries are made, it is desirable to use a hybrid pair of variables $(\underset{\sim}{x}^*,\underset{\sim}{y}^*)$ in order to simplify the algebraic formulations. The mixed vector $\underset{\sim}{x}^*$ consists of the voltages of the soldering-iron type ports, and the currents of pliers-type ports. The other mixed variable vector $\underset{\sim}{y}^*$ contains the remaining variables.

The admissible pairs of R in terms of these hybrid variables are then described by the following equations:

$$\begin{aligned}
&1)\quad (v,i) \text{ are admissible pairs of the resistors of N.}\\
&2)\quad Ai = C_1^T y^*\\
&3)\quad A^T u = v + C_2^T y^*\\
&4)\quad x^* = C_1 u + C_2 i
\end{aligned} \tag{41}$$

A. <u>Symmetric multiterminal resistor derived from a symmetric network.</u>

<u>Theorem 4</u>. Given a (π,σ,ρ)-symmetric resistive network N and a terminal selection operator $\mathcal{B}$, the n-terminal resistor $\mathcal{R}$ derived from the network N by $\mathcal{B}$ is μ-symmetric if at any selected node ⓘ

$$\mathcal{B}(\rho(ⓘ)) = \underline{\mu \circ \mathcal{B}(ⓘ)} \tag{42}$$

where $\circ$ denotes the composition operation, and where $k = \mathcal{B}(ⓘ)$ is equivalent to $\bar{k} = \mathcal{B}(\bar{ⓘ})$.

Observe that (42) implies that $BP(\rho) = P(\mu)B$.

<u>Proof</u>: (v^*,i^*) is a solution of (38). Applying the network permutation (π,σ,ρ), we obtain a new network N and a resistor $\mathcal{R}$ derived from $\hat{N}$ described by:

$$\begin{aligned}
&1)\quad \big(P(\pi,\sigma)v, P(\pi,\sigma)i\big) \text{ is an admissible pair of the resistors of } \hat{N},\\
&2)\quad \big(P(\rho)AP^T(\pi,\sigma)\big)\big(P(\pi,\sigma)i\big) = P(\rho)B^T i^*,\\
&3)\quad \big(P(\pi,\sigma)A^T P^T(\rho)\big)\big(P(\rho)u\big) = P(\pi,\sigma)v,\\
&4)\quad v^* = \big(BP^T(\rho)\big)P(\rho)u
\end{aligned} \tag{43}$$

Using the symmetry of N and (42), this becomes

$$\begin{aligned}
&1)\quad \big(P(\pi,\sigma)v, P(\pi,\sigma)i\big) \text{ is an admissible pair of the resistors of N},\\
&2)\quad AP(\pi,\sigma)i = P(\rho)B^T i^* = B^T P(\mu)i^*,\\
&3)\quad A^T P(\rho)u = P(\pi,\sigma)v,\\
&4)\quad v^* = P^T(\mu)BP(\rho)u \text{ or } P(\mu)v^* = BP(\rho)u.
\end{aligned} \tag{44}$$

Comparing (44) and (38) we see that $\left(\underset{\sim}{P}(\mu)\underset{\sim}{v}^*,\underset{\sim}{P}(\mu)\underset{\sim}{i}\right)$ is an admissible pair of $\mathcal{R}$. ¤

Observe that (42) requires that only selected nodes of N are permuted among each other by ρ. Examples of how this theorem can be used to synthesize symmetric multiterminal resistors are given in Fig. 19. The network in Fig. 19(a) is symmetric with ρ = (① ② ③)(④ ⑤ ⑥)(⑦)(⑧). The selection operator is (① ② ③) → (1 2 3) and the resulting 3-terminal resistor is (1 2 3)-symmetric. Similarly, the resulting 4-terminal resistor of Fig. 19(b) is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & - & - \\ 2 & 3 & 4 & 1 \end{pmatrix}$-symmetric.

B. <u>Symmetric multiport resistor derived from a symmetric network</u>

Again it is easy to find a method to derive a symmetric multiport resistor from a symmetric network by appropriately choosing the port entries via the selection operator $\mathcal{C}$.

<u>Theorem 5</u>. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric resistive network N, then the multiport resistor, derived from N by the port selection operator $\mathcal{C}$, is μ-symmetric if at any selected soldering-iron port entry at node pair ⓘ ⓙ, we have

$$\mathcal{C}(\rho(ⓘ),\rho(ⓙ)) = \mu \circ \mathcal{C}(ⓘ,ⓙ) \tag{45a}$$

and at any selected pliers-type entry in port or terminal i of resistor j, we have

$$\mathcal{C}(\pi^{(j)}(i),\sigma(j)) = \mu \circ \mathcal{C}(i,j)$$

or (45b)

$$\mathcal{C}[\pi^{(j)}(i),\sigma(j)] = \mu \circ \mathcal{C}[i,j].$$

Moreover, condition (45) now becomes

$$\underset{\sim}{C}_1\underset{\sim}{P}(\rho) = \underset{\sim}{P}(\mu)\underset{\sim}{C}_1 \tag{46a}$$

$$\underset{\sim}{C}_2\underset{\sim}{P}(\underset{\sim}{\pi},\sigma) = \underset{\sim}{P}(\mu)C_2 \tag{46b}$$

Proof: Using (41) and (46), the proof is analogous to that of Theorem 4. ⌑

Corollary 1. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric resistive network $\mathcal{N}$. Make a soldering-iron port entry at nodes ⓘ and ⓙ, or make a pliers-type port entry in a terminal of a two-terminal resistor whose terminals are connected to nodes ⓘ and ⓙ. Then the equivalent one-port resistor is complementary symmetric if the cyclic decomposition of ρ contains the cycle (ⓘ ⓙ), or the cycles (ⓘ $\bar{ⓘ}$)(ⓙ $\bar{ⓙ}$).

Since the set of admissible pairs (v,i) form the driving-point plot (DP plot) of the one port, this results in an odd DP plot. From this corollary Properties 22, 25 and 26 of [9] on DP plots and our Prop. 1 in Sec. 1 can be easily derived. Among other things, this proves that any one port resistor obtained by making a port entry in a network composed of complementary symmetric elements is complementary symmetric.

The hypothesis of Cor. 1 is useful in many cases. We call it the odd node conditions in the next definition.

Definition 10. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N, then two nodes ⓘ and ⓙ of N are said to satisfy the odd node conditions if either

1) ⓙ $= \rho($ⓘ$)$ and ⓘ $= \rho($ⓙ$)$ (47a)

or

2) $\bar{ⓘ}$ $= \rho($ⓘ$)$ and $\bar{ⓙ}$ $= \rho($ⓙ$)$ (47b)

They satisfy the even node conditions if either

1) ⓘ $= \rho($ⓘ$)$ and ⓙ $= \rho($ⓙ$)$ (48a)

or

2) $\bar{Ⓙ} = \rho(Ⓘ)$ and $\bar{Ⓘ} = \rho(Ⓙ)$ (48b)

Corollary 2. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric resistive network. Let a port be created via a soldering-iron entry across nodes ⓘ and ⓙ , or via a pliers-type entry in a terminal of a two-terminal resistor connecting nodes ⓘ and ⓙ. Let another port be created in the same way but using nodes ⓚ and ⓛ. Then the equivalent two-port resistor is $\begin{pmatrix}1 & 2\\ \bar{1} & \bar{2}\end{pmatrix}$ -symmetric, or complementary symmetric, if 1) nodes ⓘ ⓙ of the first port satisfy the odd node conditions and 2) also nodes ⓚ ⓛ of the second port satisfy the odd node conditions. The equivalent two-port resistor is $\begin{pmatrix}1 & 2\\ \bar{1} & 2\end{pmatrix}$ -symmetric if 1) nodes ⓘ ⓙ of the first port satisfy the odd node conditions and 2) nodes ⓚ ⓛ of the second port satisfy the even node conditions.

Properties 23, 25 and 26 of [9] on TC plots, and our Prop. 2 in Sec. 1 follow from Cor. 2 as special cases: Two ports are extracted and the first is driven by a voltage source v_{in}, and the second by a zero-valued current source. The voltage v_o at the second port is measured as a function of v_{in} (TC plot).

Many practical circuits make use of this corollary. The push-pull amplifier discussed in [20] is a case in point. Another example is the rectifier circuit shown in Fig. 20 with symmetric node permutation ρ = (① ②)(③ ④)(⑤)(⑥). Two port entries are made in this network via the operator $\mathcal{C}$: {(1,2),(⑤ ⑥)} → {1,2}. In other words a pliers-type port entry is inserted through a terminal of R_1 connected to nodes ① and ②. The second port entry is created via a soldering-iron type across nodes ⑤ and ⑥. The quivalent two-port resistor is $\begin{pmatrix}1 & 2\\ \bar{1} & 2\end{pmatrix}$ -symmetric.

We will now pose the following problem to illustrate a nice application of Cor. 2 of Thm. 5. Given a general bridge circuit (Fig. 21) where two port entries are made via the operator $\mathcal{C}$: $\{(①\ ②),(③\ ④)\} \to \{1,2\}$, find conditions on the resistors such that the equivalent two-port resistor is $\begin{pmatrix} 1 & 2 \\ \bar{1} & 2 \end{pmatrix}$-symmetric. As a result of Cor. 2, we see that this condition is satisfied, if there exists a network symmetry permutation such that ①, ② (resp. ③, ④) satisfy the odd (resp. even) node conditions. Since the network contains one connected set of nodes any network permutation either complements all or no resistors and nodes. There are two possible symmetry permutations. The first is $\pi^{(j)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $j = 1,\ldots,4$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, and $\rho = \begin{pmatrix} ① & ② & ③ & ④ \\ ② & ① & ③ & ④ \end{pmatrix}$. This results in the condition $R_1 = R_2$ and $R_3 = R_4$. The second is $\pi^{(j)} = \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$ for $j = 1,\ldots,4$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ and $\rho = \begin{pmatrix} ① & ② & ③ & ④ \\ \bar{①} & \bar{②} & \bar{④} & \bar{③} \end{pmatrix}$. This produces the second case $R_1 = R_3$ and $R_2 = R_4$. It can be checked that no other possibilities exist.

In Thms. 4 and 5 symmetric multiterminal and multiport resistors are derived from symmetric resistive networks by a suitable choice of the terminal and port entries. It is however not necessary at all for a network to be symmetric in order to be able to derive from it a symmetric multiterminal or multiport resistor. A simple example is given in Fig. 22. If we make a soldering-iron port entry to this network we obtain a complementary symmetric one port.

In fact the definitions of a symmetric multiterminal and multiport resistor derived from a network imply that the whole network can be replaced by one equivalent symmetric resistor. In many cases, however, it is not necessary to replace the entire network by one equivalent resistor. In other words, some equivalence at an intermediate level is sufficient. Some parts of the network are replaced by their equivalent, and the resulting network is symmetric, and hence Thms. 4 and 5 can be applied.

To illustrate this idea, Fig. 23 shows a 2-port resistor which exhibits $(1\ \bar{1})(2)$-symmetry. The original network does not exhibit a symmetry satisfying Def. 6. However, after the series connection of branches ③ ⑤ and ⑤ ④ is replaced by one equivalent bilateral resistor $\tilde{R}$, the network exhibits a symmetry with symmetry node permutation (①)(②)(③ ④). If we now apply Thm. 5 it follows that the resulting two-port is indeed $(1\ \bar{1})(2)$-symmetric.

V. REDUCTION OF A SYMMETRIC NETWORK

By making use of the symmetry of a network, the computation for obtaining the unique solution can be greatly simplified. Most of the results in the literature are described for linear networks [1]-[7]. Essentially two techniques are applied in the nonlinear case. The first [10] uses symmetry to reduce the number of network equations to be solved. The second [8] derives from the given network a new network whose solution is easier to compute. The solution of the original network can then be derived from the solution of the new network. This second technique is an adaptation of the bisection technique [2] for nonlinear networks which is only valid for involution symmetric networks (i.e. for ν-symmetric networks with $\nu^2 = I$). Here we unify the two approaches and present a technique which applies to all possible symmetries. As always, we assume that the network has a unique solution.

The procedure for finding the reduced network will be introduced via simple examples.

Network reduction examples

The network shown in Fig. 24(a) has a symmetry node permutation $\rho = (①\ ③)(②)(④)$. Since it has a unique solution, it follows from Thm. 5 that the node voltages are symmetric with respect to ρ. So nodes ① and ③ can be connected together and we obtain Fig. 24(b). Now identical resistors which are in parallel can be replaced by an equivalent resistor and we obtain Fig. 24(c). This network has the same voltage distribution as Fig. 24(a), and a current which is doubled. Fig. 24(c) is called the reduced network of Fig. 24(a) and is clearly easier to analyze. The other networks Fig. 24(d)(g)(j) can be respectively reduced to Fig. 24(f)(i)(ℓ) by observing that their symmetry node permutations are given respectively by $(①\ ③)(②)(④)$, $(①\ ③)(②\ ④)$ and $(①\ ③)(②\ ④)$. Observe that the reduced network in many cases contains open branches (Fig. 24(c)) and hinged loops (Fig. 24(ℓ)).

It is instructive to consider also an example (Fig. 25(a)) whose symmetry operation involves some node complementations. From previous derivations we know that the symmetry node operation is

$$\rho = \begin{pmatrix} ① & ② & ③ & ④ \\ \bar{②} & \bar{①} & \bar{④} & \bar{③} \end{pmatrix} = (①\ \bar{②})(③\ \bar{④}) \text{ if } R^{(2)} = R^{(3)} \text{ and } R^{(4)} = R^{(5)}.$$

Since nodes $\bar{②}$ and $\bar{④}$ are not directly available, we introduce the complementation artifice (Fig. 7(g,h)) to obtain the network diagram shown in Fig. 25(b). By introducing a phase inverting ideal transformer we obtain a reference node (node ⑤) which is compatible with the symmetry (Fig. 25(c)). By Thm. 5 nodes ① and $\bar{②}$ and nodes ③ and $\bar{④}$ are at the same voltage and can thus be connected to each other (Fig. 25(c)). Remember that the complementation element is connected to the reference node and thus the voltages at the two terminals of any complementation element are symmetric with respect to the zero reference voltage. This implies that the voltage at node ① $\bar{②}$ is V/2 and at node ③ $\bar{④}$ it is E/2. Consequently, the voltage at node $\bar{③}$ ④ is -E/2. If $R^{(2)}$ nad $R^{(4)}$ are described by $i = f^{(2)}(v)$ and $i = f^{(4)}(v)$ respectively, we have $i^{(2)} = f^{(2)}\ (V/2-E/2)$ and $i^{(4)} = f^{(4)}\ (V/2+E/2)$. The circuit diagram reduces to Fig. 25(d) where $R_o^{(2)} = R^{(2)} /\!/ R^{(3)}$ is the parallel connection of resistors $R^{(2)}$ and $R^{(3)}$; and analogously for $R_0^{(4)}$. Observe that a different way of writing the cyclic decomposition of ρ, such as (① $\bar{②}$)(④ $\bar{③}$), would have produced a slightly different reduced network. All possible reduced networks can be obtained from each other by complementating some nodes (just complement nodes ③ $\bar{④}$ in the case of (① $\bar{②}$)(④ $\bar{③}$)).

We extend the definition of the "port or terminal - node incidence matrix" of Section II as follows to include the case of <u>complemented terminals</u> as in Fig. 25(d). The same rules as before apply, but if a terminal is complemented we have an additional multiplication of the entry by -1. This implies that if the terminals (i,j) and $(\bar{i},j)$ of multiport resistor j are connected to the same node ⓚ, then the entry corresponding to port [i,j] and node ⓚ is 2.

In general an sxt matrix $\underset{\sim}{A}_0$ is an incidence matrix of a network with s nodes, where t is the total number of ports and terminals (some of the terminals may be complemented), if the following conditions are satisfied: 1) Each column of $\underset{\sim}{A}_0$ corresponding to a <u>terminal</u> contains only one nonzero entry (either 1 or -1) and 2) In each column of $\underset{\sim}{A}_0$ corresponding to a <u>port</u>, the only nonzero entries consist of {1,-1}, or {1,1}, or {-1,-1}, or {2} or {-2}. It is easily seen that under these conditions we can immediately derive from $\underset{\sim}{A}_0$ the interconnections and the eventual complementations of the terminals, and vice versa. It is easy to check that KVL and KCL are given by the same equations (5), where $\underset{\sim}{A}_0$ replaces $\underset{\sim}{A}$. Call $\underset{\sim}{A}$ (resp. $\underset{\sim}{A}_0$) the incidence matrix of Fig. 25(a) (resp. 25(d)), then we have

$$\underset{\sim}{A} = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \quad , \quad \underset{\sim}{A}_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix} \begin{array}{c} ①\,\bar{②} \\ ③\,\bar{④} \end{array}$$

The node voltages at ① $\bar{②}$ and ③ $\bar{④}$ can be easily derived using $\underset{\sim}{A}_0$ and KVL: $u_{①\bar{②}} = V/2$, $u_{③\bar{④}} = E/2$. The currents can be found using the constitutive equations of $R_0^{(2)}$ and $R_0^{(4)}$.

We will now describe in general the reduction technique <u>algebraically</u>, thereby giving an algorithm for finding the incidence matrix $\underset{\sim}{A}_0$ and the constitutive relations of the resistors of the reduced network. It will be shown that a reinterpretation of this algebraic reduction technique allows us to devise a general combinatorial and graphical procedure for obtaining the reduced network.

<u>Definition 11</u>. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric resistive network N characterized by: 1) a set of admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ for the resistors i.e. $\underset{\sim}{v} = [\underset{\sim}{v}^{(1)T}\ldots\underset{\sim}{v}^{(m)T}]^T$ and $\underset{\sim}{i} = [\underset{\sim}{i}^{(1)T}\ldots\underset{\sim}{i}^{(m)T}]^T$ with $(\underset{\sim}{v}^{(j)},\underset{\sim}{i}^{(j)})$ an admissible pair of the j-th resistor and 2) an incidence matrix $\underset{\sim}{A}$. Construct matrices $\underset{\sim}{S}(\rho)$ and $\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$ from the cyclic decomposition of the nodes and of the terminals (see Def. 4 of [11]). Observe that these matrices are unique up to an unessential permutation of the columns

and a multiplication of any column by -1. Then the reduced resistive network N_0 is characterized by: 1) the set of admissible pairs $(\underset{\sim}{v}_0, \underset{\sim}{i}_0)$ for the resistors of N_0 such that[4]

$$\left(\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)\underset{\sim}{v}_0, \underset{\sim}{S}(\underset{\sim}{\pi},\sigma)[\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma)\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)]^{-1}\underset{\sim}{i}_0\right) \tag{49}$$

is an admissible pair of the resistors of N, and 2) the reduced incidence matrix:

$$\underset{\sim}{A}_0 = \underset{\sim}{S}^T(\rho)\underset{\sim}{A}\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)[\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma)\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)]^{-1} \tag{50}$$

Let us show that the reduced incidence matrix defined by (50) is indeed the incidence matrix of a network containing multiport and multi-terminal resistors where some terminals may be complemented. From (50) we derive (using the fact that the columns of $\underset{\sim}{S}\ (\underset{\sim}{\pi},\rho)$ are linearly independent).

$$\underset{\sim}{A}_0\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma) = \underset{\sim}{S}^T(\rho)\underset{\sim}{A}.$$

We claim that the right hand side of this equation satisfies the conditions for such an incidence matrix. Indeed each column of $\underset{\sim}{S}^T(\rho)\underset{\sim}{A}$ corresponding to a port can only contain {1,-1}, or {1,1}, or {-1,-1}, or {2}, or {-2} as nonzero entries, and each column corresponding to a terminal can only contain {1} or {-1} as nonzero entry. It follows from the structure of $\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma)$ that the matrix $\underset{\sim}{A}_0$ must inherit these properties from $\underset{\sim}{A}_0\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma)$. The solution of the reduced network N_0 is given by:

1) $(\underset{\sim}{v}_0, \underset{\sim}{i}_0)$ is an admissible pair for the resistors of N_0. (51a)

2) $\underset{\sim}{A}_0\underset{\sim}{i}_0 = \underset{\sim}{0}$ (51b)

3) $\underset{\sim}{A}_0^T\underset{\sim}{u}_0 = \underset{\sim}{v}_0$ (51c)

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[4]Observe that the matrix $\underset{\sim}{S}^T(\underset{\sim}{\pi},\sigma)\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$ is always invertible by Cor. of Prop. 9 in [11].

The most important property of the reduced network is that from its solution we can obtain the solution of the original network.

Theorem 6. Let N be a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network and let the reference nodes be compatible with the symmetry permutation. Let N have a unique solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$. Then also N_0 has a unique solution $(\underset{\sim}{v}_0,\underset{\sim}{i}_0,\underset{\sim}{u}_0)$. The solution of N can be derived from the solution of the reduced network N_0 by using

$$\underset{\sim}{v} = \underset{\sim}{S}(\underset{\sim}{\pi},\sigma)\underset{\sim}{v}_0 \tag{52a}$$

$$\underset{\sim}{i} = \underset{\sim}{S}(\underset{\sim}{\pi},\sigma)[\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)^T\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)]^{-1}\underset{\sim}{i}_0 \tag{52b}$$

$$\underset{\sim}{u} = \underset{\sim}{S}(\rho)\underset{\sim}{u}_0. \tag{52c}$$

Proof. From Thm. 3 we know that the solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ of N satisfies (35). This implies that $\underset{\sim}{v}$ and $\underset{\sim}{i}$ (resp. $\underset{\sim}{u}$) are eigenvectors of $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)$ (resp. $\underset{\sim}{P}(\rho)$) associated with the eigenvalue 1. From Prop. 9 of [11] the columns of the matrices $\underset{\sim}{S}(\rho)$ (resp. $\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$) form a complete set of linearly independent eigenvectors of $\underset{\sim}{P}(\rho)$ (resp. $\underset{\sim}{P}(\underset{\sim}{\pi},\sigma)$) associated with the eigenvalue 1. Thus there exists a unique set of vectors $(\underset{\sim}{v}_0,\underset{\sim}{i}_0$ $\underset{\sim}{u}_0)$ such that (52) is satisfied. Substituting (52) into (5) we see that $(\underset{\sim}{v}_0,\underset{\sim}{i}_0,\underset{\sim}{u}_0)$ satisfies (51) and thus is a solution of N_0. Suppose on the contrary that there is a second solution $(\underset{\sim}{v}_0',\underset{\sim}{i}_0',\underset{\sim}{u}_0')$ of N_0, then (52) generates a second solution of N which is different from $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ since the columns of $\underset{\sim}{S}(\rho)$ and $\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$ are linearly independent. This is clearly impossible. ◻

This theorem is very useful because it allows us to devise an algorithm for solving a symmetric network by solving the reduced network which has a smaller number of nodes, resistors and terminals.

Symmetry Reduction Algorithm. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N with admissible pairs $(\underset{\sim}{v},\underset{\sim}{i})$ of the resistors $\underset{\sim}{v},\underset{\sim}{i} \in \mathbb{R}^r$ and with $n{\cdot}r$ incidence matrix $\underset{\sim}{A}$. It is also known that N has a unique solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$.

1) Decompose $(\underset{\sim}{\pi},\sigma)$ into cyclic components. Identify the normal-order cyclic components and label them consecutively $\gamma_1,\gamma_2,\ldots,\gamma_t$. Analogously we find for the directed permutation ρ of the nodes the cycles of normal order $\delta_1,\delta_2,\ldots,\delta_s$.

2) Form the rxt matrix $\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$ and the nxs matrix $\underset{\sim}{S}(\rho)$ by inspection using Def. 4 of [11] and the cyclic components $\gamma_1,\gamma_2,\ldots,\gamma_t$ and $\delta_1,\delta_2,\ldots,\delta_s$.

3) Find the set of admissible pairs of N_0, i.e. the set of $(\underset{\sim}{v}_0,\underset{\sim}{i}_0)$ with $\underset{\sim}{v}_0,\underset{\sim}{i}_0 \in \mathbb{R}^t$ satisfying (49) and the sxt reduced incidence matrix $\underset{\sim}{A}_0$ in (50).

4) Find the unique solution of the reduced network, i.e. find the $(\underset{\sim}{v}_0,\underset{\sim}{i}_0,\underset{\sim}{u}_0)$ satisfying (51).

5) Substitute the solution $(\underset{\sim}{v}_0,\underset{\sim}{i}_0,\underset{\sim}{u}_0)$ in (52) in order to obtain the solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ of N.

Let us apply this algorithm to our previous example; namely, the symmetric lattice network in Fig. 25(a).

1) From the previously derived symmetry permutation we obtain the cyclic decompositions:

$$\rho = (1\ \bar{2})(3\ \bar{4})$$

$$(\underset{\sim}{\pi},\sigma) = ([1,1])([1,2],[1,3])([1,4],[1,5])([1,6])$$

2) We find the matrices

$$\underset{\sim}{S}(\rho) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \underset{\sim}{S}(\underset{\sim}{\pi},\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3) The set of admissible pairs $(\underset{\sim}{v}_0,\underset{\sim}{i}_0)$ of N_0 is the set of $\underset{\sim}{v}_0,\underset{\sim}{i}_0 \in \mathbb{R}^4$ such that

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{v}_0, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{i}_0 \right)$$

is an admissible pair of N. Observe that this is equivalent to a parallel connection of two identical resistors $R^{(1)}$ and $R^{(2)}$, and also the parallel connection of two resistors $R^{(3)}$ and $R^{(4)}$. The reduced incidence matrix $\underset{\sim}{A}_0$ is the same as that found before.

4) The solution of the reduced network is given as follows: From $\underset{\sim}{v}_0 = \underset{\sim}{A}_0^T\underset{\sim}{u}_0$ and the branch characteristics we obtain $\underset{\sim}{u}_0 = [V/2,E/2]^T$. From $\underset{\sim}{u}_0$ all the branch voltages and currents can be determined.

5) The solution of the original network N can be found by using (52).

Observe that it is often difficult to check the uniqueness condition. In the case of two or more solutions one can easily prove that from any solution $(\underset{\sim}{v}_0,\underset{\sim}{i}_0,\underset{\sim}{u}_0)$ of N_0, a solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$ of N can be found via (52). However the existence of a solution of N_0 for any solution of N cannot be guaranteed.

Let us now derive from this algebraic procedure, a step-by-step graphical reduction method which has already been introduced via the examples at the beginning of this section. This technique is better suited for graphical reduction of a symmetric network diagram, or for a network whose combinatorial characterization $(R^{(j)}, \mathcal{R}^{(j)}, \mathcal{N})$ is given.

Algorithm. Graphical and combinatorial construction of a reduced network. Given a $(\underset{\sim}{\pi},\sigma,\rho)$-symmetric network N, which has a unique solution $(\underset{\sim}{v},\underset{\sim}{i},\underset{\sim}{u})$, find a reduced network N_0.

1) Decompose the directed permutations ρ, $(\underset{\sim}{\pi},\sigma)$, and the permutation σ into cyclic components. Identify the normal-order cyclic components of σ as $\beta_1,\beta_2,\ldots,\beta_q$, those of $(\underset{\sim}{\pi},\sigma)$ as $\gamma_1,\gamma_2,\ldots,\gamma_t$ and those of ρ as $\delta_1,\delta_2,\ldots,\delta_s$.

2) Make all nodes available as they appear in the cycles of ρ, i.e., if a node appears complemented in a cycle of ρ then complement it using Fig. 7(g,h). For each cycle of ρ interconnect the nodes to each other in the form in which they appear in the cycle. This implies among other things that the nodes of a double-order cycle are all connected to each other and to their complement. Thus this implies that these nodes are at the reference voltage of the corresponding connected part(s).

3) As a result of step 2 and of the symmetry each cycle β_i of σ consists of identical resistors all connected to the same nodes with the corresponding terminals. Let ℓ_i be the order of β_i. Moreover all terminals or ports of these resistors correspond to all terminals or ports of some cycles in $(\underset{\sim}{\pi},\sigma)$. Call this set of cycles ε_i. Connect all ports (resp. terminals) of these resistors which appear in double-order cycles of ε_i to nullators (resp. via nullators to the

reference voltage of the connected part). Let there be k_i single-order cycles in ε_i and let μ_i be the directed permutation with these cycles as cyclic components. We now replace the ℓ_i multiport (resp. multiterminal) resistors of β_i by one equivalent resistor with k_i ports (resp. k_i terminals). This can happen in two simple steps. First replace the ℓ_i parallel connected identical resistors by one resistor. The new ports (resp. terminals) can be labeled the same as those of any of the ℓ_i resistors. This new resistor may have more than k_i ports (resp. terminals). It is symmetric with respect to the directed permutation $\mu_i^{\ell_i}$. Observe that some of the cycles of $\mu_i^{\ell_i}$ act on empty ports (resp. terminals) and can therefore be eliminated. In the second step we reduce the number of ports (resp. terminals) to k_i by observing that the ports (resp. terminals) of this resistor which belong to the same cycle of $\mu_i^{\ell_i}$ are connected to the same node. Since all the ports of (resp. terminals) of this resistor belonging to the same cycle of $\mu_i^{\ell_i}$ are connected together we may replace them by one port (terminal). Observe that this second step is the same as the reduction for multiport (resp. multiterminal) resistors described in [11]. Repeat for all other cycles of σ.

This algorithm can be easily proved either directly from Cor. 1 of Thm. 3 or from the algebraic reduction algorithm and the structure of the matrices $\underset{\sim}{S}(\rho)$ and $\underset{\sim}{S}(\underset{\sim}{\pi},\sigma)$.

A general remark about the need for choosing reference nodes in this algorithm is in order. There is no need to find a compatible set of reference nodes at the onset of the algorithm. Of course as soon as some terminals have to be complemented in step 2, we assume that there is a reference node, which need not be further specified. The same is true in Step 3. If as a result of Step 3 the reduced network N_0 contains some complementations, then the reference node for the corresponding connected set has to be the common third terminal of all complementations. If there are no complementations in a connected set, then the reference node of this connected is free. It can be shown that any set of reference nodes found for N_0 generates a compatible set of reference nodes for N, by choosing as reference nodes of N any node of the cycle of ρ corresponding to a reference node of N_0.

To conclude this section let us derive the bisection technique of Prop. 3 directly from this algorithm. In the case of Fig. 2(a) the symmetry node permutation is $\rho = \begin{pmatrix} ① & ② & ③ \\ ① & ③ & ② \end{pmatrix} = (①)(②\ ③)$. In Step 2 nodes ② and ③ are connected to each other. Step 3 shows that the main task is the computation of the voltages and the currents of the reduced network N_0, which is the parallel connection of two identical 3-terminal resistors N' with terminal 1 open and terminals 2 and 3 connected to each other. The currents in the terminals of any N' in Fig. 2(a) are half the corresponding currents of the reduced network N_0 and the corresponding voltages are the same. This implies Prop. 3. Analogously in the case of Fig. 2(c) the symmetry node permutation is $\rho = \begin{pmatrix} ① & ② & ③ & ④ \\ ④ & ② & ③ & ① \end{pmatrix} = (①\ ④)(②)(③)$. Here the nodes ① and ④ can be connected together. The reduced network N_0 consists of two identical parallel connected four-terminal resistors N' with terminals 1 and 4 connected to each other and terminals 2 and 3 open. This amounts to solving the network of Fig. 2(d) and implies Prop. 3.

VI. CONCLUSIONS

In this paper we have presented the first general algebraic definition of symmetry of nonlinear resistive networks. It is defined as an invariance with respect to a network permutation and includes as special cases all previous ad hoc definitions based on geometric transformations and complementations. It is believed that this definition of symmetry is the most general that could be devised, since any relaxation of the conditions for a network permutation results in the undesirable situation where the solutions of a network and of the permuted network have no direct relationship.

The advantages of an algebraic definition are its generality and independence from drawings. Moreover, the symmetry properties of the solution can be easily derived. In this respect the cyclic decomposition of a directed permutation [11] is a particularly useful vehicle.

In the course of the paper we solve the following problems: (1) Find the group of all symmetries of a network. (2) Find reference nodes for a symmetric network such that its unique solution exhibits a symmetry. (3) Synthesize a symmetric multiport (resp. multiterminal) resistor by making suitable port (resp. terminal) entries in a symmetric network. (4) Use the network symmetry to simplify the network to be analyzed or to reduce the number of equations to be solved. The solutions to these problems generalize many results which had been derived for special circuits or special symmetries or which have been in use among circuit designers without rigorous justifications.

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FIGURE CAPTIONS

Fig. 1. (a) A reflection-symmetric network, (b) a complementary symmetric network and (c) a network which is symmetric with respect to a reflection followed by a complementation.

Fig. 2. Bisection of reflection symmetric and 180° rotational symmetric circuits.

Fig. 3. A symmetric lattice network.

Fig. 4. (a) A multiport and (b) A multiterminal resistor with the associated reference directions.

Fig. 5. The interconnections of the components in a network.

Fig. 6. The network diagram of a resistive network.

Fig. 7. (a) The phase-inverting ideal transformer, (b) a simplified symbol and its use as complementation element for the following items: (c) a terminal, (d,e) a port, (f) a multiterminal resistor and (g,h) a node.

Fig. 8. A step-by-step derivation of the network diagram of a transformed network of the network of Fig. 6.

Fig. 9. Simple examples of symmetric networks.

Fig. 10. (a) The symmetric push-pull amplifier and (b,c) a step-by-step graphical analysis of the symmetry.

Fig. 11. Graphical detection of symmetries in a network.

Fig. 12. A symmetric network which does not allow a symmetric drawing.

Fig. 13. The symmetric lattice network with an artificial invariant node 5.

Fig. 14. The Eccles-Jordan multivibrator.

Fig. 15. Rotational symmetric network without invariant nodes.

Fig. 16. An example, illustrating the choice of a set of reference nodes compatible with a symmetry.

Fig. 17. A 3-terminal resistor derived from a network N by making terminal entries according to the terminal selection operator $\mathcal{B}$: ⑥, ②, ⑤ $\rightarrow$ 1,2,3.

Fig. 18. A 2-port resistor derived from a network N by making a soldering-iron port entry between nodes ① and ⑥ and a pliers-type port entry in terminal 1 of the one-port resistor $R^{(5)}$.

Fig. 19. Symmetric multiterminal resistor derived from symmetric networks.

Fig. 20. The symmetric rectifier considered as a symmetric resistive network with two port entries.

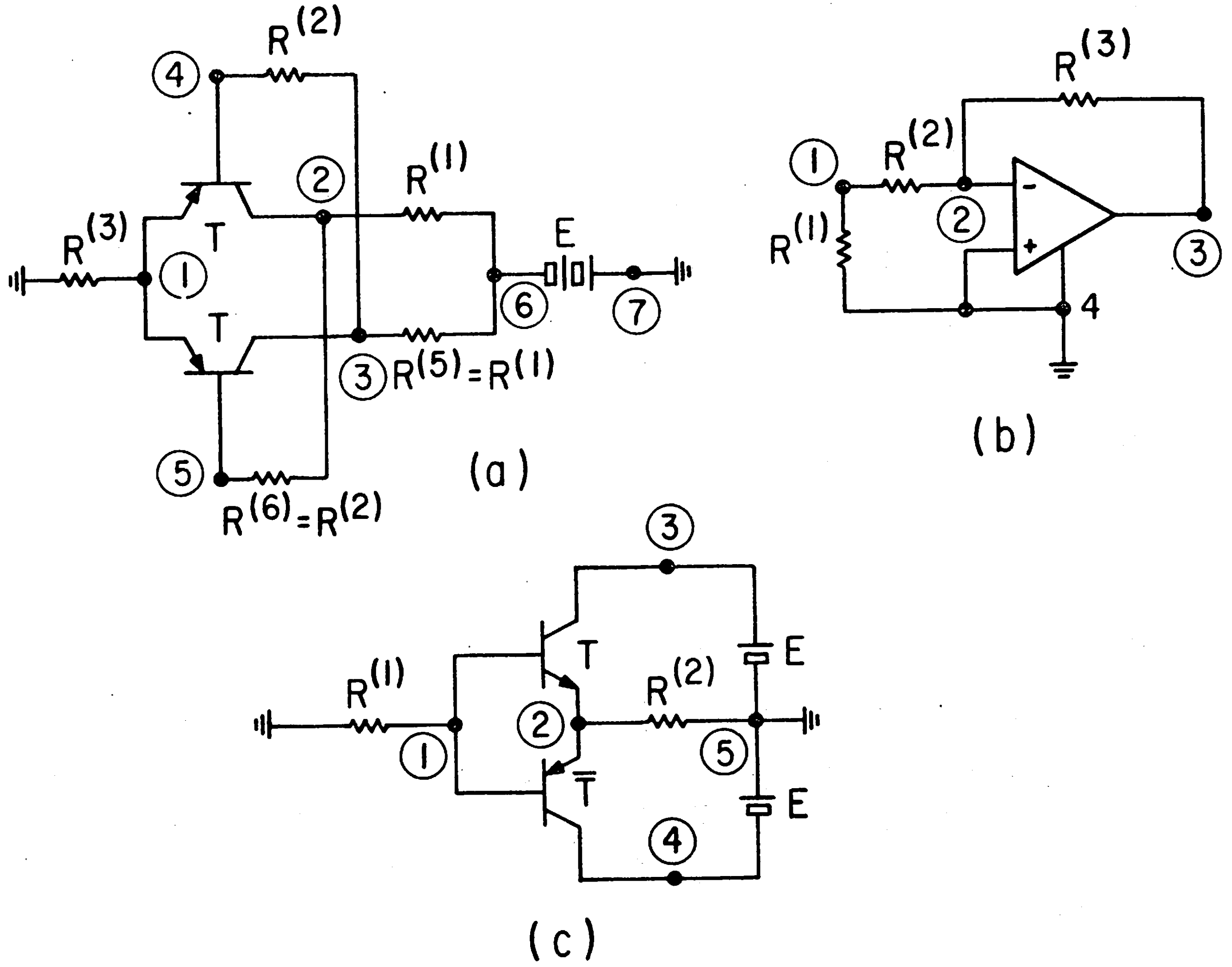
Fig. 21. The nonlinear bridge circuit with two port entries.

Fig. 22. A symmetric one-port resistor derived from a network which is not symmetric.

Fig. 23. A $(1\ \bar{1})(2)$-symmetric two-port resistor derived from a network which exhibits some symmetry when the part of the network enclosed in a box is replaced by an equivalent bilateral resistor $\tilde{R}$.

Fig. 24. Simple examples of reduction. The networks of (a)(d)(g)(j) are reduced to those of (c)(f)(i)(ℓ), respectively.

Fig. 25. Graphical reduction of a symmetric lattice network.



R(2)
4
2
R(1)
R(3)
T
1
T
E
6
7
3 R(5)=R(1)
5
R(6)=R(2)
(a)
R(3)
1
R(2)
2
R(1)
-
+
3
4
(b)
3
T
E
R(1)
R(2)
2
5
1
T
E
4
(c)

Fig. 1

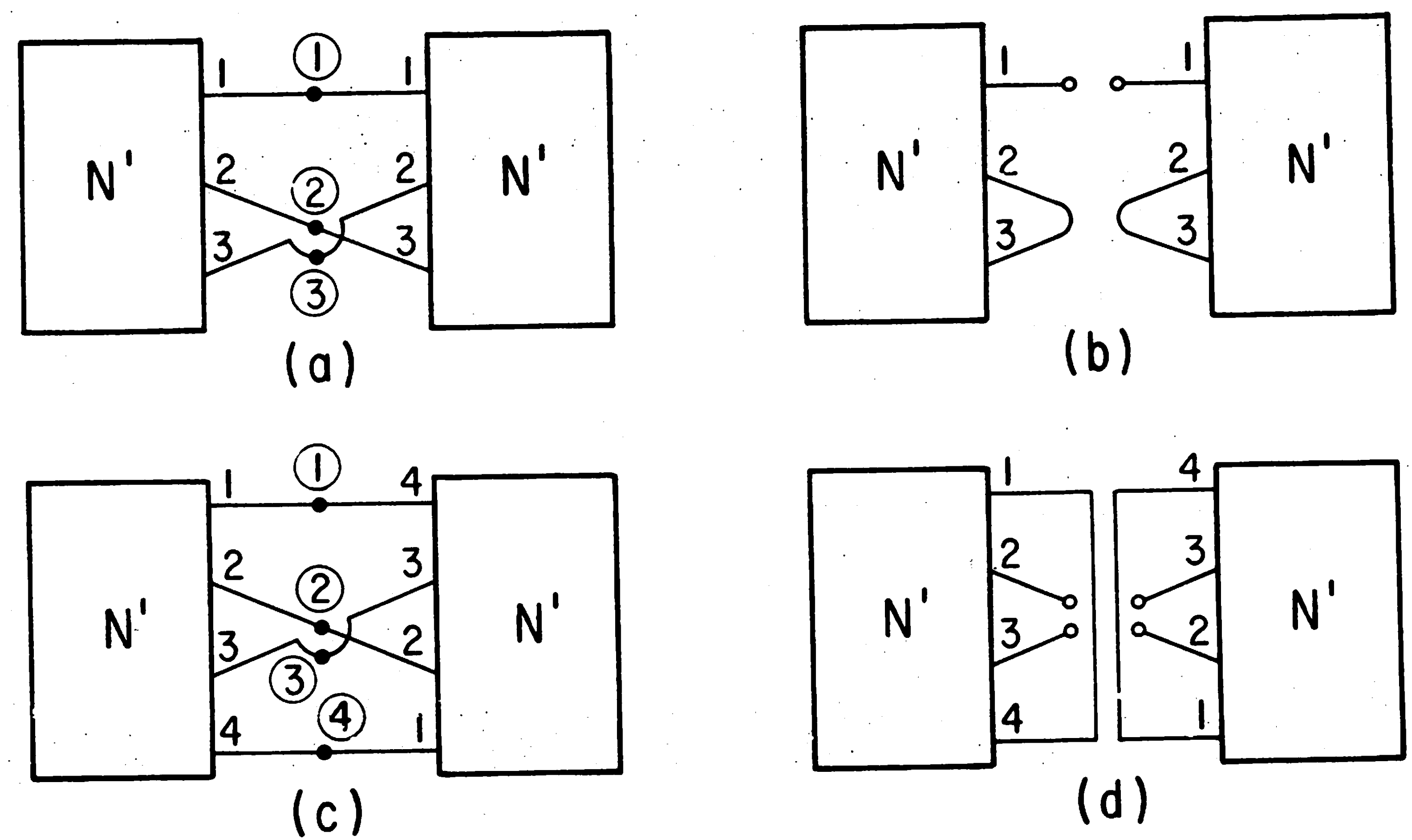


Fig. 2

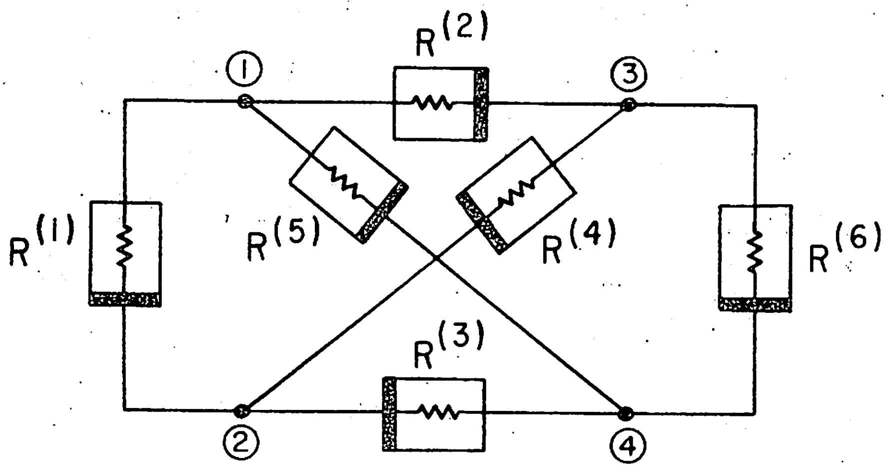


Fig. 3

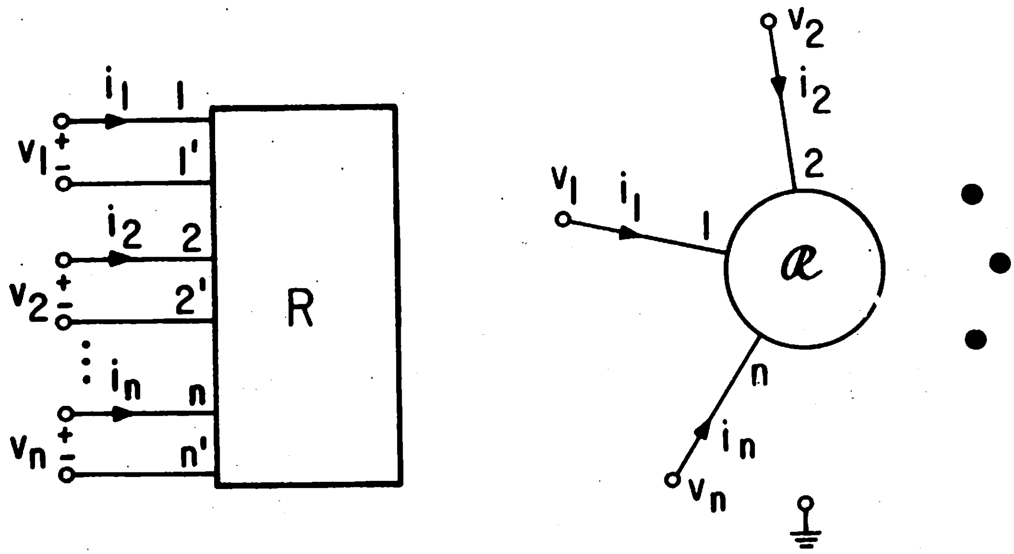
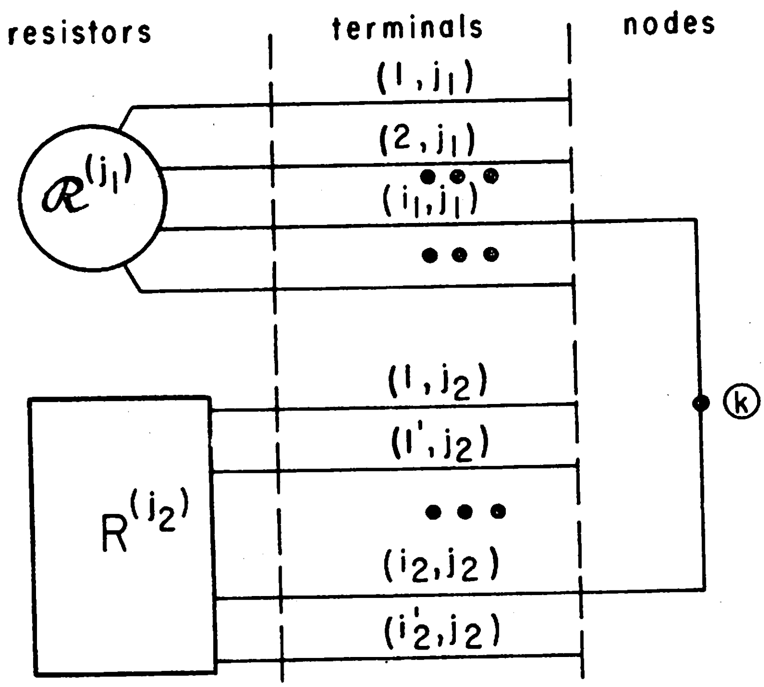


Fig. 4



resistors
terminals
nodes
(1, j1)
(2, j1)
(i1, j1)
R(j1)
(1, j2)
(1', j2)
R(j2)
(i2, j2)
(i'2, j2)
k

Fig. 5

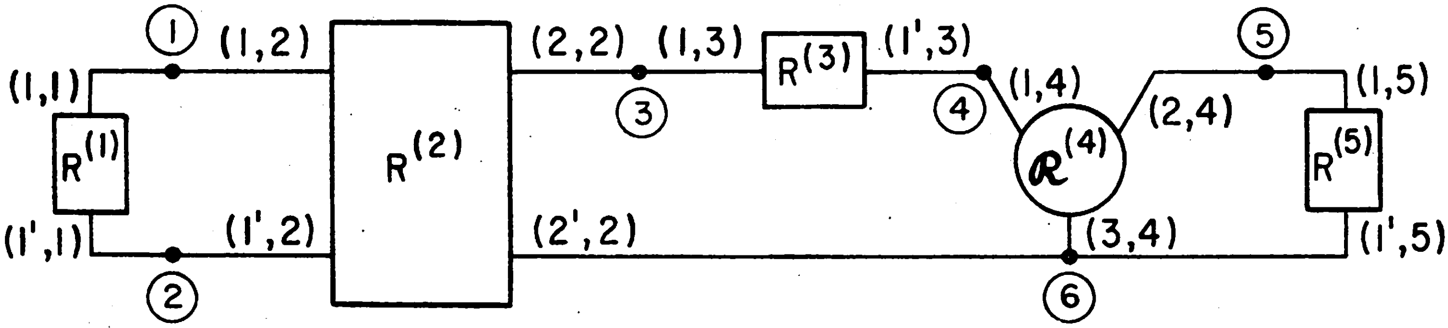
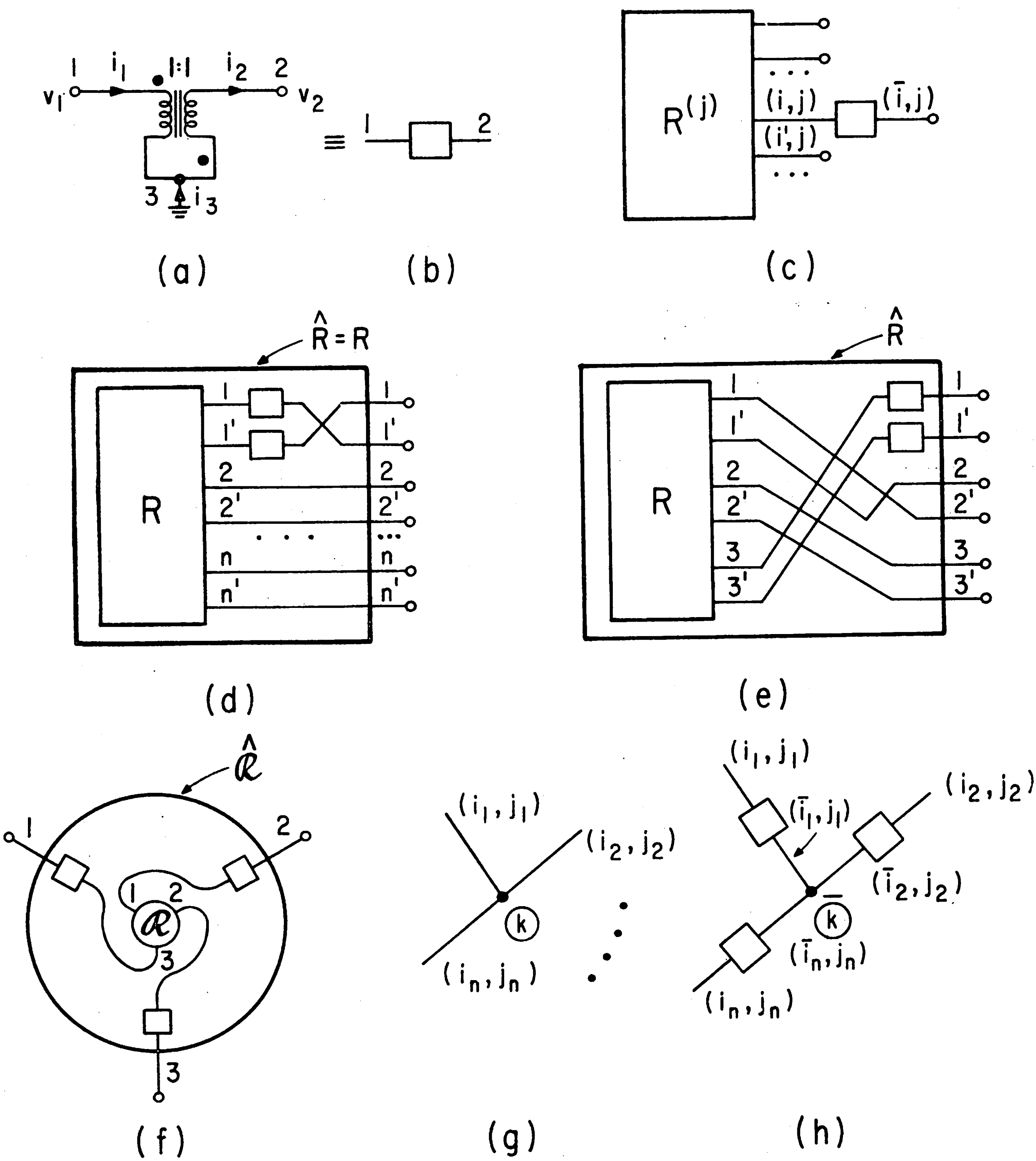
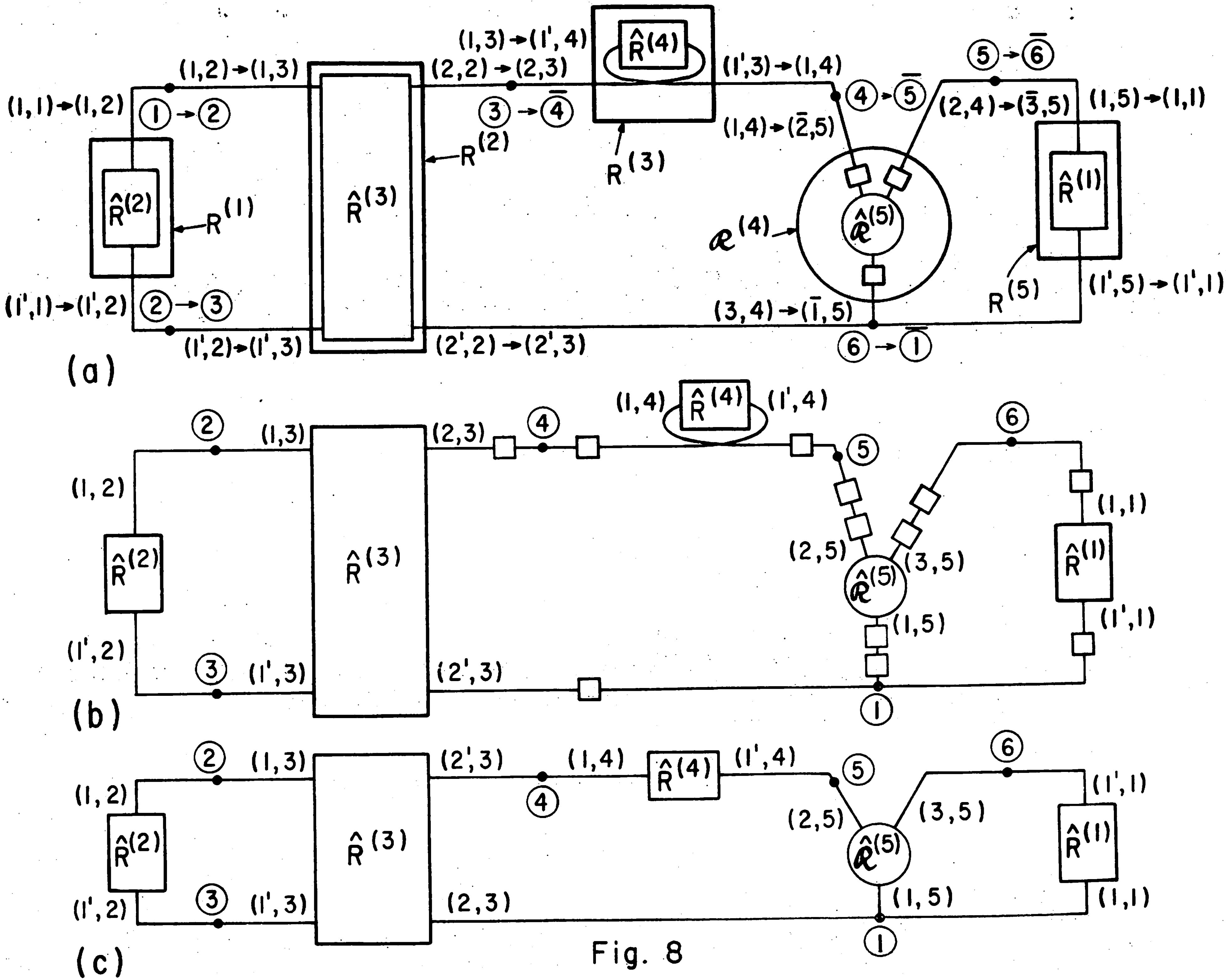


Fig. 6



1 i_1 1:1 i_2 2
v_1 v_2
3 i_3
≡ 1 2
$R^{(j)}$
(i,j) $(\bar{i},j)$
(i',j)
(a) (b) (c)
$\hat{R}=R$
R
1 1' 2 2' n n'
$\hat{R}$
1 1' 2 2' 3 3'
(d) (e)
$\hat{\mathcal{R}}$
$\mathcal{R}$
1 2 3
(i_1, j_1) (i_2, j_2) k (i_n, j_n)
$(\bar{i}_1, j_1)$ $(\bar{i}_2, j_2)$ $\bar{k}$ $(\bar{i}_n, j_n)$
(f) (g) (h)

Fig. 7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(1,2)→(1,3)
1 → 2
(1',1)→(1',2)
2 → 3
(1',2)→(1',3)
R(1)
R(2)
(2,2)→(2,3)
(1,3)→(1',4)
3 → 4
R(3)
(1',3)→(1,4)
(1,4)→(2,5)
4 → 5
5 → 6
(2,4)→(3,5)
(1,5)→(1,1)
R(4)
R(5)
(3,4)→(1,5)
(1',5)→(1',1)
(2',2)→(2',3)
6 → 1
(a)
(b)
(c)

Fig. 8

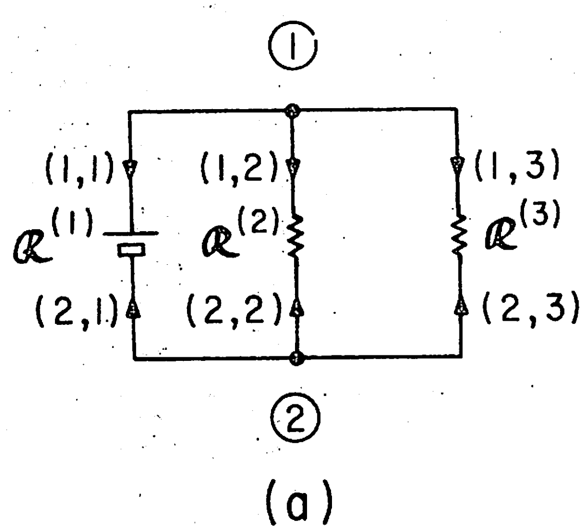
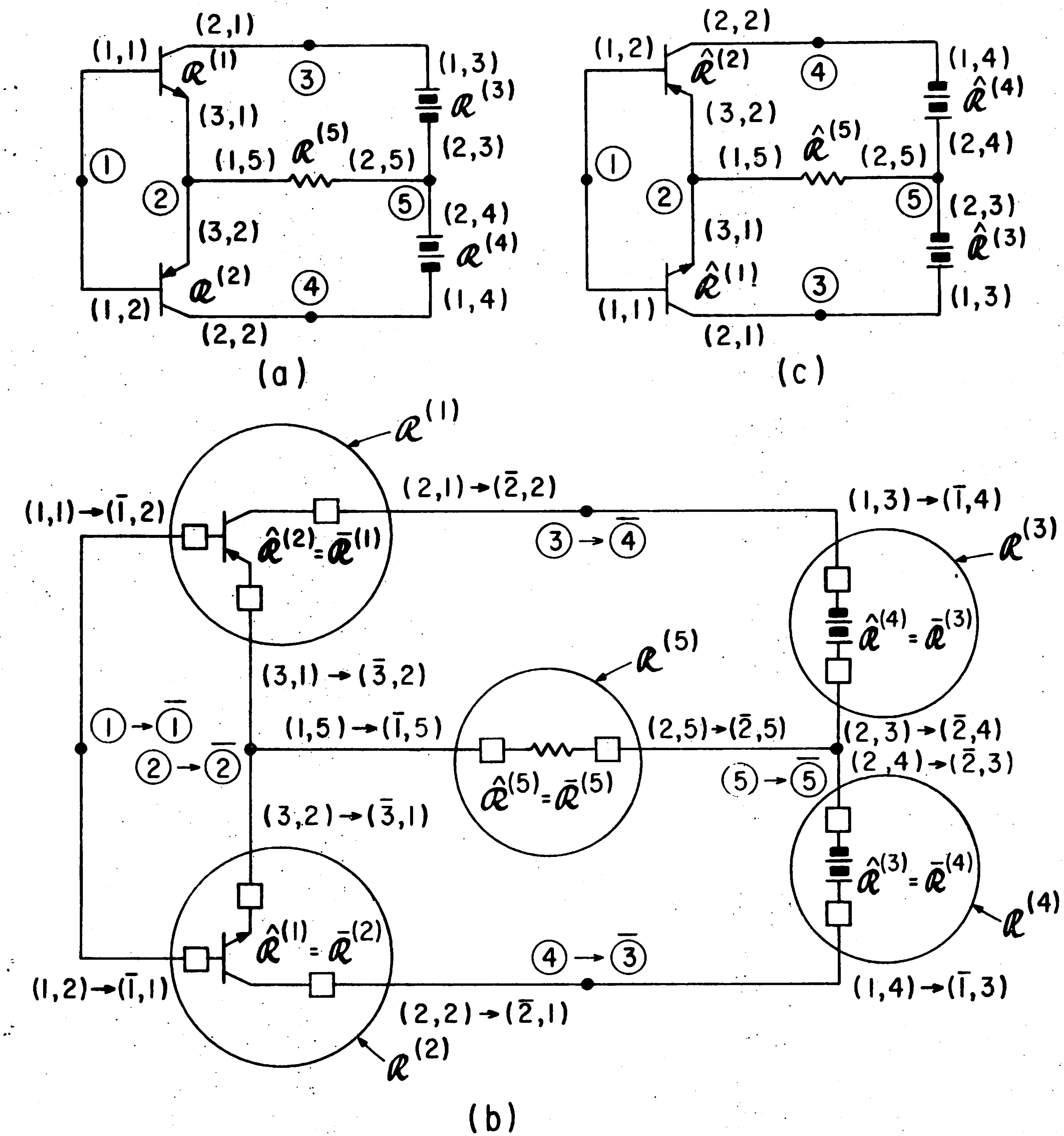


Fig. 9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(2,4)
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$\mathcal{R}^{(4)}$
$\mathcal{R}^{(2)}$
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(a)
(2,2)
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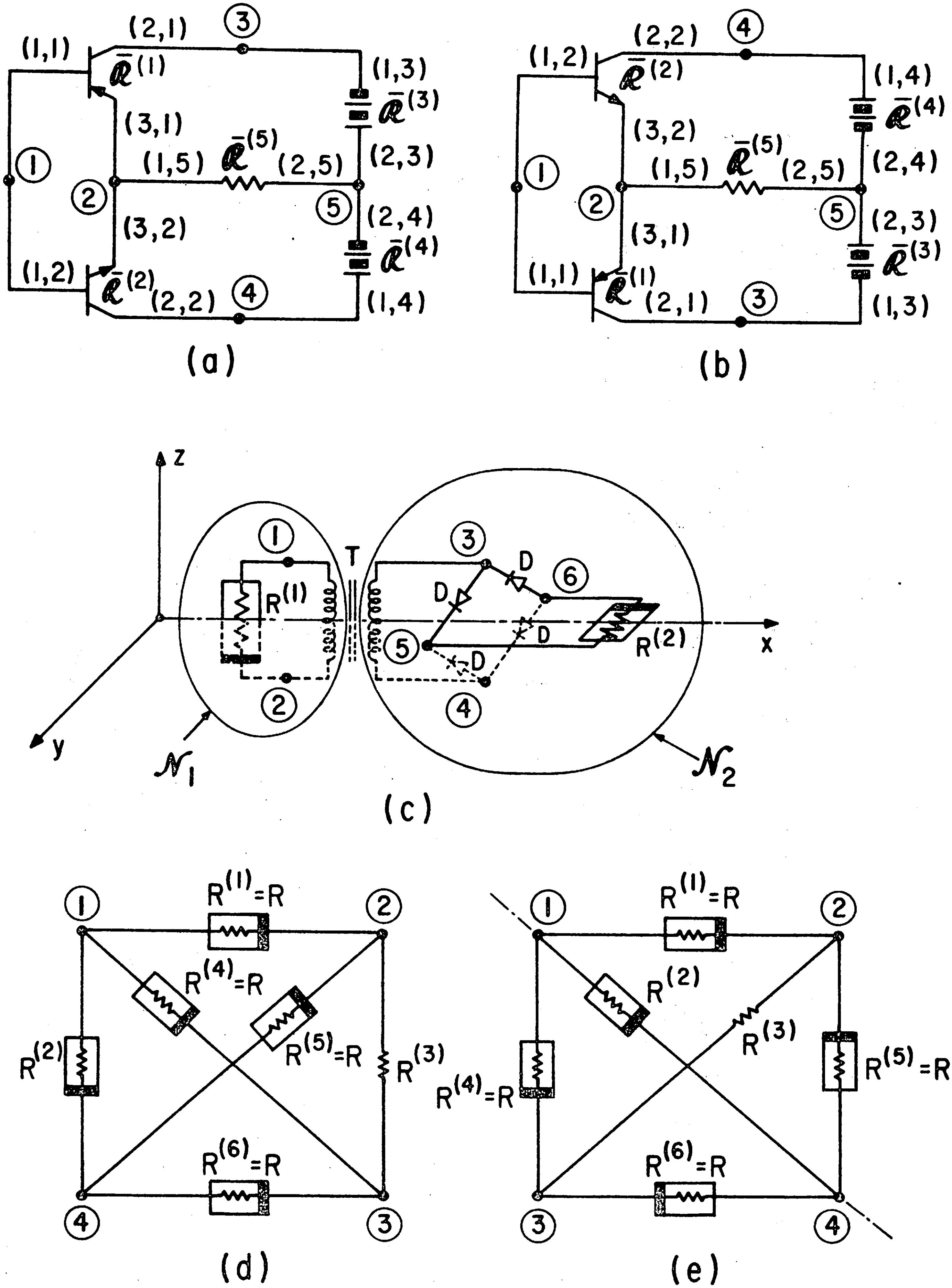
$\hat{\mathcal{R}}^{(2)}$
(1,4)
$\hat{\mathcal{R}}^{(4)}$
(3,2)
$\hat{\mathcal{R}}^{(5)}$
(1,5)
(2,5)
(2,4)
(2,3)
(3,1)
$\hat{\mathcal{R}}^{(3)}$
$\hat{\mathcal{R}}^{(1)}$
(1,1)
(1,3)
(2,1)
(c)
$\mathcal{R}^{(1)}$
$(2,1) \rightarrow (\bar{2},2)$
$(1,1) \rightarrow (\bar{1},2)$
$(1,3) \rightarrow (\bar{1},4)$
$\hat{\mathcal{R}}^{(2)} = \bar{\mathcal{R}}^{(1)}$
$3 \rightarrow \bar{4}$
$\mathcal{R}^{(3)}$
$\hat{\mathcal{R}}^{(4)} = \bar{\mathcal{R}}^{(3)}$
$\mathcal{R}^{(5)}$
$(3,1) \rightarrow (\bar{3},2)$
$1 \rightarrow \bar{1}$
$(1,5) \rightarrow (\bar{1},5)$
$(2,5) \rightarrow (\bar{2},5)$
$(2,3) \rightarrow (\bar{2},4)$
$2 \rightarrow \bar{2}$
$(2,4) \rightarrow (\bar{2},3)$
$5 \rightarrow \bar{5}$
$\hat{\mathcal{R}}^{(5)} = \bar{\mathcal{R}}^{(5)}$
$(3,2) \rightarrow (\bar{3},1)$
$\hat{\mathcal{R}}^{(3)} = \bar{\mathcal{R}}^{(4)}$
$\mathcal{R}^{(4)}$
$\hat{\mathcal{R}}^{(1)} = \bar{\mathcal{R}}^{(2)}$
$4 \rightarrow \bar{3}$
$(1,2) \rightarrow (\bar{1},1)$
$(1,4) \rightarrow (\bar{1},3)$
$(2,2) \rightarrow (\bar{2},1)$
$\mathcal{R}^{(2)}$
(b)

Fig. 10



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4
(1,4)
(a)
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4
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(2,3)
(3,1)
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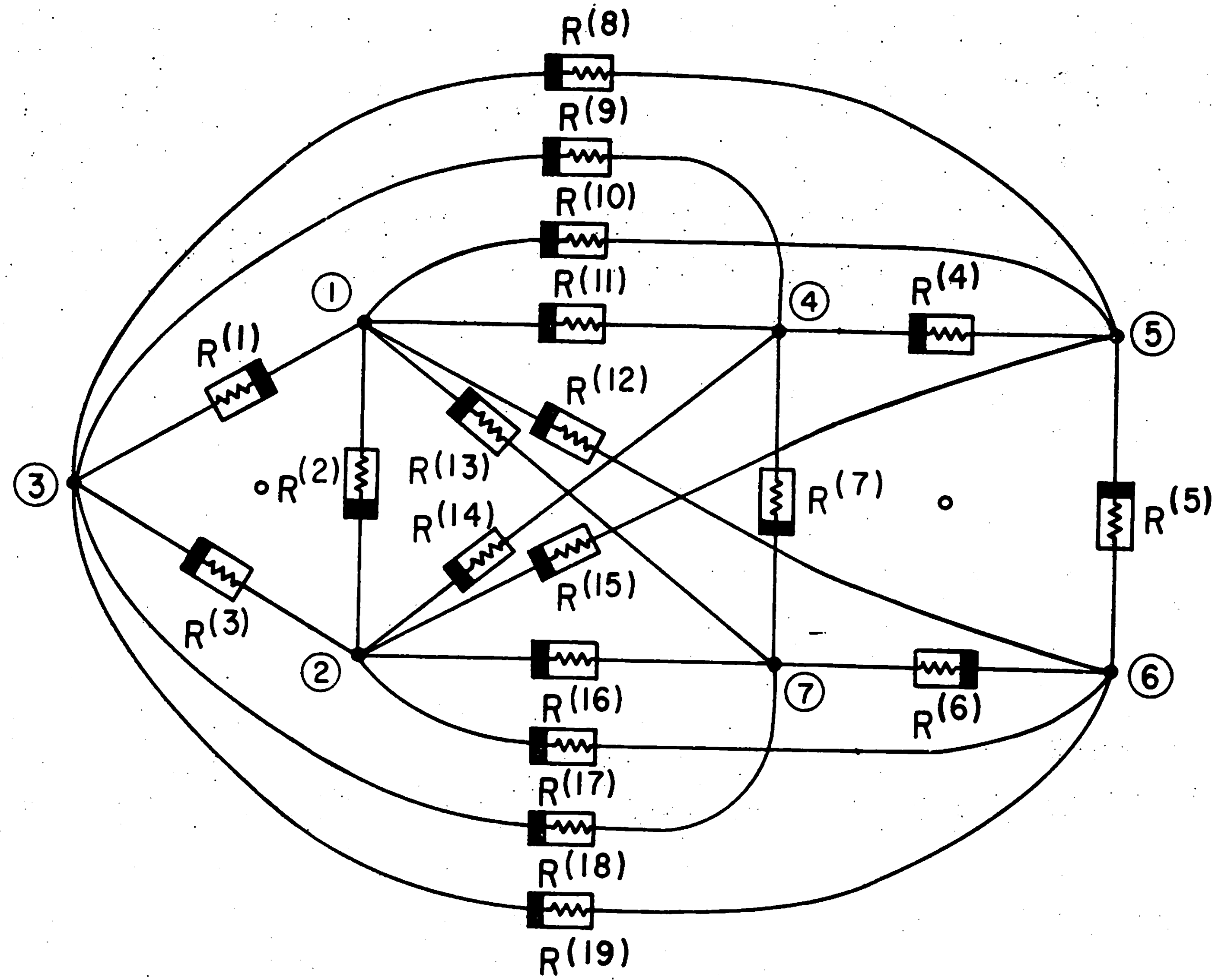
3
(1,3)
(b)
z
T
1
6
D
R(1)
R(2)
x
y
2
4
5
N1
N2
(c)
R(1)=R
R(4)=R
R(5)=R
R(2)
R(3)
R(6)=R
(d)
R(1)=R
R(2)
R(3)
R(4)=R
R(5)=R
R(6)=R
(e)

Fig. 11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R(9)
R(10)
R(11)
R(4)
R(12)
R(1)
R(2)
R(13)
R(7)
R(5)
R(14)
R(15)
R(3)
R(16)
R(6)
R(17)
R(18)
R(19)
1
2
3
4
5
6
7

Fig. 12

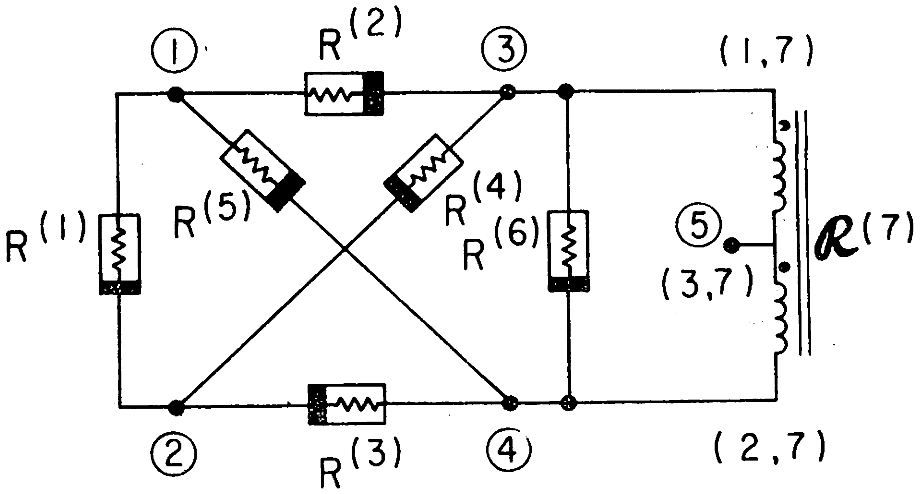


Fig. 13

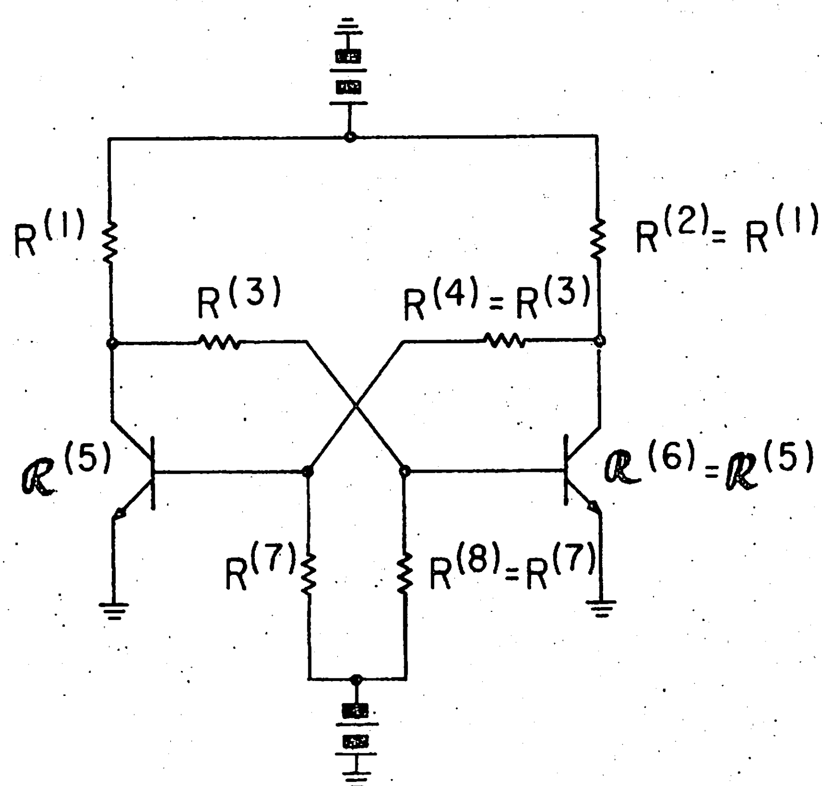


Fig. 14

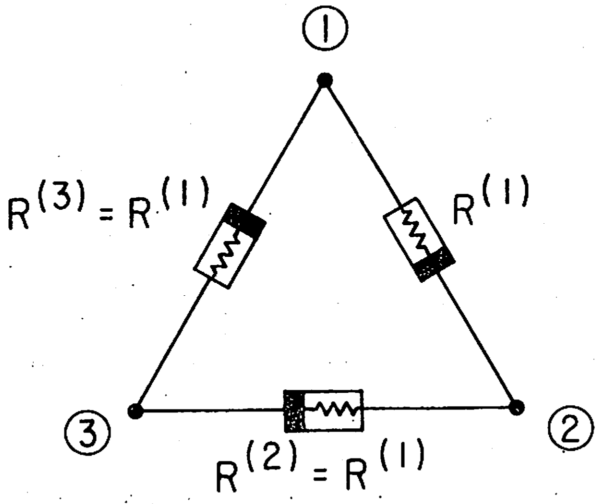
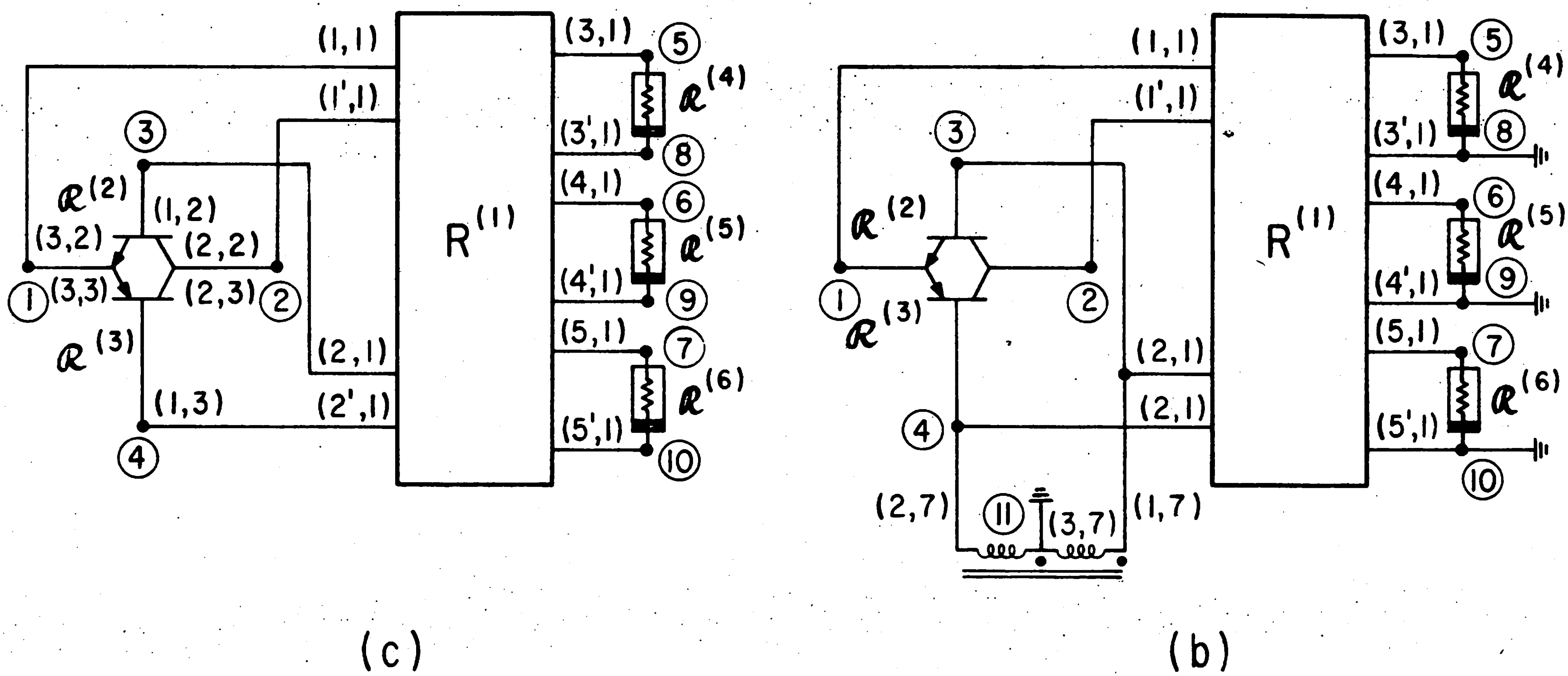


Fig. 15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R(1)
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R(3)
R(4)
R(5)
R(6)
1
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(c)
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R(1)
R(2)
R(3)
R(4)
R(5)
R(6)
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2
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4
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6
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9
10
11
(b)

Fig. 16

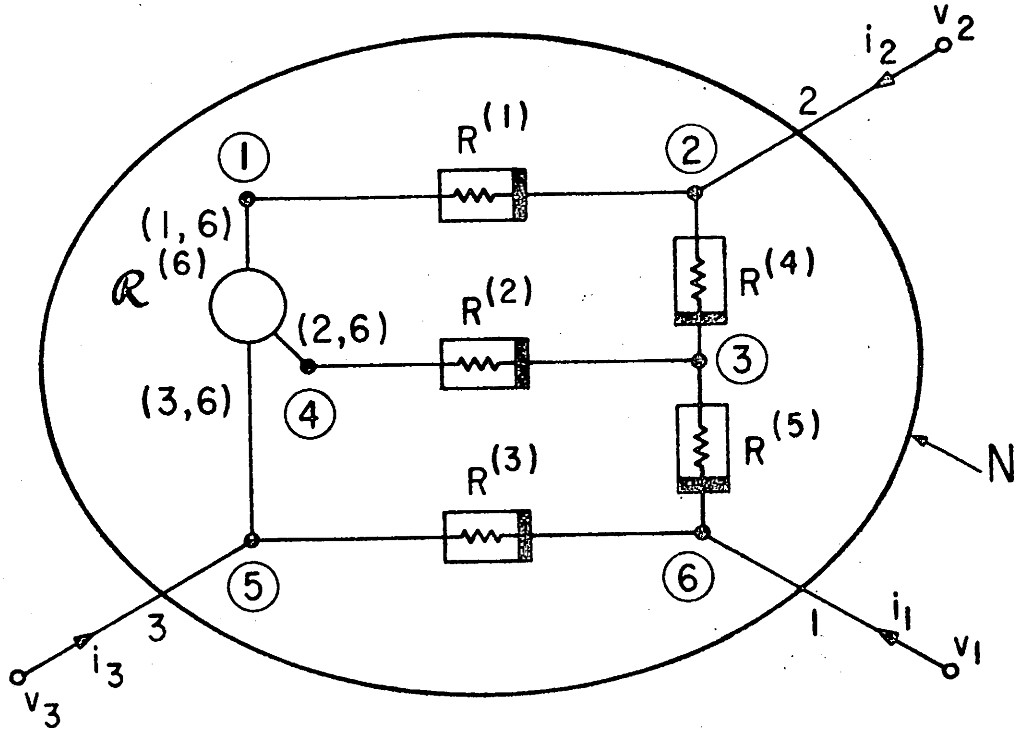


Fig. 17

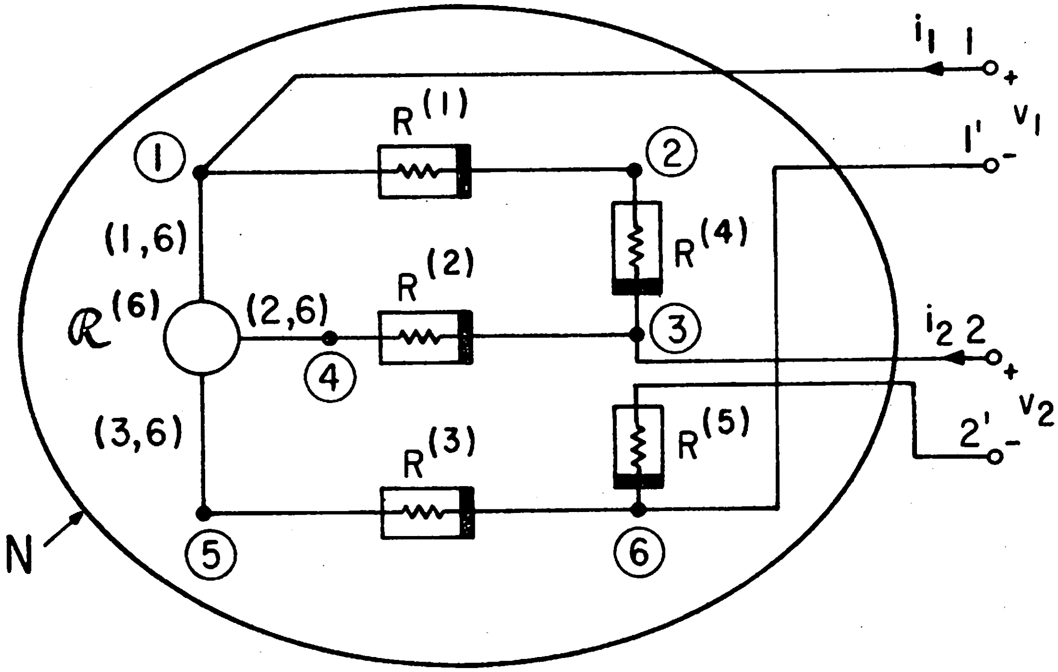
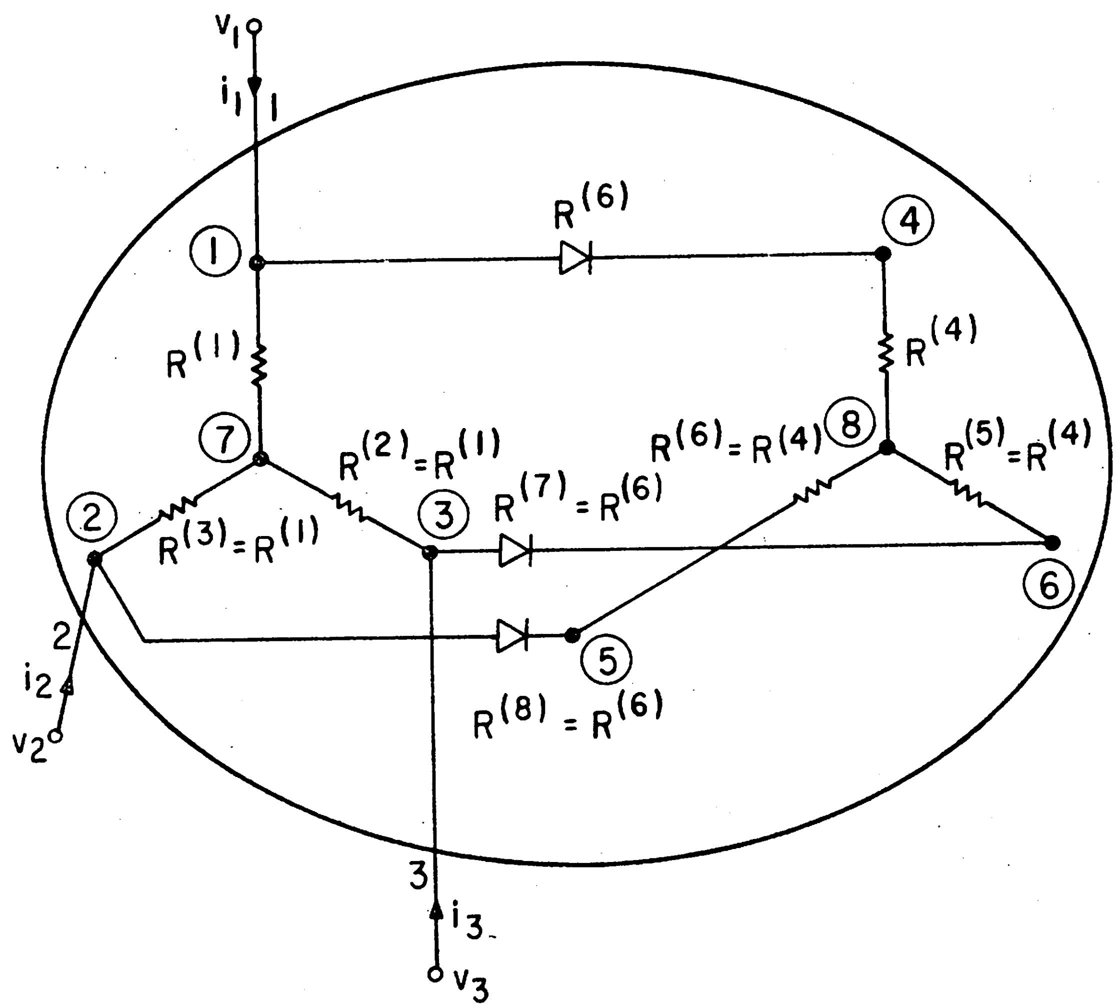
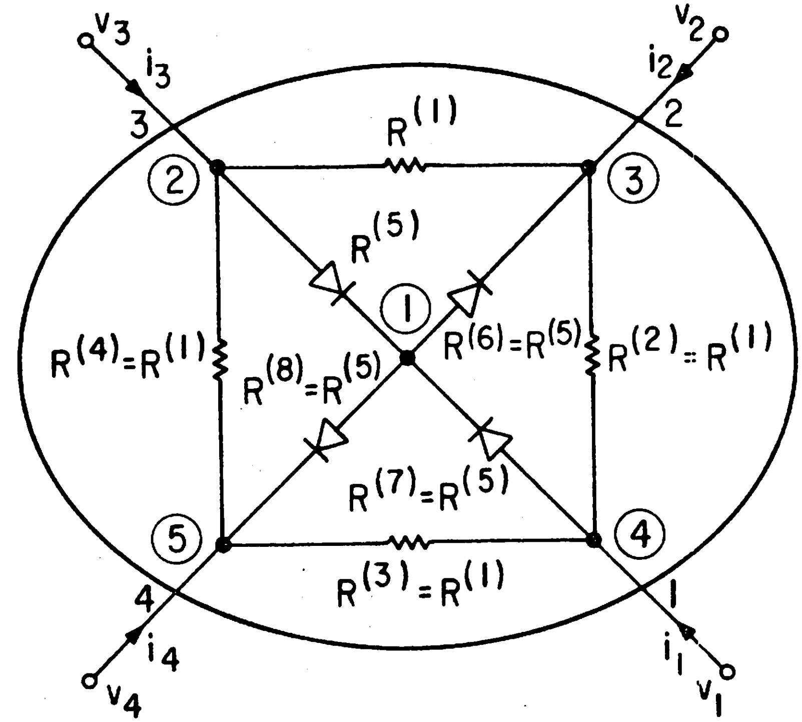


Fig. 18



v1
i1
1
2
i2
v2
3
i3
v3
R(6)
R(1)
R(4)
R(2)=R(1)
R(3)=R(1)
R(7)=R(6)
R(6)=R(4)
R(5)=R(4)
R(8)=R(6)
1
2
3
4
5
6
7
8

(a)



v3
i3
3
v2
i2
2
v4
i4
4
v1
i1
1
R(1)
R(5)
R(4)=R(1)
R(8)=R(5)
R(6)=R(5)
R(2)=R(1)
R(7)=R(5)
R(3)=R(1)
1
2
3
4
5

(b)

Fig. 19

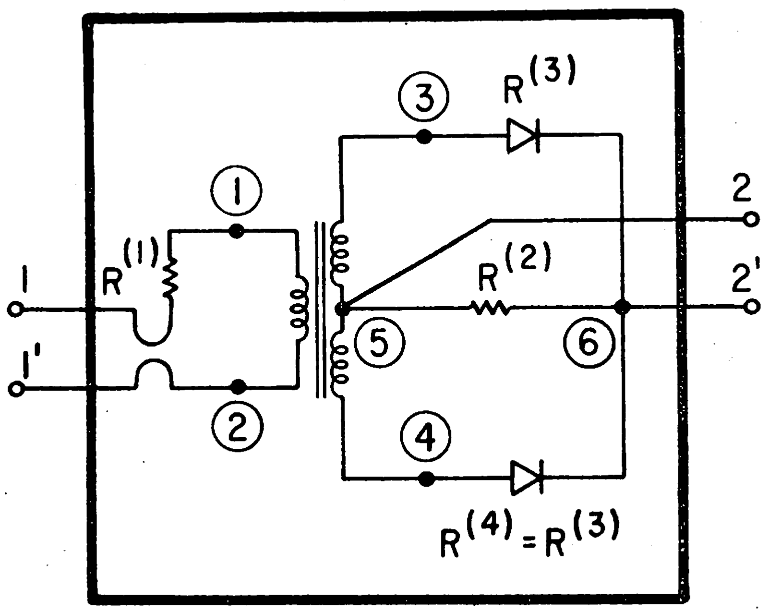


Fig. 20

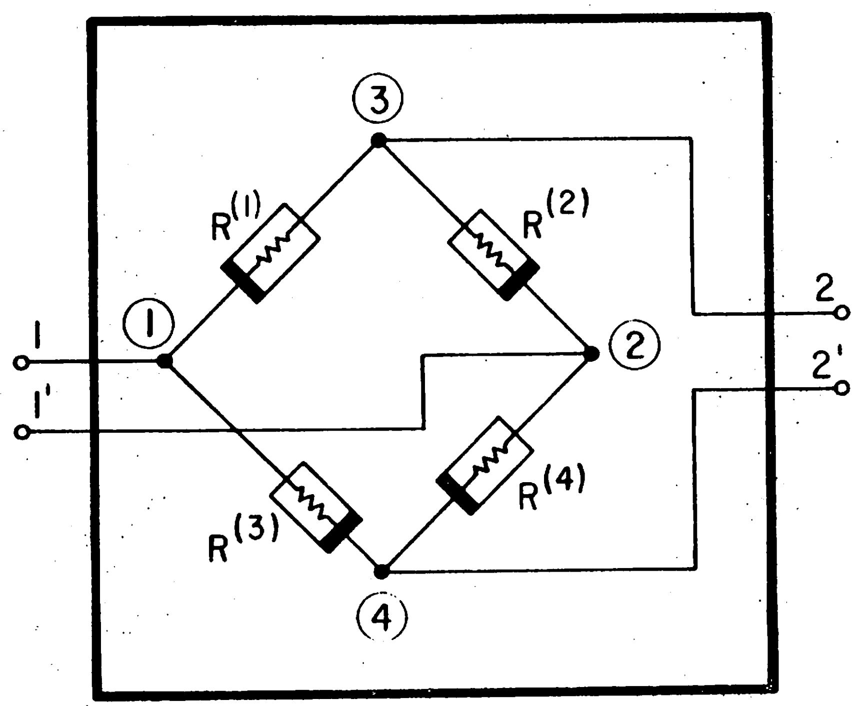


Fig. 21

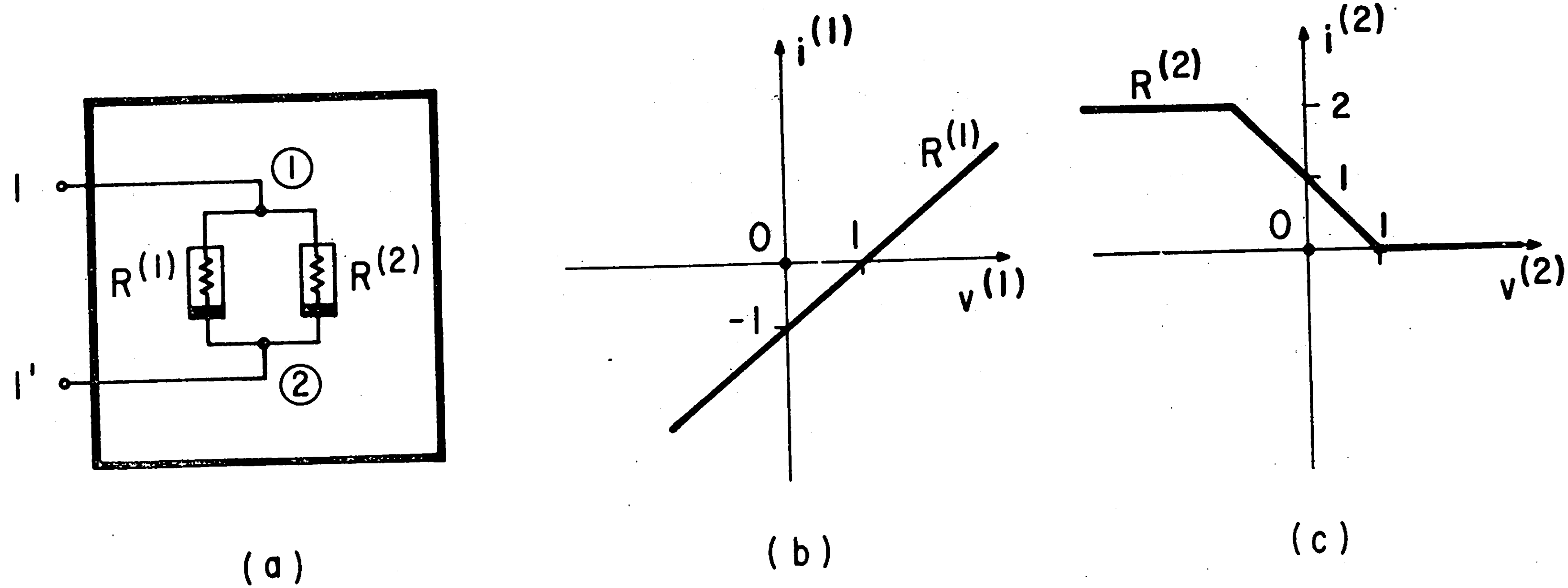


Fig. 22

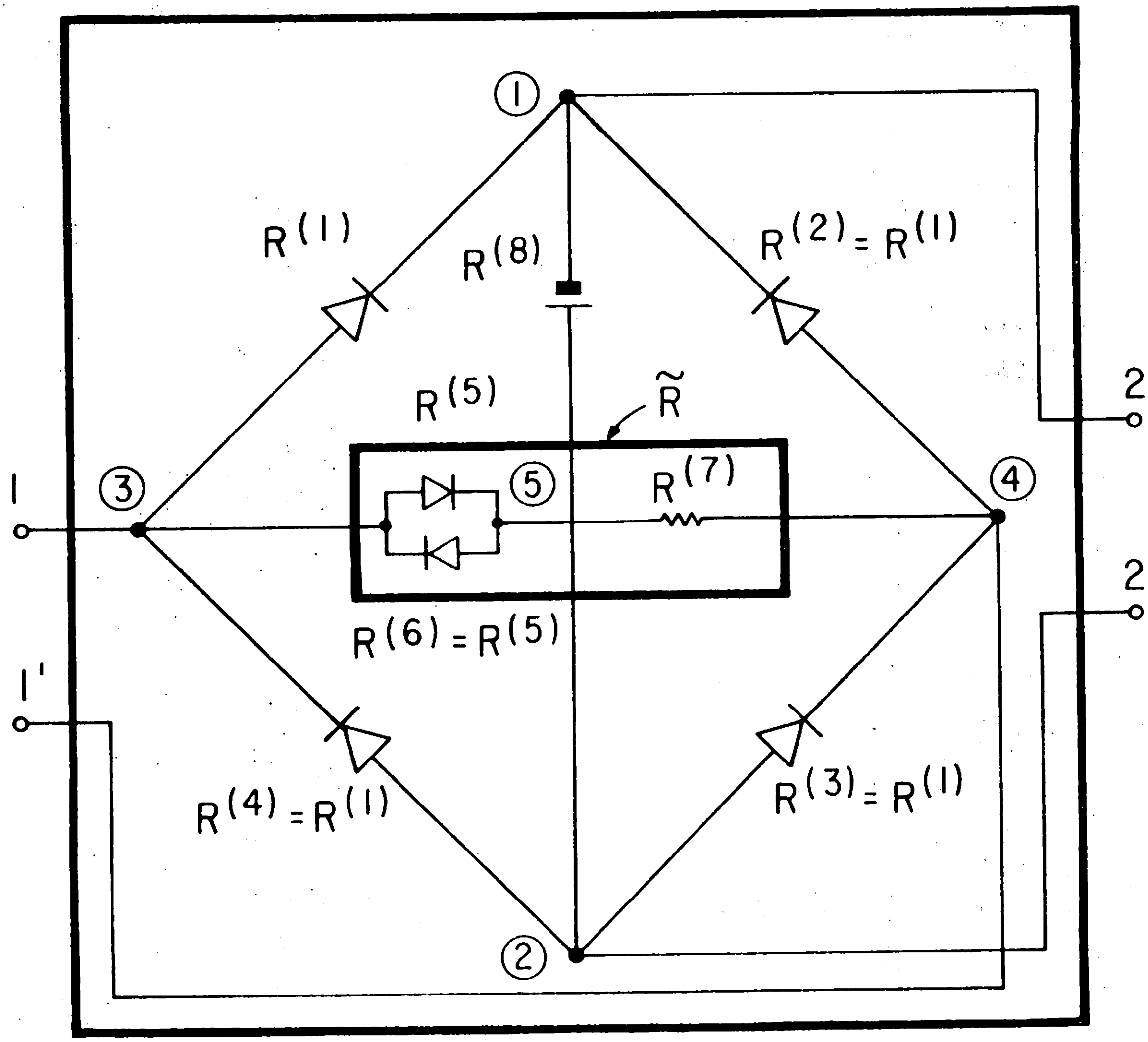


Fig. 23

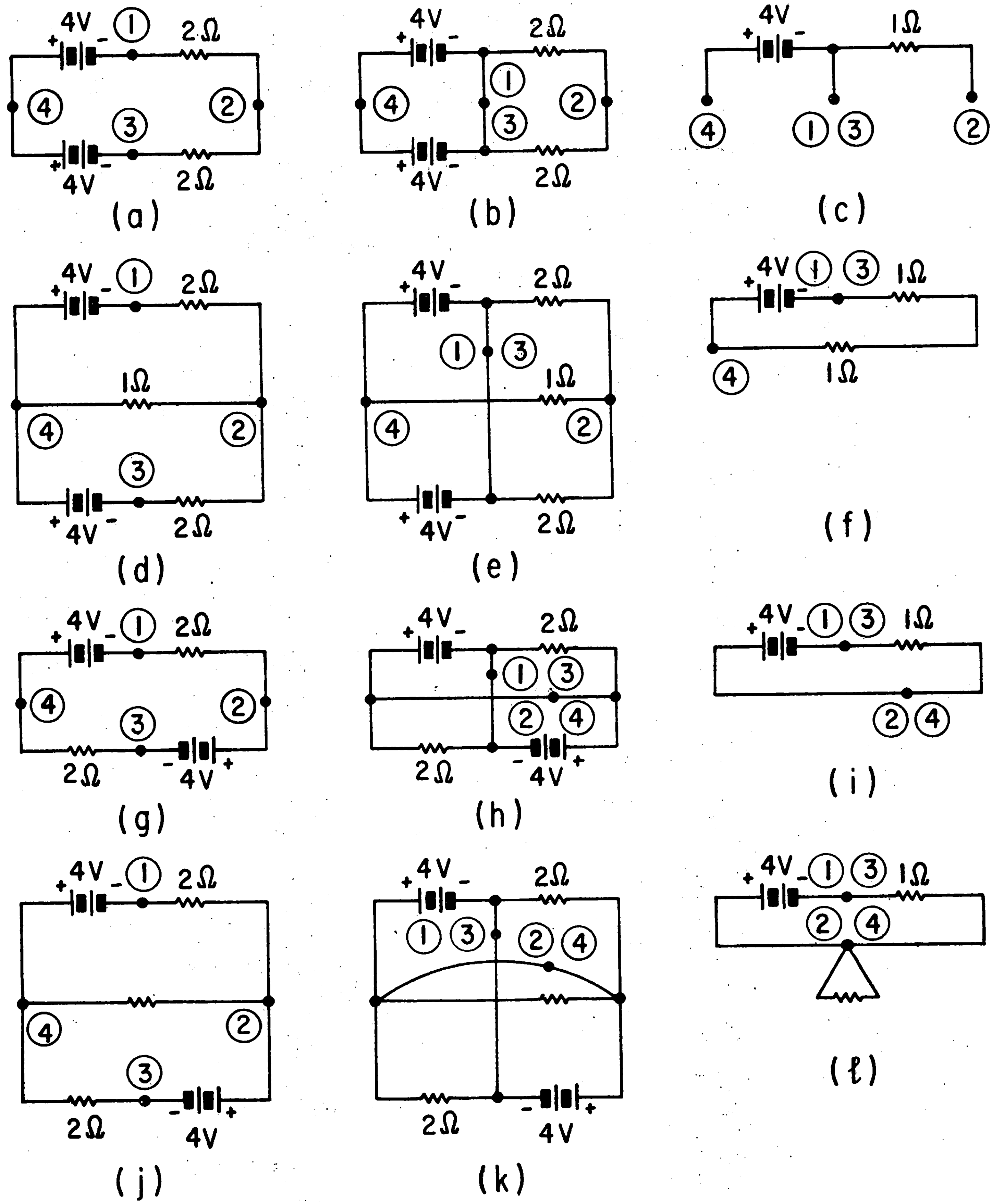
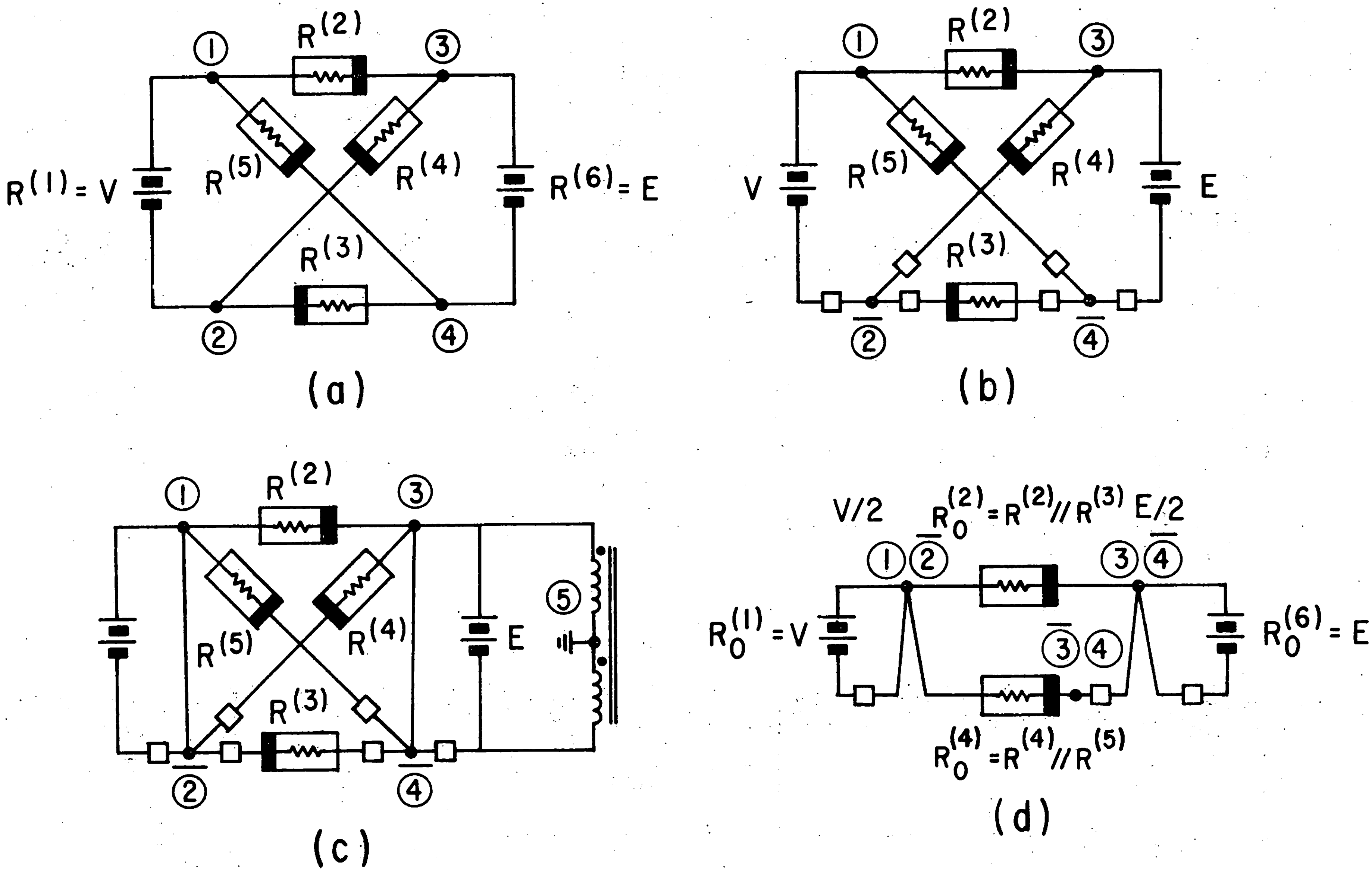

4V ① 2Ω
④ ②
③
4V 2Ω
(a)
4V 2Ω
① ③
④ ②
4V 2Ω
(b)
4V 1Ω
④ ① ③ ②
(c)
4V ① 2Ω
1Ω
④ ②
③
4V 2Ω
(d)
4V 2Ω
① ③
1Ω
④ ②
4V 2Ω
(e)
4V ① ③ 1Ω
④ 1Ω
(f)
4V ① 2Ω
④ ②
③
2Ω 4V
(g)
4V 2Ω
① ③
② ④
2Ω 4V
(h)
4V ① ③ 1Ω
② ④
(i)
4V ① 2Ω
④ ②
③
2Ω 4V
(j)
4V 2Ω
① ③ ② ④
2Ω 4V
(k)
4V ① ③ 1Ω
② ④
(ℓ)

Fig. 24



1
$R^{(2)}$
3
$R^{(1)} = V$
$R^{(5)}$
$R^{(4)}$
$R^{(6)} = E$
$R^{(3)}$
2
4
(a)
1
$R^{(2)}$
3
V
$R^{(5)}$
$R^{(4)}$
E
$R^{(3)}$
$\overline{2}$
$\overline{4}$
(b)
1
$R^{(2)}$
3
5
E
$R^{(5)}$
$R^{(4)}$
$R^{(3)}$
$\overline{2}$
$\overline{4}$
(c)
V/2
$R_0^{(2)} = R^{(2)} // R^{(3)}$
E/2
1 $\overline{2}$
3 $\overline{4}$
$R_0^{(1)} = V$
$\overline{3}$ 4
$R_0^{(6)} = E$
$R_0^{(4)} = R^{(4)} // R^{(5)}$
(d)

Fig. 25