WHEN IS A FLOW IN A QUEUING NETWORK POISSONIAN?

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ABSTRACT

Feedback networks of M/M/1 queuing systems are considered. In equilibrium, the flow of customers from node i to node j is Poissonian if and only if there is no path through the network feeding back from node j to node i. Furthermore, when this condition holds, the flow is independent of the number of customers at every node k which is not j nor connected by a path from j.

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1. Introduction and Summary

Consider a feedback network consisting of n nodes, $i = 1,\ldots,n$. During the time interval $[0,t]$ A_t^i customers arrive at node i from outside the network. $(A_t^i, t \geq 0)$ is an independent Poisson process with rate γ_i. Node i is an M/M/1 queuing system with service parameter μ_i. Upon completing service at i a customer either immediately joins the queue at j with probability r_{ij}, or leaves the network with probability $r_{i0} = 1 - \sum_{j=1}^{n} r_{ij}$. Denote the number of customers in $[0,t]$ who move from i to j by S_t^{ij} and the number who leave the network by S_t^{i0}. Let λ_i be the average total rate of flow of customers into i. Then

$$\lambda_i = \gamma_i + \sum_{j=1}^{n} r_{ji}\lambda_j, \quad i = 1,\ldots,n. \tag{1}$$

It is assumed that (1) yields a unique solution $\{\lambda_i > 0\}$, and that the stability condition $\rho_i = \lambda_i\mu_i^{-1} < 1$ holds.

Let $X = \mathbb{N}^n$ where $\mathbb{N}$ denotes the nonnegative integers. At time t the state of the network is $X_t = (X_t^1,\ldots,X_t^n)$ where X_t^i is the number of customers in queue (including the one in service) at node i. Jackson [1] has shown that, in equilibrium, the probability distribution of the state is the same as if the total arrivals into each node i formed an independent Poisson process with rate λ_i,

$$P(X_t=x) = P_1(x^1)\ldots P_n(x^n), \quad P_i(x^i) = (1-\rho_i)\rho_i^{x^i}. \tag{2}$$

This is surprising in view of the fact [2,3] that for a single M/M/1 queue with feedback, the number of customers who complete service, the number fed back and the total number of arrivals are all not Poisson. However, the number of customers who leave the feedback queue do form a Poisson process. This last result, known as the Output Theorem for an M/M/1 queue [4,5], was recently extended by the authors [6]. We showed in particular that for a network of the kind introduced above customers who leave the network from any node form a Poisson process. In terms of the previous notation, the process $(S_t^{i0}, t \geq 0)$ is Poisson; moreover it is independent of the state of the network X_t.

The statistical characterization of the "internal" flows that is the processes (S_t^{ij}), $1 \leq j \leq n$, is given here. Say that ℓ is a descendent of k if there is a path through the network from k to ℓ i.e., there is a sequence $i_1, i_2, \ldots, i_m$ such that

$$r_{ki_1} r_{i_1 i_2} \cdots r_{i_{m-1} i_m} r_{i_m \ell} > 0.$$

Theorem 1 If $(S_t^{ij}, t \geq 0)$ is Poisson then i is not a descendent of j.

Theorem 2 If i is not a descendent of j then (S_t^{ij}) is Poisson; moreover $\{S_u^{ij}; u \leq t\}$ is independent of the queue length X_t^k at every node k which is neither j nor a descendent of j.

Theorem 1 is proved in the next section and Theorem 2 in section 3.

2. Proof of Theorem 1

The idea of the proof is very simple. If (S_t^{ij}) is Poisson then the probability that a customer will move from i to j during the time $[\tau, \tau+t]$ is the same whether or not a similar move was previously observed at time τ. However, if there is a path back from j to i, then the customer who moved at time τ will, with positive probability, reappear at i thereby increasing the chance of a subsequent move during $[\tau, \tau+t]$. We now give a formal proof.

Let X_0 be a random variable, the initial state. Let (A_t^i), $(\bar{S}_t^{ij})$, $1 \le i \le n$, $0 \le j \le n$ be right-continuous, independent Poisson processes with rates γ_i, $r_{ij}\mu_i$ respectively. These processes are also independent of X_0. The state process (X_t) is then given as the solution of the integral equation

$$X_t = X_0 + A_t^i + \sum_{j=1}^{n} S_t^{ji} - \sum_{j=0}^{n} S_t^{ij}$$

$$= X_0 + A_t^i + \sum_{j=1}^{n} \int_0^t 1(X_{u-}^j > 0)\, d\bar{S}_u^{ji} - \sum_{j=0}^{n} \int_0^t 1(X_{u-}^i > 0)\, d\bar{S}_u^{ij}. \qquad (3)$$

Here $1(\cdot)$ is the indicator function of the set $(\cdot)$.

Define the following transition functions.

Arrivals. $U_i : X \to X$, where $U_i(x^1,\ldots,x^n) = (x^1,\ldots x^i+1,\ldots x^n)$, $1 \le i \le n$.

Internal transitions. $T_{ij} : E_i \to X$, where $E_i = \{x \in X | x^i > 0\}$ and $T_{ij}(x) = (x^1,\ldots x^i-1,\ldots x^j+1,\ldots,x^n)$, $1 \le i,j \le n$.

Departures. $T_{i0} : E_i \to X$, where $T_{i0}(x) = (x^1,\ldots,x^i-1,\ldots x^n)$, $1 \le i \le n$.

For $B \subset X$ let

$$P_t(B) = \text{Prob}\{X_t \in B\},\ P_t(x) = P_t(\{x\}).$$

With the notation introduced above the differential equation governing P_t is readily seen to be

$$\dot{P}_t(x) = \sum_{i=1}^{n} \gamma^i [P_t(U_i^{-1}x) - P_t(x)] + \sum_{i=1}^{n}\sum_{j=0}^{n} \mu_i r_{ij} [P_t(T_{ij}^{-1}x) - P_t(x \cap E_i)]. \qquad (4)$$

Here $T_{ij}^{-1}x = \{y \in E_i | T_i(y) = x\}$, $x \cap E_i = \{x\} \cap E_i$. Observe that $\dot{P}_t(x) \equiv 0$ when $P_t(x) \equiv P(x)$ is the distribution given by (2).

In the remaining discussion assume that $P_t(x) \equiv P(x)$. Fix i,j and suppose that $(S_t^{ij}, t \geq 0)$ is a Poisson process. Since λ_i, given by (1), is the average rate of flow into node i and r_{ij} is the probability that a customer departing from i joins the queue at j, therefore the process (S_t^{ij}) has rate $\lambda_i r_{ij}$. Let

$$\mathcal{F}_t = \sigma(S_u^{ij}, u \leq t).$$

Then, since a Poisson process has independent increments,

$$\text{Prob}(S_{t+\Delta}^{ij} - S_t^{ij} = 1 | \mathcal{F}_t) = \lambda_i r_{ij} \Delta + o(\Delta). \tag{5}$$

Since $\bar{S}^{ij}$ is Poisson with rate $r_{ij}\mu_i$, (3) gives

$$\text{Prob}(S_{t+\Delta}^{ij} - S_t^{ij} = 1 | \mathcal{F}_t) = \text{Prob}(X_t^i > 0 | \mathcal{F}_t) r_{ij} \mu_i \Delta + o(\Delta). \tag{6}$$

From (5), (6)

$$\text{Prob}(X_j^i > 0 | \mathcal{F}_t) = \lambda_i \mu_i^{-1} = \rho_i,$$

and since this is constant it must also equal the unconditional probability,

$$\text{Prob}(X_t^i > 0 | \mathcal{F}_t) = P(X_t^i > 0) = P(E_i) = \rho_i. \tag{7}$$

Suppose now that τ is a jump time of (S_t^{ij}), that is a transition of type T_{ij} occurs at τ so that $S_\tau^{ij} - S_{\tau-}^{ij} = 1$. We wish to evaluate

$$\pi_t(x) = \text{Prob}(X_{t+\tau} = x | dS_\tau^{ij} = 1)$$

Since (X_t) is Markovian, $\pi_t(x)$ obeys the differential equation (4) but with the initial condition

$$\pi_0(x) = \text{Prob}(X_\tau = x | dS_\tau^{ij} = 1)$$

For later reference observe that as a consequence of (7) we still have

$$\pi_t(E_i) = \text{Prob}(X_{t+\tau} > 0 | dS_\tau^{ij} = 1) = P(E_i) \tag{8}$$

The initial condition $\pi_0(x)$ can be evaluated as follows. Since $X_\tau = T_{ij}(X_{\tau-})$ and $X_{\tau-} \in E_i$ therefore applying Bayes' rule, and using (7),

$$\pi_0(x) = \text{Prob}(X_\tau = x | X_{\tau-} \in E_i) = P(T_{ij}^{-1}x)[P(E_i)]^{-1} = P(T_{ij}^{-1}x)\rho_i^{-1}. \tag{9}$$

$$T_{ij}^{-1}x = (x^1,..x^i+1,..x^j-1,..,x^n) \text{ if } x^j > 0,\ T_{ij}^{-1}x = \phi \text{ if } x^j = 0, \tag{10}$$

therefore from (2) we get

$$P(T_{ij}^{-1}x) = \rho_i\rho_j^{-1}\ 1(x \in E_j)P(x). \tag{11}$$

From (9), (11) follows

$$\pi_0(x) = \rho_j^{-1}\ P(x)1(x \in E_j). \tag{12}$$

On the other hand,

$$U_j^{-1}x = (x^1,..,x^j-1,..x^n) \text{ if } x^j > 0,\ U_j^{-1}x = \phi \text{ if } x^j = 0,$$

and so, from (2),

$$P(U_j^{-1}x) = \rho_j^{-1}P(x)1(x \in E_j) = \pi_0(x). \tag{13}$$

Thus the distribution π_0 is the same as that obtained from P after an external customer arrives at j.

<u>Lemma</u> Suppose that i is a descendent of j i.e. there is a sequence of nodes $k_1,..,k_m$ so that

$$r_{jk_1}r_{k_1k_2}..r_{k_{m-1}k_m}r_{k_mi} > 0 \tag{14}$$

Then $\pi_t(E_i) > P(E_i)$ for $t > 0$.

<u>Proof</u> Let X_0 be the initial state distributed as before as $\text{Prob}(X_0 = x) = P(x)$ and let $\tilde{X}_0 = U_j(X_0)$ so that, from (13), $\text{Prob}(\tilde{X}_0 = x) = \pi_0(x)$. Let (X_t), $(\tilde{X}_t)$ be the solutions of (3)

corresponding to the initial conditions X_0, $\tilde{X}_0$ respectively. Then clearly

$$\text{Prob}(X_t=x) = P(x), \ \text{Prob}(\tilde{X}_t=x) = \pi_t(x), \ t \geq 0. \tag{15}$$

We show first that

$$Z_t = \tilde{X}_t - X_t \geq 0, \text{ for } t \geq 0. \tag{16}$$

To see this it is convenient to define the n-dimensional processes (θ_t), $(\tilde{\theta}_t)$ where $\theta_t^k = 1(X_{t-}^k > 0)$, $\tilde{\theta}_t^k = 1(\tilde{X}_{t-}^k > 0)$ and the nxn-dimensional process $(\bar{\bar{S}}_t)$ where

$$\bar{\bar{S}}_t^{k\ell} = \bar{S}_t^{\ell k} - \delta_{k\ell} \sum_{j=0}^{n} \bar{S}_t^{kj}.$$

$\delta_{k\ell}$ is the Kronecker index. Then from (3),

$$X_t = X_0 + A_t + \int_0^t d\bar{\bar{S}}_u \theta_u,$$

$$\tilde{X}_t = \tilde{X}_0 + A_t + \int_0^t d\bar{\bar{S}}_u \tilde{\theta}_u,$$

and so

$$Z_t = Z_0 + \int_0^t d\bar{\bar{S}}_u (\tilde{\theta}_u - \theta_u). \tag{17}$$

Now let $0 < \tau_1 < \tau_2 < ..$ be the jump times of $\bar{\bar{S}}$. Clearly

$$Z_{\tau_1-} = Z_0 = \tilde{X}_0 - X_0 = U_j(X_0) - X_0 \geq 0,$$

and suppose, as induction hypothesis, that $Z_{\tau_m-} \geq 0$. Then $\tilde{\theta}_{\tau_m} - \theta_{\tau_m} \geq 0$ since $\tilde{X}^k_{\tau_m-} \geq X^k_{\tau_m-}$ entails $1(\tilde{X}^k_{\tau_m-} > 0) \geq 1(X^k_{\tau_n-} > 0)$; moreover $Z_{\tau_m-} \geq \tilde{\theta}_{\tau_m} - \theta_{\tau_m}$ since $\tilde{X}^k_{\tau_m-} - X^k_{\tau_m-} \geq 0$ entails $\tilde{X}^k_{\tau_m-} - X^k_{\tau_m-} \geq 1(\tilde{X}^k_{\tau_m-} > 0) - 1(X^k_{\tau_m-} > 0)$. Finally, from (17),

$$Z_{\tau_{m+1}^-} - Z_{\tau_m} = dZ_{\tau_m^-} = d\bar{\bar{S}}_{\tau_m}(\tilde{\theta}_{\tau_m} - \theta_{\tau_m}),$$

and so

$$Z_{\tau_{m+1}^-} = Z_{\tau_m^-} + d\bar{\bar{S}}_{\tau_m}(\tilde{\theta}_{\tau_m} - \theta_{\tau_m}) \geq \tilde{\theta}_{\tau_m} - \theta_{\tau_m} + d\bar{\bar{S}}_{\tau_m}(\tilde{\theta}_{\tau_m} - \theta_{\tau_m})$$

$$= [I + d\bar{\bar{S}}_{\tau_m}](\tilde{\theta}_{\tau_m} - \theta_{\tau_m}),$$

where I is the identity matrix. The entries of the matrix $I + d\bar{\bar{S}}_{\tau_m}$ and of the vector $\tilde{\theta}_{\tau_m} - \theta_{\tau_m}$ are nonnegative. So $Z_{\tau_{m+1}^-} \geq 0$, and (16) follows. A particular consequence of (16) is that

$$\text{Prob}(X_t^i > 0 \text{ and } \tilde{X}_t^i = 0) = 0. \tag{18}$$

We claim next that

$$\text{Prob}(X_t^i = 0 \text{ and } \tilde{X}_t^i > 0) > 0. \tag{19}$$

Consider the event $H = \{X_0 = 0\}$ so that in H, $\tilde{X}_0 = U_j(X_0) = (0,..,1,0)$ with 1 in the jth component. $\text{Prob}(H) = (1-\rho_1)..(1-\rho_n) > 0$ according to (2). Also consider the event $G = \{0<\tau_1<..<\tau_{m+1}<t<\tau_{m+2}$ and τ_1 is a jump of $\bar{S}^{jk_1}$, τ_2 is a jump of $\bar{S}^{k_1k_2}$,..,τ_m is a jump of $\bar{S}^{k_{m-1}k_m}$, τ_{m+1} is a jump of $\bar{S}^{k_m i}\}$. Since the $\bar{S}^{k\ell}$ are independent Poisson processes with positive rates by (14), therefore $\text{Prob}(G) > 0$. Also since these processes are independent of the initial state therefore $\text{Prob}(G \cap H) = \text{Prob}(G)\text{Prob}(H) > 0$. But clearly on $G \cap H$ we have $X_t^i = 0$ and $\tilde{X}_t^i > 0$, so

$$\text{Prob}(X_t^i = 0, \tilde{X}_t^i > 0) \geq \text{Prob}(G)\text{Prob}(H) > 0$$

and (19) is proved. From (15), (18), (19) it is immediate that

$$\text{Prob}(\tilde{X}_t^i > 0) = \pi_t(E_i) > \text{Prob}(X_t^i > 0) = P(E_i),$$

and the lemma is proved. ◻

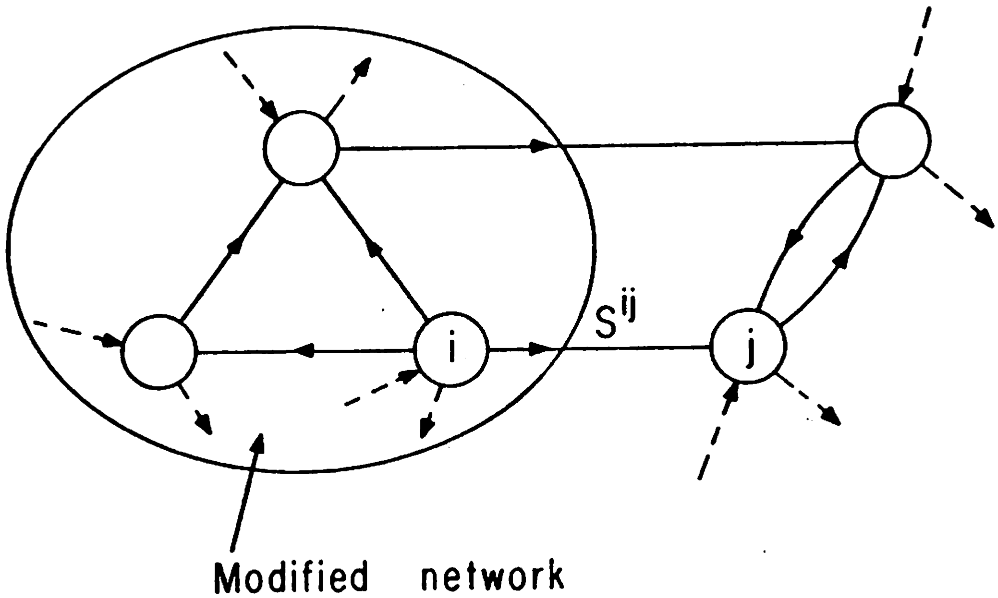

Fig. 1 Illustration for Theorem 2.

We can now complete the proof of Theorem 1. Since (S_t^{ij}) is Poisson therefore, according to (8), $\pi_t(E_i) \equiv P(E_i)$. Hence by Lemma 1 i cannot be a descendent of j.

3. Proof of Theorem 2.

The graph of the network consists of the nodes $N = \{1,..,n\}$ and the directed edges $\mathcal{E} = \{(k,\ell) \in N \times N | r_{k\ell} > 0\}$. Fix j. Let $D = \{k|k = j$ or k is a descendent of $j\}$, and $C = N-D$. Express $\mathcal{E}$ as the disjoint union of the sets $\mathcal{E}_C$, $\mathcal{E}_D$ and $\mathcal{E}_{CD}$ where $\mathcal{E}_C = \mathcal{E} \cap \{CxC\}$, $\mathcal{E}_D = \mathcal{E} \cap \{DxD\}$ and $\mathcal{E}_{CD} = \mathcal{E} - \{\mathcal{E}_C \cup \mathcal{E}_D\}$. It is easy to see that $\mathcal{E}_{CD}$ cannot contain any edge going from a node in D to a node in C i.e., $\mathcal{E}_{CD} \cap \{DxC\} = \phi$.

Now fix i, suppose $r_{ij} > 0$ and i is not a descendent of j. Then $i \in C$ and $(i,j) \in \mathcal{E}_{CD}$. Consider the modified network consisting only of the nodes in C and of the edges in $\mathcal{E}_C$ and $\mathcal{E}_{CD}$, regarding the edges in $\mathcal{E}_{CD}$ as corresponding to flows of customers who depart from this modified network. (This is illustrated in Figure 1 where the dashed arrows correspond to arrivals or departures in the original network.) It is obvious that the number of customers S_t^{ij} and the queue lengths X_t^k for $k \in C$ are the same in the original and modified networks. But in the modified network they are external departures. It follows from [6] that, in equilibrium, (S_t^{ij}) is Poisson and $\{S_u^{ij}, u \leq t\}, \{X_t^k, k \in \dot{C}\}$ are independent. The theorem is proved.

4. Conclusion

The set of flows in a Jacksonian network which in equilibrium are Poisson have been characterized. Briefly, the number of customers who leave i and go to j is Poisson if and only if the edge (i,j) is not part of a loop in the graph of the network. Although each node was assumed to be an M/M/1 queuing system it is easily seen, following the argument given here and in [6], that the result holds even when the service rate at a node depends upon the queue length at the same node. In particular, the result holds when node i is an $M/M/m_i$ queuing system.

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