THE COLORED BRANCH THEOREM: A SIMPLE CIRCUIT THEORETIC PROOF AND ITS APPLICATIONS

by

J. Vandewalle and L.O. Chua

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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J. Vandewalle[†] and L. O. Chua

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

The colored branch theorem (Minty 1960 [1]) is a result in graph theory, which essentially says that the existence (resp., non-existence) of a certain loop immediately implies the non-existence (resp., existence) of a certain cut set.

Its relevance and use in circuit theory, however, has only recently been recognized. Since it is expected that many more applications in circuit theory will follow, the theorem is interpreted and proved in a network setting. Many graph-theoretic corollaries are derived, which may facilitate later use. It is illustrated that many results in circuit theory can be simplified or given a simpler proof using this theorem and its corollaries.

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[†]Presently on leave from the Katholieke Universiteit Leuven, Belgium.

I. Introduction

In 1960 Minty obtained a powerful result [1] in graph theory, which is often called "the colored branch theorem". It essentially says that the existence (resp., the non-existence) of a certain loop in a graph immediately implies the non-existence (resp., existence) of a certain cut set, and vice versa. Like Tellegen's theorem this does not depend on the type of elements or the coupling of elements in a network $\mathcal{N}$; it only depends on the graph $\mathcal{G}$ associated with $\mathcal{N}$.

Although Minty used his theorem first in his algorithm for solving monotone networks [1,2] its main application was originally found in transportation and network flow problems [3], as well as in matroid theory [4,5]. In recent years, however, it has been used more and more in circuit theory [6-11] and it seems likely that many more applications will follow.

Some of the reasons why this theorem is not so well known among electrical engineers are: (1) The theorem is usually formulated only in a graph-theoretic framework and therefore it is difficult to adapt to circuit problems. (2) The use of this theorem in several research papers is in many cases obscured by the mainstream of ideas in the paper.

In this paper we wish to convince the reader that the colored branch theorem is also a simple and powerful theorem in circuit theory. In particular, the following is presented: (1) The theorem is made plausible to the electrical engineer by substituting the branches by simple elements, such as open and short circuits, ideal diodes, or a resistor battery series combination.

This interpretation produces a simple mnemonic tool for remembering the theorem. (2) A rigorous proof is given directly using this network. (3) It is illustrated that this theorem is very general: It is valid for both linear and nonlinear networks as well as for resistive and dynamic networks. Even couplings among elements are allowed. It shares with KCL, KVL, and Tellegen's theorem the distinction of being among the most general theorems in circuit theory. (4) The great generality of the theorem still leaves much freedom to the user. We will show that some particular choices lead to many nice and surprising properties. In many cases this theorem can be used to give a painless proof of a known result, or to generate equivalent conditions, which are easier to check.

We only assume basic knowledge of the theory of directed graphs [12]. We also assume that the number of branches and the number of nodes are finite. Given a graph $\mathcal{G}$, we say that a branch b is removed if it is erased from the graph $\mathcal{G}$. We say that a branch b is coalesced or shrunk if the two nodes to which b is connected are made coincident and b is then erased. Given a finite set A, we call any set of subsets of A, (i.e. $\{A_1, A_2, \ldots, A_n\}$ with $A_i \subset A$ for $1 \leq i \leq n$), such that $A = A_1 \cup A_2 \ldots \cup A_n$, and $A_i \cap A_j = \phi$ for $i \neq j$ a partition of A, where $\cup$ denotes the union, $\cap$ denotes the intersection and ϕ denotes the empty set. In other words $\{A_1, A_2, \ldots, A_n\}$ is a partition of A if each element of A belongs to just one A_i.

II. The Colored Branch Theorem and Its Proof

Given a directed graph $\mathcal{G}$ let it be colored as follows: every branch of the graph is painted in one of three colors: red, blue or green.

Moreover, one green branch is singled out by coloring it dark green[1]. It is clear that any directed graph $\mathcal{G}$ can be colored in many different ways. Any such graph will henceforth be called a directed colored graph $\hat{\mathcal{G}}$. Given a directed colored graph and a loop L, any subset of branches in L is said to be "similarly directed" if they are all oriented in the same direction. Likewise, given a cut set C, any subset of branches in C are said to be "similarly directed" if they are all oriented in the same direction.

The general colored branch theorem is then stated as follows[2] [1]:

Theorem 1. Let $\hat{\mathcal{G}}$ be a (not necessarily connected) directed colored graph, then exactly one, but not both, of the following properties must hold:

(1) The dark green branch forms a loop exclusively with the green and/or red branches. Moreover, the green branches in the loop are similarly directed.

(2) The dark green branch forms a cut set exclusively with the green and/or blue branches. Moreover, the green branches in the cut set are similarly directed.

It is worthwhile to pause here for a while to assess the degree of generality of this theorem. The branches of the graph can be arbitrarily colored as long as (1) each branch is colored red, blue or green

[1] The sets of red, blue and green branches form a partition of the set of all branches of the graph. Any set may be empty except the set of green branches, which must contain at least the dark green branch.

[2] For simplicity, the theorem is stated for a directed colored graph. It will be clear in the context that only the green branches need to be oriented.

and (2) exactly one branch is colored dark green. Under these two conditions the existence (resp., non-existence) of a loop satisfying condition (1) immediately guarantees the non-existence (resp., existence) of a cut set satisfying condition (2). Observe that the theorem does not say that such a loop (resp., cut set) is unique; in fact, there may exist many such loops (resp., cut sets). It is baffling that the absence of a loop satisfying condition (1) (resp., cut set satisfying condition (2)) implies the existence of a cut set satisfying condition (2) (resp., loop satisfying condition (1)). From an intuitive point of view it seems likely that counterexamples of this theorem can be contrived as follows. Starting from a given colored graph such that condition (1) is satisfied but condition (2) is not, we re-color one of the red branches of the loop in blue and repeat this for all loops satisfying (1). So condition (1) must eventually be violated, while we expect condition 2 will remain violated. The fact stands however, that in whatever way we choose such a seemingly harmless recoloring scheme, we always end up in the process with a cut set satisfying condition (2), as predicted by the theorem! The reader is encouraged to work out some examples to convince himself.

This theorem however is not counterintuitive if we look at the following network interpretation. Given a directed colored graph $\hat{\mathcal{G}}$, we replace the red branches by short circuits, the blue branches by open circuits and the dark green branch by a series connection of a 1Ω resistor and a 1V battery with the positive terminal of the battery located at the arrowhead of the dark green branch. Let the remaining green branches be replaced by ideal diodes such that positive current flows in the same direction as that defined by the green branch. As an illustration we derive in Fig. 1(b) a network interpretation for the graph of Fig. 1(a), where branches r_3 and r_9 are colored red, branches b_6 and b_7 are colored blue and branches g_2, g_4, g_5 and g_8 are colored green with g_1^* chosen as the dark green branch. It is clear in this example (see Fig. 1(a)) that the branches $\{g_1^*,g_2,g_4,g_8\}$ in the loop formed by $\{g_1^*,g_2,r_3,g_4,g_8,r_9\}$ are similarly directed. This implies in Fig. 1(b) that the branches g_2, r_3, g_4, g_8 and r_9 offer no resistance to the battery in g_1^*, and the voltage across g_1^* is zero so that a current of 1A must flow through the 1Ω resistor. This implies immediately that there is no cut set of g_1^*, diodes and open circuits such that all diodes and g_1^* are similarly directed. For, suppose there exists such a cut set then every diode of the cut set must convey a non-negative current, while g_1^* carried a current of 1A, thereby contradicting KCL.

In general, given any directed colored graph there corresponds a network containing open circuits, short circuits, ideal diodes, and a 1Ω resistor 1V battery series combination, and vice versa. In terms of this network, the colored branch theorem suggests the following simple and illuminating conclusion: The first possibility corresponds to the case where the branch g^*, consisting of the 1Ω resistor in series with the 1V battery, supplies no current to the remaining branches, so that there cannot exist a loop containing g^*, forward-biased diodes, and short circuits. In this case there must exist a cut set containing reversed-biased diodes and open circuits. The second possibility corresponds to the case where the branch g^* supplies the maximum current of 1A to the remaining branches, so that there exists a loop containing only forward-biased diodes and short circuits which carry the 1A current.

In this case there cannot exist a cut set containing g*, open circuits and reversed-biased diodes.

Applications of this theorem to hydraulic networks and street traffic networks have been mentioned by Minty [2]. The applications to street network routings is particularly enlightening. The branches here are two-way streets, one-way streets and blocked streets. The conclusion of the theorem is then trivial: a point B can either be reached from a point A or not at all. We will give shortly a completely rigorous proof of this electrical network interpretation of the theorem and thereby prove the colored branch theorem. If the branch g* consisting of the 1Ω resistor 1V battery series combination forms a loop with ideal diodes and short circuits such that the diodes in this loop allow a positive current to flow out of the positive terminal of the battery, then we say that the diodes of the loop are similarly directed. Analogously a cut set containing g*, ideal diodes and open circuits is said to have similarly directed ideal diodes if the diodes of this cut set allow a positive voltage to exist at all nodes $n_1^+ \ldots n_m^+$ with respect to the nodes $n_1^- \ldots n_m^-$, where the removal of the branches of the cut set creates two components: one component containing the node at the positive terminal of branch g* and all the nodes $n_1^+ \ldots n_m^+$, and the second component containing the node at the negative side of branch g* and all the nodes $n_1^- \ldots n_m^-$.

Theorem 2. Given a network $\mathcal{N}$ (Fig. 2(a)) consisting of a branch g* made up of a 1Ω resistor 1V battery series connection and other branches made up of ideal diodes, open circuits and short circuits, then exactly one but not both of the following properties hold:

(1) There exists a loop containing only g*, diodes and short circuits, where all diodes are similarly directed. In this case i = 1A and v = 0V.

(2) There exists a cut set containing only g* diodes and open circuits, where all diodes are similarly directed. In this case i = 0A and v = -1V.

Proof: The basic strategy of the proof is to reduce the number of internal nodes and branches of the one-port $\mathcal{N}_1$ (Fig. 2(a)) repeatedly to arrive at one of the four cases listed in Fig. 2(b). It is easy to see that the theorem is satisfied in each of the four cases. The two circuits on the left contain a loop made up of the branch g* and a short circuit, and a similarly directed diode, respectively. Observe that there is no cut set formed by g* with an open circuit and/or a similarly directed diode. On the other hand, the two circuits on the right of Fig. 2(b) exhibit the opposite property. We will show that during this reduction process, the existence or absence of a loop satisfying condition (1), or a cut set satisfying condition (2) is preserved. We perform the following reduction algorithm.

Step 1. Remove all open circuit branches and shrink (coalesce) all short circuit branches. Replace each parallel or series connection of two ideal diodes by its equivalent, which is again an ideal diode, an open circuit, or a short circuit. Remove branches which form hinged loops or self cut sets. Repeat these reduction techniques until there are no more open or short circuit branches, and until there are no more parallel or series connections of ideal diodes.

Step 2. Reduce the number of internal nodes in $\mathcal{N}_1$ by one by substituting $\mathcal{N}_1$ with an equivalent one-port $\mathcal{N}_2$ as follows. Single out the internal node "n" to be removed (Fig. 2(c)). Node n is connected

connected to nodes n_1^+, $n_2^+ \ldots n_k^+$ with the diodes pointing toward n, and to nodes n_1^-, $n_2^- \ldots n_\ell^-$ with the diodes pointing away from n. It is easy to check that $\mathcal{N}_1''$ of Fig. 2(c) is equivalent to $\mathcal{N}_2''$ of Fig. 2(d) (see for example [13,14]). This equivalence transformation removes node n and may introduce some new parallel or series connections of diodes.

Step 3. Return to Step 1 as long as there remain any parallel or series connections of diodes. Return to Step 2 as long as there remain any internal nodes. If there are no more internal nodes, parallel and series connections of diodes, hinged loops, or self cut sets, in $\mathcal{N}_1$, then the resulting network must necessarily assume one of the four cases shown in Fig. 2(b).

It remains for us to show that during the reduction steps the validity of the theorem is preserved. Only the equivalence transformation in Step 2 requires additional investigation. Of course, a loop satisfying condition (1) (resp., a cut set satisfying condition (2)) and containing no branches of $\mathcal{N}_1''$ is preserved in $\mathcal{N}_2$. It follows from the comparison of Fig. 2(c) and (d) that for a loop which contains the diode between nodes n_1^+ and n and the diode between nodes n and n_j^- in Fig. 2(c), there exists a loop containing the diode between n_i^+ and n_j^- in Fig. 2(d). Moreover, if the first loop satisfies condition (1), then likewise the second loop must satisfy the same condition. A cut set satisfying condition (2) and containing diodes of $\mathcal{N}_1''$ can only contain some diodes connected to nodes $n_1^+ \ldots n_k^+$, or some diodes connected to nodes $n_1^- \ldots n_\ell^-$, but not both, since the diodes in each group are similarly directed. Assume that the diodes connecting $n_1^+ \ldots n_{i_s}^+$ to n belong to a cut set of Fig. 2(c) satisfying condition (2). Now, by definition, the removal of all branches of this cut set creates two connected components. One connected component containing, among other nodes, the "+" node of g* and nodes $n^+_{i_1} \ldots n^+_{i_s}$. The other component contains, among other nodes, node n and the "-" node of g*. It is now easy to see that a cut set of Fig. 2(d) satisfying condition (2) is obtained if all diodes of $\mathcal{N}''_2$ connecting nodes $n^+_{i_1} \ldots n^+_{i_s}$ to nodes $n^-_1 \ldots n^-_\ell$ are members of the cut set formed by g* and those branches of $\mathcal{N}'_1$ belonging to the original cut set in Fig. 2(c).

Conversely, the existence of a loop satisfying condition (1) in Fig. 2(d) implies the existence of a loop satisfying condition (1) in Fig. 2(c), and analogously for a cut set. ¤

Observe that the number of loops satisfying condition (1) (resp., cut sets satisfying condition (2)) in $\mathcal{N}_2$ can be smaller than in $\mathcal{N}_1$. Hence in general a network may have several loops satisfying condition (1) (resp., cut sets satisfying condition (2)).

The algorithm described in theorem 2 can be rendered more efficient by the following two rules: (1) If $\mathcal{N}_1$ contains a loop of diodes all biased in the same direction, then all branches of this loop can be coalesced. (2) If $\mathcal{N}_1$ contains a cut set of diodes all biased from one component to the other then only that component which contains branch g* needs to be considered in the algorithm.

However, if one is only interested in the existence of a loop satisfying condition (1) (resp., cut set satisfying condition (2)) and not in the equivalent one-port $\mathcal{N}_1$, one can proceed as follows [12].

Algorithm: Check whether the branch g* made up of the 1Ω resistor 1 volt battery series combination forms a loop satisfying condition (1), or a cut set satisfying condition (2) of theorem 2.

Step 1. Label the "+" node of branch g* with an asterisk "*".

Step 2. Consider all nodes which can be reached from a labelled node via a short circuit, or via an ideal diode pointing away from the labelled node. Label all these nodes also with an "*".

Step 3. Repeat Step 2 until no more nodes can be labelled or until the "-" node of branch g* is labelled.

In the first case there is no loop satisfying condition (1) and in the second case such a loop exists. The existence or non-existence of a cut set satisfying condition (2) then follows from theorem 2.

To illustrate this algorithm, consider the graph shown in Fig. 3. Applying the algorithm, we found that Step 3 terminates without labelling the "-" node of branch g_1^* and hence there is no loop satisfying condition (1). By theorem 2 there must exist a cut set satisfying condition (2). It is easily seen that $\{g_1^*, b_4, g_5, g_6, g_{10}\}$ is such a cut set.

To conclude this section recall that the graph in theorems 1 and 2 were not required to be connected. It follows, however, from the formulation of these theorems, and from the definitions of "loop" and "cut set", that only the connected component containing branch g* can contain a loop satisfying condition (1), or a cut set satisfying condition (2).

III. Graph-theoretic Corollaries of the Colored Branch Theorem

The colored branch theorem is extremely general. By a clever coloring (partition) of the branches and eventually a repeated use of the theorem, one can derive many surprising properties often painlessly.

Corollary 1: Colored branch corollary 1. Let g* be any branch of a graph $\mathcal{G}$[3] and label it the dark green branch. Color each of the remaining branches arbitrarily either in red or in blue. Let B denote the set of the blue branches, and let R denote the set of the red branches. Then branch g* either forms a loop exclusively with branches of R, or a cut set exclusively with branches of B, but not both.

Proof: If we choose the set of green branches in Theorem 1 to consist of the single branch g*, then g* must be identified as the dark green branch. Since there are no other green branches, the "similarly directed" requirements on the green branches in Theorem 1 become superfluous here. Hence the corollary follows by default. ¤

Corollary 2: Colored branch corollary 2. Let $\mathcal{G}$ be a directed graph and let g* be any branch of $\mathcal{G}$. Then g* either forms a loop with branches of $\mathcal{G}$ such that all branches of the loop are similarly directed, or a cut set with branches of $\mathcal{G}$ such that all branches of the cut set are similarly directed, but not both.

Proof: Color all branches of $\mathcal{G}$ green and color g* dark green. Applying Theorem 1 with the sets of red and blue branches both empty, we obtain the above result. ¤

It is important to distinguish the difference between the conclusions of the colored branch corollaries 1 and 2: The branches of the loop and cut set in corollary 1 need not be similarly directed. Observe that except for g*, the branches in the loop and those in the cut set must come from different sets. In corollary 2, on the other hand, all branches come from the same set. In this case, however, the branches

[3] The graph $\mathcal{G}$ can be either directed or undirected.

in the loop, and those in the cut set, must be similarly directed. For example, consider the graph shown in Fig. 4(a). If we choose $R = \{r_2, r_3, r_4\}$ and $B = \{b_5, b_6\}$ as the sets of blue and red branches respectively, and identify the dark green branch as g_1^*, then the colored branch corollary 1 asserts only that g_1^* must either form a loop with branches in R, or a cut set with branches in B. In this case the former holds, namely $\{g_1^*, r_2, r_3, r_4\}$ form a loop. Notice that this loop is not similarly directed. On the other, all branches are colored green in order to apply the colored branch corollary 2. Identify g_1^* again as the dark green branch in Fig. 4(b). Observe that the branches of the loop $\{g_1^*, g_2, g_6\}$ are similarly directed. It can be checked exhaustively that there is no cut set containing g_1^* such that all branches are similarly directed, as predicted by the colored branch corollary 2.

A repeated application of the colored branch theorem allows us to derive the following result:

Corollary 3: Colored branch corollary 3. Partition the branches of a directed graph $\mathcal{G}$ arbitrarily into three sets: A, B and C. We define four subsets of A as follows: (1) Let D_1 be the maximal subset of A such that each branch of D_1 forms a loop (resp., cut set) exclusively with branches of A and/or C, and such that all branches of A in this loop are similarly directed. (2) Let D_2 be the maximal subset of A, such that each branch of D_2 does not form a cut set (resp., loop) exclusively with branches of A and/or B, and such that all branches of A in this cut set are similarly directed. (3) Let E_1 be the maximal subset of A such that each branch of E_1 forms a cut set (resp., loop) exclusively with branches of A and/or B and such that all branches of A in this cut set are similarly directed. (4) Let E_2 be the maximal subset of A such that each branch of E_2 does not form a loop (resp., cut set) exclusively with branches of A and/or C and such that all branches of A in this loop are similarly directed. Then we have

$$D_1 = D_2 \triangleq D, \quad E_1 = E_2 \triangleq E \tag{1}$$

and

$$D \cup E = A, \quad D \cap E = \phi. \tag{2}$$

where ϕ is the empty set. In other words, A is partitioned into D and E.

Proof: First it has to be proved that the sets D_1, D_2, E_1 and E_2 exist and are unique. We prove it, for example, for D_1. Let D_1' and D_1'' be any two subsets of A such that each branch of D_1' (resp., D_1'') forms a loop exclusively with branches of A and/or C and such that all branches of A in the loop are similarly directed. Then, $D_1' \cup D_2''$ also satisfies this property. So the maximal subset D_1 is the union of all subsets of A satisfying the property, and is therefore unique. There exists at least one subset of A satisfying the property since the empty set ϕ satisfies the property by default. Hence D_1 exists.

In order to prove $D_1 = D_2$ and $E_1 = E_2$, we show that $D_1 \supset D_2$, $D_1 \subset D_2$, $E_1 \supset E_2$, and $E_1 \subset E_2$. Let us prove, for example, $D_1 \supset D_2$. Choose any branch g* in D_2 and color it dark green. Color the remaining branches of A in green, the branches of B in blue and the branches of C in red. Then g* does not form a cut set with the green and/or blue branches such that all green branches of the cut set are similarly directed. Hence, by the colored branch theorem it forms a loop with the green and/or red branches such that the green branches in the loop are similarly directed. Since D_1 is maximal this implies $g^* \in D_1$.

It remains to prove (2). Choose any branch of D. Since this branch belongs to $D = D_2$ it does not form a cut set with branches of A and/or B such that the branches of A in the cut set are similarly directed. Since E_1 is maximal this branch does not belong to $E_1 = E$. Analogously it can be proved that branches in E cannot be present in D. Hence $D \cap E = \phi$. In order to prove $D \cup E = A$, choose any branch of A which does not belong to D. Since $D = D_2$ is maximal, this branch forms a cut set exclusively with branches of A and/or B such that all branches of A in the cut set are similarly directed. Hence this branch belongs to $E_1 = E$. Analogously it can be shown that any branch which does not belong to E belongs to D. ¤

A similar corollary can be derived from the colored branch corollary 1.

Corollary 4: Colored branch corollary 4. Partition the branches of a graph $\mathcal{G}$ arbitrarily into three sets: A, B and C. We define four subsets of A as follows: (1) Let D_1 be the maximal subset of A such that each branch of D_1 forms a loop (resp., cut set) exclusively with branches of A and/or C. (2) Let D_2 be the maximal subset of A such that each branch of D_2 does not form a cut set (resp., loop) exclusively with branches of B. (3) Let E_1 be the maximal subset of A such that each branch of E_1 forms a cut set (resp., loop) exclusively with branches of B. (4) Let E_2 be the maximal subset of A, such that each branch E_2 does not form a loop (resp., cut set) exclusively with branches of A and/or C.

Then, we have

$$D_1 = D_2 \triangleq D, \quad E_1 = E_2 \triangleq E \tag{3}$$

and

$$D \cup E = A, \quad D \cap E = \phi \tag{4}$$

In other words, each branch of A belongs either to $D_1 = D_2 \triangleq D$ or to $E_1 = E_2 \triangleq E$, but not to both.

Proof: The proof is analogous to that of the colored branch corollary 3. We mention only the differences. In the proof of $D_1 \supset D_2$, all branches of A are colored red, except for the branch g* which is still dark green. The colored branch corollary 1 is now invoked in order to guarantee that g* forms a loop with branches of A and/or C. In all instances the "similarly directed" requirements are dropped and the cut sets mentioned in the definition of D_2 and E_1 consist only of branches from B and thus cannot contain branches from A. ¤

Let us now compare corollaries 3 and 4. Assume that a directed graph has been partitioned into three sets A, B and C as specified in corollaries 3 and 4. It is clear from the definitions that set D_1 in corollary 3 must satisfy more conditions (all branches of A in the loop must be similarly directed) than set D_1 in corollary 4. Thus set D in corollary 3 is a subset of set D in corollary 4. Analogously set E in corollary 4 is a subset of set E in corollary 3, as they should.

In order to assess the generality of the colored branch corollaries 3 and 4 and for further reference, we state some direct consequences of these corollaries.

Corollary 5. Partition the branches of a directed (resp. undirected) graph $\mathcal{G}$ into three sets A, B and C.

(a) Call D the set D_1 defined in (1) of Corollary 3 (resp. Corollary 4), then D is equal to D_2, and $E \triangleq A-D$ is equal to both E_1 and E_2, where D_2, E_1 and E_2 are defined in (2), (3) and (4) of Corollary 3 (resp. Corollary 4).

(b) Call D the set D_2 defined in (2) of Corollary 3 (resp., Corollary 4), then D is equal to D_1, and $E \triangleq A-D$ is equal to both E_1 and E_2, where D_1, E_1 and E_2 are defined in (1), (3) and (4) of Corollary 3 (resp., Corollary 4).

(c) Call E the set E_1 defined in (3) of Corollary 3 (resp., Corollary 4), then E is equal to E_2, and $D \triangleq A-E$ is equal to both D_1 and D_2, where E_2, D_1 and D_2 are defined in (4), (1) and (2) of Corollary 3 (resp., Corollary 4).

(d) Call E the set E_2 defined in (4) of Corollary 3 (resp., Corollary 4), then E is equal to E_1, and $D \triangleq A-E$ is equal to both D_1 and D_2, where E_1, D_1 and D_2 are defined in (3), (1) and (2) of Corollary 3 (resp., Corollary 4).

This corollary allows us to derive a nice property concerning the existence of certain loops and cut sets obtained by partitioning the twigs of a tree and the links of a cotree in a particular way. There are three alternative formulations.

Corollary 6a. Partition the twigs of a tree $\mathcal{T}$ arbitrarily into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\mathcal{L}_1$ be the maximal set of links of the corresponding cotree $\mathcal{L}$ such that each link of $\mathcal{L}_1$ forms a loop exclusively with twigs of $\mathcal{T}_1$. Let $\mathcal{L}_2$ be the set of the remaining links of $\mathcal{L}$.

Then any twig of $\mathcal{T}_2$ forms a cut set exclusively with links of $\mathcal{L}_2$.

Proof: Set A = $\mathcal{L}$, B = $\mathcal{T}_1$ and C = $\mathcal{T}_2$. It follows from part (c) of Corollary 5 (with $E_1 = \mathcal{L}_1$ in definition (3) of Corollary 4) that $D_1 = \mathcal{L}_2$ is the maximal subset of $\mathcal{L}$ such that each branch of $\mathcal{L}_2$ forms a cut set exclusively with branches of $\mathcal{L}$ and $\mathcal{T}_2$. However each twig of $\mathcal{T}_2$ forms a fundamental cut set with $\mathcal{L}$. Since $\mathcal{L}_2$ is maximal, each twig of $\mathcal{T}_2$ forms a cut set exclusively with links of $\mathcal{L}_2$. ⌑

An alternative statement of Corollary 6a can be formulated as follows:

Corollary 6b. Partition the links of a cotree $\mathcal{L}$ arbitrarily into two sets $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\mathcal{T}_2$ be the maximal set of twigs of the corresponding tree $\mathcal{T}$, such that each twig of $\mathcal{T}_2$ forms a cut set exclusively with links of $\mathcal{L}_2$. Let $\mathcal{T}_1$ be the set of the remaining twigs of $\mathcal{T}$. Then any link of $\mathcal{L}_1$ forms a loop exclusively with twigs of $\mathcal{T}_1$.

Proof: The proof is the dual of that given in Corollary 6a and is therefore omitted. ⌑

A third formulation of this property does not assume the tree or cotree to be given.

Corollary 6c. Partition the branches of a graph $\mathcal{G}$ arbitrarily into two sets S_1 and S_2. Let T_1 be the maximal subset of S_1 such that each branch of T_1 does not form a loop exclusively with branches of S_1. Let L_1 be the maximal subset of S_1 such that each branch of L_1 does not form a cut set exclusively with branches of S_2. Let T_2 be the maximal subset of S_2 such that each branch of T_2 does not form a loop exclusively with branches of S_1. Let L_2 be the maximal subset of S_2 such that each branch of L_2 does not form a cut set exclusively with branches of S_2. Then we have:

(1) $$S_1 = T_1 \cup L_1, \quad T_1 \cap L_1 = \phi \qquad (5)$$

$$S_2 = T_2 \cup L_2, \quad T_2 \cap L_2 = \phi \qquad (6)$$

(2) Each branch of L_1 forms a loop exclusively with branches of T_1 and each branch of T_2 forms a cut set exclusively with branches of L_2.

(3) Let $\mathcal{T}_1 \supset T_1$ be any maximal subset of S_1 such that each branch of $\mathcal{T}_1$ does not form a loop exclusively with branches of $\mathcal{T}_1$. Call $\mathcal{L}_1$ the remaining branches of S_1. Let $\mathcal{L}_2 \supset L_2$ be any maximal subset of S_2 such that each branch of $\mathcal{L}_2$ does not form a cut set exclusively with branches of $\mathcal{L}_2$. Call $\mathcal{T}_2$ the remaining branches of S_2. Then $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ form a tree and cotree of the graph $\mathcal{G}$. Also each branch of $\mathcal{L}_1$ forms a loop exclusively with branches of $\mathcal{T}_1$ and each branch of $\mathcal{T}_2$ forms a cut set exclusively with branches of $\mathcal{L}_2$.

Proof: (1) Apply the Colored branch corollary 4. Identify $A = S_1$, $B = S_2$ and $C = \phi$. Then (5) immediately follows. By identifying $A = S_2$, $B = S_1$ and $C = \phi$ and applying the Colored branch corollary 4 we obtain (6).

(2) This follows also from the application of the Colored branch corollary 4 in (1).

(3) We prove that $\mathcal{T} \triangleq \mathcal{T}_1 \cup \mathcal{T}_2$ does not contain any loop and $\mathcal{L} \triangleq \mathcal{L}_1 \cup \mathcal{L}_2$ does not contain any cut set. Suppose the contrary that there exists a cut set in $\mathcal{L}$. Then since $\mathcal{T}_1$ is a maximal

subset of S_1 having no loops, any branch of $\mathcal{L}_1 = S_1 - \mathcal{T}_1$ must form a loop with $\mathcal{T}_1$. Hence, by the Colored branch corollary 1 any branch of $\mathcal{L}_1$ cannot form a cut set with branches of $\mathcal{T}_2 \cup \mathcal{L}_1 \cup \mathcal{L}_2$. Thus any cut set in $\mathcal{L}$ can only contain branches of $\mathcal{L}_2$. But this is impossible by the definition of $\mathcal{L}_2$. Analogously it can be proved that $\mathcal{T}$ contains no loops. The properties about the branches of $\mathcal{L}_1$ and $\mathcal{T}_2$ follow from part (2). ◻

Observe that by labelling the branches of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ consecutively, we obtain in each of the three cases a <u>fundamental loop matrix</u> $\underset{\sim}{B}$ having the following structure [15]:

$$\underset{\sim}{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathfrak{T}_1 & \mathfrak{T}_2 \\ \left[\begin{array}{c} \underset{\sim}{1}_{\mathcal{L}_1\mathcal{L}_1} \\ \underset{\sim}{0}_{\mathcal{L}_2\mathcal{L}_1} \end{array}\right. & \begin{array}{c} \underset{\sim}{0}_{\mathcal{L}_1\mathcal{L}_2} \\ \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} \end{array} & \begin{array}{c} \underset{\sim}{B}_{\mathcal{L}_1\mathfrak{T}_1} \\ \underset{\sim}{B}_{\mathcal{L}_2\mathfrak{T}_1} \end{array} & \left.\begin{array}{c} \underset{\sim}{0}_{\mathcal{L}_1\mathfrak{T}_2} \\ \underset{\sim}{B}_{\mathcal{L}_2\mathfrak{T}_2} \end{array}\right] \end{array} \begin{array}{c} \mathcal{L}_1 \\ \mathcal{L}_2 \end{array}, \tag{7}$$

where the upper right submatrix is always a zero matrix.

Corollary 6c guarantees the existence of a tree and a cotree having certain properties in the case where the branches of a graph are partitioned into two sets. The next corollary deals with the case that the branches are partitioned into three sets.

<u>Corollary 7</u>. Let the branches of a graph $\mathcal{G}$ be partitioned <u>arbitrarily</u> into three sets X, Y and Z. Then there exists a tree $\mathcal{T}$ and a cotree $\mathcal{L}$ of $\mathcal{G}$ such that $\mathcal{T}$ contains n_X branches of X, where n_X is equal to the number of branches of X minus the number of independent loops in X; and $\mathcal{L}$ contains n_Z branches of Z, where n_Z is equal to the number of branches of Z minus the number of independent cut sets in Z. Let $X_2\mathcal{T} \subset \mathcal{T}$ and $Z_1\mathcal{L} \subset \mathcal{L}$, where $X_2\mathcal{T}$ is the maximal subset of branches of X, such that each branch of $X_2\mathcal{T}$ <u>does not</u> form a loop exclusively

with branches of X, and where $Z_1\mathcal{L}$ is the maximal subset of Z, such that each branch of $Z_1\mathcal{L}$ <u>does not</u> form a <u>cut set</u> exclusively with branches of Z. Partition the branches of $\mathcal{T}$ into the following four sets $X_1\mathcal{T}, X_2\mathcal{T}$, $Y\mathcal{T}$, and $Z\mathcal{T}$ such that:

$$X_1\mathcal{T} \cup X_2\mathcal{T} = X \cap \mathcal{T} \text{ and } X_1\mathcal{T} \cap X_2\mathcal{T} = \phi \tag{8}$$

$$Y\mathcal{T} = Y \cap \mathcal{T} \tag{9}$$

$$Z\mathcal{T} = Z \cap \mathcal{T}, \tag{10}$$

Partition the branches of the corresponding cotree $\mathcal{L}$ into the following four sets $X\mathcal{L}$, $Y\mathcal{L}$, $Z_1\mathcal{L}$, and $Z_2\mathcal{L}$:

$$X\mathcal{L} = X \cap \mathcal{L} \tag{11}$$

$$Y\mathcal{L} = Y \cap \mathcal{L} \tag{12}$$

$$Z_1\mathcal{L} \cup Z_2\mathcal{L} = Z \cap \mathcal{L} \text{ and } Z_1\mathcal{L} \cap Z_2\mathcal{L} = \phi \tag{13}$$

Then each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$, and each branch of $Z\mathcal{T}$ forms a cut set exclusively with branches of $Z_2\mathcal{L}$.

<u>Proof</u>: Recall that a tree $\mathcal{T}$ and the corresponding cotree $\mathcal{L}$ form a partition of the branches of a graph such that $\mathcal{T}$ contains no loops and $\mathcal{L}$ contains no cut sets. Therefore the number n_X (resp., n_Z) of branches of X (resp., Z) in any tree (resp., cotree) as defined in the corollary is the maximum that can be attained. We construct an arbitrary tree satisfying the specifications of Corollary 7 by partitioning first the branches of X into three sets as follows: Remove any branch from each independent loop of X, thereby eliminating all loops in X. Denote this set of removed branches by $X\mathcal{L}$. The remaining branches of X of course contain the set $X_2\mathcal{T}$ defined in the corollary,

and can therefore be considered as the union of two disjoint sets $X_1\mathcal{T}$ and $X_2\mathcal{T}$. Thus X is partitioned into $X_1\mathcal{T}$, $X_2\mathcal{T}$ and $X\mathcal{L}$. Next we partition the branches of Z into three corresponding sets as follows: Shrink any branch from each independent cut set of Z thereby eliminating all cut sets in Z. Denote this set of shrunk branches by $Z\mathcal{T}$. The remaining branches of course, contain the set $Z_1\mathcal{L}$ defined in the corollary, and therefore can be considered as the union of two disjoint sets $Z_1\mathcal{L}$ and $Z_2\mathcal{L}$. Thus Z is partitioned into $Z\mathcal{T}$, $Z_1\mathcal{L}$ and $Z_2\mathcal{L}$. It follows then from these partitionings of X and Z and from the definition of $Z_1\mathcal{L}$ and $X_2\mathcal{T}$, that each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$ and that each branch of $Z\mathcal{T}$ forms a cut set exclusively with branches of $Z_2\mathcal{L}$. It remains for us to prove that there exists a tree $\mathcal{T}$ containing all branches of $X_1\mathcal{T} \cup X_2\mathcal{T} \cup Z\mathcal{T}$, and a corresponding cotree $\mathcal{L}$ containing all branches of $X\mathcal{L} \cup Z_1\mathcal{L} \cup Z_2\mathcal{L}$. First we claim that $\mathcal{T}^1 \triangleq X_1\mathcal{T} \cup X_2\mathcal{T} \cup Z\mathcal{T}$ does not contain any loop and that $\mathcal{L}^1 \triangleq X\mathcal{L} \cup Z_1\mathcal{L} \cup Z_2$ does not contain any cut set. We will prove only the first claim, since the second follows by duality. Since each branch of $Z\mathcal{T}$ forms a cut set exclusively with branches of $Z_2\mathcal{L}$, by the Colored branch corollary 1, it cannot form a loop with branches of $X_1\mathcal{T} \cup X_2\mathcal{T} \cup Z\mathcal{T}$. Since $X_1\mathcal{T} \cup X_2\mathcal{T}$ does not contain any loop by construction, $\mathcal{T}^1$ does not contain any loop. We now construct a tree and a corresponding cotree inductively starting from $\mathcal{T} = \mathcal{T}^1$ and $\mathcal{L} = \mathcal{L}^1$ by adding branches of Y to $\mathcal{T}$ or $\mathcal{L}$ as follows. Consider one branch y of Y at a time. If y forms no loop with $\mathcal{T}$ add it to $\mathcal{T}$. If y forms a loop with $\mathcal{T}$, then by the Colored branch corollary 1 it forms no cut set with $\mathcal{L}$ and hence y is added to $\mathcal{L}$. Finally we obtain a partitioning $\mathcal{T} \cup \mathcal{L}$ of the branches of the graph such that $\mathcal{T}$ contains no loop and $\mathcal{L}$ contains no cut set. ¤

Observe that by labelling the branches of $X\mathcal{L}$, $Y\mathcal{L}$, $Z_1\mathcal{L}$, $Z_2\mathcal{L}$, $X_1\mathcal{T}$, $X_2\mathcal{T}$, $Y\mathcal{T}$ and $Z\mathcal{T}$ consecutively the <u>fundamental loop matrix</u> $\underset{\sim}{B}$ assumes the following form:

$$B = \begin{array}{c} \\ \end{array}\begin{array}{cccccccc} X\mathcal{L} & Y\mathcal{L} & Z_1\mathcal{L} & Z_2\mathcal{L} & X_1\mathcal{T} & X_2\mathcal{T} & Y\mathcal{T} & Z\mathcal{T} \\ \end{array}$$

$$B = \left[\begin{array}{cccccccc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{15} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{25} & \underset{\sim}{B}_{26} & \underset{\sim}{B}_{27} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{35} & \underset{\sim}{B}_{36} & \underset{\sim}{B}_{37} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{45} & \underset{\sim}{B}_{46} & \underset{\sim}{B}_{47} & \underset{\sim}{B}_{48} \end{array}\right] \begin{array}{l} X\mathcal{L} \\ Y\mathcal{L} \\ Z_1\mathcal{L} \\ Z_2\mathcal{L} \end{array} \quad (14)$$

where the zero submatrices in the upper right part of $\underset{\sim}{B}$ are caused by the fact that each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$ and each branch of $Z\mathcal{T}$ forms a cut set exclusively with branches of $Z_2\mathcal{L}$.

Before we close this section, a general comment about equivalent formulations of the statement of these graph-theoretic results is in order. In many cases it turns out that a "reduced" graph obtained by shrinking some branches and removing some others can just as well be used to express these results. This equivalent formulation may be conceptually simpler or may enhance the efficiency of the associated algorithms.

For example, we could check the Colored branch theorem immediately on the reduced graph $\mathcal{G}_r$ obtained by shrinking the red branches and removing the blue branches. If we do the same in Colored branch corollary 1, then $\mathcal{G}_r$ only contains one branch and this branch forms either a self-loop or a self-cut-set. For Colored branch corollaries 3 and 4, we would obtain the same sets D_1, D_2, E_1 and E_2 if the branches of B (resp., C) were removed and those of C (resp., B) were shrunk. In defining T_1, L_1, $\mathcal{T}_1$ and $\mathcal{L}_1$, in Corollary 6c, the branches of S_2 can be removed from the graph, such that $\mathcal{T}_1$ and $\mathcal{L}_1$ are merely a tree and a cotree in the resulting graph. Analogously the branches of S_1 can be shrunk from the graph in defining T_2, L_2, $\mathcal{T}_2$ and $\mathcal{L}_2$. This corresponds to the formulation in [22].

IV. Applications of the Colored Branch Theorem and its Corollaries in Circuit Theory

In this section we compile a list of applications of the Colored branch theorem in circuit theory. Some applications are new while other allow simpler proofs of known properties. Whenever an application or a proof is new, a detailed treatment is given. Otherwise, we simply refer to the relevant literature.

1. The Colored branch theorem and its corollaries 1 and 2, imply that the existence (resp., non-existence) of certain loops or cut sets is equivalent to the non-existence (resp., existence) of certain cut sets or loops. Now, in an algorithm or in the formulation of a property, one of the two equivalent properties may turn out to be simpler. This observation allows us to find equivalent conditions for the non-existence of a loop of capacitors or inductors (or a cut set of capacitors or inductors) in an RLC circuit. It also allows us to check whether a digital filter has a delay free loop or not. One simple illustration of this idea is as follows. Check graphically whether a branch g* forms a cut set with the blue branches of a non-planar colored graph.

A simple example is given in Fig. 5. It is difficult to see at first sight that g_1^* forms a cut set with the blue branches b_2, b_4, b_6, b_7 and b_9. However it is trivial to see that g_1^* forms no loop with any subset of the red branches $\{r_3, r_5, r_8, r_{10}\}$.

2. The Colored branch theorem is the key for obtaining a no-gain property [6,7,16,17] used in finding fundamental limits of dc-to-dc converters and in deriving qualitative properties of resistive networks containing three-terminal [10] and multiterminal elements [7]. We describe here the generalized formulation found in [7] and present a unified proof.

Proposition 1: No-gain property for networks containing two-terminal elements. Given a network containing independent voltage and current sources, positive linear two-terminal resistors[4], short circuit elements which do not form loops among each other, and open circuit elements which do not form cut sets among each other. Then for any solution of this network we have:

(1) The current magnitude through any element is not greater than the sum of the current magnitude through all independent voltage and current sources.

(2) The voltage magnitude across any element is not greater than the sum of the voltage magnitudes across all independent voltage and current sources.

Proof: Consider one solution of the network. The proposition is trivially satisfied in the case where the branch considered (henceforth called branch ℓ) is a voltage source or a current source, or where its

[4] A positive linear resistor has a resistance R restricted to $0 < R < \infty$.

branch voltage $v_\ell = 0$ or branch current $i_\ell = 0$. In order to prove part (1) of the proposition for the case where branch ℓ is a resistor, an open circuit, or a short circuit, and has $v_\ell \neq 0$ or $i_\ell \neq 0$, we proceed as follows. All branches with $v = 0$ and $i = 0$ can be removed, since they contribute nothing to the sum in (1). Moreover, they do not alter the solution nor do they introduce any new loops. Let S be the set of the remaining sources and let X be the set of the remaining resistors, short circuit and open circuit elements. Since all resistances are nonnegative we can define associated reference directions for all branches in X such that the currents and the voltages in these branches are nonnegative. The set up for the Colored branch theorem is now as follows: The branches in X are colored green, those in S blue while there are no red branches. We now prove that branch ℓ does not form a similarly directed loop exclusively with branches of X. If $v_\ell > 0$ the voltages along such a loop would violate KVL. If $v_\ell = 0$, then by KVL all branches of such a loop have zero voltage. Since none of them has zero current, all must be short circuit elements. But this is impossible since there are no loops containing only short circuit elements. Thus by the Colored branch theorem there exists a similarly directed cut set C containing branch ℓ and branches from X and S. Applying KCL to this cut set, we obtain

$$\sum_{k \in X \cap C} i_k + \sum_{k \in S \cap C} \delta_k i_k = 0, \tag{15}$$

for some $\delta_k = \pm 1$. Since all i_k with $k \in X$ are nonnegative, we have

$$|i_\ell| = i_\ell \leq \sum_{k \in X \cap C} i_k = - \sum_{k \in S \cap C} \delta_k i_k \leq \sum_{k \in S \cap C} |i_k|. \tag{16}$$

□

3. Wolaver's three-basket theorem [16,17] is a similar result concerning a partitioning of the branches of a network into 3 sets called baskets. Using the Colored branch theorem a considerably simpler proof than that of [16,17] can be given.

<u>Proposition 2: Wolaver's three-basket theorem</u>. Let the branches of a resistive network be partitioned into three sets (baskets) S_1, S_2 and S_3, such that each branch k of S_3 has $v_k i_k \geq 0$. Then for any solution of the network and any branch ℓ of S_3, both of the following expressions cannot be simultaneously satisfied:

$$\text{(a)} \quad |i_\ell| > \sum_{k \in S_1} |i_k| \tag{17a}$$

$$\text{(b)} \quad |v_\ell| > \sum_{k \in S_2} |v_k|. \tag{17b}$$

<u>Proof</u>: Consider a solution of this network. Reverse the reference directions of any branch in S_3 if necessary until all voltages and currents of branches in S_3 are nonnegative. This entails no loss of generality since $v_k i_k \geq 0$ for $k \in S_3$. Let the branches of S_1 be colored blue, those of S_2 red, and those of S_3 green with branch ℓ being the dark green branch. Then it follows from the Colored branch theorem that branch ℓ forms either a similarly directed loop L exclusively with branches in S_2 and S_3, or a similarly directed cut set C exclusively with branches in S_1 and S_3, but not both. If the first property holds, then by applying KVL along the loop L we obtain

$$\sum_{k \in L \cap S_3} v_k + \sum_{k \in L \cap S_2} \delta_k v_k = 0, \tag{18}$$

for some $\delta_k = \pm 1$. Using the fact that $v_k \geq 0$ for $k \in S_3$, we have

$$|v_\ell| \le v_\ell + \sum_{\substack{k \in L \cap S_3 \\ k \ne \ell}} = - \sum_{k \in L \cap S_2} \delta_k v_k \le \sum_{k \in S_2} |v_k|, \tag{19}$$

which clearly contradicts (17b). Analogously it can be proved that the existence of a similarly directed cut set C contradicts (17a). Hence (17a) and (17b) cannot both be satisfied. ⌑

This result can be refined as follows. By combining the proof of this proposition with the Colored branch corollary 3, a graph-theoretic method can be devised for partitioning the branches of S_3 into two sets S_3' and S_3'' such that each branch of S_3' violates (17a) and each branch of S_3'' violates (17b). Assume that the polarity of the voltage and current in all branches of S_3 for a solution is given. Let S_3' (resp., S_3'') be the maximal subset of branches of S_3 such that each branch of S_3' (resp., S_3'') forms a cut set (resp., loop) exclusively with branches of S_3 and/or S_1 (resp., S_2) and such that the branches of S_3 in the cut set (resp., loop) are similarly directed. It follows from Colored branch corollary 3 that the branches of S_3' and S_3'' form a partitioning of S_3. From the proof of Proposition 2 it follows that each branch of S_3' violates (17a) and each branch of S_3'' violates (17b).

4. In circuit theory it is important to know whether a certain property P is closed under interconnection, i.e., an arbitrary interconnection of multiports, each having property P results in a multiport which inherits this property. For some useful properties such as strict passivity, this is in general not true; but by imposing an additional topological condition on the ports, the closure property is valid [8,11]. The Colored branch corollary 1 is essential for proving this conditional closure property.

5. In order to find properties of RLC networks it is important to be able to express the voltages and currents associated with the inductor and capacitor ports as a <u>function</u> of the voltages and currents at the remaining ports (resp., as an <u>operator</u> of the voltage and current waveforms at the remaining ports). It is shown in [8] that it is sufficient to require that the inductor and capacitor ports form no loops or cut sets among themselves (resp., satisfy a topological condition called the LC-hypothesis). The proof is once again based on the Colored branch corollary 1 (resp., Colored branch corollary 4). The reader will easily recognize the repeated implicit use that is made of the Colored branch corollary 4 in the proof of Theorem 10 of [8].

6. In many cases state equations have to be derived for RLC networks from KVL, KCL and the constitutive relations. Such a derivation is usually based on the existence of a tree containing the maximum number of capacitors (resp., inductors) and a corresponding cotree containing a maximum number of inductors (resp., capacitors). Such a tree is called a <u>C-normal tree</u> [18] (resp., <u>L-normal tree</u>) [9,19]. The existence of such a tree and its properties constitute a direct application of corollary 7. As usual, we assume that the independent voltage sources do not form loops exclusively with each other or with capacitors (resp., inductors) and that the independent current sources do not form cut sets exclusively with each other or with inductors (resp., capacitors). Identify the sets X, Y and Z of Corollary 7 as follows: Choose X to be the set consisting of all capacitors (resp., inductors) and all independent voltage sources, choose Y to be the set of consisting of all resistors, and choose Z to be the set consisting of all inductors (resp., capacitors) and all independent current sources.

It follows then from the preceding assumptions that all independent voltage sources (resp., current sources) are included in $X_2\mathcal{T}$ (resp., $Z_1\mathcal{L}$). Hence, it follows from Corollary 7 that there exists a tree which contains the maximum number of capacitors (resp., inductors), all independent voltage sources and a minimum number of inductors (resp., capacitors). Similarly, there exists a corresponding cotree which contains the maximum number of inductors (resp., capacitors), all independent current sources and the minimum number of capacitors (resp., inductors).

7. Corollaries 6a, 6b and 6c provide 3 equivalent methods for generating a <u>complete set of variables</u> for a graph in the following sense. A set of branch voltages <u>and</u> currents is called a complete set of variables for a graph if they can be assigned arbitrary values without violating KVL and KCL and if they determine at least one of the two variables (the voltage or the current) in each branch by using VKL and KCL and without invoking the element constitutive equations [20-21]. Such a complete set of variables is essential in the hybrid analysis of a nonlinear network. By labelling the branch voltages and currents in $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{L}_1$, $\mathcal{L}_2$ as $\underset{\sim}{v}_{\mathcal{T}_1}$, $\underset{\sim}{v}_{\mathcal{T}_2}$, $\underset{\sim}{v}_{\mathcal{L}_1}$, $\underset{\sim}{v}_{\mathcal{L}_2}$ and $\underset{\sim}{i}_{\mathcal{T}_1}$, $\underset{\sim}{i}_{\mathcal{T}_2}$ $\underset{\sim}{i}_{\mathcal{L}_1}$, $\underset{\sim}{i}_{\mathcal{L}_2}$ respectively, and using the fundamental loop matrix $\underset{\sim}{B}$ of (7), $\underset{\sim}{v}_{\mathcal{T}_1}$ and $\underset{\sim}{i}_{\mathcal{L}_2}$ emerge as a complete set of variables. Indeed KVL and KCL are not violated by assigning arbitrary values to $\underset{\sim}{v}_{\mathcal{T}_1}$ and $\underset{\sim}{i}_{\mathcal{L}_2}$. Moreover, $\underset{\sim}{v}_{\mathcal{L}_1}$ and $\underset{\sim}{i}_{\mathcal{T}_2}$ are determined uniquely via KVL and KCL as follows:

$$\underset{\sim}{v}_{\mathcal{L}_1} = -\underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1}\underset{\sim}{v}_{\mathcal{T}_1}, \quad \underset{\sim}{i}_{\mathcal{T}_2} = \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{T}_2}\underset{\sim}{i}_{\mathcal{L}_2}, \tag{20}$$

where the submatrices of $\underset{\sim}{B}$ are defined in (7).

The hybrid analysis proceeds then as follows: If the resistors in $\mathcal{T}_1$ and $\mathcal{L}_1$ are voltage controlled and those in $\mathcal{T}_2$ and $\mathcal{L}_2$ are current controlled, then the KVL and KCL equations, and the constitutive relations completely determine the solution of the resistive network via a set of $n_{\mathcal{T}_1} + n_{\mathcal{L}_2}$ equations with $n_{\mathcal{T}_1} + n_{\mathcal{L}_2}$ unknown variables $\underset{\sim}{v}_{\mathcal{T}_1}$ and $\underset{\sim}{i}_{\mathcal{L}_2}$, where $n_{\mathcal{T}_1}$ (resp., $n_{\mathcal{L}_2}$) is the dimension of $\underset{\sim}{v}_{\mathcal{T}_1}$ (resp., $\underset{\sim}{i}_{\mathcal{L}_2}$). The three different techniques for finding a complete set of variables, which follow from Corollary 6a, 6b and 6c have been used extensively. The approach taken in Corollary 6a and 6b corresponds with that of [20,15,21], while the approach of Corollary 6c corresponds to that of [22]. Observe that the notion of a "hybrid tree" [23] amounts to another equivalent method of finding a partition of the branches with the same properties. In general there are many distinct ways for partitioning a network into $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{L}_1$, and $\mathcal{L}_2$. Any partition which minimizes the number of $n_{\mathcal{T}_1} + n_{\mathcal{L}_2}$ also minimizes the number of equations required in the hybrid analysis. This unique number is called the topological degree of freedom [22] and is completely determined by the graph of the network.

V. Concluding Remarks

This paper is entirely devoted to a simple, yet not so well known result called the Colored branch theorem. Perhaps the easiest method for applying this theorem is via its network interpretations. In particular, consider a network $\mathcal{N}$ (Fig. 2a) consisting of branch g* in parallel with a one-port $\mathcal{N}_1$ containing only diodes, open circuit elements, and short circuit elements. Then exactly one, but not both, of the following properties hold:

(1) $\mathcal{N}_1$ offers no resistance to the current i, in which case g* forms a loop with short circuit elements and forward-biased diodes. (2) $\mathcal{N}_1$ offers an infinite resistance to the current i, in which case there exists a cut set of g* with open circuit elements and reverse-biased diodes. To obtain the theorem, one simply replaces the short circuit elements of $\mathcal{N}$ by red branches, the open circuit elements by blue branches, the diodes by "oriented" green branches, and g* by an "oriented" dark green branch.

Due to its graph-theoretic nature this theorem is very general: It applies to both linear and nonlinear networks, as well as to resistive dynamic networks. Even couplings among elements allowed. A clever use of the freedom in the choice of the sets of red, blue and green branches allowed us to derive many useful graph-theoretic corollaries. These corollaries greatly simplify the use of the Colored branch theorem in circuit theory. Some new applications are discussed and simpler proofs are given for existing properties. Many more applications are expected to follow in nonlinear circuit theory, analog and digital filter theory, and power networks. In other fields such as mechanical, thermodynamic, and hydraulic systems, the corollaries are equally applicable. In short, we conclude that the Colored branch theorem and its corollaries are among the most general and fundamental tools in circuit theory. We believe that the Colored branch theorem has not yet been fully exploited and that its potential for future unconventional applications in Circuit Theory should be quite promising.

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FIGURE CAPTIONS

Fig. 1. (a) A directed colored graph: branch g_1^* is colored dark green, branches g_2, g_4, g_5 and g_8 are colored green branches r_3 and r_9 are colored red, and branches b_6 and b_7 are colored blue.

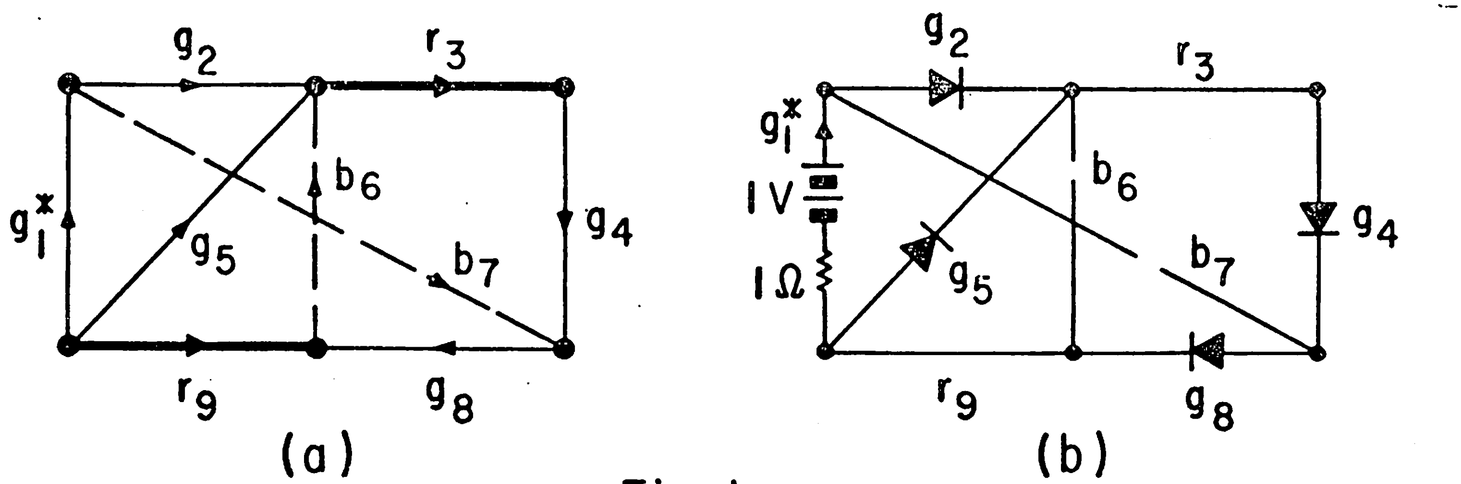
(b) Network interpretation of the directed colored graph in (a).

Fig. 2. (a) The network $\mathcal{N}$ corresponding to a general directed colored graph. (b) The four possible equivalents of $\mathcal{N}$. (c) and (d) By replacing $\mathcal{N}_1''$ of (c) by its equivalent $\mathcal{N}_2''$ of (d) the internal node n is eliminated.

Fig. 3. Example illustrating the algorithm for detecting the existence (resp., non-existence) of a loop satisfying condition (1), or a cut set satisfying condition (2).

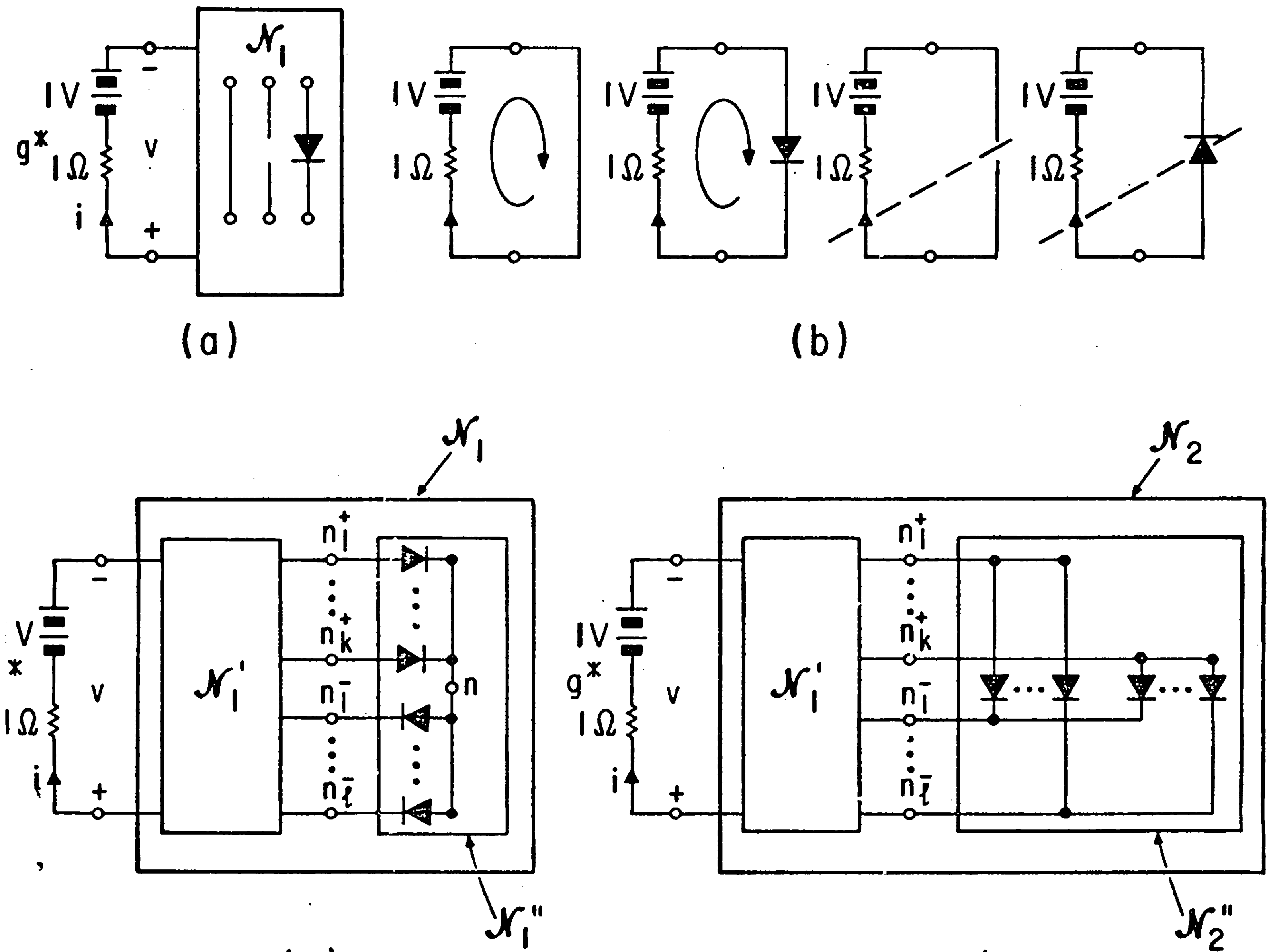
Fig. 4. An example illustrating the differences between Colored branch corollary 1 for the graph in (a) and colored branch corollary 2 for the graph in (b).

Fig. 5. By simply checking that g_1^* forms no loop with the set of red branches $\{r_3, r_5, r_8, r_{10}\}$, it follows that g_1^* forms a cut set with the set of blue branches $\{b_2, b_4, b_6, b_7, b_9\}$.



g2
r3
g1*
b6
g4
g5
b7
r9
g8
1V
1Ω
(a)
(b)

Fig. 1



1V
g*
1Ω
v
i
−
+
𝒩1
(a)
(b)
𝒩1'
𝒩1''
𝒩2
𝒩2''
n1+
nk+
n1−
nl−
n
(c)
(d)

Fig. 2

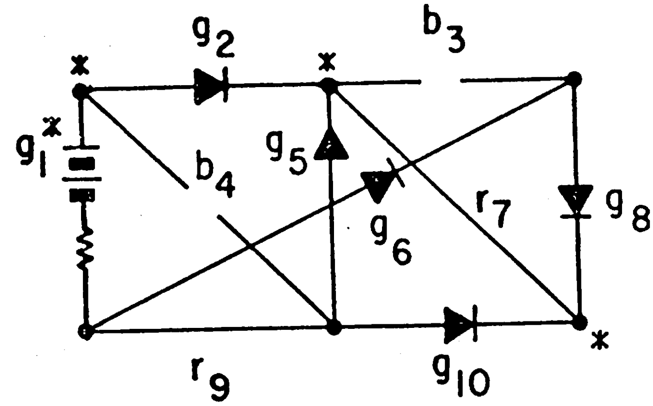


Fig. 3

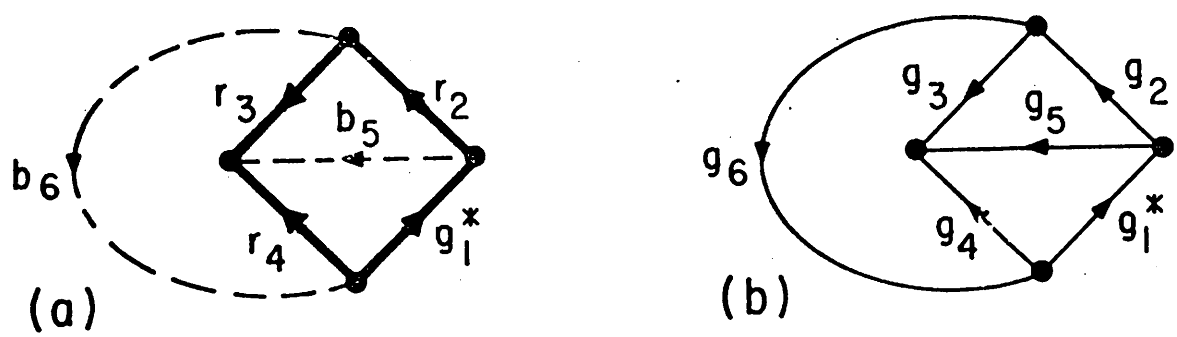


Fig. 4

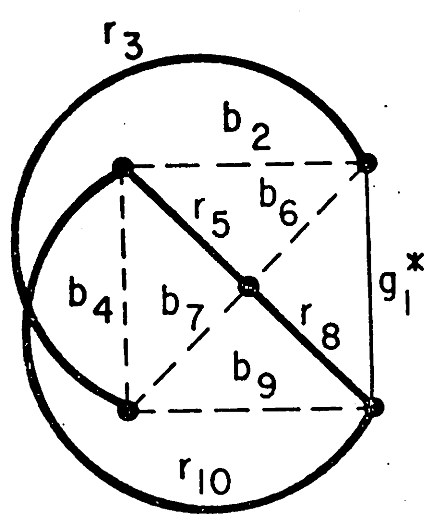


Fig. 5