Copyright © 1978, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

SYNTHESIS OF RECIPROCAL PIECEWISE-LINEAR N-PORT RESISTORS

by

Leon O. Chua and David J. Curtin

Memorandum No. UCB/ERL M78/84

20 December 1978

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

SYNTHESIS OF RECIPROCAL PIECEWISE-LINEAR N-PORT RESISTORS

Leon O. Chua and David J. Curtin

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Every reciprocal n-port resistor represented by a continuous n-dimensional piecewise-linear function is shown to be realizable by a circuit containing only 2-terminal piecewise-linear resistors and a (p+q)-port transformer. An explicit circuit realization is given along with illustrative examples. The necessary and sufficient conditions under which this realization contains only passive elements are also given.

Research sponsored by the Office of Naval Research Contract N00014-76-C-0572.

I. INTRODUCTION

The problem of synthesizing a dynamic nonlinear RLC n-port can be easily reduced to that of synthesizing a resistive nonlinear n-port [1]. Using the decomposition technique given recently in [2], this resistive nonlinear n-port can in turn be realized using only reciprocal nonlinear n-port resistors and a non-reciprocal linear n-port resistor. Since the latter can be realized by well-known techniques [3], the most fundamental problem that remains to be solved in the area of nonlinear network synthesis is that of realizing reciprocal nonlinear n-port resistors using only "2-terminal nonlinear resistors" and "ideal transformers." The solution of this basic synthesis problem will have far reaching significance not only for nonlinear network theory [1], but also for device modeling. Observe that our building blocks exclude controlled sources since we are concerned only with reciprocal n-ports here. On the other hand, it is essential to include the ideal transformer as a building block because the class of reciprocal nonlinear n-port resistors which can be realized using only 2-terminal nonlinear resistors is extremely small. To appreciate how restrictive this class is, one only needs to observe that the class of nonlinear 2-ports that admits a Δ-Y equivalent transformation is almost nil [4]! Consequently, any general reciprocal nonlinear network synthesis technique must allow at least one reciprocal linear n-port, such as the ideal transformer, as a building block.

The above basic synthesis problem is known to be rather formidable and would probably require many years of concentrated research before a completely general synthesis technique could be developed. This paper presents a solution to this problem for a subclass of nonlinear n-ports characterized by multidimensional piecewise-linear functions which are affine over convex polyhedral regions bounded by linear partitions [5]. This subclass is important in view of the following two observations. First, every n-port made up of 2-terminal piecewise-linear resistors and ideal transformers is described by such a multidimensional piecewise-linear function [6]. Second, the most practical method for building a 2-terminal nonlinear resistor is to approximate the resistor's i-v curve by means of a piecewise-linear function [7]. Therefore, it is desirable in practice to have an n-port specification represented by a multi-dimensional piecewise-linear function as in [5].

For the sake of clarity, the following theorems are stated and proved first for two-ports, followed by a generlization to n-ports essentially using the same techniques of proof. In Section II, the necessary and sufficient conditions on the "coefficients" defining a multi-dimensional piecewise-linear n-port resistor N are given for N to be reciprocal. This reciprocity criterion is then used to develop a synthesis technique using only 2-terminal piecewise-linear resistors and a (p+q)-port transformer in Section III. The necessary and sufficient conditions under which this realization is also passive are derived in Section IV.

II. RECIPROCITY CONDITIONS

For simplicity, we begin by considering a 2-port voltage-controlled resistor described by 2-dimensional continuous piecewise-linear functions which are affine over convex polyhedral regions bounded by linear partitions. Such a 2-port may be described, as shown in [5], by:

$$\boxed{\begin{aligned} i_1 &= a_1 + b_{1_1} v_1 + b_{1_2} v_2 + \sum_{k=1}^{m_1} g_{1k} |\alpha_{1k_1} v_1 + \alpha_{1k_2} v_2 - \beta_{1k}| \\ i_2 &= a_2 + b_{2_1} v_1 + b_{2_2} v_2 + \sum_{k=1}^{m_2} g_{2k} |\alpha_{2k_1} v_1 + \alpha_{2k_2} v_2 - \beta_{2k}| \end{aligned}} \tag{1}$$

A voltage-controlled 2-port resistor is reciprocal if, and only if, its incremental conductance matrix is symmetric at all operating points [1]. Consequently, our 2-port will be reciprocal if, and only if:

$$\frac{\partial i_1}{\partial v_2} = \frac{\partial i_2}{\partial v_1} \tag{2}$$

for all v_1, v_2 except at those isolated points for which the incremental conductance matrix is undefined due to the absolute value signs in (1); i.e., at the breakpoints. Substitution into (2) yields:

$$\begin{aligned} & b_{1_2} + \sum_{k=1}^{m_1} \left\{ g_{1k}\alpha_{1k_2} \left[\operatorname{sgn}(\alpha_{1k_1} v_1 + \alpha_{1k_2} v_2 - \beta_{1k}) \right] \right\} \\ &\quad = b_{2_1} + \sum_{k=1}^{m_2} \left\{ g_{2k}\alpha_{2k_1} \left[\operatorname{sgn}(\alpha_{2k_1} v_1 + \alpha_{2k_2} v_2 - \beta_{2k}) \right] \right\} \end{aligned} \tag{3}$$

where

$$\operatorname{sgn}(x) \triangleq \begin{cases} +1, & x > 0 \\ \text{undefined}, & x = 0 \\ -1, & x < 0 \end{cases} \tag{4}$$

Theorem 1. Two-port reciprocity criteria.

A piecewise-linear 2-port described by (1) is reciprocal if, and only if, its constitutive relation can be rewritten such that:

a) $b_{1_2} = b_{2_1}$ (5)

b) $m_1 = m_2$ (6)

c) $\left.\begin{aligned} \alpha_{1k_1} &= \alpha_{2k_1} \qquad k = 1,2,\ldots,m_1 \\ \alpha_{1k_2} &= \alpha_{2k_2} \qquad k = 1,2,\ldots,m_1 \end{aligned}\right\}$ (7)

$\beta_{1k} = \beta_{2k} \qquad k = 1,2,\ldots,m_1$ (8)

d) $g_{1k}\alpha_{1k_2} = g_{2k}\alpha_{2k_1} \quad k = 1,2,\ldots,m_1$ (9)

Proof.

This result follows directly from the fact that a 2-port described by (1) is reciprocal if, and only if, (3) is satisfied for all v_1, v_2 except at those points for which the sgn function is undefined. ¤

Example 1.

Consider a 2-port described by:

$$\left.\begin{aligned} i_1 &= 13 + 6v_1 + 4v_2 + 2|6v_1 - 18v_2 + 22| + 6|14v_1 + 7v_2| - 17|v_1 - 4| \\ i_2 &= 22 + 4v_1 + 21|2v_1 + v_2| - 12|3v_1 - 9v_2 + 11| + 18|8v_2 + 6| \end{aligned}\right\} \quad (10)$$

To check the conditions in Theorem 1, we rewrite (10) as

$$\left.\begin{aligned} i_1 &= 13 + 6v_1 + 4v_2 + 4|3v_1 - 9v_2 + 11| + 6|14v_1 + 7v_2 - 0| \\ &\qquad - 17|v_1 + 0v_2 - 4| + 0|0v_1 + 8v_2 + 6| \\ i_2 &= 22 + 4v_1 + 0v_2 - 12|3v_1 - 9v_2 + 11| + 3|14v_1 + 7v_2 - 0| \\ &\qquad + 0|v_1 + 0v_2 - 4| + 18|0v_1 + 8v_2 + 6| \end{aligned}\right\} \quad (11)$$

Clearly (5)-(9) are all satisfied, indicating that the 2-port described by (10) is reciprocal.

Let us now expand our results to the n-port case. Such an n-port may be described, as shown in [5], by:

$$\boxed{\begin{aligned} i_1 &= a_1 + b_{1_1} v_1 + \ldots + b_{1_n} v_n + \sum_{k=1}^{m_1} g_{1k} \left| \alpha_{1k_1} v_1 + \ldots + \alpha_{1k_n} v_n - \beta_{1k} \right| \\ &\vdots \\ i_n &= a_n + b_{n_1} v_1 + \ldots + b_{n_n} v_n + \sum_{k=1}^{m_n} g_{nk} \left| \alpha_{nk_1} v_1 + \ldots + \alpha_{nk_n} v_n - \beta_{nk} \right| \end{aligned}} \tag{12}$$

A voltage-controlled n-port is reciprocal if, and only if, its incremental conductance matrix is symmetric at all operating points. Consequently, our n-port will be reciprocal if, and only if:

$$\frac{\partial i_j}{\partial v_\ell} = \frac{\partial i_\ell}{\partial v_j} \tag{13}$$

for all $j,\ell \in \{1,2,\ldots,n\}$ and for all $v_1,\ldots,v_n$ except at those isolated points for which the incremental conductance matrix is undefined due to the absolute value signs in (12); i.e., at the breakpoints. Substitution into (13) yields:

$$\begin{aligned} & b_{j_\ell} + \sum_{k=1}^{m_j} \left\{ g_{jk}\alpha_{jk_\ell} \left[\operatorname{sgn}(\alpha_{jk_1} v_1 + \ldots + \alpha_{jk_n} v_n - \beta_{jk}) \right] \right\} \\ & \quad = b_{\ell_j} + \sum_{k=1}^{m_\ell} \left\{ g_{\ell k}\alpha_{\ell k_j} \left[\operatorname{sgn}(\alpha_{\ell k_1} v_1 + \ldots + \alpha_{\ell k_n} v_n - \beta_{\ell k}) \right] \right\} \end{aligned} \tag{14}$$

for all $j,\ell \in \{1,2,\ldots,n\}$, where $\operatorname{sgn}(x)$ is defined as in (4).

<u>Theorem 2</u>. <u>n-port reciprocity criteria</u>

An n-port described by (12) is <u>reciprocal</u> if, and only if, its constitutive relation may be rewritten such that:

a) $b_{j_\ell} = b_{\ell_j}$ (15)

b) $m_j = m_\ell$ (16)

c) $$\left.\begin{aligned} \alpha_{jk_1} &= \alpha_{\ell k_1} \qquad k = 1,2,\ldots,m_j \\ &\vdots \qquad\qquad\quad \vdots \\ \alpha_{jk_n} &= \alpha_{\ell k_n} \qquad k = 1,2,\ldots,m_j \end{aligned}\right\} \tag{17}$$

$$\beta_{jk} = \beta_{\ell k} \qquad k = 1,2,\ldots,m_j \tag{18}$$

d) $g_{jk}\alpha_{jk_\ell} = g_{\ell k}\alpha_{\ell k_j}$ (19)

for all $j,\ell \in \{1,2,\ldots,n\}$.

<u>Proof</u>.

This theorem is a direct extension of Theorem 1 and follows directly from the fact than an n-port described by (12) is reciprocal if, and only if, (14) is satisfied for all $v_1,\dots,v_n$ except at those points for which the sgn function is undefined. ¤

Example 2.

Consider a 4-port described by:

$$\left.\begin{aligned}
i_1 &= 1 + v_1 + v_2 - \tfrac{3}{2}v_3 + \tfrac{1}{2}v_4 - 9|8v_1 - 5| + 2|-v_1 + 13v_3| \\
&\quad - 5|5v_1 - 7v_2 + 8v_4 + 6| + 3|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| \\
i_2 &= v_1 + v_2 + \tfrac{1}{2}v_3 - \tfrac{3}{2}v_4 + 8|12v_2 - 6v_4 + 16| + 7|5v_1 - 7v_2 + 8v_4 \\
&\quad + 6| - 22|11v_2 - 3v_3 + 5v_4 - 8| - 6|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| \\
i_3 &= -4 - \tfrac{3}{2}v_1 + \tfrac{1}{2}v_2 + v_3 + v_4 + 7|9v_3| - 26|-v_1 + 13v_3| \\
&\quad + 6|11v_2 - 3v_3 + 5v_4 - 8| + 6|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| \\
i_4 &= 15 + \tfrac{1}{2}v_1 - \tfrac{3}{2}v_2 + v_3 + v_4 - 4|12v_2 - 6v_4 + 16| - 8|5v_1 - 7v_2 + 8v_4 \\
&\quad + 6| - 10|11v_2 - 3v_3 + 5v_4 - 8| + 9|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1|
\end{aligned}\right\} \quad (20)$$

To check the conditions in Theorem 2, we rewrite (20) as:

$$\left.\begin{aligned}
i_1 &= 1 + v_1 + v_2 - \tfrac{3}{2}v_3 + \tfrac{1}{2}v_4 - 9|8v_1 + 0v_2 + 0v_3 + 0v_4 - 5| \\
&\quad + 2|-v_1 + 0v_2 + 13v_3 + 0v_4| - 5|5v_1 - 7v_2 + 0v_3 + 8v_4 + 6| \\
&\quad + 3|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| + 0|0v_1 + 12v_2 + 0v_3 - 6v_4 + 16| \\
&\quad + 0|0v_1 + 11v_2 - 3v_3 + 5v_4 - 8| + 0|0v_1 + 0v_2 + 9v_3 + 0v_4 - 0| \\
i_2 &= 0 + v_1 + v_2 + \tfrac{1}{2}v_3 - \tfrac{3}{2}v_4 + 0|8v_1 + 0v_2 + 0v_3 + 0v_4 - 5| \\
&\quad + 0|-v_1 + 0v_2 + 13v_3 + 0v_4 - 0| + 7|5v_1 - 7v_2 + 0v_3 + 8v_4 + 6| \\
&\quad - 6|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| + 8|0v_1 + 12v_2 + 0v_3 - 6v_4 + 16| \\
&\quad - 22|0v_1 + 11v_2 - 3v_3 + 5v_4 - 8| + 0|0v_1 + 0v_2 + 9v_3 + 0v_4 - 0| \\
i_3 &= -4 - \tfrac{3}{2}v_1 + \tfrac{1}{2}v_2 + v_3 + v_4 + 0|8v_1 + 0v_2 + 0v_3 + 0v_4 - 5| \\
&\quad - 26|-v_1 + 0v_2 + 13v_3 + 0v_4 - 0| + 0|5v_1 - 7v_2 + 0v_3 + 8v_4 + 6| \\
&\quad + 6|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| + 0|0v_1 + 12v_2 + 0v_3 - 6v_4 + 16| \\
&\quad + 6|0v_1 + 11v_2 - 3v_3 + 5v_4 - 8| + 7|0v_1 + 0v_2 + 9v_3 + 0v_4 - 0| \\
i_4 &= 15 + \tfrac{1}{2}v_1 - \tfrac{3}{2}v_2 + v_3 + v_4 + 0|8v_1 + 0v_2 + 0v_3 + 0v_4 - 5| \\
&\quad + 0|-v_1 + 0v_2 + 13v_3 + 0v_4 - 0| - 8|5v_1 - 7v_2 + 0v_3 + 8v_4 + 6| \\
&\quad + 9|2v_1 - 4v_2 + 4v_3 + 6v_4 - 1| - 4|0v_1 + 12v_2 + 0v_3 - 6v_4 + 16| \\
&\quad - 10|0v_1 + 11v_2 - 3v_3 + 5v_4 - 8| + 0|0v_1 + 0v_2 + 9v_3 + 0v_4 - 0|
\end{aligned}\right\} \quad (21)$$

It is easily verified that (15)-(19) are satisfied for all combinations of $j,\ell \in \{1,2,\ldots,n\}$, indicating that the 4-port described by (20) is reciprocal.

Remark.

For the n-port case, we have continually stated "for all $j,\ell \in \{1,2,\ldots,n\}$." However, if we check, say, $j = 1$, $\ell = 2$, then we needn't check $j = 2$, $\ell = 1$, since all this does is reverse the equality in (13). Also, when $j = \ell$, (13) is obviously always satisfied. Hence, we could instead write "for all $j,\ell \in \{1,2,\ldots,n\}$, where $\ell > j$."

III. SYNTHESIS TECHNIQUE FOR RECIPROCAL N-PORTS

As suggested in [1], it is reasonable to assume that the synthesis of reciprocal n-ports will, in general, require the use of ideal transformers. Consequently, during the development of a circuit topology which could be used to synthesize any 2-port satisfying the conditions of Theorem 1 (resp., n-port satisfying the conditions of Theorem 2), a (p+q)-port transformer [1,3] was introduced into the synthesis, where q = 2 (resp., q = n). Connected to this transformer are "p" 2-terminal continuous piecewise-linear resistors, the 1-dimensional analogs to our 2-ports described by (1) (resp., n-ports described by (12)). To simplify our work, we further constrained these resistors to contain only 2 segments in their respective i-v characteristic curves.[1] Such resistors can be described by [8]:

$$R_{a_r} : i_{a_r} = a_{a_r} + b_{a_r} v_{a_r} + g_{a_r} |v_{a_r} - \beta_{a_r}| \ , \ r \in \{1,2,\ldots,p\} \tag{22}$$

For the 2-port case, the following theorem results.

Theorem 3. Synthesis of reciprocal two-ports.[2]

A reciprocal 2-port described by (1) can be synthesized exactly using a (p+q)-port transformer and p resistors described by (22), where p = m+3 and q = 2.

Proof.

Consider the configuration depicted in Fig. 1. For the (p+q)-port transformer we have:

---

[1] In fact, it can be shown that resistors with no more than 2 segments must be used, in general, if an exact synthesis is to be performed. Due to its nontrivial nature, however, the proof of this statement is not included since all we desire to prove is that we can indeed synthesize reciprocal n-ports described by (12).

[2] $m = m_1 = m_2$ when the equations describing the 2-port are rewritten so as to satisfy the conditions of Theorem 1.

KVL:
$$\begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_p} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{21} \\ k_{12} & k_{22} \\ \vdots & \vdots \\ k_{1p} & k_{2p} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{23}$$

KCL:
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & \cdots & k_{2p} \end{bmatrix} \begin{bmatrix} i_{a_1} \\ i_{a_2} \\ \vdots \\ i_{a_p} \end{bmatrix} \tag{24}$$

where $k_{j\ell}$ denotes the turns ratio of the $\underline{\ell}$th winding of the $\underline{j}$th transformer.

Choose the coefficients of the p resistors described by (22) and the p×q transformer turns ratios as follows:

$$a_{a_2} = a_{a_3} = \cdots = a_{a_p} = 0 \tag{25}$$

a_{a_1}, k_{11}, and k_{21} so as to satisfy:[3]

$$k_{11}a_{a_1} = a_1 \qquad k_{21}a_{a_1} = a_2 \tag{26}$$

$$b_{a_1} = b_{a_4} = b_{a_5} = \cdots = b_{a_p} = 0 \tag{27}$$

b_{a_2}, b_{a_3}, k_{12}, k_{13}, k_{22}, k_{23}, so as to satisfy:[4]

$$\left.\begin{aligned} &b_{a_2}(k_{12})^2 + b_{a_3}(k_{13})^2 = b_{1_1} \\ &b_{a_2}k_{12}k_{22} + b_{a_3}k_{13}k_{23} = b_{1_2} = b_{2_1} \\ &b_{a_2}(k_{22})^2 + b_{a_3}(k_{23})^2 = b_{2_2} \end{aligned}\right\} \tag{28}$$

$$g_{a_1} = g_{a_2} = g_{a_3} = 0 \tag{29}$$

[3]For example, $a_{a_1} = 1$, $k_{11} = a_1$, and $k_{21} = a_2$ will always work.

[4]These variables may always be chosen as shown in Appendix 2 for n = 2.

$$g_{a_{r+3}} = \frac{g_{1r}}{\alpha_{1r_1}} = \frac{g_{2r}}{\alpha_{2r_2}} \qquad r = 1,2,\ldots,m \tag{30}$$

$$\beta_{a_{r+3}} = \beta_{1r} = \beta_{2r} \qquad r = 1,2,\ldots,m \tag{31}$$

$$k_{1,r+3} = \alpha_{1r_1} = \alpha_{2r_1} \qquad r = 1,2,\ldots,m \tag{32}$$

$$k_{2,r+3} = \alpha_{1r_2} = \alpha_{2r_2} \qquad r = 1,2,\ldots,m \tag{33}$$

Observe, from Theorem 1, that whenever α_{1r_1} (resp., α_{2r_2}) equals zero, g_{1r} (resp., g_{2r}) will also equal zero, thus yielding zero over zero in (30) which is simply undefined.

Substituting (25), (27), and (29)-(31), along with footnotes (3) and (4), into (22) yields:

$$\left.\begin{aligned}
&R_{a_1} : i_{a_1} = 1 \\
&R_{a_2} : i_{a_2} = \lambda_1 v_{a_2} \\
&R_{a_3} : i_{a_3} = \lambda_2 v_{a_3} \\
&R_{a_{r+3}} : i_{a_{r+3}} = (g_{1r}/\alpha_{1r_1})|v_{a_{r+3}} - \beta_{1r}| \qquad r = 1,2,\ldots,m
\end{aligned}\right\} \tag{34}$$

Substituting (23) into (34), noting the turns ratios given in footnotes 3 and 4 along with (32) and (33), and then substituting the resulting equations into (24) yields the general form of a reciprocal 2-port described by (1). ¤

<u>Remark</u>.

Observe that the first resistor R_{a_1} in (34) is simply a dc current source, whereas R_{a_2}, R_{a_3}, are linear resistors.

[5]Recall that dc current sources and linear resistors are just special cases of <u>nonlinear</u> resistors.

Corollary.

The synthesis suggested in Theorem 3 can be performed by choosing the coefficients of the p resistors and (p+q)-port transformer's turns ratios as given in (25)-(33).

Proof.

This result follows immediately from the Proof of Theorem 3. ⌑

Example 3.

Recall the 2-port given in Example 1; i.e., equation (11). Following the synthesis technique of the above Corollary we obtain the circuit given in Fig. 2, where:

$$\left.\begin{aligned}
R_{a_1} &: i_{a_1} = 1 \\
R_{a_2} &: i_{a_2} = 8v_{a_2} \\
R_{a_3} &: i_{a_3} = -2v_{a_3} \\
R_{a_4} &: i_{a_4} = (4/3)\,|v_{a_4} + 11| \\
R_{a_5} &: i_{a_5} = (3/7)\,|v_{a_5}| \\
R_{a_6} &: i_{a_6} = -17\,|v_{a_6} - 4| \\
R_{a_7} &: i_{a_7} = (9/4)\,|v_{a_7} + 6|
\end{aligned}\right\} \quad (35)$$

and where $k_{11} = 13$, $k_{21} = 22$, $k_{12} = 2/\sqrt{5}$, $k_{22} = 1/\sqrt{5}$, $k_{13} = 1/\sqrt{5}$, $k_{23} = -2/\sqrt{5}$, $k_{14} = 3$, $k_{24} = -9$, $k_{15} = 14$, $k_{25} = 7$, $k_{16} = 1$, $k_{26} =$ [illegible], $k_{17} = 0$, $k_{27} = 8$.

It is easily verified that this synthesis does indeed produce a 2-port described by (11).

The results presented in Theorem 3 are expanded to the n-port case in the following theorem.

Theorem 4. Synthesis of reciprocal n-ports[6]

A reciprocal n-port described by (12) can be synthesized exactly using a (p+q)-port transformer and p resistors described by (22), where $p = m + n + 1$ and $q = n$.

[6] $m = m_1 = m_2 = \ldots = m_n$ when the equations describing the n-port are rewritten so as to satisfy the conditions of Theorem 2.

Due to its length, the Proof of Theorem 4 is given in Appendix 1.

Corollary.

The synthesis suggested in Theorem 4 can be performed by choosing the coefficients of the p resistors and the (p+q)-port transformer's turns ratios as given in (A.3)-(A.10).

Proof.

This result follows immediately from the Proof of Theorem 4. ⌑

Example 4.

Recall the 4-port given in Example 2; i.e., equation (21). Following the synthesis technique of the above Corollary we obtain the circuit given in Fig. 4 where:

$$\left.\begin{aligned}
R_{a_1} &: i_{a_1} = 1\\
R_{a_2} &: i_{a_2} = v_{a_2}\\
R_{a_3} &: i_{a_3} = 3v_{a_3}\\
R_{a_4} &: i_{a_4} = -2v_{a_4}\\
R_{a_5} &: i_{a_5} = 2v_{a_5}\\
R_{a_6} &: i_{a_6} = -(9/8)|v_{a_6} -5|\\
R_{a_7} &: i_{a_7} = -2|v_{a_7}|\\
R_{a_8} &: i_{a_8} = -1|v_{a_8} +6|\\
R_{a_9} &: i_{a_9} = (3/2)|v_{a_9} -1|\\
R_{a_{10}} &: i_{a_{10}} = (2/3)|v_{a_{10}} +16|\\
R_{a_{11}} &: i_{a_{11}} = -2|v_{a_{11}} -8|\\
R_{a_{12}} &: i_{a_{12}} = (7/9)|v_{a_{12}}|
\end{aligned}\right\} \quad (36)$$

and where $k_{11} = 1$, $k_{21} = 0$, $k_{31} = -4$, $k_{41} = 15$, $k_{12} = \frac{1}{2}$, $k_{22} = \frac{1}{2}$, $k_{32} = \frac{1}{2}$, $k_{42} = \frac{1}{2}$, $k_{13} = \frac{1}{2}$, $k_{23} = \frac{1}{2}$, $k_{33} = -\frac{1}{2}$, $k_{43} = -\frac{1}{2}$, $k_{14} = \frac{1}{2}$, $k_{24} = -\frac{1}{2}$, $k_{34} = \frac{1}{2}$, $k_{44} = -\frac{1}{2}$, $k_{15} = \frac{1}{2}$, $k_{25} = -\frac{1}{2}$, $k_{35} = -\frac{1}{2}$, $k_{45} = \frac{1}{2}$, $k_{16} = 8$, $k_{26} = 0$, $k_{36} = 0$, $k_{46} = 0$, $k_{17} = -1$, $k_{27} = 0$, $k_{37} = 13$, $k_{47} = 0$, $k_{18} = 5$, $k_{28} = -7$, $k_{38} = 0$, $k_{48} = 8$, $k_{14} = 2$, $k_{29} = -4$, $k_{39} = 4$, $k_{49} = 6$, $k_{1,10} = 0$, $k_{2,10} = 12$, $k_{3,10} = 0$, $k_{4,10} = -6$, $k_{1,11} = 0$, $k_{2,11} = 11$, $k_{3,11} = -3$, $k_{4,11} = 5$, $k_{1,12} = 0$, $k_{2,12} = 0$, $k_{3,12} = 9$, $k_{4,12} = 0$.

It is easily verified that this synthesis does indeed produce a 4-port described by (21).

Remark.

Notice that the synthesis of reciprocal n-ports presented does not actually use p nonlinear resistors described by (22), but instead uses 1 dc current source, n linear resistors, and only m nonlinear resistors described by (22).

The specifications in both Examples 3 and 4 are active, thereby requiring at least one active component in the circuit realization; namely, the dc current source. If the specifications were passive, however, it would be desirable that only passive components be used. Our objective in the next section is to investigate the conditions under which this property is satisfied.

IV. PASSIVITY CONDITIONS

We begin our discussion by determining the criteria under which a one-port (i.e., two-terminal) continuous piecewise-linear voltage-controlled resistor is passive. Since our synthesis technique in the preceding section makes use of only 2-terminal resistors with a maximum of two-segments in their respective i-v curves, let us derive first the passivity conditions for a piecewise-linear resistor described by (22). Now a two-terminal resistor is passive if, and only if, its i-v curve lies exclusively in the first and third quadrants [1]. This directly leads to the following theorem.

Theorem 5. One-port passivity criteria

A two-terminal continuous piecewise-linear voltage-controlled resistor described by (22) is passive if, and only if:

a) $a_{a_r} = -g_{a_r} |-\beta_{a_r}|$ (37)

b) $0 \leq b_{a_r} + g_{a_r} < +\infty$ (38)

c) $0 \leq b_{a_r} - g_{a_r} < +\infty$ (39)

Proof.

From [8] we have that

$$b_{a_r} = \frac{1}{2}(m_{0_r} + m_{1_r}) \tag{40}$$

$$g_{a_r} = \frac{1}{2}(m_{1_r} - m_{0_r}) \tag{41}$$

$$a_{a_r} = i_{a_r}(0) - g_{a_r}|\beta_{a_r}| \tag{42}$$

where m_{0_r} and m_{1_r} are the slopes of the first and last segments of the i-v curve of R_{a_r}, respectively.

(a) Necessity. The i-v curve of a passive resistor must pass through the origin, i.e., $i_{a_r}(0) = 0$. Substituting this into (22) yields:

$$0 = a_{a_r} + g_{a_r}|-\beta_{a_r}| \tag{43}$$

which is simply (37). Furthermore, it is clear that if the i-v curve is to be restricted to the first and third quadrants only, then we must satisfy:

$$0 \leq m_{0_r} < +\infty \tag{44}$$

$$0 \leq m_{1_r} < +\infty \tag{45}$$

From (40) and (41) we find that:

$$m_{0_r} = b_{a_r} - g_{a_r} \tag{46}$$

$$m_{1_r} = g_{a_r} + b_{a_r} \tag{47}$$

Substituting (46) and (47) into (44) and (45), respectively, yields (39) and (38), respectively.

(b) Sufficiency. Substitution of (37) into (22), and evaluating this expression at $v_{a_r} = 0$, yields:

$$i_{a_r}(0) = 0 \tag{48}$$

Substituting (38) and (39) into (47) and (46), respectively, yields (45) and (44), respectively. Thus, our resistor's i-v curve lies exclusively in the first and third quadrants only (and passes through the origin) implying that it is passive.

¤

Our next step is to expand our results to the case of reciprocal two-ports described by (1). Such a two-port must satisfy the conditions of Theorem 1 for it to be reciprocal. Now an algebraic n-port resistor is passive if, and only if, $\langle \underset{\sim}{v}, \underset{\sim}{i} \rangle \geq 0$ for all $(\underset{\sim}{v}, \underset{\sim}{i})$ satisfying the constitutive relations of the resistor [1]. Hence, for the two-port case we have:

$$v_1 i_1 + v_2 i_2 \geq 0 \tag{49}$$

for all v_1, v_2, i_1, i_2 satisfying the constitutive relation of the two-port resistor. Substituting (1) into (49) yields a complex expression from which it is difficult to extract the necessary and sufficient conditions under which the inequality is satisfied. Consequently, an alternate approach is used.

If our reciprocal two-port can be synthesized using <u>only</u> passive elements, then clearly it too must be passive. Connecting p resistors satisfying Theorem 5 to a (p+q)-port transformer (q=2), as in Fig. 1, and examining the resulting equations for the two-port yields the following theorem.

<u>Theorem 6</u>. <u>Reciprocal passive two-port synthesis criteria</u>.

A <u>reciprocal</u> continuous piecewise-linear voltage-controlled two-port resistor described by (1) (i.e., a two-port resistor satisfying Theorem 1) may be synthesized using <u>only</u> passive resistors described by (22) and a (p+q)-port transformer in the configuration given in Fig. 1 if, and only if, i_1 and i_2 can be recast in the following manner:

$$\left.\begin{aligned}
i_1 &= \sum_{k=1}^{p} a_{1_k} + \sum_{k=1}^{p} b_{1_{1k}} v_1 + \sum_{k=1}^{p} b_{1_{2k}} v_2 \\
&\quad + \sum_{k=1}^{p} g_{1k} |\alpha_{k_1} v_1 + \alpha_{k_2} v_2 - \beta_k| \\
i_2 &= \sum_{k=1}^{p} a_{2_k} + \sum_{k=1}^{p} b_{2_{1k}} v_1 + \sum_{k=1}^{p} b_{2_{2k}} v_2 \\
&\quad + \sum_{k=1}^{p} g_{2k} |\alpha_{k_1} v_1 + \alpha_{k_2} v_2 - \beta_k|
\end{aligned}\right\} \tag{50}$$

where:[7]

$$\left.\begin{aligned} &\text{a)}\quad a_{1_k} = -g_{1k}|-\beta_k| \qquad k = 1,2,\ldots,p \\ &\phantom{\text{a)}}\quad a_{2_k} = -g_{2k}|-\beta_k| \qquad k = 1,2,\ldots,p \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} &\text{b)}\quad 0 \le b_{1_{1k}} + g_{1k}\alpha_{k_1} < +\infty \qquad k = 1,2,\ldots,p \\ &\phantom{\text{b)}}\quad 0 \le b_{2_{2k}} + g_{2k}\alpha_{k_2} < +\infty \qquad k = 1,2,\ldots,p \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} &\text{c)}\quad 0 \le b_{1_{1k}} - g_{1k}\alpha_{k_2} < +\infty \qquad k = 1,2,\ldots,p \\ &\phantom{\text{c)}}\quad 0 \le b_{2_{2k}} - g_{2k}\alpha_{k_2} < +\infty \qquad k = 1,2,\ldots,p \end{aligned}\right\} \quad (53)$$

$$\text{d)}\quad b_{1_{2k}} = b_{2_{1k}} = \alpha_{k_2}\frac{b_{1_{1k}}}{\alpha_{k_1}} = \alpha_{k_1}\frac{b_{2_{2k}}}{\alpha_{k_2}} \qquad k = 1,2,\ldots,p \qquad (54)$$

Proof.

(a) Sufficiency. We must show that if the conditions of Theorem 6 are met, then the reciprocal two-port described by (50) can be synthesized using only passive elements. Since our two-port is reciprocal, we know from Theorem 1 that:

$$g_{1k}\alpha_{k_2} = g_{2k}\alpha_{k_1} \qquad k = 1,2,\ldots,m \qquad (55)$$

where $m = m_1 = m_2$. Furthermore, since both (1) and (50) describe the same two-port, it is easily argued that one may always choose:

$$g_{1k} = g_{2k} = 0 \qquad k = m+1, m+2, \ldots., p \qquad (56)$$

without affecting the ability of the two-port to satisfy the conditions of Theorem 6. Consequently, we have:

$$g_{1k}\alpha_{k_2} = g_{2k}\alpha_{k_1} \qquad k = 1,2,\ldots,p \qquad (57)$$

The coefficients of the p resistors described by (22) are chosen as follows:

[7] Note that condition (d) can still be satisfied when α_{k_1} (resp., α_{k_2}) equals zero, if $b_{1_{1k}}$ (resp., $b_{2_{2k}}$) is also zero, thus yielding zero over zero which is simply undefined. If both α_{k_1} and α_{k_2} are zero, an unrealistic if not contrived situation, then $b_{1_{2k}} = b_{2_{1k}}$ must be equal to zero.

$$\boxed{\begin{aligned}
&\beta_{a_r} = \beta_k \qquad r = k = 1,2,\dots,p && (58)\\
&a_{a_r} = \frac{a_{1_k}}{\alpha_{k_1}} = \frac{a_{2_k}}{\alpha_{k_2}} \qquad r = k = 1,2,\dots,p && (59)\\
&b_{a_r} = \frac{b_{1_{1k}}}{(\alpha_{k_1})^2} = \frac{b_{2_{2k}}}{(\alpha_{k_2})^2} \qquad r = k = 1,2,\dots,p && (60)\\
&g_{a_r} = \frac{g_{1k}}{\alpha_{k_1}} = \frac{g_{2k}}{\alpha_{k_2}} \qquad r = k = 1,2,\dots,p && (61)
\end{aligned}}$$

As it initially appears that there may be some problems with zeros appearing in the denominators of (59)-(61), we now discuss these equations individually.

The second equality in (59) may be rewritten as:

$$a_{1_k}\alpha_{k_2} = a_{2_k}\alpha_{k_1} \qquad k = 1,2,\dots,p \tag{62}$$

Substituting (51) into (62) yields (57), when multiplied through by $-|-\beta_k|$, which we know always holds. In addition, whenever α_{k_1} (resp., α_{k_2}) is zero, a_{1_k} (resp., a_{2_k}) is also zero by (51), (56), and (57). Therefore, if a_{1_k}/α_{k_1} is zero over zero, then a_{a_k} is chosen to be the value of a_{2_k}/α_{k_2}, and vice-versa.[8]

The second equality in (60) always holds, since multiplying it through by $\alpha_{k_1}\alpha_{k_2}$ yields (54), which we have assumed is true. Footnote 7 explains why we will never have α_{k_1} (resp., α_{k_2}) equal to zero without $b_{1_{1k}}$ (resp., $b_{2_{2k}}$) also equal to zero. As before, if $b_{1_{1k}}/(\alpha_{k_1})^2$ is zero over zero, then b_{a_k} is chosen to be the value of $b_{2_{2k}}/(\alpha_{k_2})^2$, and vice-versa.[9]

Finally, the second equality in (61) will always hold in light of (57), while g_{1k} (resp., g_{2k}) will always be zero whenever α_{k_1} (resp., α_{k_2}) is zero for reasons previously discussed. Consequently, if g_{1k}/α_{k_1} is zero over zero, then g_{a_k} is chosen to be the value of g_{2k}/α_{k_2} and vice versa.[10]

The turns ratios of the (p+q)-port transformer (q=2) are chosen as follows:

---

[8] If both α_{k_1} and α_{k_2} are zero, an unrealistic if not contrived situation, then a_{a_k} must be set equal to zero.

[9] If both α_{k_1} and α_{k_2} are zero, then b_{a_k} must be chosen to be zero.

[10] If both α_{k_1} and α_{k_2} are zero, then g_{a_k} must be chosen to be zero.

$$\boxed{\begin{array}{ll} k_{1k} = \alpha_{k_1} & k = 1,2,\ldots,p \\ k_{2k} = \alpha_{k_2} & k = 1,2,\ldots,p \end{array}} \tag{63}$$

Substituting (63), (23), and (58)-(61) into (22) for $r = 1,2,\ldots,p$, and then substituting the resulting set of equations into (24), yields (50). Therefore, at least, our synthesis is correct. It remains to be shown that all p resistors are passive. Substituting (58), (59), and (61) into (37) yields (51), which we have assumed is true. Substituting (60) and (61) into both (38) and (39) yields (52) and (53), respectively, which we have also assumed is true. In conclusion, our p resistors all satisfy the conditions of Theorem 5 and are therefore passive.

(b) Necessity. Here we must show that if we connect any p passive resistors described by (22) to a (p+q)-port transformer (q=2) as in Fig. 1, then the conditions of Theorem 6 <u>can be</u> satisfied. It is easily verified that connecting such resistors and the transformer as in Fig. 1 yields a two-port

$$\left.\begin{aligned} i_1 &= \sum_{k=1}^{p} k_{1k} a_{a_k} + \sum_{k=1}^{p} b_{a_k} (k_{1k})^2\, v_1 + \sum_{k=1}^{p} b_{a_k} k_{1k} k_{2k}\, v_2 \\ &\quad + \sum_{k=1}^{p} k_{1k} g_{a_k} \left| k_{1k} v_1 + k_{2k} v_2 - \beta_{a_k} \right| \\ i_2 &= \sum_{k=1}^{p} k_{2k} a_{a_k} + \sum_{k=1}^{p} b_{a_k} k_{1k} k_{2k}\, v_1 + \sum_{k=1}^{p} b_{a_k} (k_{2k})^2\, v_2 \\ &\quad + \sum_{k=1}^{p} k_{2k} g_{a_k} \left| k_{1k} v_1 + k_{2k} v_2 - \beta_{a_k} \right| \end{aligned}\right\} \tag{64}$$

where (37)-(39) are satisfied for $r = k = 1,2,\ldots,p$. Observe that (64) is in exactly the same form as (50). Thus, we may write:

$$\left.\begin{aligned} a_{1_k} &= k_{1k} a_{a_k} \qquad k = 1,2,\ldots,p \\ a_{2_k} &= k_{2k} a_{a_k} \qquad k = 1,2,\ldots,p \end{aligned}\right\} \tag{65}$$

$$b_{1_{1k}} = b_{a_k} (k_{1k})^2 \qquad k = 1,2,\ldots,p \tag{66}$$

$$b_{1_{2k}} = b_{2_{1k}} = b_{a_k} k_{1k} k_{2k} \qquad k = 1,2,\ldots,p \tag{67}$$

$$b_{2_{2k}} = b_{a_k} (k_{2k})^2 \qquad k = 1,2,\ldots,p \tag{68}$$

$$g_{1k} = k_{1k} g_{a_k} \qquad k = 1,2,\ldots,p$$
$$g_{2k} = k_{2k} g_{ak} \qquad k = 1,2,\ldots,p \tag{69}$$

$$\alpha_{k_1} = k_{1k} \qquad k = 1,2,\ldots,p \tag{70}$$

$$\alpha_{k_2} = k_{1k} \qquad k = 1,2,\ldots,p \tag{71}$$

$$\beta_k = \beta_{a_k} \qquad k = 1,2,\ldots,p \tag{72}$$

Substituting (65), (69), and (72) into (51) yields (37), which we have assumed holds. Substituting (66) and (68)-(71) into (52) and (53) yields (38) and (39), respectively, which we have assumed are true. Notice that the inequality signs are unchanged since $(k_{1k})^2$ and $(k_{2k})^2$ are, of course, always positive. Finally, substituting (66)-(68) into (54) shows that (54) is clearly satisfied. Thus, the combination of p passive resistors described by (22) and a (p+q)-port transformer (q=2) as in Fig. 1 always yields a two-port whose constitutive relation can be recast such that (51)-(54) (i.e., the conditions of Theorem 6) are satisfied. ¤

The following corollaries result directly from Theorem 6 and its proof.

Corollary 1. Two-port passivity criteria.

A reciprocal continuous piecewise-linear voltage-controlled two-port resistor described by (1) is passive if the conditions of Theorem 6 are met.

Proof.

The combination of only passive elements results in another passive element. ¤

Corollary 2. Passive two-port synthesis technique.

The class of passive two-ports given in Corollary 1 to Theorem 6 may be synthesized using the circuit interconnection of Fig. 1 by choosing the coefficients of the p resistors as in (58)-(61) and the transformer's turns ratios as in (63).

Proof.

This corollary follows immediately from the "sufficiency" proof of Theorem 6. ¤

Remark.

In footnotes 7-10 we state that $\alpha_{k_1} = \alpha_{k_2} = 0$ for some k is an unnatural situation. This is so because if we substitute this into (1), then we obtain a term in the expression for i_1 (resp., i_2) which is just $g_{1k}|-\beta_k|$ (resp., $g_{2k}|-\beta_k|$). Such a term is simply a constant and would normally be included within the a_1 (resp., a_2) term in all but a contrived case.

To conclude this section, we extend Theorem 6 to reciprocal n-ports described by (12).

Theorem 7. Reciprocal passive n-port synthesis criteria.

A reciprocal continuous piecewise-linear voltage-controlled n-port resistor described by (12) (i.e., an n-port resistor satisfying Theorem 2) may be synthesized using only passive resistors described by (22) and a (p+q)-port transformer in the configuration given in Fig. 3 if, and only if, $i_1, i_2, \ldots, i_n$ can be recast as follows:

$$\left.\begin{aligned}
i_1 &= \sum_{k=1}^{p} a_{1_k} + \sum_{k=1}^{p} b_{1_{1k}} v_1 + \ldots + \sum_{k=1}^{p} b_{1_{nk}} v_n \\
&\quad + \sum_{k=1}^{p} g_{1k} |\alpha_{k_1} v_1 + \ldots + \alpha_{k_n} v_n - \beta_k| \\
&\;\vdots \\
i_n &= \sum_{k=1}^{p} a_{n_k} + \sum_{k=1}^{p} b_{n_{1k}} v_1 + \ldots + \sum_{k=1}^{p} b_{n_{nk}} v_n \\
&\quad + \sum_{k=1}^{p} g_{nk} |\alpha_{k_1} v_1 + \ldots + \alpha_{k_n} v_n - \beta_k|
\end{aligned}\right\} \quad (73)$$

where:[11]

$$\left.\begin{aligned}
\text{a)}\quad a_{1_k} &= -g_{1k}|-\beta_k| \quad k = 1,2,\ldots,p \\
&\;\vdots \\
a_{n_k} &= -g_{nk}|-\beta_k| \quad k = 1,2,\ldots,p
\end{aligned}\right\} \quad (74)$$

$$\left.\begin{aligned}
\text{b)}\quad 0 &\le b_{1_{1k}} + g_{1k}\alpha_{k_1} < +\infty \quad k = 1,2,\ldots,p \\
&\;\vdots \\
0 &\le b_{n_{nk}} + g_{nk}\alpha_{k_n} < +\infty \quad k = 1,2,\ldots,p
\end{aligned}\right\} \quad (75)$$

$$\left.\begin{aligned}
\text{c)}\quad 0 &\le b_{1_{1k}} - g_{1k}\alpha_{k_1} < +\infty \quad k = 1,2,\ldots,p \\
&\;\vdots \\
0 &\le b_{n_{nk}} - g_{nk}\alpha_{k_n} < +\infty \quad k = 1,2,\ldots,p
\end{aligned}\right\} \quad (76)$$

[11] Note that condition (d) can still be satisfied when α_{k_j} (resp., α_{k_ℓ}) equals zero, if $b_{j_{jk}}$ (resp., $b_{\ell_{\ell k}}$) is also zero, thus yielding zero over zero which is simply undefined. If both α_{k_j} and α_{k_ℓ} are zero, then $b_{j_{\ell k}} = b_{\ell_{jk}}$ must be equal to zero.

d) $b_{j_{\ell k}} = b_{j_{\ell k}} \dfrac{b_{j_{jk}}}{\alpha_{k_j}} = \alpha_{k_j} \dfrac{b_{\ell_{\ell k}}}{\alpha_{k_\ell}} \qquad k = 1,2,\ldots,p \qquad (77)$

for all $j,\ell \in \{1,2,\ldots,n\}$.[12]

Proof.

This proof is stepwise identical to the proof of Theorem 6, with the minor changes being obvious at each stage. ¤

Corollary 1. n-port passivity criteria.

A reciprocal continuous piecewise-linear voltage-controlled n-port resistor described by (12) is passive if the conditions of Theorem 7 are met.

Proof.

The combination of only passive elements results in another passive element. ¤

Corollary 2. Passive n-port synthesis technique.

The class of passive n-ports given in Corollary 1 to Theorem 7 may be synthesized using the circuit interconnection of Fig. 3 by choosing the coefficients of the p resistors as:

$$\boxed{\begin{array}{ll} \beta_{a_r} = \beta_k & r = k = 1,2,\ldots,p \\ a_{a_r} = \dfrac{a_{1_k}}{\alpha_{k_1}} = \ldots = \dfrac{a_{n_k}}{\alpha_{k_n}} & r = k = 1,2,\ldots,p \\ b_{a_r} = \dfrac{b_{1_{1k}}}{(\alpha_{k_1})^2} = \ldots = \dfrac{b_{n_{nk}}}{(\alpha_{k_n})^2} & r = k = 1,2,\ldots,p \\ g_{a_r} = \dfrac{g_{1k}}{\alpha_{k_1}} = \ldots = \dfrac{g_{nk}}{\alpha_{k_n}} & r = k = 1,2,\ldots,p \end{array}}$$

(78), (79), (80), (81)

and the transformer's turns ratios as:

[12] Recall the Remark following Example 2 which states that we may actually write "for all $j,\ell \in \{1,2,\ldots,n\}$, where $\ell > j$" to save computations when checking this condition.

$$\boxed{\begin{array}{ll} k_{1k} = \alpha_{k_1} & k = 1,2,\ldots,p \\ \vdots & \\ k_{nk} = \alpha_{k_n} & k = 1,2,\ldots,p \end{array}} \tag{82}$$

Proof.

This corollary follows immediately from the "sufficiency" proof of Theorem 7. ¤

Remark.

Mathematically speaking, note that conditions (b) and (c) of Theorem 7 could be combined and rewritten as:

$$\left.\begin{array}{ll} b_{1_{1k}} \geq |g_{1k}\alpha_{k_1}| & k = 1,2,\ldots,p \\ \vdots & \\ b_{n_{nk}} \geq |g_{nk}\alpha_{k_n}| & k = 1,2,\ldots,p \end{array}\right\} \tag{83}$$

Similar changes could also be made in Theorems 5 and 6. Implementing these changes, however, would unnecessarily make the associated proofs more complex.

Observe that any resistor described by (22) which satisfies the conditions of Theorem 5 is not only passive, but also locally passive since the resistor's constitutive relation contains only 2 segments in its i-v curve. Therefore, any n-port synthesized by using these type resistors and a (p+q)-port transformer as in Theorem 7 will be locally passive as well as passive.

An example demonstrating the use of Theorem 7 and its corollaries is included in Appendix 3 and Appendix 4, respectively.

V. CONCLUDING REMARKS

We have shown that every reciprocal n-port resistor represented by a continous multi-dimensional piecewise-linear function as in (12) can be realized by a circuit containing only 2-terminal continous piecewise-linear resistors and a (p+q)-port transformer. We have also derived the necessary and sufficient conditions for our n-port realization in Fig. 3 (where the resistors are all described by Eq.22) to be passive. Since we have as yet been unable to prove that these same conditions must also apply to all other circuit configurations corresponding to other methods of realization, our conditions so far are only sufficient for a reciprocal, passive, and locally passive n-port to be realizable using only passive and locally passive components. We have good reasons to conjecture, however, that these conditions are necessary as well.

From the <u>practical</u> synthesis point of view, the 2-terminal piecewise-linear resistors can be easily realized with high precision using the techniques developed in [7]. The (p+q)-port transformer, however, would be highly impractical if one insists on using an iron-core device, which is highly frequency dependent. However, a (p+q)-port transformer can itself be simulated by relatively inexpensive OP AMP circuits over a wide frequency and dynamic range.

APPENDIX

1. Proof of Theorem 4.

Consider the configuration shown in Fig. 3. For the (p+q)-port transformer we have:

KVL:
$$\begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_p} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & & \vdots \\ k_{1p} & k_{2p} & \cdots & k_{np} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{A.1}$$

KCL:
$$\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & \cdots & k_{2p} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{np} \end{bmatrix} \begin{bmatrix} i_{a_1} \\ i_{a_2} \\ \vdots \\ i_{a_p} \end{bmatrix} \tag{A.2}$$

where $k_{j\ell}$ denotes the turns ratio of the ℓth winding of the jth transformer.

Choose the coefficients of the p resistors described by (22) and the pxq transformer turns ratios as follows:

$$a_{a_2} = a_{a_3} = \ldots = a_{a_p} = 0 \tag{A.3}$$

$a_{a_1}, k_{11}, k_{21}, \ldots, k_{n1}$ so as to satisfy:[13]

$$k_{11}a_{a_1} = a_1,\ k_{21}a_{a_1} = a_2, \ldots, k_{n1}a_{a_1} = a_n \tag{A.4}$$

$$b_{a_1} = b_{a_{n+2}} = b_{a_{n+3}} = \ldots = b_{a_p} = 0 \tag{A.5}$$

$b_{a_2}, b_{a_3}, \ldots, b_{a_{n+1}},\ k_{12}, k_{13}, \ldots, k_{1,n+1}, k_{22}, k_{23}, \ldots, k_{2,n+1}, \ldots, k_{n2}, k_{n3}, \ldots, k_{n,n+1}$

[13]For example, $a_{a_1} = 1$, $k_{11} = a_1$, $k_{21} = a_2, \ldots, k_{n1} = a_n$ will always work.

so as to satisfy:[14]

$$\left.\begin{array}{l}
b_{a_2}(k_{12})^2 + b_{a_3}(k_{13})^2 + \ldots + b_{a_{n+1}}(k_{1,n+1})^2 = b_{1_1} \\
b_{a_2}k_{12}k_{22} + b_{a_3}k_{13}k_{23} + \ldots + b_{a_{n+1}}k_{1,n+1}k_{2,n+1} = b_{1_2} = b_{2_1} \\
\vdots \\
b_{a_2}k_{12}k_{n2} + b_{a_3}k_{13}k_{n3} + \ldots + b_{a_{n+1}}k_{1,n+1}k_{n,n+1} = b_{1_n} = b_{n_1} \\
b_{a_2}(k_{22})^2 + b_{a_3}(k_{23})^2 + \ldots + b_{a_{n+1}}(k_{2,n+1})^2 = b_{2_2} \\
b_{a_2}k_{22}k_{32} + b_{a_3}k_{23}k_{33} + \ldots + b_{a_{n+1}}k_{2,n+1}k_{3,n+1} = b_{2_3} = b_{3_2} \\
\vdots \\
b_{a_2}k_{22}k_{n2} + b_{a_3}k_{23}k_{n3} + \ldots + b_{a_{n+1}}k_{2,n+1}k_{n,n+1} = b_{2_n} = b_{n_2} \\
\vdots \\
b_{a_2}(k_{n2})^2 + b_{a_3}(k_{n3})^2 + \ldots + b_{a_{n+1}}(k_{n,n+1})^2 = b_{n_n}
\end{array}\right\} \quad (A.6)$$

$$g_{a_1} = g_{a_2} = \ldots = g_{a_{n+1}} = 0 \qquad (A.7)$$

$$g_{a_{r+n+1}} = \frac{g_{1r}}{\alpha_{1r_1}} = \frac{g_{2r}}{\alpha_{2r_2}} = \ldots = \frac{g_{nr}}{\alpha_{nr_n}} \qquad r = 1,2,\ldots,m \qquad (A.8)$$

$$\beta_{a_{r+n+1}} = \alpha_{1r} = \alpha_{2r} = \ldots = \alpha_{nr} \qquad r = 1,2,\ldots,m \qquad (A.9)$$

$$\begin{array}{ll}
k_{1,r+n+1} = \alpha_{1r_1} = \alpha_{2r_1} = \ldots = \alpha_{nr_1} & r = 1,2,\ldots,m \\
k_{2,r+n+1} = \alpha_{1r_2} = \alpha_{2r_2} = \ldots = \alpha_{nr_2} & r = 1,2,\ldots,m \\
\vdots & \\
k_{n,r+n+1} = \alpha_{1r_n} = \alpha_{2r_n} = \ldots = \alpha_{nr_n} & r = 1,2,\ldots,m
\end{array} \qquad (A.10)$$

Observe from Theorem 2 that whenever α_{1r_1} (resp., $\alpha_{2r_2},\alpha_{3r_3},\ldots,\alpha_{nr_n}$) equals zero, g_{1r} (resp., $g_{2r},g_{3r},\ldots,g_{nr}$) will also equal zero, thus yielding zero over zero for that particular term in (30), which is simply undefined.

[14] One possible solution to (A.6) is presented in Appendix 2.

Substituting (A.3), (A.5), and (A.7)-(A.10), along with footnote 13 and Appendix 2, into (22) yields:

$$R_{a_1} : i_{a_1} = 1$$

$$R_{a_{j+1}} : i_{a_{j+1}} = \lambda_j \; v_{a_{j+1}} \qquad j = 1,2,\ldots,n \tag{A.11}$$

$$R_{a_{r+n+1}} : i_{a_{r+n+1}} = (g_{1r}/\alpha_{1r_1})\,|v_{a_{r+n+1}} - \beta_{1r}| \quad r = 1,2,\ldots,m$$

Substituting (A.1) into (A.11), noting the turns ratios given in footnote 13, Appendix 2, and (A.10), and then substituting the resulting equations into (A.2) yields the general form of a reciprocal n-port described by (12). ¤

2. <u>One possible solution to (A.6)</u>.

Rewriting (A.6) in matrix form gives us:

$$\underbrace{\begin{bmatrix} k_{12} & k_{13} & \cdots & k_{1,n+1} \\ k_{22} & k_{23} & \cdots & k_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n2} & k_{n3} & \cdots & k_{n,n+1} \end{bmatrix}}_{\underset{\sim}{K}^T} \underbrace{\begin{bmatrix} b_{a_2} & 0 & \cdots & 0 \\ 0 & b_{a_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{a_{n+1}} \end{bmatrix}}_{\underset{\sim}{\Lambda}} \underbrace{\begin{bmatrix} k_{12} & k_{22} & \cdots & k_{n2} \\ k_{13} & k_{23} & \cdots & k_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n+1} & k_{2,n+1} & \cdots & k_{n,n+1} \end{bmatrix}}_{\underset{\sim}{K}}$$

$$= \underbrace{\begin{bmatrix} b_{1_1} & b_{1_2} & \cdots & b_{1_n} \\ b_{2_1} & b_{2_2} & \cdots & b_{2_n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_1} & b_{n_2} & \cdots & b_{n_n} \end{bmatrix}}_{\underset{\sim}{B}} \tag{A.12}$$

We know from (15) that $\underset{\sim}{B}$ is <u>symmetric</u> (and of course real). Therefore, $\underset{\sim}{B}$ is a <u>normal</u> matrix for which we can always find an <u>orthonormal</u> set of n <u>eigenvectors</u> [9]. Denoting these eigenvectors as $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2, \ldots \underset{\sim}{\xi}_n$, we may write:

$$\underset{\sim}{\xi}_j^T \, \underset{\sim}{\xi}_k = \delta_{jk} \qquad j,k \in \{1,2,\ldots,n\} \tag{A.13}$$

where:

$$\delta_{jk} \triangleq \begin{cases} 1 & ,j=k \\ 0 & ,j\neq k \end{cases} \tag{A.14}$$

Next, let us denote the eigenvalues of $\underset{\sim}{B}$ as $\lambda_1,\lambda_2,\ldots,\lambda_n$ such that eigenvector $\underset{\sim}{\xi}_\ell$ is associated with eigenvalue λ_ℓ, $\ell \in \{1,2,\ldots,n\}$.

$$\therefore \underset{\sim}{B}\underset{\sim}{\xi}_j = \lambda_j \underset{\sim}{\xi}_j \qquad j = 1,2,\ldots,n \tag{A.15}$$

We now write:

$$\left[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \left[\underset{\sim}{\xi}_1\lambda_1 \vdots \underset{\sim}{\xi}_2\lambda_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\lambda_n\right] \tag{A.16}$$

Substituting (A.15) into (A.16) yields:

$$\left[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \underset{\sim}{B}\left[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\right] \tag{A.17}$$

If we postmultiply both sides of (A.17) by $[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n]^T$ and observe (A.13) we get:

$$\left[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \left[\underset{\sim}{\xi}_1 \vdots \underset{\sim}{\xi}_2 \vdots \cdots \vdots \underset{\sim}{\xi}_n\right]^T = \underset{\sim}{B} \tag{A.18}$$

Comparing (A.18) and (A.12), we see that we may always choose the k's in $\underset{\sim}{K}^T$ and the b_a's in $\underset{\sim}{\Lambda}$ as follows:

$$[k_{1,j+1} \; k_{2,j+1} \; \cdots \; k_{n,j+1}]^T = \underset{\sim}{\xi}_j \qquad j = 1,2,\ldots,n \tag{A.19}$$

$$b_{a_{j+1}} = \lambda_j \qquad j = 1,2,\ldots,n \tag{A.20}$$

3. <u>An example illustrating the use of Theorem 7</u>.

Consider a 4-port, similar to the one in Example 2, described by:

$$\left.\begin{aligned}
i_1 &= 72 + 114v_1 - 74v_2 - 50v_3 + 55v_4 - 9|8v_1-5| \\
&\quad + 2|-v_1+13v_3| - 5|5v_1-7v_2+8v_4+6| \\
&\quad + 3|2v_1-4v_2+4v_3+6v_4-1| \\
i_2 &= 12 - 74v_1 + 492v_2 + 18v_3 - 21v_4 + 8|12v_2-6v_4+16| \\
&\quad + 7|5v_1-7v_2+8v_4+6| - 22|11v_2-3v_3+5v_4-8| \\
&\quad - 6|2v_1-4v_2+4v_3+6v_4-1| \\
i_3 &= -54 - 50v_1 + 18v_2 + 587v_3 + 18v_4 + 7|9v_3| \\
&\quad - 26|-v_1+13v_3| + 6|11v_2-3v_3+5v_4-8| \\
&\quad + 6|2v_1-4v_2+4v_3+6v_4-1| \\
i_4 &= 183 + 55v_1 - 21v_2 + 18v_3 + 193v_4 - 4|12v_2-6v_4+16| \\
&\quad - 8|5v_1 - 7v_2 + 8v_4 + 6|-10|11v_2 - 3v_3 + 5v_4 - 8| \\
&\quad + 9|2v_1-4v_2+4v_3+6v_4-1|
\end{aligned}\right\} \tag{A.21}$$

Following the same procedure as in Example 2, it is easily determined that this 4-port is reciprocal. To check the conditions of Theorem 7, we recast (A.21) as follows:

$$
\begin{aligned}
i_1 = {} & \sum_{k=1}^{8} a_{1_k} + \left(\sum_{k=1}^{8} b_{1_{1k}}\right)v_1 + \left(\sum_{k=1}^{8} b_{1_{2k}}\right)v_2 + \left(\sum_{k=1}^{8} b_{1_{3k}}\right)v_3 + \left(\sum_{k=1}^{8} b_{1_{4k}}\right)v_4 \\
& - 9|8v_1+0v_2+0v_3+0v_4-5| + 2|-v_1+0v_2+13v_3+0v_4-0| \\
& - 5|5v_1-7v_2+0v_3+8v_4+6| + 3|2v_1-4v_2+4v_3+6v_4-1| \\
& + 0|0v_1+12v_2+0v_3-6v_4+16| + 0|0v_1+11v_2-3v_3+5v_4-8| \\
& + 0|0v_1+0v_2+9v_3+0v_4-0| + 0|3v_1-9v_2-12v_3-v_4-0| \\
i_2 = {} & \sum_{k=1}^{8} a_{2_k} + \left(\sum_{k=1}^{8} b_{2_{1k}}\right)v_1 + \left(\sum_{k=1}^{8} b_{2_{2k}}\right)v_2 + \left(\sum_{k=1}^{8} b_{2_{3k}}\right)v_3 + \left(\sum_{k=1}^{8} b_{2_{4k}}\right)v_4 \\
& + 0|8v_1+0v_2+0v_3+0v_4-5| + 0|-v_1+0v_2+13v_3+0v_4-0| \\
& + 7|5v_1-7v_2+0v_3+8v_4+6| - 6|2v_1-4v_2+4v_3+6v_4-1| \\
& + 8|0v_1+12v_2+0v_3-6v_4+16| - 22|0v_1+11v_2-3v_3+5v_4-8| \\
& + 0|0v_1+0v_2+9v_3+0v_4-0| + 0|3v_1-9v_2-12v_3-v_4-0| \\
i_3 = {} & \sum_{k=1}^{8} a_{3_k} + \left(\sum_{k=1}^{8} b_{3_{1k}}\right)v_1 + \left(\sum_{k=1}^{8} b_{3_{2k}}\right)v_2 + \left(\sum_{k=1}^{8} b_{3_{3k}}\right)v_3 + \left(\sum_{k=1}^{8} b_{3_{4k}}\right)v_4 \\
& + 0|8v_1+0v_2+0v_3+0v_4-5| - 26|-v_1+0v_2+13v_3+0v_4-0| \\
& + 0|5v_1-7v_2+0v_3+8v_4+6| + 6|2v_1-4v_2+4v_3+6v_4-1| \\
& + 0|0v_1+12v_2+0v_3-6v_4+16| + 6|0v_1+11v_2-3v_3+5v_4-8| \\
& + 7|0v_1+0v_2+9v_3+0v_4-0| + 0|3v_1-9v_2-12v_3-v_4-0| \\
i_4 = {} & \sum_{k=1}^{8} a_{4_k} + \left(\sum_{k=1}^{8} b_{4_{1k}}\right)v_1 + \left(\sum_{k=1}^{8} b_{4_{2k}}\right)v_2 + \left(\sum_{k=1}^{8} b_{4_{3k}}\right)v_3 + \left(\sum_{k=1}^{8} b_{4_{4k}}\right)v_4 \\
& + 0|8v_1+0v_2+0v_3+0v_4-5| + 0|-v_1+0v_2+13v_3+0v_4-0| \\
& - 8|5v_1-7v_2+0v_3+8v_4+6| + 9|2v_1-4v_2+4v_3+6v_4-1| \\
& - 4|0v_1+12v_2+0v_3-6v_4+16| - 10|0v_1+11v_2-3v_3+5v_4-8| \\
& + 0|0v_1+0v_2+9v_3+0v_4-0| + 0|3v_1-9v_2-12v_3-v_4-0|
\end{aligned}
\tag{A.22}
$$

where the terms within each summation are listed in Table 1.

Having recast (A.21) as in (A.22) and Table 1, it is easy to see that the conditions of Theorem 7 are all satisfied, thus implying that our 4-port can be synthesized using only passive elements.

From Corollary 1 to Theorem 7 we know that this 4-port is of course passive.

4. An example illustrating the use of Corollary 2 to Theorem 7.

Corollary 2 to Theorem 7 gives us the actual technique by which we can synthesize passive reciprocal n-ports described by (12) using only passive elements. We now demonstrate this technique by synthesizing our 4-port in (A.21). The coefficients of the p = 8 resistors described by (22) are determined from (78)-(81) and are given in Table 2. Similarly, the (p = 8) x (q = n = 4) = 32 transformer's turns ratios are determined from (82) and are also listed in Table 2.

Substituting into (22) yields:

$$\left.\begin{aligned}
R_{a_1} &: i_{a_1} = \frac{45}{8} + \frac{9}{8} v_{a_1} - \frac{9}{8} |v_{a_1} - 5| \\
R_{a_2} &: i_{a_2} = 2v_{a_2} - 2|v_{a_2}| \\
R_{a_3} &: i_{a_3} = 6 + v_{a_3} - |v_{a_3} + 6| \\
R_{a_4} &: i_{a_4} = -\frac{3}{2} + \frac{3}{2} v_{a_4} + \frac{3}{2} |v_{a_4} - 1| \\
R_{a_5} &: i_{a_5} = -\frac{32}{3} + \frac{2}{3} v_{a_5} + \frac{2}{3} |v_{a_5} + 16| \\
R_{a_6} &: i_{a_6} = 16 + 2v_{a_6} - 2|v_{a_6} - 8| \\
R_{a_7} &: i_{a_7} = \frac{7}{9} v_{a_7} + \frac{7}{9} |v_{a_7}|
\end{aligned}\right\} \quad (A.23)$$

$$R_{a_8} : i_{a_8} = v_{a_8}$$

Fig. 5 shows the final circuit with the resistor's constitutive relations given in (A.23) and the transformer's turns ratios given in Table 2. Substituting (A.1) into (A.23) above, and then substituting the resulting set of equations into (A.2) yields (A.21), thus verifying that our synthesis is indeed correct. Using Theorem 5, it is easily verified that each resistor in (A.23) is passive.

Note that the 4-port in (A.21) could also have been synthesized using the technique given in the corollary to Theorem 4. Such a synthesis would not have used only passive elements, however, and consequently would not have been nearly as instructive.

ACKNOWLEDGEMENT

The authors would like to thank Professor Joos Vandewalle for his many useful comments and discussions.

REFERENCES

[1] L. O. Chua, "Nonlinear Circuit Theory," Proceeding of the 1978 European Conference on Circuit Theory and Design, Vol. II, Guest Lectures, September 4-8, 1978, pp. 65-172.

[2] J. L. Wyatt, L. O. Chua, and G. F. Oster, "Nonlinear n-port decomposition via the Laplace operator," IEEE Trans. on Circuits and Systems, Special Issue on the Mathematical Foundations of System Theory, September 1978, pp. 741-754.

[3] B. D. O. Anderson and S. Vongpanitlerd, Network Analysis and Synthesis, Prentica-Hall, Englewood Cliff, New Jersey, 1973.

[4] L. O. Chua, "Δ-Y and Y-Δ transformation for nonlinear networks," Proc. of the IEEE, Vol. 59, No. 3, March 1971, pp. 417-419.

[5] S. M. Kang and L. O. Chua, "A global representation of multidimensional piecewise-linear functions with linear partitions," IEEE Trans. on Circuits and Systems, Vol. 25, No. 11, November 1978, pp. 938-940.

[6] L. O. Chua, Introduction to Nonlinear Network Theory, McGraw-Hill Book Company, New York, New York, 1969.

[7] L. O. Chua and S. Wong, "Synthesis of piecewise-linear networks," IEE Journal on Electronic Circuits and Systems, July 1978, pp. 102-108.

[8] L. O. Chua and S. M. Kang, "Sectionwise piecewise-linear functions: canonical representation, properties, and applications," Proc. of IEEE, Special Issue on Multidimensional Systems, Vol. 65, No. 6, June 1977, pp. 915-929.

[9] B. Noble and J.W. Daniel, Applied Linear Algebra, Prentice Hall, Englewood Cliffs, N.J., 1977.

FIGURE CAPTIONS

Fig. 1. General circuit interconnection used in the synthesis of reciprocal 2-ports described by (1).

Fig. 2. Synthesis of the reciprocal 2-port in Example 3.

Fig. 3. General circuit interconnection used in the synthesis of reciprocal n-ports described by (12).

Fig. 4. Synthesis of the reciprocal 4-port in Example 4.

Fig. 5. Synthesis of the passive reciprocal 4-port in Appendix 4.

TABLE 1

k	a_{1_k}	a_{2_k}	a_{3_k}	a_{4_k}	$b_{1_{1k}}$	$b_{2_{1k}}$	$b_{3_{1k}}$	$b_{4_{1k}}$	$b_{1_{2k}}$	$b_{2_{2k}}$	$b_{3_{2k}}$	$b_{4_{2k}}$	$b_{1_{3k}}$	$b_{2_{3k}}$	$b_{3_{3k}}$	$b_{4_{3k}}$	$b_{1_{4k}}$	$b_{2_{4k}}$	$b_{3_{4k}}$	$b_{4_{4k}}$
1	45	0	0	0	72	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	2	0	-26	0	0	0	0	0	-26	0	338	0	0	0	0	0
3	30	-42	0	48	25	-35	0	40	-35	49	0	-56	0	0	0	0	40	-56	0	64
4	-3	6	-6	-9	6	-12	12	18	-12	24	-24	-36	12	-24	24	36	18	-36	36	54
5	0	-128	0	64	0	0	0	0	0	96	0	-48	0	0	0	0	0	-48	0	24
6	0	176	-48	80	0	0	0	0	0	242	-66	110	0	-66	18	-30	0	110	-30	50
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	63	0	0	0	0	0
8	0	0	0	0	9	-27	-36	-3	-27	81	108	9	-36	108	144	12	-3	9	12	1

TABLE 2

k	${}^{\beta}a_k$	${}^{a}a_k$	${}^{b}a_k$	${}^{g}a_k$	k_{1k}	k_{2k}	k_{3k}	k_{4k}
1	5	$\frac{45}{8}$	$\frac{9}{8}$	$-\frac{9}{8}$	8	0	0	0
2	0	0	2	-2	-1	0	13	0
3	-6	6	1	-1	5	-7	0	8
4	1	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	2	-4	4	6
5	-16	$-\frac{32}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	12	0	-6
6	8	16	2	-2	0	11	-3	5
7	0	0	$\frac{7}{9}$	$\frac{7}{9}$	0	0	9	0
8	0	0	1	0	3	-9	-12	-1

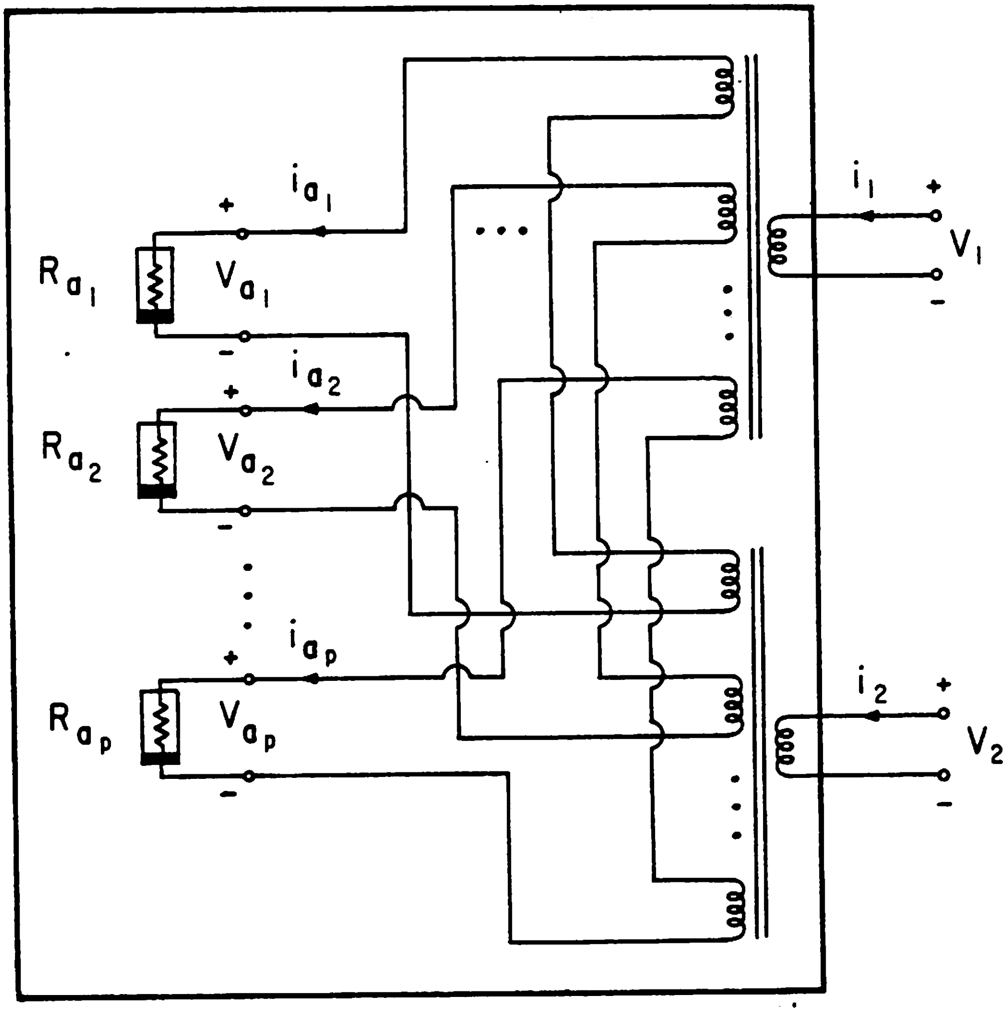


Fig. 1

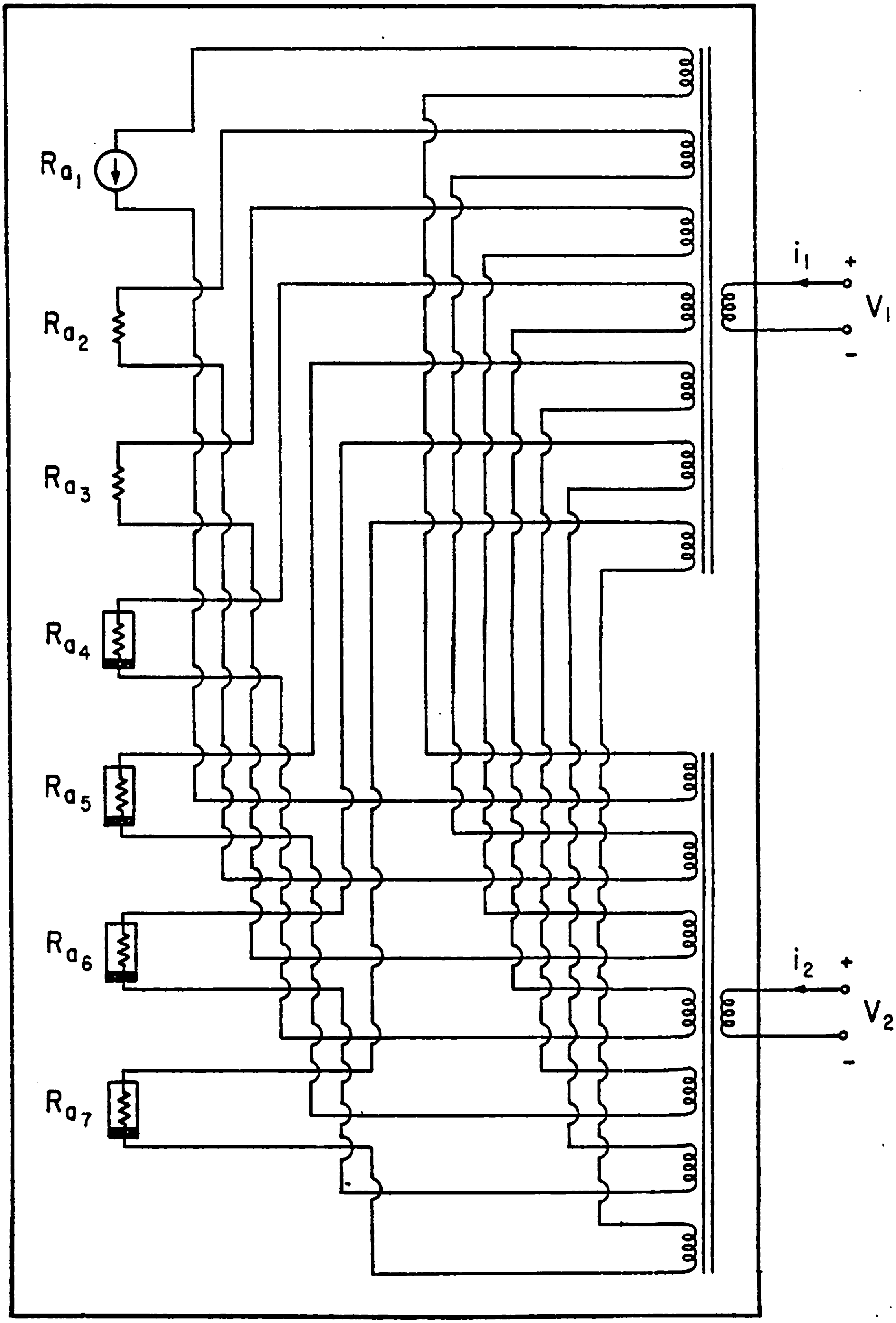


Fig. 2

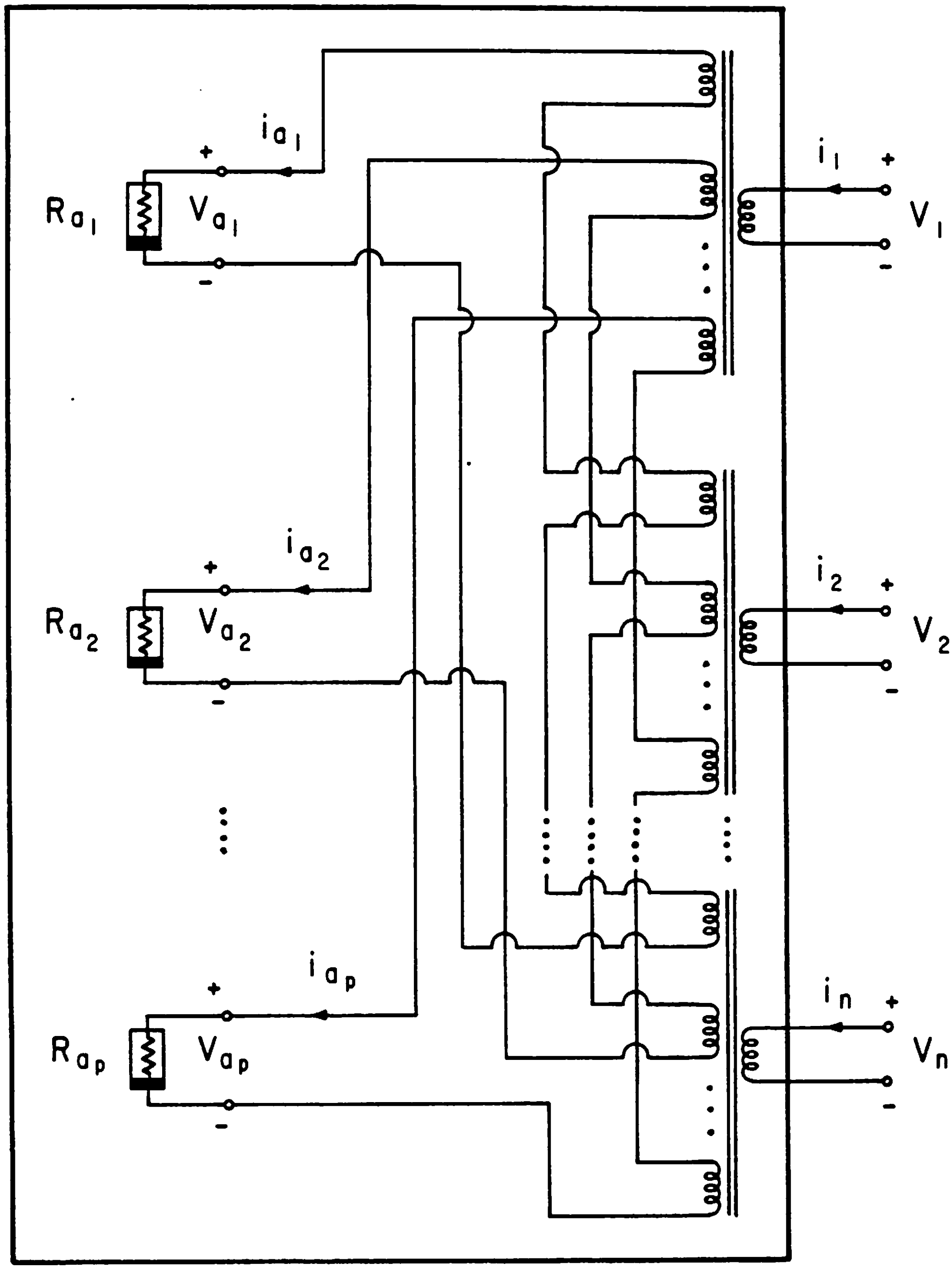
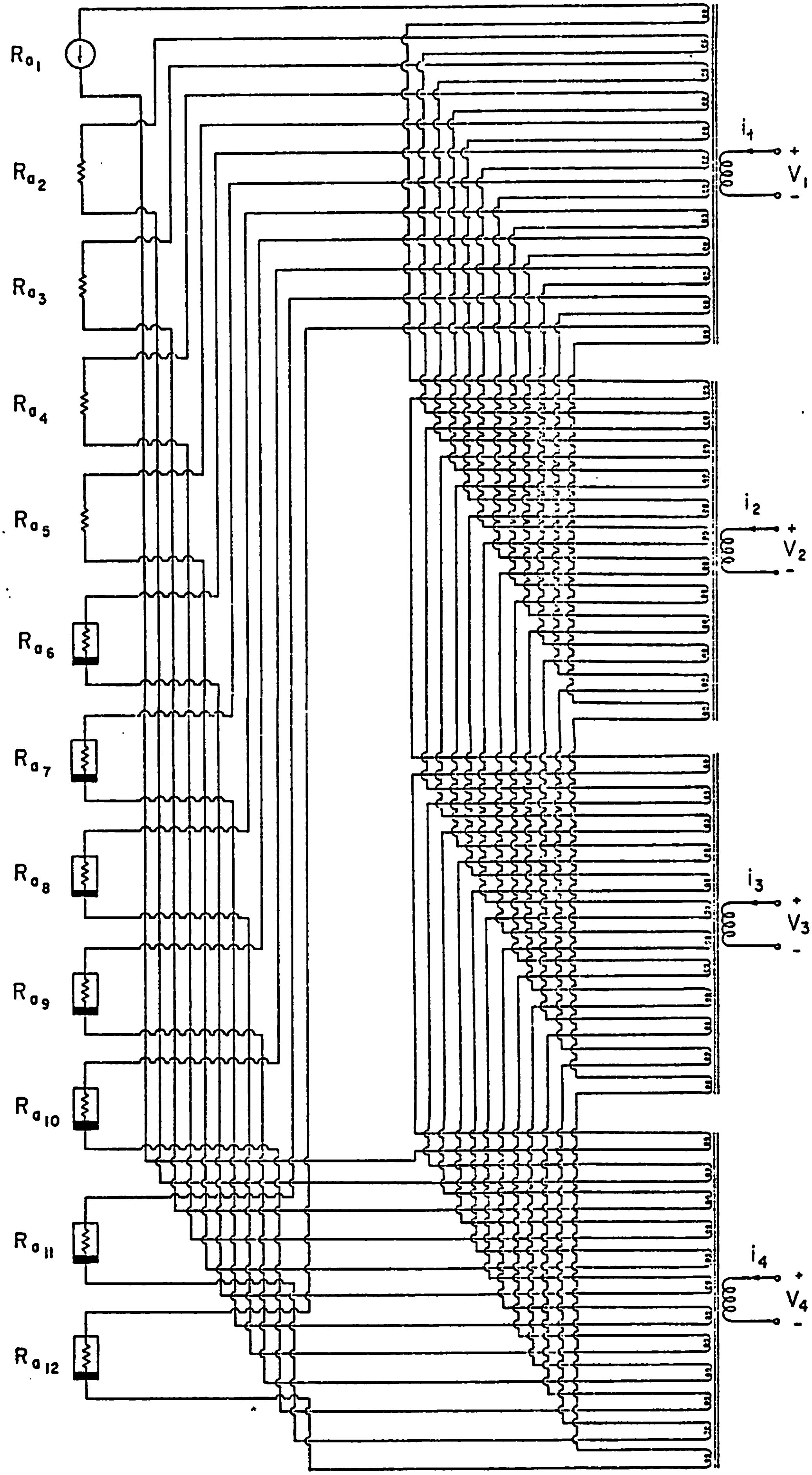


Fig. 3



R_{a_1}
R_{a_2}
R_{a_3}
R_{a_4}
R_{a_5}
R_{a_6}
R_{a_7}
R_{a_8}
R_{a_9}
$R_{a_{10}}$
$R_{a_{11}}$
$R_{a_{12}}$
i_1
+ V_1 -
i_2
+ V_2 -
i_3
+ V_3 -
i_4
+ V_4 -

Fig. 4

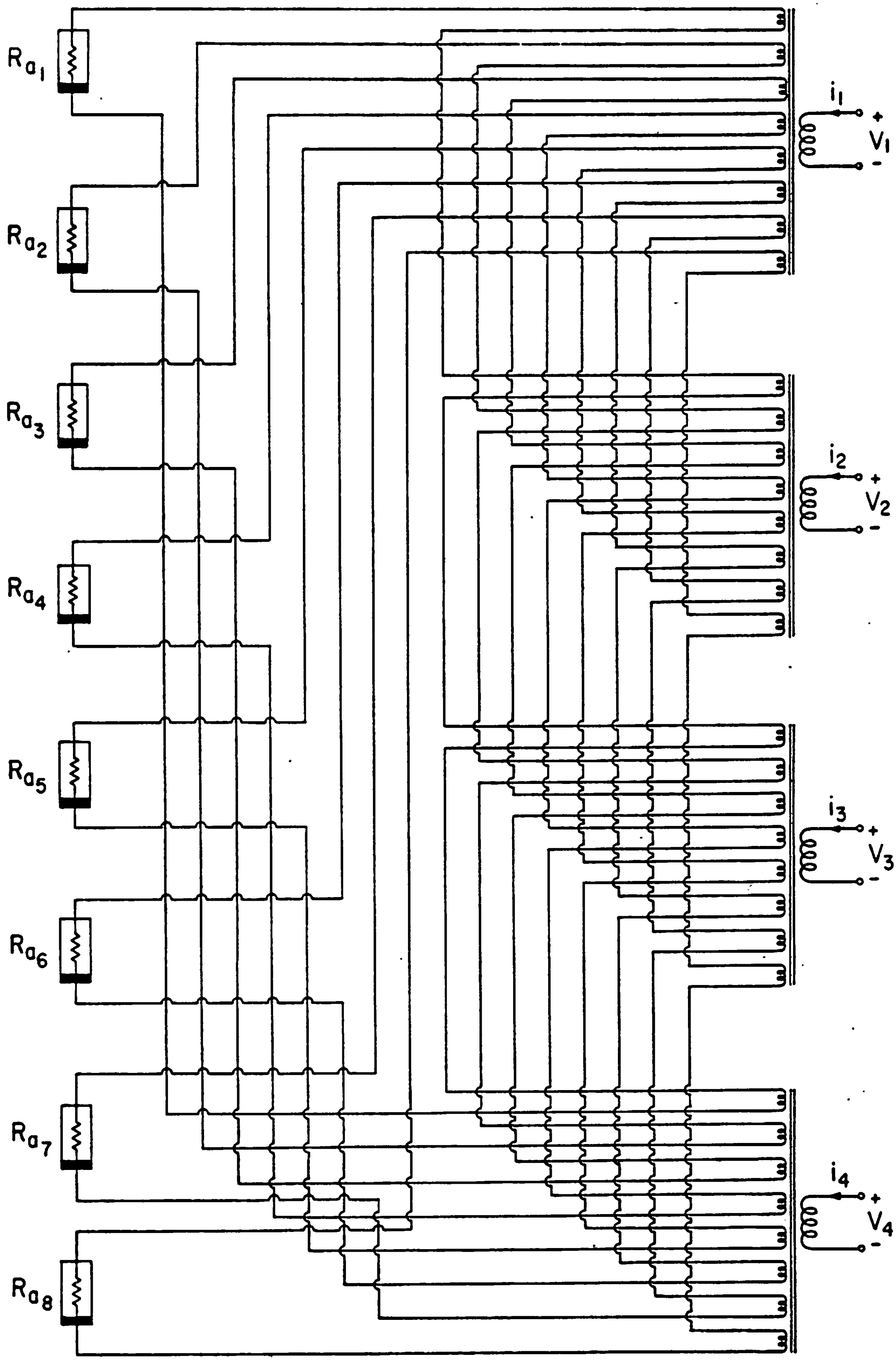


Fig. 5