FREQUENCY DOMAIN ANALYSIS OF NONLINEAR SYSTEMS:

FORMULATION OF TRANSFER FUNCTIONS

by

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ABSTRACT

Explicit and recursive formulas for obtaining nth order transfer functions of composite nonlinear systems are presented. A recursive method for obtaining the nth order output of a nonlinear circuit by solving a linear circuit n times is derived. Each time different input sources are used. Recursive formulas for obtaining the nth order transfer functions of nonlinear circuits are then generated. These results are used to obtain formulas for nth order transfer functions of cascade systems, as well as inverse systems. Methods for synthesizing nonlinear circuits and inverse systems via feedback configurations are given. Finally, the general structures of transfer functions for a large class of nonlinear systems are also derived.

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1. Introduction

This is a sequel to a recent paper [1] on frequency domain nonlinear system analysis via the Volterra series. In particular the results developed in section 4 of [1] will be used extensively in this paper. In section 2 we derive the nth order transfer function of a nonlinear circuit by solving a linear circuit repeatedly — each time with different sources. This problem has previously been investigated in [3] using node analysis. However, not all nonlinear elements can be expressed in voltage-controlled "admittance" form and hence a more general method of analysis is desirable. Moreover, the use of exponential inputs assume apriori that all transfer functions generated by the solution process are symmetric. In this section, we derive rigorously recursive formulas for obtaining the nonlinear transfer functions. In section 3, we apply these recursive formulas to derive explicit formulas and recursive formulas for obtaining the transfer functions of cascade systems and inverse systems, respectively. Synthesis of nonlinear circuits and inverse systems via feedback configurations are also presented. Finally, the general structures of transfer functions for a large class of nonlinear systems are derived.

All systems considered in this paper are assumed to have a single input. We adopt the same notations and definitions introduced in [1], except that the zeroth order term for each Volterra series is assumed to be identically zero. Thus, for an analytic system F, the output w(t) is an analytic functional [2] F[x(t)] of its input x(t) and can be expressed as a Volterra functional series; namely,

$$w(t) = F[x(t)] = \sum_{m=1}^{\infty} F_m[x(t)] = \sum_{m=1}^{\infty} w_m(t) \tag{1a}$$

where

$$w_m(t) \triangleq F_m[x(t)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) \prod_{i=1}^{m} x(t-\tau_i)d\tau_i \tag{1b}$$

The mth order term can be found from the input-output relation as follows [2]:

$$w_m(t) = F_m[x(t)] = \frac{1}{m!}\left(\frac{d^m}{dt^m} F[\varepsilon x(t)]\right)_{\varepsilon=0} \tag{2}$$

For the purpose of this paper, it suffices to use the relationship

$$w(t) = F_m[x(t)] = \text{coefficient of } \varepsilon^m \text{ in } F[\varepsilon x(t)] \tag{3}$$

All results from [1], especially those contained in the section on symmetrization, will be applied in this paper without detailed explanation.

2. Formulation of Transfer Functions for Nonlinear Networks

In this section, we will generalize the method in [3,11] for finding the overall nonlinear transfer functions of single-input nonlinear networks. The method is a recursive one which involves only solving a modified linear circuit repeatedly. Let each nonlinear element of the network be expressed as a Volterra series of the form:

$$w(t) = \sum_{m=1}^{\infty} w_m(t) \tag{4a}$$

where

$$w_m(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) \prod_{i=1}^{m} x(t-\tau_i)d\tau_i \tag{4b}$$

For a two-terminal element, $(x(t),w(t))$ can either be $(v(t),i(t))$ or $(i(t),v(t))$, where $v(t)$ and $i(t)$ are the voltage across and the current through the nonlinear element, respectively. If $(x(t),w(t))$ is $(v(t),i(t))$, then we represent the mth order transfer function by $Y_m(s_1,s_2,\ldots,s_m)$ and call it the mth order admittance of the voltage-controlled "admittance" element. If $(x(t),w(t))$ is $(i(t),v(t))$, then we represent the mth order transfer function by $Z_m(s_1,s_2,\ldots,s_m)$ and call it the mth order impedance of the current-controlled "impedance" element. Equation (4) can also be used to represent the four types of two-port nonlinear controlled sources. In these cases $x(t)$ can be either $v(t)$ or $i(t)$, and $w(t)$ can also be either $v(t)$ or $i(t)$. Since (4) is a general representation of the preceding six types of nonlinear elements, we will henceforth call $x(t)$ the controlling variable, and $w(t)$ the controlled variable of the nonlinear element. We will also make the following four assumptions which define the class of nonlinear networks being investigated in this section:

1. The nonlinear elements in the circuit consist of the six types described above.

Before stating assumption 2, let us decompose each nonlinear element into a linear component and a strictly nonlinear component as shown in Fig. 1. Modify the circuit by embedding the linear component of each nonlinear element into the linear circuit to which the nonlinear element was originally attached. Thus each modified nonlinar element assumes the form:

$$w_N(t) = w(t) - w_1(t) = \sum_{m=2}^{\infty} w_m(t), \qquad (5)$$

where $w_N(t)$ is the new controlled variable with the controlling variable $x(t)$ remaining unchanged.

2. The output $q(t)$ of the circuit, the controlled variable $w_N(t)$, and the controlling variable $x(t)$, of each nonlinear element of the modified circuit, are analytic functionals of the input and hence can be represented by Volterra series.

3. The modified network has a unique solution for all branch voltages and branch currents corresponding to each input of the input ensemble.

4. The modified network still has a unique solution for all its branch voltages and branch currents after replacing the branch of each nonlinear element corresponding to $w_N(t)$ with an independent voltage (resp., current) source if $w_N(\cdot)$ is a voltage (resp., current) waveform[1].

Remark:

Assumptions 3 and 4 can be interpreted as a generalization of the well-known "substitution theorem".

For each input waveform $u(\cdot)$, let $[\underset{\sim}{v}_N(\cdot) \;\; \underset{\sim}{i}_N(\cdot)]^T$ and $[\underset{\sim}{x}_v(\cdot) \;\; \underset{\sim}{x}_i(\cdot)]^T$ denote the waveforms of the controlled variables and the corresponding controlling variables, respectively, of the modified nonlinear elements as shown in Fig. 1[2]. Let $q(\cdot)$ be the corresponding output waveform. It follows from Assumption 2 that each component of $[\underset{\sim}{v}_N(\cdot) \;\; \underset{\sim}{i}_N(\cdot)]^T$ and $[q(\cdot) \;\; \underset{\sim}{x}_v(\cdot) \;\; \underset{\sim}{x}_i(\cdot)]^T$ can be expressed as a Volterra series with input $u(\cdot)$. In vector form, we can write:

$$[\underset{\sim}{v}_N(\cdot) \;\; \underset{\sim}{i}_N(\cdot)]^T = \sum_{n=1}^{\infty} [\underset{\sim}{v}_{Nn}(\cdot) \;\; \underset{\sim}{i}_{Nn}(\cdot)]^T \qquad (6)$$

and

$$[q(\cdot) \;\; \underset{\sim}{x}_v(\cdot) \;\; \underset{\sim}{x}_i(\cdot)]^T = \sum_{n=1}^{\infty} [q_n(\cdot) \;\; \underset{\sim}{x}_{vn}(\cdot) \;\; \underset{\sim}{x}_{in}(\cdot)]^T \qquad (7)$$

where $[\underset{\sim}{v}_{Nn}(\cdot) \;\; \underset{\sim}{i}_{Nn}(\cdot)]^T$ and $[q_n(\cdot) \;\; \underset{\sim}{x}_{vn}(\cdot) \;\; \underset{\sim}{x}_{in}(\cdot)]^T$ are the nth order terms with respect to $u(\cdot)$. With the same input waveform $u(\cdot)$, replace the branches of

---

[1] Thus, all branches corresponding to v_N in Fig. 1 a,b and c are replaced with independent voltage sources, while those corresponding to i_N in Fig. 1 d,e and f are replaced with independent current sources.

[2] Observe that $\underset{\sim}{v}_N(\cdot)$ and $\underset{\sim}{i}_N(\cdot)$ pertain to different elements.

the modified nonlinear elements that would have produced $[\underset{\sim}{v}_N(\cdot) \quad \underset{\sim}{i}_N(\cdot)]^T$ with the corresponding independent sources with the same waveforms $[\underset{\sim}{v}_N(\cdot) \quad \underset{\sim}{i}_N(\cdot)]^T$. Let us call these independent sources the <u>nonlinear sources</u> and their <u>n</u>th order components $[\underset{\sim}{V}_{Nn} \quad \underset{\sim}{i}_{Nn}]^T$ as shown in (6) the <u>nth order nonlinear sources</u>. It follows from the substitution theorem that all branch voltage waveforms and branch current waveforms remain the same as if no substitution had been made. Observe that after these substitutions have been made, we are left with a <u>linear</u> circuit with sources attached to it. The following facts will show how the <u>n</u>th order output $q_n(\cdot)$ can be found recursively by solving a series of "recursive" <u>linear</u> circuits of this type.

<u>Facts</u>:

1. There are no <u>1</u>st order nonlinear sources, i.e., $[\underset{\sim}{v}_{N1}(\cdot) \quad \underset{\sim}{i}_{N1}(\cdot)]^T = 0$.
2. $[q_1(\cdot) \quad \underset{\sim}{x}_{v1}(\cdot) \quad \underset{\sim}{x}_{i1}(\cdot)]^T$ is due to $u(\cdot)$ alone and can be found by setting all nonlinear sources to zero.
3. For $n > 1$, $[q_n(\cdot) \quad \underset{\sim}{x}_{vn}(\cdot) \quad \underset{\sim}{x}_{in}(\cdot)]^T$ is due to the <u>n</u>th order nonlinear sources $[\underset{\sim}{v}_{Nn}(\cdot) \quad i_{Nn}(\cdot)]^T$ alone and can be found by setting $u(\cdot)$ to zero.
4. For $n > 1$, each element of $[v_{Nn}(\cdot) \quad i_{Nn}(\cdot)]^T$ is a functional of the corresponding element of $[\underset{\sim}{x}_{vk}(\cdot) \quad \underset{\sim}{x}_{ik}(\cdot)]^T$ for all $k < n$.

<u>Remark</u>:

The above facts clearly suggest a recursive algorithm for finding $q(\cdot)$. Indeed, it follows from Fact 2 that the <u>1</u>st order term $[q_1(\cdot) \quad \underset{\sim}{x}_{v1}(\cdot) \quad \underset{\sim}{x}_{i1}(\cdot)]^T$ can be obtained by solving a linear circuit with input $u(\cdot)$. Using fact 4 with $n = 2$, we can obtain the <u>2</u>nd order nonlinear sources $[\underset{\sim}{v}_{N2}(\cdot) \quad \underset{\sim}{i}_{N2}(\cdot)]^T$ in terms of the <u>1</u>st order components $[\underset{\sim}{x}_{v1}(\cdot) \quad \underset{\sim}{x}_{i1}(\cdot)]^T$, since $k = 1$. It then follows from Fact 3 that $[q_2(\cdot) \quad x_{v2}(\cdot) \quad x_{i2}(\cdot)]^T$ can be obtained by solving a linear circuit with sources $[\underset{\sim}{v}_{N2}(\cdot) \quad \underset{\sim}{i}_{N2}(\cdot)]^T$. A repetition of the above algorithm will allow us to find all higher-order solutions, each solution requiring only the analysis of a <u>linear</u> circuit with independent sources whose waveforms are prescribed by the solutions of the previous analysis.

<u>Proof of Facts 1 and 4</u>:

Before extracting the linear component, each nonlinear element is described by (4)[3]. After embedding the linear component of each nonlinear element into the linear circuit, each modified nonlinear element assumes the form given by (5). It follows from assumption 2 that both $w_N(\cdot)$ and $x(\cdot)$ can be expressed as a

[3] Notice that $w_m(t)$ is an <u>m</u>th order term <u>with respect to x(t) only</u> and not the input $u(\cdot)$ of the circuit.

Volterra series with input u(·); namely,

$$w_N(t) = \sum_{n=1}^{\infty} w_{Nn}(t) \tag{8}$$

$$x(t) = \sum_{n=1}^{\infty} x_n(t) \tag{9}$$

where $w_{Nn}(t)$ and $x_n(t)$ are nth order terms with respect to u(t). Substituting (9) into (4b), we obtain

$$\begin{aligned} w_m(t) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) \prod_{i=1}^{m} \left(\sum_{k=1}^{\infty} x_k(t-\tau_i) \right) d\tau_i \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) x_{k_1}(t-\tau_1) x_{k_2}(t-\tau_2) \ldots x_{k_m}(t-\tau_m) d\tau_1 \ldots d\tau_m \end{aligned} \tag{10}$$

Equation (10) represents a sum of components, each having the form:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) \prod_{i=1}^{m} x_{k_i}(t-\tau_i) d\tau_i \tag{11}$$

Recall that (11) is due to the input u(t). Now if we replace u(t) with $\varepsilon u(t)$, then (11) would assume the following corresponding form:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m) \prod_{i=1}^{m} \varepsilon^{k_i} x_{k_i}(t-\tau_i) d\tau_i .$$

Hence, it follows from (3) that each component in the summation of (10) is of order $k_1+k_2+\ldots+k_m$ with respect to the input u(t). As k_i, $\forall i = 1,\ldots,m$, can take on integer values from 1 to ∞, $k_1+k_2+\ldots+k_m \geq m$. Since $w_N(t) = \sum_{m=2}^{\infty} w_m(t)$, i.e. $m \geq 2$, $w_N(t)$ can contain no 1st order component with respect to u(t). This proves Fact 1. Furthermore, $k_1+k_2+\ldots+k_m > k_i$ for all $i = 1,2,\ldots,m$. Thus (11) is a component of w_{Nn}, where $n = k_1+k_2+\ldots+k_m$, and involves only the terms $x_{k_i}(t)$ with $k_i < n$. This proves Fact 4. ⌑

Example 1.

Let us recast $w_N(t) = \sum_{m=2}^{\infty} w_m(t)$ into the form of (8)[4]:

$$
\begin{aligned}
w_N(t) = & \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(\tau_1,\tau_2)x_1(t-\tau_1)x_1(t-\tau_2)d\tau_1 d\tau_2\right\} \\
& + \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(\tau_1,\tau_2)[x_1(t-\tau_1)x_2(t-\tau_2)+x_2(t-\tau_1)x_1(t-\tau_2)]d\tau_1 d\tau_2\right. \\
& + \left.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_3(\tau_1,\tau_2,\tau_3)x_1(t-\tau_1)x_1(t-\tau_2)x_1(t-\tau_3)d\tau_1 d\tau_2 d\tau_3\right\} \\
& + \cdots
\end{aligned}
\tag{12}
$$

Proof of Facts 2 and 3:

It follows from assumptions 3 and 4 that we can use the substitution theorem to find $[q(\cdot)\ \ \underset{\sim}{x}_v(\cdot)\ \ \underset{\sim}{x}_i(\cdot)]^T$. Thus

$$
\begin{bmatrix} q(\cdot) \\ \underset{\sim}{x}_v(\cdot) \\ \underset{\sim}{x}_i(\cdot) \end{bmatrix} = \mathcal{L} \begin{bmatrix} u(\cdot) \\ \underset{\sim}{v}_N(\cdot) \\ \underset{\sim}{i}_N(\cdot) \end{bmatrix} = \mathcal{L} \begin{bmatrix} u(\cdot) \\ \sum_{j=2}^{\infty} \underset{\sim}{v}_{Nj}(\cdot) \\ \sum_{j=2}^{\infty} \underset{\sim}{i}_{Nj}(\cdot) \end{bmatrix}
$$

where $\mathcal{L}$ is a linear operator. Now let $u(\cdot)$ be changed to $\varepsilon u(t)$, so that

$$
\begin{bmatrix} q(\cdot) \\ \underset{\sim}{x}_v(\cdot) \\ \underset{\sim}{x}_i(\cdot) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \varepsilon u(\cdot) \\ \sum_{j=2}^{\infty} \varepsilon^j \underset{\sim}{v}_{Nj}(\cdot) \\ \sum_{j=2}^{\infty} \varepsilon^j \underset{\sim}{i}_{Nj}(\cdot) \end{bmatrix} = \varepsilon\mathcal{L} \begin{bmatrix} u(\cdot) \\ 0 \\ 0 \end{bmatrix} + \sum_{j=2}^{\infty} \varepsilon^j \mathcal{L} \begin{bmatrix} 0 \\ \underset{\sim}{v}_{Nj}(\cdot) \\ \underset{\sim}{i}_{Nj}(\cdot) \end{bmatrix}
$$

---

[4]Recall that $w_m(t)$ is the mth order output with respect to the controlling variable $x(t)$, whereas $w_{Nn}(t)$ is the nth order term with respect to the input $u(t)$.

Fact 2 and 3 then follow immediately from (3). ¤

Now let us solve the problem using the frequency domain approach in order to get rid of the convolution type of integrals. Observe that (11) represents a typical term of order $k_1+k_2+\ldots+k_m$ with respect to $u(t)$. In the "multiple" time domain, (11) assumes the following form [1]:

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_m(\tau_1,\tau_2,\ldots,\tau_m)x_{k_1}(t_1-\tau_1,t_2-\tau_1,\ldots,t_{k_1}-\tau_1)x_{k_2}(t_{k_1+1}-\tau_2,\ldots,t_{k_1+k_2}-\tau_2)\cdots$$

$$x_{k_m}(t_{k_1+\ldots+k_{m-1}+1}-\tau_m,\ldots,t_{k_1+\ldots+k_m}-\tau_m)d\tau_1 d\tau_2\ldots d\tau_m \qquad (13)$$

Taking the multiple Laplace transform of (13), we obtain

$$F_m\left(\sum_{i=1}^{k_1} s_i,\sum_{i=k_1+1}^{k_1+k_2} s_i,\ldots,\sum_{i=k_1+\ldots+k_{m-1}+1}^{k_1+\ldots+k_m} s_i\right)X_{k_1}\left(s_1,s_2,\ldots,s_{k_1}\right)X_{k_2}\left(s_{k_1+1},\ldots,s_{k_1+k_2}\right)\cdots X_{k_m}\left(s_{k_1+\ldots+k_{m-1}+1},\ldots,s_{k_1+\ldots+k_m}\right) \qquad (14)$$

Example 2.

Transforming the two terms enclosed by $\{\cdot\}$ in (12) into the frequency domain, we obtain

$$W_{N2}(s_1,s_2) \doteq F_2(s_1,s_2)X_1(s_1)X_1(s_2) \qquad (15)$$

$$W_{N3}(s_1,s_2,s_3) \doteq F_2(s_1,s_2+s_3)X_1(s_1)X_2(s_2,s_3)+F_2(s_1+s_2,s_3)X_2(s_1,s_2)X_1(s_3)$$
$$+ F_3(s_1,s_2,s_3)X_1(s_1)X_1(s_2)X_1(s_3)$$

Since[5] $F_2(s_1+s_2,s_3)X_2(s_1,s_2)X_1(s_3) \doteq F_2(s_2+s_3,s_1)X_2(s_2,s_3)X_1(s_1)$, we can write $W_{N3}(s_1,s_2,s_3)$ as follows:

$$W_{N3}(s_1,s_2,s_3) \doteq [F_2(s_1,s_2+s_3)+F_2(s_2+s_3,s_1)]X_1(s_1)X_2(s_2,s_3)$$
$$+ F_3(s_1,s_2,s_3)X_1(s_1)X_1(s_2)X_1(s_3)$$
$$\doteq 2\bar{F}_2(s_1,s_2+s_3)X_1(s_1)X_2(s_2,s_3)+F_3(s_1,s_2,s_3)X_1(s_1)X_1(s_2)X_1(s_3) \qquad (16)$$

Notice that $F_2(s_1,s_2)$ in (15) as well as $\bar{F}_2(s_1,s_2)$ and $F_3(s_1,s_2,s_3)$ in (16) all satisfy the partial symmetric requirement on $\Lambda(\cdot)$ as defined in Lemma 4 of section 4 of [1]. This self partial-symmetry property also holds for the higher order terms $W_{Nn}(s_1,s_2,\ldots,s_n)$. Thus, for convenience we will use the symmetric forms $\bar{f}_m(\tau_1,\tau_2,\ldots,\tau_m)$ and $\bar{F}_m(s_1,s_2,\ldots,s_m)$. To simplify the unwieldy notation in (14) we introduce the abbreviation $\bar{F}_m X_{k_1} X_{k_2} \ldots X_{k_m}$. Since $\bar{F}_m$ is symmetric, it follows that if $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is a permutation of $(k_1,k_2,\ldots,k_m)$, then[5]

$$\bar{F}_m X_{k_1} X_{k_2} \ldots X_{k_m} \doteq \bar{F}_m X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_m} \tag{17}$$

It follows from (17) that if p of the variables of $\{X_{k_1}, X_{k_2}, \ldots, X_{k_m}\}$ in $\bar{F}_m X_{k_1} X_{k_2} \ldots X_{k_m}$ are equal to X_{k_i}, then we can group them together into a single term $(X_{k_i})^p$. With this notation (15) becomes $W_{N2}(s_1,s_2) \doteq \bar{F}_2(X_1)^2$ and (16) becomes $W_{N3}(s_1,s_2,s_3) \doteq 2\bar{F}_2(X_1)^1(X_2)^1 + \bar{F}_3(X_1)^3$.

<u>Lemma</u>

The <u>n</u>th order nonlinear source $W_{Nn}(s_1,s_2,\ldots,s_n)$ of (8) is given by

$$\boxed{W_{Nn}(s_1,s_2,\ldots,s_n) \doteq \sum_{m_{n-1}=0}^{1} \sum_{m_{n-2}=0}^{a_{n-2}} \cdots \sum_{m_2=0}^{a_2} \left(\frac{m!}{\prod_{i=1}^{n-1} m_i!} \right) \bar{F}_m(X_1)^{m_1}(X_2)^{m_2} \ldots (X_{n-1})^{m_{n-1}}, \quad n \geq 2} \tag{18}$$

where $m_1 = n - \sum_{i=2}^{n-1} im_i$, $m = \sum_{i=1}^{n-1} m_i$, and $a_p = In\left(\frac{n - \sum_{i=p+1}^{n-1} im_i}{p} \right)$, $\forall p = 2,\ldots,n-2$.

<u>Proof</u>: See Appendix 1.

<u>Remarks</u>:

1. The symbol $In(\cdot)$ in (18) denotes an integer-valued function whose value $In(r)$ is obtained by deleting the fractional part of r. For example, $In(5.3) = 5$ and $In(2.9) = 2$.

2. Notice that the number of summation signs in (18) is n-2. Thus, for n = 2, there is no summation sign and $W_{N2}(s_1,s_2) \doteq \bar{F}_m(X_1)^{m_1}$ with $m_1 = 2$ and $m = 2$.

---

[5] This follows from Lemma 1 in section 4 of [1].

Example 3.

Applying (18) to find $W_{N4}(s_1,s_2,s_3,s_4)$, we obtain (with n=4)

$$
\begin{aligned}
W_{N4}(s_1,s_2,s_3,s_4) &\doteq \sum_{m_3=0}^{1} \sum_{m_2=0}^{a_2} \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3} \\
&\doteq \sum_{m_2=0}^{2} \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=0} + \sum_{m_2=0}^{0} \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=1} \\
&\doteq \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=0,m_2=0,m_1=4} + \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=0,m_2=1,m_1=2} \\
&\quad + \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=0,m_2=2,m_1=0} + \frac{m!}{m_1!m_2!m_3!} \bar{F}_m (X_1)^{m_1}(X_2)^{m_2}(X_3)^{m_3}\Big|_{m_3=1,m_2=0,m_1=1} \\
&\doteq \bar{F}_4(X_1)^4 + 3\bar{F}_3(X_1)^2(X_2)^1 + \bar{F}_2(X_2)^2 + 2\bar{F}_2(X_1)^1(X_3)^1
\end{aligned}
$$

We are now ready to present an algorithm for obtaining the nth order output transform and the nth order transfer function of a nonlinear circuit.

Algorithm:

1. For the modified linear circuit with input source vector $[u(\cdot) \;\; v_N(\cdot) \;\; i_N(\cdot)]^T$. Determine $\underset{\sim}{H}(s)$, the transfer function matrix of the linear circuit, such that

$$
\begin{bmatrix} Q(s) \\ \underset{\sim}{X}_v(s) \\ \underset{\sim}{X}_i(s) \end{bmatrix} = \underset{\sim}{H}(s) \begin{bmatrix} U(s) \\ \underset{\sim}{V}_N(s) \\ \underset{\sim}{I}_N(s) \end{bmatrix}
$$

2. Set n = 1 and obtain
$$
\begin{bmatrix} Q_n(s_1) \\ \underset{\sim}{X}_{vn}(s_1) \\ \underset{\sim}{X}_{in}(s_1) \end{bmatrix} = \underset{\sim}{H}(s_1) \begin{bmatrix} U(s_1) \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \qquad (19)
$$

3. Extract the nth order output transform $Q_n(s_1,s_2,\ldots,s_n)$, thus yielding the nth order transfer function $\dfrac{Q_n(s_1,s_2,\ldots,s_n)}{U(s_1)U(s_2)\ldots U(s_n)}$.

4. If n equals the required value, stop. Otherwise, set n = n+1.

5. Use (18) to obtain each element of the nth order nonlinear source vector
$$
\begin{bmatrix} \underset{\sim}{V}_{Nn}(s_1,s_2,\ldots,s_n) \\ \underset{\sim}{I}_{Nn}(s_1,s_2,\ldots,s_n) \end{bmatrix}
$$

6. Obtain

$$\begin{bmatrix} Q_n(s_1,s_2,\ldots,s_n) \\ \underset{\sim}{X}_{vn}(s_1,s_2,\ldots,s_n) \\ \underset{\sim}{X}_{in}(s_1,s_2,\ldots,s_n) \end{bmatrix} \doteq \underset{\sim}{H}(s_1,s_2,\ldots,s_n) \begin{bmatrix} 0 \\ \underset{\sim}{V}_{Nn}(s_1,s_2,\ldots,s_n) \\ \underset{\sim}{I}_{Nn}(s_1,s_2,\ldots,s_n) \end{bmatrix} \tag{20}$$

7. Go to 3.

Remarks:

1. Equation (19) (resp., (20)) follows from Fact 2 (resp., Fact 3) and the multiple Laplace transform, (14) of (13) with m = 1, since the equation in step 1 represents convolution in the time domain.

2. Instead of finding the transfer function matrix $\underset{\sim}{H}(s)$, we can solve the circuit by using any linear circuit analysis method in the frequency domain [13,14]. The only difference here is that we must change s to $s_1+s_2+\ldots+s_n$ and define the nth order nonlinear source vector by

$$\begin{bmatrix} \underset{\sim}{V}_{Nn}(s_1,s_2,\ldots,s_n) \\ \underset{\sim}{I}_{Nn}(s_1,s_2,\ldots,s_n) \end{bmatrix}.$$

3. It can be seen from the algorithm that the existence of the nth order transfer function depends only on the existence of a unique $\underset{\sim}{H}(s)$. Observe, however, that the algorithm itself does not imply that a solution of the original circuit exists. In fact, the series so obtained may not even converge.

We conclude this section with an example.

Example 4.

Let us use the above algorithm to find the nonlinear input impedances of the series-parallel nonlinear circuit shown in Fig. 2a[6]. Assume the time domain and frequency domain characterization of each nonlinear element is as follows:

Capacitor

time domain: $v_C(t) = \sum_{n=1}^{\infty} a_n q_C^n(t)$, where q_C denotes the capacitor charge

frequency domain: impedances $\bar{Z}_{Cn}(s_1,s_2,\ldots,s_n) = \dfrac{a_n}{s_1 s_2 \ldots s_n}$, $\forall n \geq 1$

[6]An example of a non-series-parallel nonlinear bridge circuit is given in Appendix 2.

Inductor

time domain: $i_L(t) = \sum_{n=1}^{\infty} b_n \phi_L^n(t)$, where ϕ_L denotes the inductor flux

frequency domain: admittance $\bar{Y}_{Ln}(s_1,s_2,\ldots,s_n) = \dfrac{b_n}{s_1 s_2 \ldots s_n}$, $\forall n \geq 1$

Resistor

time domain: $i_R(t) = \sum_{n=1}^{\infty} g_n v_R^n(t)$

frequency domain: admittance $\bar{Y}_{Rn}(s_1,s_2,\ldots,s_n) = g_n$, $\forall n \geq 1$

In order to find the input impedances of the circuit, we choose current i(t) as the input and the associated port voltage v(t) as the output. Let $v_{N_C}(\cdot)$, $i_{N_L}(\cdot)$, and $i_{N_R}(\cdot)$ denotes the waveforms of the nonlinear sources associated with the capacitor. inductor, and resistor, respectively. The modified linear circuit with the input and the nonlinear sources is shown in Fig. 2b. Following the above algorithm, we write

$$\begin{bmatrix} V(s) \\ I_C(s) \\ V_L(s) \\ V_R(s) \end{bmatrix} = \begin{bmatrix} \frac{a_1}{s} + \frac{s}{b_1+g_1 s} & 1 & \frac{-s}{b_1+g_1 s} & \frac{-s}{b_1+g_1 s} \\ 1 & 0 & 0 & 0 \\ \frac{s}{b_1+g_1 s} & 0 & \frac{-s}{b_1+g_1 s} & \frac{-s}{b_1+g_1 s} \\ \frac{s}{b_1+g_1 s} & 0 & \frac{-s}{b_1+g_1 s} & \frac{-s}{b_1+g_1 s} \end{bmatrix} \begin{bmatrix} I(s) \\ V_{N_C}(s) \\ I_{N_L}(s) \\ I_{N_R}(s) \end{bmatrix} \tag{21}$$

Thus

$$[V_1(s_1) \quad I_{C1}(s_1) \quad V_{L1}(s_1) \quad V_{R1}(s_1)]^T = \begin{bmatrix} \frac{a_1}{s_1} + \frac{s_1}{b_1+g_1 s_1} & 1 & \frac{s_1}{b_1+g_1 s_1} & \frac{s_1}{b_1+g_1 s_1} \end{bmatrix}^T I(s_1) \tag{22}$$

The 1st order input impedance is

$$Z_1(s_1) \triangleq \frac{V_1(s_1)}{I(s_1)} = \frac{a_1}{s_1} + \frac{s_1}{b_1+g_1 s_1} \tag{23}$$

It follows from (18) that the 2nd order nonlinear sources are given by:

$$\begin{bmatrix} V_{N_C 2}(s_1,s_2) \\ \\ I_{N_L 2}(s_1,s_2) \\ \\ I_{N_R 2}(s_1,s_2) \end{bmatrix} \doteq \begin{bmatrix} \bar{Z}_{C2}(s_1,s_2)I_{C1}(s_1)I_{C1}(s_2) \\ \\ \bar{Y}_{L2}(s_1,s_2)V_{L1}(s_1)V_{L1}(s_2) \\ \\ \bar{Y}_{R2}(s_1,s_2)V_{R1}(s_1)V_{R1}(s_2) \end{bmatrix} = \begin{bmatrix} \frac{a_2}{s_1 s_2} \\ \\ (\frac{b_2}{s_1 s_2})(\frac{s_1}{b_1+g_1 s_1})(\frac{s_2}{b_1+g_1 s_2}) \\ \\ g_2(\frac{s_1}{b_1+g_1 s_1})(\frac{s_2}{b_1+g_1 s_2}) \end{bmatrix} I(s_1)I(s_2) \tag{24}$$

It follows from (20) that

$$\begin{bmatrix} V_2(s_1,s_2) \\ \\ I_{C2}(s_1,s_2) \\ \\ V_{L2}(s_1,s_2) \\ \\ V_{R2}(s_1,s_2) \end{bmatrix} \doteq \begin{bmatrix} 1 & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} \\ \\ 0 & 0 & 0 \\ \\ 0 & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} \\ \\ 0 & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} & \frac{-(s_1+s_2)}{b_1+g_1(s_1+s_2)} \end{bmatrix} \begin{bmatrix} V_{N_C 2}(s_1,s_2) \\ I_{N_L 2}(s_1,s_2) \\ I_{N_R 2}(s_1,s_2) \end{bmatrix} \tag{25}$$

Substituting (24) into (25), we obtain the following 2nd order input impedance:

$$Z_2(s_1,s_2) \doteq \frac{V_2(s_1,s_2)}{I(s_1)I(s_2)} \doteq \frac{a_2}{s_1 s_2} - \left(\frac{b_2}{s_1 s_2} + g_2\right)\left(\frac{s_1}{b_1+g_1 s_1}\right)\left(\frac{s_2}{b_1+g_1 s_2}\right)\left(\frac{s_1+s_2}{b_1+g_1(s_1+s_2)}\right) \tag{26}$$

Higher order transfer functions can be obtained by applying the algorithm repeatedly.

3. Some Frequency Domain Applications

A. Cascade Systems

The derivation of nonlinear transfer functions of composite cascade systems is usually quite involved. Here, we derive an explicit formula for the nth order transfer function $H_n(s_1,s_2,\ldots,s_n)$ of the cascade system $H = F * K$ shown in Fig. 3. If we let $K_n(s_1,s_2,\ldots,s_n)$ and $F_n(s_1,s_2,\ldots,s_n)$

be the nth order transfer functions of systems K and F respectively, for all $n \geq 1$, then we have

$$H_n(s_1,s_2,\ldots,s_n) \doteq \sum_{m_n=0}^{1} \sum_{m_{n-1}=0}^{a_{n-1}} \cdots \sum_{m_2=0}^{a_2} \left(\frac{m!}{\prod_{i=1}^{n} m_i!} \right) \bar{F}_m(K_1)^{m_1}(K_2)^{m_2}\ldots(K_n)^{m_n} \tag{27}$$

where $m_1 = n - \sum_{i=2}^{n} im_i$, $m = \sum_{i=1}^{n} m_i$, and $a_p = In\left(\frac{n - \sum_{i=p+1}^{n} im_i}{p}\right)$,

$\forall p = 2,3,\ldots,n-1$.

Proof:

Decompose F into a linear component F_1 and a strictly nonlinear component F_N, such that $F = F_1+F_N$ as shown in Fig. 4. It is clear that the system F_N has exactly the same form given by (5). Therefore the nth order output transform of F_N with respect to u(t) is given by (18). The nth order output transform component of F_1 with respect to u(t) is $F_1(s_1+s_2+\ldots+s_n)X_n(s_1,s_2,\ldots,s_n)$. Adding this term to (18) gives the total nth order output transform for the composite system with respect to u(t). Substituting $X_i(s_1,s_2,\ldots,s_i) \doteq K_i(s_1,s_2,\ldots,s_i)U(s_1)U(s_2)\ldots U(s_i)$ into the above sum for all $i \leq n$ and divide the sum by $U(s_1)U(s_n)\ldots U(s_n)$, we obtain (27). ¤

B. Inverse Systems

Inverse systems have been investigated in [5-7, 10-12]. The cascade of a system and its inverse gives the identity system. To derive the nonlinear transfer functions of an inverse system in terms of the original system's transfer functions is often a complicated process, especially for high orders. Here, we will derive a recursive formula for calculating the nth order transfer function of an inverse system. Let the nth order transfer function of a system Y be $Y_n(s_1,s_2,\ldots,s_n)$, for all $n \geq 1$. Let its inverse system $Z = Y^{-1}$ be characterized by the transfer functions $Z_n(s_1,s_2,\ldots,s_n)$, for all $n \geq 1$. The nonlinear transfer functions of the inverse system are then given by:

$$Z_1(s_1) = \frac{1}{Y_1(s_1)} \tag{28a}$$

$$Z_n(s_1,s_2,\ldots,s_n) = \sum_{m_{n-1}=0}^{1} \sum_{m_{n-2}=0}^{a_{n-2}} \cdots \sum_{m_2=0}^{a_2} \left(\frac{-m!}{\prod_{i=1}^{n-1} m_i!} \right) \frac{\bar{Y}_m(Z_1)^{m_1}(Z_2)^{m_2}\ldots(Z_{n-1})^{m_{n-1}}}{Y_1(s_1,s_2,\ldots,s_n)}, \quad n \geq 2 \tag{28b}$$

where $m_1 = n - \sum_{i=2}^{n-1} im_i$, $m = \sum_{i=1}^{n-1} m_i$, and $a_p = In\left(\frac{n - \sum_{i=p+1}^{n-1} im_i}{p}\right)$,

$\forall p = 2,3,\ldots,n-2$.

Remark:

For n = 2, the number of summation signs in (28b) is zero. Hence

$$Z_2(s_1,s_2) = \frac{Y_2(Z_1)^2}{Y_1(s_1+s_2)} .$$

Proof:

Let a nonlinear element be characterized by nonlinear admittances $Y_n(s_1,s_2,\ldots,s_n)$, for all $n \geq 1$. By considering this element as a single-element nonlinear circuit, we can use the formulas from the preceding section to find the associated nonlinear impedances $Z_n(s_1,s_2,\ldots,s_n)$ for all $n \geq 1$. The resulting expressions are precisely given by (28). ¤

Remark:

It can be seen from (28), or from Remark 3 of the preceding algorithm, that the existence of all the transfer functions of the inverse of a system characterized by the transfer functions $Y_n(s_1,s_2,\ldots,s_n)$, for all $n \geq 1$, depends only on the existence of the inverse $\frac{1}{Y_1(s)}$ of the linear component. Nevertheless, the Volterra series so obtained may not necessarily converge.

Example 5.

Given the nth order admittances $Y_n(s_1,s_2,\ldots,s_n)$, n = 1,2,3, the impedances of the first three orders of the inverse system follow from (28):

$$Z_1(s_1) = \frac{1}{Y_1(s_1)} \tag{29a}$$

$$Z_2(s_1,s_2) \doteq -\frac{\bar{Y}_2(s_1,s_2)}{Y_1(s_1,s_2)Y_1(s_1)Y_1(s_2)} \tag{29b}$$

$$Z_3(s_1,s_2,s_3) \doteq \frac{2\bar{Y}_2(s_1,s_2+s_3)\bar{Y}_2(s_2,s_3)-\bar{Y}_3(s_1,s_2,s_3)Y_1(s_2+s_3)}{Y_1(s_1+s_2+s_3)Y_1(s_2+s_3)Y_1(s_1)Y_1(s_2)Y_1(s_3)} \tag{29c}$$

Example 6.

Here, let us derive the input impedances of the circuit shown in Fig. 2a. This has already been done in the last section using a different approach. First, notice that the total nth order admittance $Y_n(s_1,s_2,\ldots,s_n)$ of two voltage-controlled "admittance" elements connected in parallel is given by $Y_n(s_1,s_2,\ldots,s_n) \doteq Y'_n(s_1,s_2,\ldots,s_n) + Y''_n(s_1,s_2,\ldots,s_n)$, where $Y'_n(s_1,s_2,\ldots,s_n)$ and $Y''_n(s_1,s_2,\ldots,s_n)$ are the nth order admittances of the two admittance elements, respectively. Similarly, the total nth order impedance $Z_n(s_1,s_2,\ldots,s_n)$ of two current-controlled "impedance" elements connected in series is $Z_n(s_1,s_2,\ldots,s_n) \doteq Z'_n(s_1,s_2,\ldots,s_n) + Z''_n(s_1,s_2,\ldots,s_n)$, where $Z'(s_1,s_2,\ldots,s_n)$ and $Z''_n(s_1,s_2,\ldots,s_n)$ are the nth order impedances of the two impedance elements, respectively.

For the parallel inductor and resistor shown in Fig. 2a, the composite 1st order admittance is given by $Y_1(s_1) = \frac{b_1}{s_1} + g_1 = \frac{b_1+g_1s_1}{s_1}$, while the composite 2nd order admittance is given by $\bar{Y}_2(s_1,s_2) = \frac{b_2}{s_1+s_2} + g_2$. It follows from (29a) and (29b) that the 1st and 2nd order impedances of this composite parallel element are given as follows:

1st order impedance: $Z_1(s_1) = \frac{1}{Y_1(s_1)} = \frac{s_1}{b_1+g_1s_1}$

2nd order impedance: $Z_2(s_1,s_2) \doteq \frac{-\bar{Y}_2(s_1,s_2)}{Y_1(s_1+s_2)Y_1(s_1)Y_1(s_2)}$

$$= -\left(\frac{b_2}{s_1s_2} + g_2\right)\left(\frac{s_1}{b_1+g_1s_1}\right)\left(\frac{s_2}{b_1+g_1s_2}\right)\left(\frac{s_1+s_2}{b_1+g_1(s_1+s_2)}\right)$$

Thus, the 1st and 2nd order input impedances of the circuit are given by:

1st order impedance: $Z_{C1}(s_1) + Z_1(s_1) = \frac{a_1}{s_1} + \frac{s_1}{b_1+g_1s_1}$

2nd order impedance: $Z_{C2}(s_1,s_2) + Z_2(s_1,s_2)$

$$\doteq \frac{a_2}{s_1s_2} - \left(\frac{b_2}{s_1s_2} + g_2\right)\left(\frac{s_1}{b_1+g_1s_1}\right)\left(\frac{s_2}{b_1+g_1s_2}\right)\left(\frac{s_1+s_2}{b_1+g_1(s_1+s_2)}\right)$$

These results agree with those obtained earlier in the last section, as they should.

C. Feedback Systems

Let system H denotes the composite feedback system shown in Fig. 5. It has been shown in [6,7] that

$$H = F*(I+K*F)^{-1} = (F^{-1}+K)^{-1}, \tag{30}$$

where I denotes the identity operator. Since (30) involves only addition, cascade, and inverse operations, the nth order transfer functions of the composite system H can be easily obtained by using the formulas presented earlier.

D. Synthesis Problems

(i) Synthesis of Nonlinear Networks Via Feedback Systems

Here, we give a compact feedback representation of nonlinear circuits using the results from the last section. First, let us look at a circuit containing only one nonlinear element. Let the modified strictly nonlinear element be denoted by F_N. Following step 1 of the preceding algorithm, we obtain

$$\begin{bmatrix} Q(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} H_u^q(s) & H_{w_N}^q(s) \\ H_u^x & H_{w_M}^x(s) \end{bmatrix} \begin{bmatrix} U(s) \\ W_N(s) \end{bmatrix}$$

where $x(t)$ and $w_N(t)$ are the input and output, respectively, of the modified nonlinear element F_N. It follows from Fact 2 of the last section that $q_1(t)$ and $x_1(t)$ are the output of the linear systems represented by $H_u^q(s)$ and $H_u^x(s)$, respectively, with input $u(t)$. It also follows from Facts 1 and 3 that $\sum_{n=2}^{\infty} q_n(t)$ and $\sum_{n=2}^{\infty} x_n(t)$ are the outputs of the linear systems represented by $H_{w_N}^q(s)$ and $H_{w_N}^x(s)$, respectively, with input $W_N(s)$. From this information, we obtain the feedback representation of the nonlinear network as shown in Fig. 6. Consider next a circuit having one more nonlinear element denoted by K. Let the strictly nonlinear component of K be denoted by K_N. After modifying the nonlinear circuit, let $b(t)$ and $p_N(t)$ be the input and output of K_N. For the modified linear circuit, let

$$\begin{bmatrix} Q(s) \\ X(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} H_u^q(s) & H_{w_N}^q(s) & H_{p_N}^q(s) \\ H_u^x(s) & H_{w_N}^x(s) & H_{p_N}^x(s) \\ H_u^b(s) & H_{w_N}^b(s) & H_{p_N}^b(s) \end{bmatrix} \begin{bmatrix} U(s) \\ W_N(s) \\ P_N(s) \end{bmatrix}$$

Following the same procedure as above, we obtain the feedback representation shown in Fig. 7. Circuits containing more than 2 nonlinear elements can be similarly represented.

Remark:

In addition to synthesis applications, this subsection shows that the stability and analyticity of nonlinear circuits may be investigated using well-established feedback system theory by first transforming the circuit into a feedback system.

(ii) Synthesis of Inverse Systems

Let F_1 and F_N be the linear and strictly nonlinear components, respectively, of a system F; i.e., $F = F_1 + F_N$. The inverse F^{-1} of F can be synthesized by any one of the three feedback systems shown in Figs. 8a,b and c, where F_1^{-1} is the inverse of F_1. Fig. 8a is obtained by considering F to be a voltage controlled "admittance" element characterized by nonlinear admittances; i.e., $F_n(s_1,s_2,\ldots,s_n) = Y_n(s_1,s_2,\ldots,s_n)$. Let F be a single-element network. Choosing a current source as the input u(t) and the associated port voltage as the output q(t), we can synthesize this network by using Fig. 6. In the present case $q(t) = x(t)$, $H_u^x(s) = \frac{1}{Y_1(s)}$, and $H_{w_N}^x(s) = -\frac{1}{Y_1(s)}$. Thus Fig. 8a is obtained. Fig. 8b[7] is obviously equivalent to Fig. 8a. Fig. 8c use F rather than F_N in the synthesis. Since F_1^{-1} is a linear system, $F_1^{-1} * F_N = F_1^{-1} * (F-F_1) = F_1^{-1} * F^{-1} - F_1^{-1} * F_1 = F_1^{-1} * F - I$. Hence, Fig. 8c follows from 8a. An easier way of deriving Fig. 8b is to apply the feedback equation (30) of Fig. 5. Equation (30) shows that if we choose F^{-1} to be F_1 and K to be F_N, then $H = (F_1+F_N)^{-1} = F^{-1}$, and hence Fig. 8b is obtained.

(iii) Structure of Transfer Functions

In dealing with synthesis problems or circuit-theoretic properties, it is often desirable to know the general structure of the associated transfer functions. The results presented in the preceding section will provide some insights into the structure of nonlinear transfer functions.

A large class of nonlinear systems can be decomposed into a linear subsystem and a nonlinear subsystem such that all memory components are

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[7] Fig. 8b has been obtained in [5,6] via a different approach.

exclusively contained in the linear part. Thus, we let all transfer functions of the nonlinear elements in a nonlinear circuit be constants. It then follows from the preceding algorithm that the 1st order transfer function of the nonlinear circuit must assume the form:

$$A_1(s_1), \tag{31}$$

where $A_1(s)$ is a linear transfer function which is assumed to be a rational function of s. Likewise, the 2nd order transfer function must assume the form:

$$\sum_{i=1}^{m} B_{i1}(s_1+s_2)C_{i1}(s_1)C_{i1}(s_2),$$

where $B_{i1}(s)$ and $C_{i1}(s)$ are linear transfer functions for all $i = 1,2,\ldots,m$. Another structure which seems to be more general in appearance, but which is actually derivable from the above expression, is given by:

$$\sum_{i=1}^{m} B_{i1}(s_1+s_2)C_{i1}(s_1)D_{i1}(s_2), \tag{32}$$

where $B_{i1}(s)$, $C_{i1}(s)$ and $D_{i1}(s)$ are linear transfer functions, for all $i = 1,2,\ldots,m$. For example, if we let $B_{1,1}(s) = -B_{2,1}(s) = -B_{3,1}(s)$, $C_{1,1} = C'_{1,1}(s) + D'_{1,1}(s)$, $C_{2,1}(s) = C'_{1,1}(s)$ and $C_{3,1}(s) = D'_{1,1}(s)$, then $\sum_{i=1}^{3} B_{i1}(s_1+s_2)C_{i1}(s_1)C_{i1}(s_2) \doteq 2B_{1,1}(s_1+s_2)C'_{1,1}(s_1)D'_{1,1}(s_2)$ which has the structure defined in (32). Similarly, to derive the general structure of higher-order transfer functions, we assume all linear transfer functions involved can be expressed as a sum of other linear transfer functions. For example, $A_1(s_1) = \sum_{i=1}^{k} A_{i1}(s_1)$, where $A_{i1}(s_1)$ is a linear transfer function for all i ($A_{i1}(s_1)$ may be identically equal to zero for some i). In deriving the general structure of order n, we will also assume that all lower-order transfer functions exhibit the structures already obtained. For example, the general structure of the 3rd order transfer function is given by:

$$\sum_{i=1}^{a} E_{i1}(s_1+s_2+s_3)F_{i2}(s_1,s_2)G_{i1}(s_3)$$
$$+ \sum_{i=1}^{b} H_{i1}(s_1+s_2+s_3)P_{i1}(s_1)R_{i1}(s_2)T_{i1}(s_3), \tag{33}$$

where $E_{i1}(s)$, $G_{i1}(s)$, $H_{i1}(s)$, $P_{i1}(s)$, $R_{i1}(s)$ and $T_{i1}(s)$ are linear transfer functions, and $F_{i2}(s_1,s_2)$ is a 2nd order transfer function having a structure given by (32). A substitution of the structure of $F_{i2}(s_1,s_2)$ into (33) gives rise to the following structure:

$$\sum_{i=1}^{c} E_{i1}(s_1+s_2+s_3)K_{i1}(s_1+s_2)U_{i1}(s_1)V_{i1}(s_2)G_{i1}(s_3) + \sum_{i=1}^{b} H_{i1}(s_1+s_2+s_3)P_{i1}(s_1)R_{i1}(s_2)T_{i1}(s_3) \tag{34}$$

where $K_{i1}(s)$, $U_{i1}(s)$ and $V_{i1}(s)$ are linear transfer functions. It can be seen from the preceding algorithm that the general structure of an nth order transfer function consists of terms like

$$\sum_{i=1}^{d} A_{i1}(s_1+s_2+\ldots+s_n)B_{ik_1}(s_1,s_2,\ldots,s_{k_1})B_{ik_2}(s_{k_1+1},\ldots,s_{k_1+k_2})\ldots B_{ik_m}(s_{k_1+\ldots+k_{m-1}+1},\ldots,s_{k_1+\ldots+k_m}) \tag{35}$$

where $k_1+k_2+\ldots+k_m = n$ and $B_{ik_j}(s_1,\ldots,s_{k_j})$, $\forall j = 1,2,\ldots,m$, are k_jth order transfer functions. Since $k_i < n$, the general structure of an nth order transfer function can be obtained recursively. In particular, synthesis of a basic term such as $B_{i1}(s_1+s_2)C_{i1}(s_1)D_{i1}(s_2)$ in the summation of (32) is shown in Fig. 9. Similarly, syntheses of $E_{i1}(s_1+s_2+s_3)F_{i2}(s_1,s_2)G_{i1}(s_3)$ and $H_{i1}(s_1+s_2+s_3)P_{i1}(s_i)R_{i1}(s_2)T_{i1}(s_3)$ from each summation of (33) are shown in Figs. 10a and b, respectively.

Remarks:

1. Notice that the syntheses of these general transfer function structures require only the syntheses of linear transfer functions, as well as such operations as addition, multiplication and cascade among them. The only operation that generates a nonlinearity from linear systems is multiplication, which is memoryless.

2. Although the general structures of transfer functions are obtained by assuming the nonlinear elements in the original circuit to be memoryless, all nonlinear transfer functions derived in the literature for different systems can be shown to be special cases of the general structures presented in this paper.

3. A survey of [1] and the present paper shows that all common system operations can be synthesized or expressed explicitly in terms of the 4 basic operations of addition, multiplication, invertion and cascading between systems. It

follows from section 4 of [1] that for any of the above 4 basic operations, and their compositions, subsystems can be replaced by the corresponding equivalent subsystems without affecting the symmetric form of the nonlinear transfer functions of the overall system.

4. Since the structures of transfer functions are invariant under the above 4 basic system operations, our general structures of nonlinear transfer functions include a rather large class of analytic systems.

APPENDIX

Appendix 1. Proof of Lemma

Expression (14) represents a term in the multiple frequency domain expression of order $k_1+k_2+\ldots+k_m$. This term has a corresponding time domain component in $w_m(t)$ in (10) of order $k_1+k_2+\ldots+k_m$ with respect to $u(t)$. Now, $W_{Nn}(s_1,s_2,\ldots,s_n)$ represents the sum of all such terms in multiple frequency domain with $k_1+k_2+\ldots+k_m = n$, for all $n \geq m \geq 2$. However, $n = k_1+k_2+\ldots+k_m \geq m \geq 2$ and $k_i \geq 1$, for all $i = 1,2,\ldots,m$, imply that $n-1 \geq k_i \geq 1$ for all $i = 1,2,\ldots,m$. Hence, it follows from (17), all nth order terms assume the structure

$$\bar{F}_m(X_1)^{m_1}(X_2)^{m_2}\ldots(X_{n-1})^{m_{n-1}} \tag{A-1}$$

If $m_i \neq 0$, then there are m_i of X_i in $X_{k_1}X_{k_2}\ldots X_{k_m}$. Thus,

$m = m_1+m_2+\ldots+m_{n-1} = \sum_{i=1}^{n-1} m_i$ and $n = k_1+k_2+\ldots+k_m = \sum_{i=1}^{n-1} im_i$. It follows from (10) that each set of distinct permutations of $(k_1,k_2,\ldots,k_m)$ in (14) corresponds to a component in $W_{Nn}(s_1,s_2,\ldots,s_n)$. It follows from (A-1) that the number of such permutations is $\frac{(m_1+m_2+\ldots+m_{n-1})!}{m_1!m_2!\ldots m_{n-1}!} = \frac{m!}{m_1!m_2!\ldots m_{n-1}!}$

Equation (17) shows that we can add these together to give:

$$\frac{m!}{m_1!m_2!\ldots m_{n-1}!}\bar{F}_m(X_1)^{m_1}(X_2)^{m_2}\ldots(X_{n-1})^{m_{n-1}} \tag{A-2}$$

where $m = \sum_{i=1}^{n-1} m_i$ and $n = \sum_{i=1}^{n-1} im_i$. We will now show that (18) is the sum of all such terms given by (A-2), each term corresponding to a distinct vector $(m_1,m_2,\ldots,m_{n-1})$[8] such that $m_1+2m_2+\ldots+(n-1)m_{n-1} = n$. Since $m \triangleq \sum_{i=1}^{n-1} m_i$ (as defined in 18), it is sufficient to show that the summation signs in (18) will

---

[8]Each distinct vector $(m_1,m_2,\ldots,m_{n-1})$ corresponds to a distinct combination of the set $\{k_1,k_2,\ldots,k_m\}$.

generate all the distinct vectors satisfying $\sum_{i=1}^{n-1} im_i = n$. Let us prove a more general case; namely, for each $n \geq 2$, and given any nonnegative integer K, the following expression generates all vectors $(m_1, m_2, \ldots, m_{n-1})$ such that $\sum_{i=1}^{n-1} im_i = K$:

$$\sum_{m_{n-1}=0}^{In(\frac{K}{n-1})} \sum_{m_{n-2}=0}^{a_{n-2}} \cdots \sum_{m_2=0}^{a_2} (m_1, m_2, \ldots, m_{n-1}) \tag{A-3}$$

where $m_1 = K - \sum_{i=2}^{n-1} im_i$ and $a_p = In\left(\frac{K - \sum_{i=p+1}^{n-1} im_i}{p}\right)$, $\forall p = 2, \ldots, n-2$.

We prove this by induction. For $n = 2$, $m_1 = K$ is obviously the only possibility. For $n = 3$, m_2 can take on integer values from 0 to $In(\frac{K}{2})$ to satisfy $m_1+2m_2 = K$. For each fixed m_2, $m_1 = K-2m_2$. Thus, $\sum_{m_2=0}^{In(\frac{K}{2})} (m_1, m_2)$, where $m_1 = K-2m_2$, generates all distinct vectors (m_1, m_2) such that $m_1+2m_2 = K$. Assume (A-3) is true for $n = r$. Consider $n = r+1$. For any <u>fixed m_r</u>, it follows from the assumption of the $n = r$ case, that all vectors $(m_1, m_2, \ldots, m_r)$ which satisfy $m_1+2m_2+\ldots+rm_r = K$ (i.e., $m_1+2m_2+\ldots+(r-1)m_{r-1} = K-rm_r$) are given by:

$$\sum_{m_{r-1}=0}^{In(\frac{K-rm_r}{r-1})} \sum_{m_{r-2}=0}^{a_{n-2}} \cdots \sum_{m_2=0}^{a_2} (m_1, m_2, \ldots, m_{r-1}, m_r), \text{ where } m_1 = K - rm_r - \sum_{i=2}^{r-1} im_i$$

and $a_p = In\left(\frac{K-rm_r - \sum_{i=p+1}^{r-1} im_i}{p}\right)$, $\forall p = 2, 3, \ldots, r-2$. Since m_r can take on integer values from 0 to $In(\frac{K}{2})$, all vectors satisfying $\sum_{i=1}^{r} im_i = K$ must be given by

$$\sum_{m_r=0}^{In(\frac{K}{r})} \sum_{m_{r-1}=0}^{In(\frac{k-rm_r}{r-1})} \sum_{m_{r-2}=0}^{a_{r-2}} \cdots \sum_{m_2=0}^{a_2} (m_1, m_2, \ldots, m_r)$$

where $m_1 = K - rm_r - \sum_{i=2}^{r-1} im_i = K - \sum_{i=2}^{r} im_i$, and $a_p = In\left(\frac{K-rm_r - \sum_{i=p+1}^{r-1} im_i}{p}\right)$,

$\forall p = 2,3,\ldots,r-2$. Since m_r can take on integer values from 0 to $In(\frac{K}{2})$, all vectors satisfying $\sum_{i=1}^{r} im_i = K$ must be given by:

$$\sum_{m_r=0}^{In(\frac{K}{r})} \sum_{m_{r-1}=0}^{In(\frac{K-rm_r}{r-1})} \sum_{m_{r-2}=0}^{a_{r-2}} \sum_{m_2=0}^{a_2} (m_1,m_2,\ldots,m_r)$$

where $m_1 = K - rm_r - \sum_{i=2}^{r-1} im_i = K - \sum_{i=2}^{r} im_i$, and $a_p = In\left(\frac{K-rm_r - \sum_{i=p+1}^{r-1} im_i}{p}\right)$,

$\forall p = 2,3,\ldots,r-2$. By setting $a_{r-1} = In\left(\frac{K-rm_r}{r-1}\right)$, we have $a_p = In\left(\frac{K - \sum_{i=p+1}^{r} im_i}{p}\right)$, $\forall p = 2,3,\ldots,r-1$. Therefore (A-3) is also true for $n = r+1$. Thus, we have proved (A-3) by induction. Finally, (18) corresponds simply to $K = n$, and hence must also be true. ∎

Appendix 2 Formulation of Transfer Functions for a Bridge Network

We will now use the algorithm presented in section 2 to find the nonlinear transfer functions of the bridge network shown in Fig. 11a. The input is a current source i(t) and the output is the voltage $v_R(t)$ across the nonlinear resistor. Let the time domain and frequency domain characterization of the two nonlinear elements be as follows:

Inductor:

time domain: $\phi_L(t) = \sum_{n=1}^{\infty} a_n i_L^n(t)$, where ϕ_L denotes the inductor flux.

frequency domain: impedance $\bar{Z}_{Ln}(s_1,s_2,\ldots,s_n) = a_n(s_1+s_2+\ldots+s_n)$, $\forall n \geq 1$

Resistor:

time domain: $i_R(t) = \sum_{n=1}^{\infty} g_n v_R^n(t)$

frequency domain: admittance $\bar{Y}_{Rn}(s_1,s_2,\ldots,s_n) = g_n$, $\forall n \geq 1$

Let[9] $a_1 = 1H$, $a_2 = \frac{1}{3}$ HA, $g_1 = \frac{1}{2}\,\Omega^{-1}$ and $g_2 = \frac{1}{5}\,\Omega^{-1}V$. The modified linear circuit with the input and nonlinear sources are shown in Fig. 11b. Let us denote the nth order transfer function to be derived by $H_{in}^{V_R}(s_1,s_2,\ldots,s_n)$. Following the 1st step of the algorithm we write:

$$\begin{bmatrix} V_R(s) \\ I_L(s) \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} -2s^2+2 & -3s^2-6 & -3s \\ 2s+4 & 3s & -2s \end{bmatrix} \begin{bmatrix} I(s) \\ I_{N_R}(s) \\ V_{N_L}(s) \end{bmatrix} \qquad \text{(A-4)}$$

where $\Delta(s) = 2s^2+3s+4$.

Following step 2 of the algorithm we have:

$$[V_{R1}(s_1) \;\; I_{L1}(s_1)]^T = \frac{1}{\Delta(s_1)} [-2s_1^2+2 \;\; 2s_1+4]^T I(s_1) \qquad \text{(A-5)}$$

Following step 3, the 1st order output transform is:

$$V_{R1}(s_1) = \frac{-2s_1^2+2}{\Delta(s_1)} I(s_1) \qquad \text{(A-6)}$$

---

[9]Notice that the unit of a_n is HA^{n-1}, where H stands for henery and A stands for amp., which the unit of g_n is $\Omega^{-1}V^{n-1}$, where V stands for voltage.

The 1st order transfer function is:

$$H_{i1}^{v_R}(s_1) = \frac{V_{R1}(s_1)}{I(s_1)} = \frac{-2s_1^2+2}{\Delta(s_1)} \quad \text{(A-7)}$$

To derive the 2nd order transfer function we follow step 5 to find the 2nd order nonlinear sources:

$$\begin{bmatrix} I_{N_R 2}(s_1,s_2) \\ V_{N_L 2}(s_1,s_2) \end{bmatrix} = \begin{bmatrix} \bar{Y}_{R2}(s_1,s_2)V_{R1}(s_1)V_{R1}(s_2) \\ \bar{Z}_{L2}(s_1,s_2)I_{L1}(s_1)I_{L1}(s_2) \end{bmatrix} = \frac{I(s_1)I(s_2)}{\Delta(s_1)\Delta(s_2)} \begin{bmatrix} \frac{1}{5}(-2s_1^2+2)(-2s_2^2+2) \\ \frac{1}{3}(s_1+s_2)(2s_1+4)(2s_2+4) \end{bmatrix}$$

(A-8)

Following step 6:

$$\begin{bmatrix} V_{R2}(s_1,s_2) \\ I_{L2}(s_1,s_2) \end{bmatrix} = \frac{I(s_1)I(s_2)}{\Delta(s_1+s_2)\Delta(s_1)\Delta(s_2)} \begin{bmatrix} -3(s_1+s_2)^2-6 & -3(s_1+s_2) \\ 3(s_1+s_2) & -2(s_1+s_2) \end{bmatrix} \begin{bmatrix} \frac{1}{5}(-2s_1^2+2)(-2s_2^2+2) \\ \frac{1}{3}(s_1+s_2)(2s_1+4)(2s_2+4) \end{bmatrix}$$

(A-9)

Following step 3, the 2nd order output transform is:

$$V_{R2}(s_1,s_2) \doteq \frac{\frac{1}{5}[-3(s_1+s_2)^2-6](-2s_1^2+2)(-2s_2^2+2)-(s_1+s_2)^2(2s_1+4)(2s_2+4)}{\Delta(s_1+s_2)\Delta s_1)\Delta(s_2)} I(s_1)I(s_2)$$

(A-10)

Therefore, the 2nd order transfer function is:

$$H_{i2}^{v_R}(s_1,s_2) \doteq \frac{V_{R2}(s_1,s_2)}{I(s_1)I(s_2)} \doteq \frac{\frac{1}{5}[-3(s_1+s_2)^2-6](-2s_1^2+2)(-2s_2^2+2)-(s_1+s_2)^2(2s_1+4)(2s_2+4)}{\Delta(s_1+s_2)\Delta(s_1)\Delta(s_2)} \quad \text{(A-11)}$$

We may continue to follow the algorithm to find higher order transfer functions.

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FIGURE CAPTIONS

Fig. 1 Decomposition of nonlinear elements into linear and nonlinear components: (a) impedance element, (b) voltage controlled voltage source, (c) current controlled voltage source, (d) admittance element, (e) voltage controlled current source, and (f) current controlled current source.

Fig. 2 (a) A nonlinear circuit whose nth order input impedances are to be found, and (b) the modified linear circuit with the input and the nonlinear sources

Fig. 3 Equivalent system resulting from cascading two systems

Fig. 4 Decomposition of F into $F_1 + F_n$ in the cascade system

Fig. 5 Equivalent system resulting from a feedback system

Fig. 6 Synthesis of a circuit with one nonlinear element

Fig. 7 Synthesis of a circuit with two nonlinear elements

Fig. 8 Three equivalent syntheses of the inverse system F^{-1}

Fig. 9 Synthesis of a typical term from the general expression of a 2nd order transfer function

Fig. 10 Synthesis of two typical terms from the general expression of a 3rd order transfer function

Fig. 11 (a) A nonlinear bridge network whose nth order transfer functions $H_{in}^{v_R}(s_1,s_2,\ldots,s_n)$ are to be found, and (b) the modified linear network with the input and the nonlinear sources.

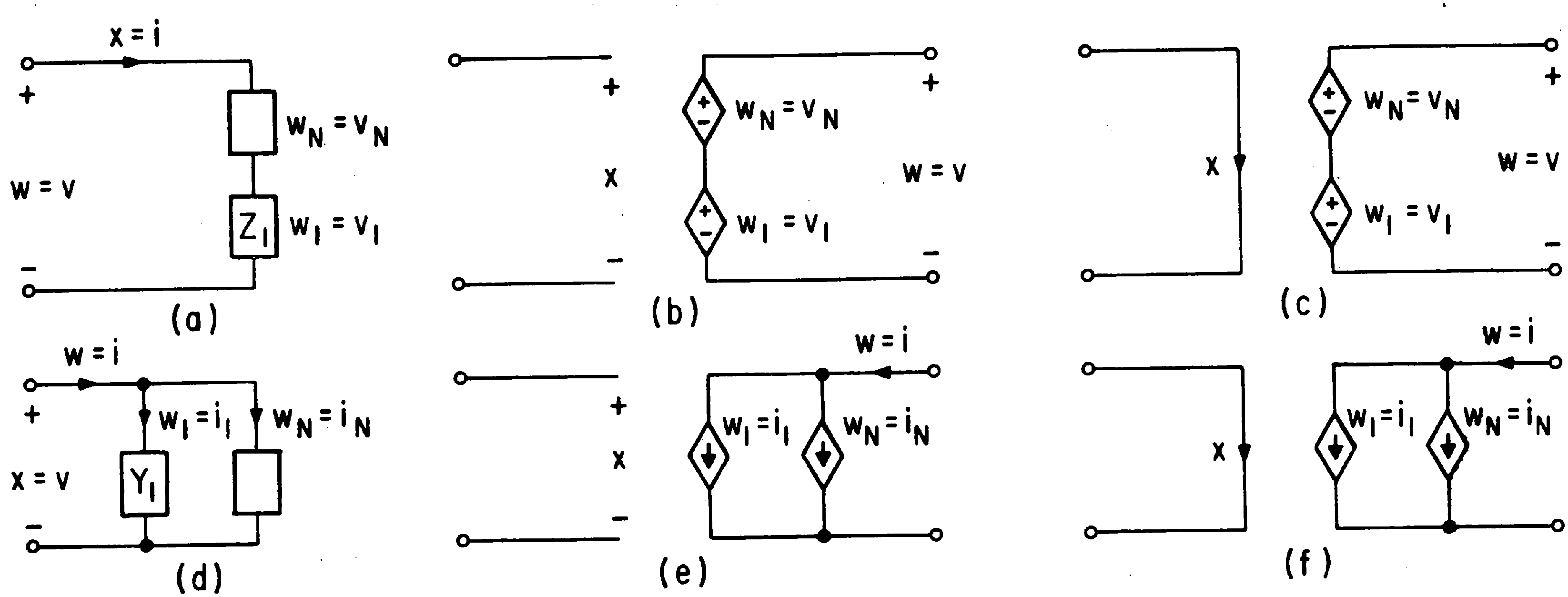

x = i
w = v
$w_N = v_N$
Z_1
$w_1 = v_1$
(a)
x
(b)
w = v
(c)
w = i
$w_1 = i_1$
$w_N = i_N$
x = v
Y_1
(d)
(e)
(f)

Fig. 1

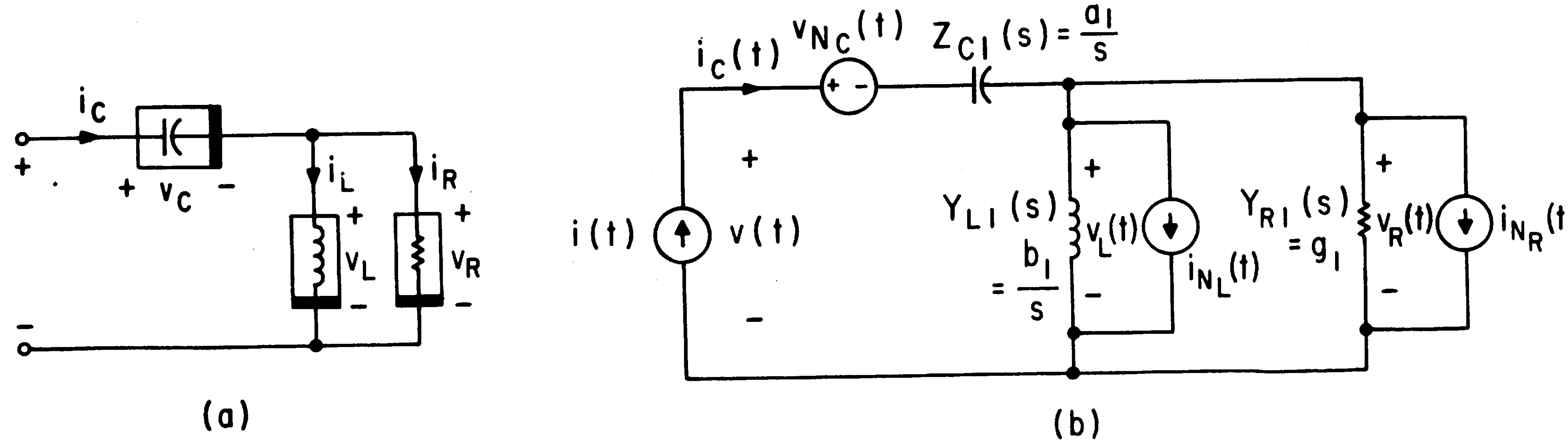

i_C
v_C
i_L
i_R
v_L
v_R
(a)
$i_C(t)$
$v_{NC}(t)$
$Z_{C1}(s) = \frac{a_1}{s}$
i(t)
v(t)
$Y_{L1}(s) = \frac{b_1}{s}$
$v_L(t)$
$i_{N_L}(t)$
$Y_{R1}(s) = g_1$
$v_R(t)$
$i_{N_R}(t)$
(b)

Fig. 2

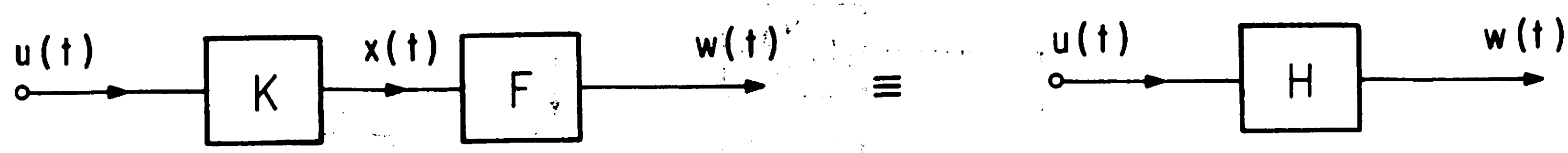


Fig. 3

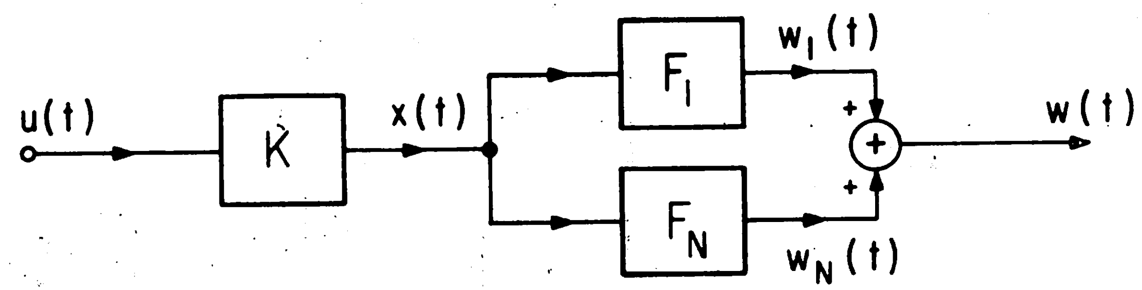


Fig. 4

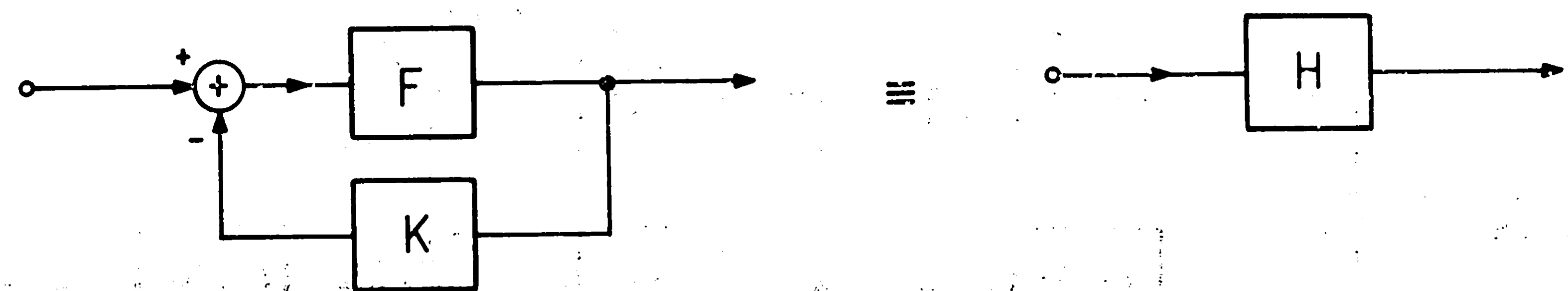
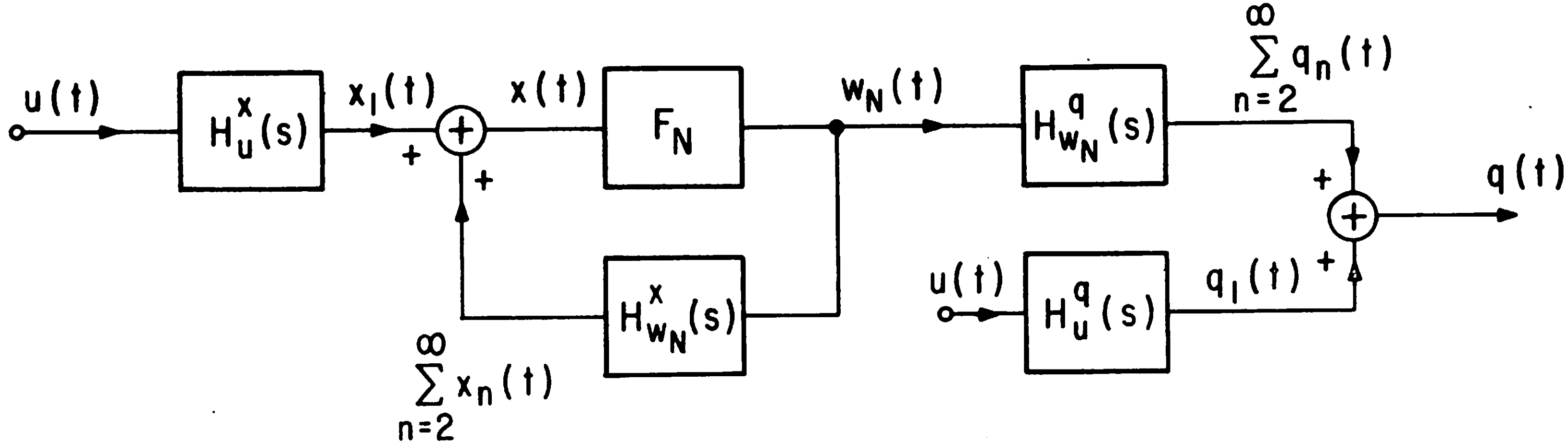
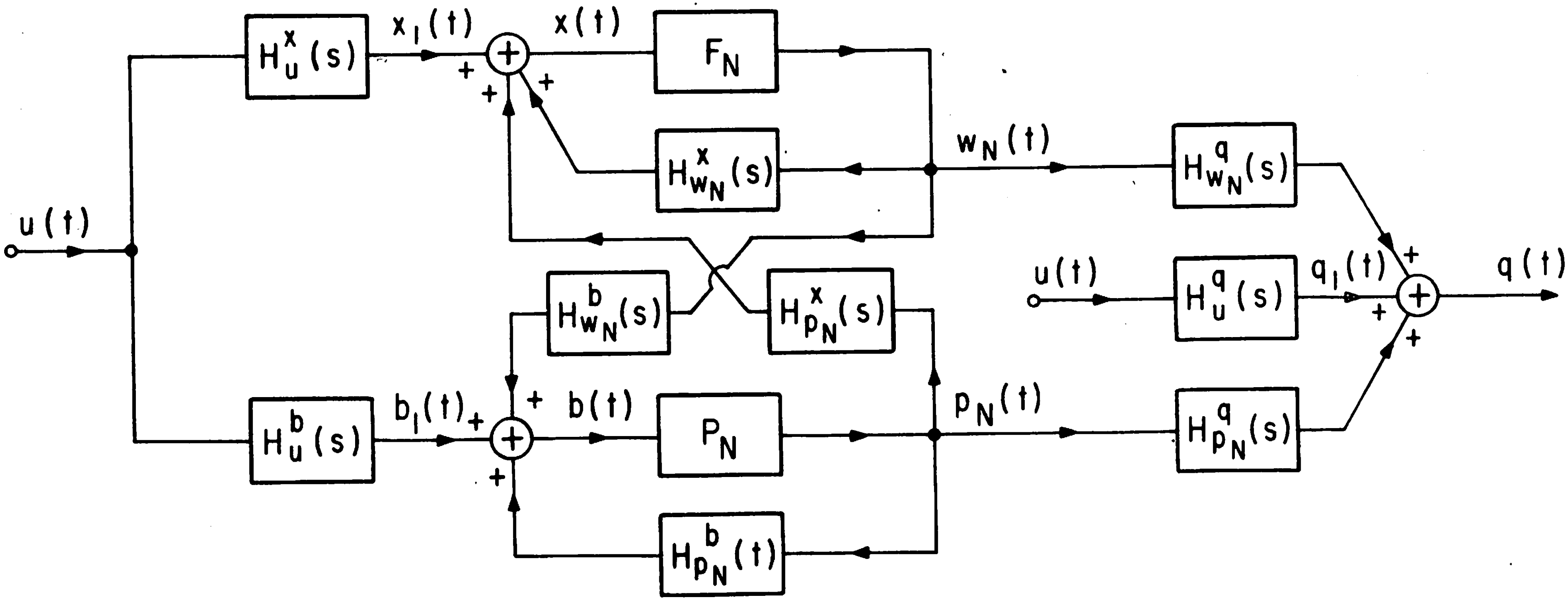


Fig. 5



u(t)
$H_u^x(s)$
$x_1(t)$
+
+
x(t)
F_N
$w_N(t)$
$H_{w_N}^x(s)$
$\sum_{n=2}^{\infty} x_n(t)$
$H_{w_N}^q(s)$
$\sum_{n=2}^{\infty} q_n(t)$
u(t)
$H_u^q(s)$
$q_1(t)$
+
+
q(t)

Fig. 6



u(t)
$H_u^x(s)$
$x_1(t)$
x(t)
F_N
$H_{w_N}^x(s)$
$w_N(t)$
$H_{w_N}^b(s)$
$H_{p_N}^x(s)$
$H_u^b(s)$
$b_1(t)$
b(t)
P_N
$p_N(t)$
$H_{p_N}^b(t)$
$H_{w_N}^q(s)$
u(t)
$H_u^q(s)$
$q_1(t)$
$H_{p_N}^q(s)$
q(t)
+

Fig. 7

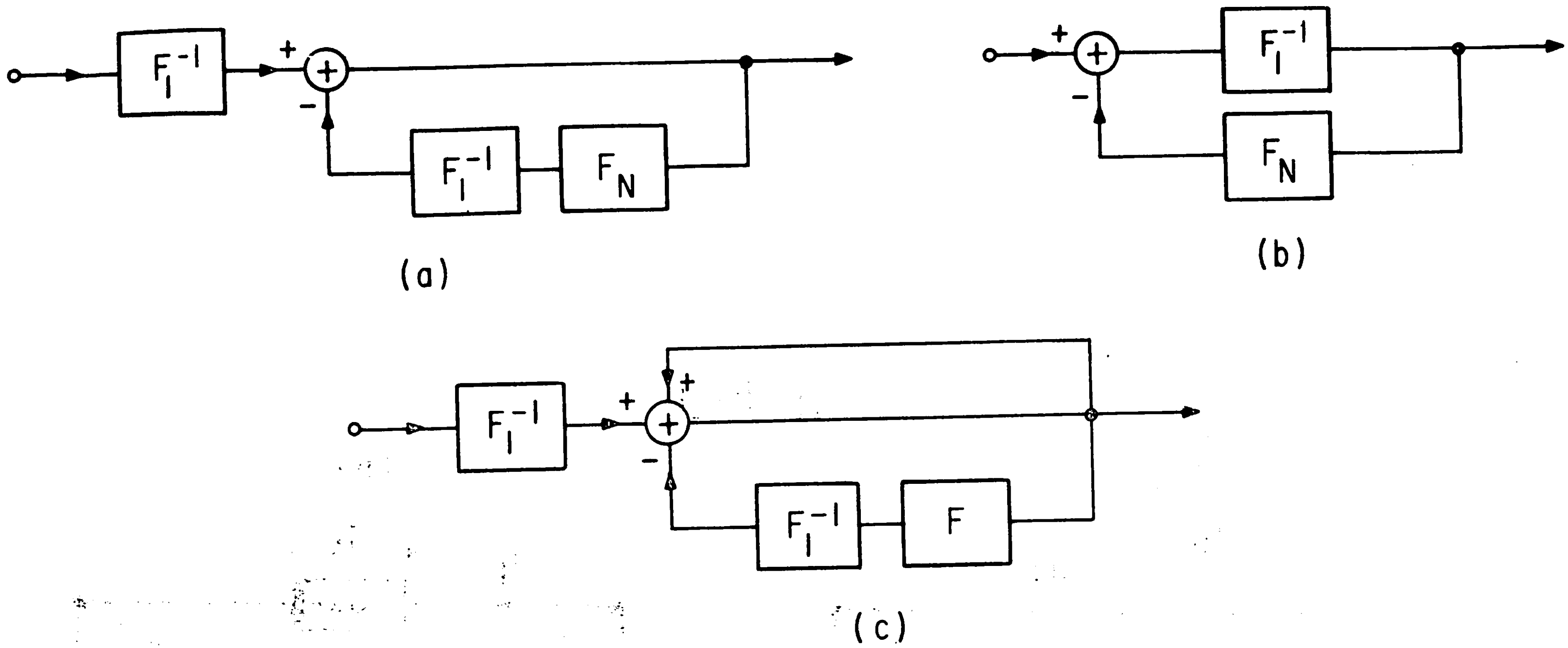


Fig. 8

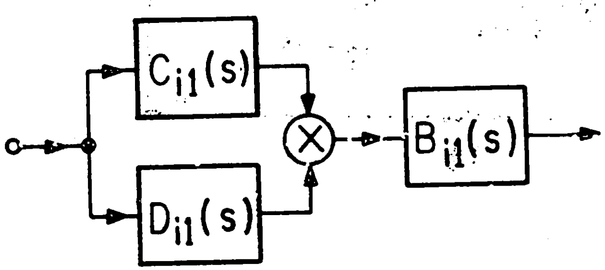


Fig. 9

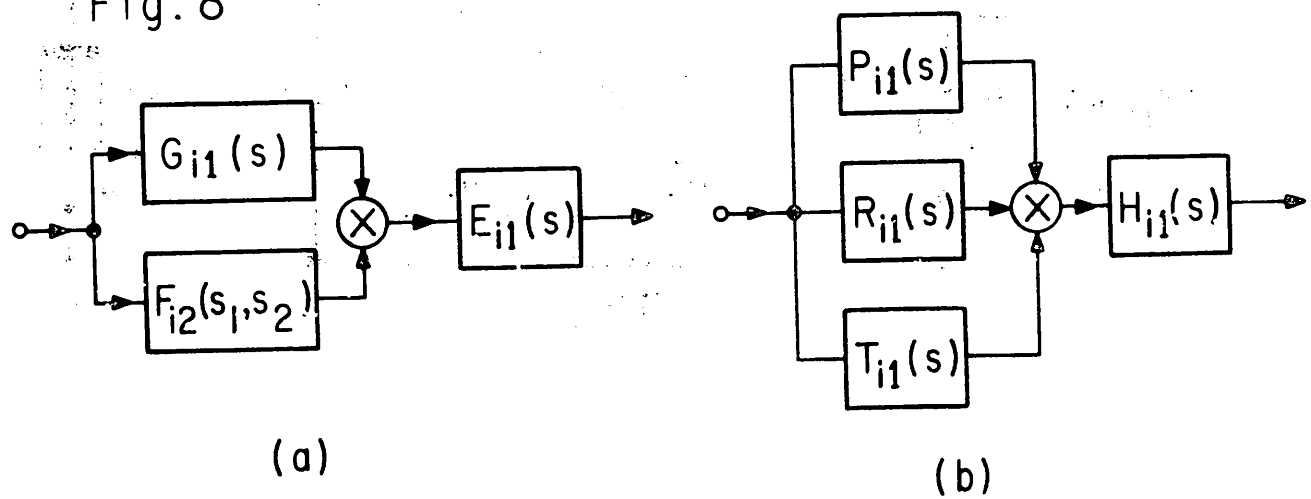


Fig. 10

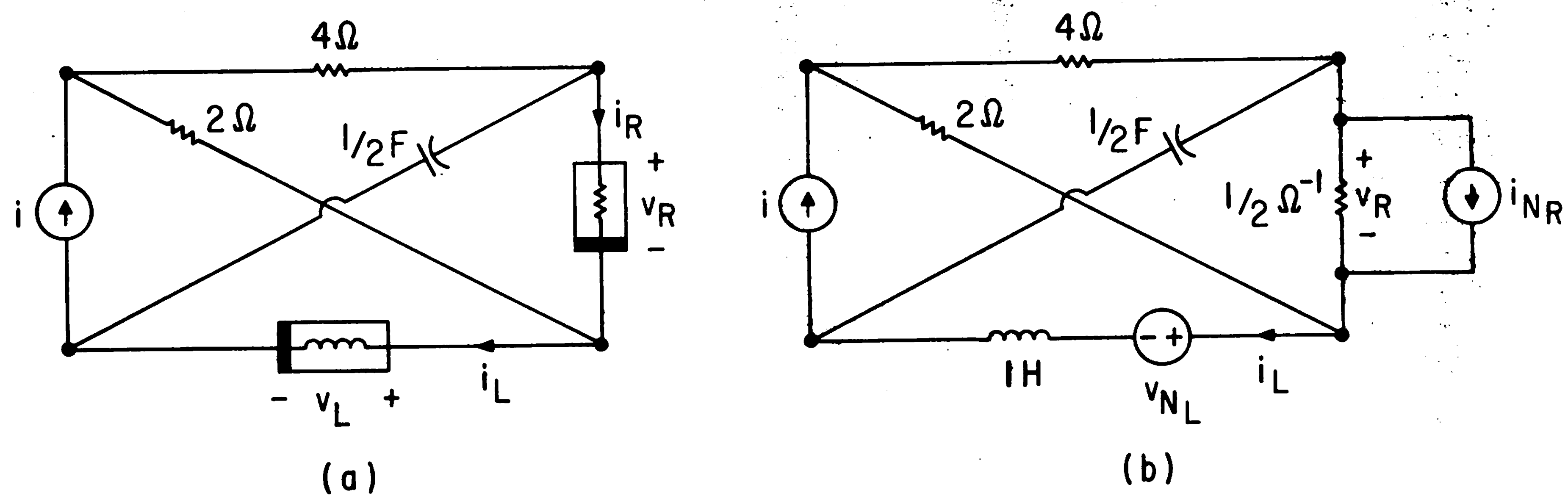


Fig. 11