LIAR'S PARADOX AND TRUTH-QUALIFICATION PRINCIPLE

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Memorandum No. UCB/ERL M79/34

18 May 1979

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1. Introduction. Stated in its "naked" and most elementary form, Liar's paradox arises as a result of a self-referential definition of a proposition p by the assertion

(1) $$p \triangleq p \text{ is false}$$

where the symbol $\triangleq$ stands for "is defined to be."

There is a voluminous literature dealing with various issues relating to self-referential definitions of the form (1). The analyses of Liar's paradox which are particularly relevant to that presented in this note are those of Bochvar [1], van Fraassen [13], Skyrms [11], Kearns [7], Herzberger [5], Martin [8], Chihara [2], Pollock [9], Swiggart [12] and Haack [4].

Our approach to Liar's paradox is in the spirit of approaches employing three-valued logic, but is more general in that (1) is treated as a special case of a self-referential definition in fuzzy logic, FL, [14], [15], [16], [3] having the form

(2) $$p \triangleq p \text{ is } \tau$$

where τ is a truth-value whose denotation is a fuzzy subset of the set of truth-values of Lukasiewicz's L_{Aleph_1} logic,[1] and p is a proposition whose meaning is characterized by a possibility distribution -- which is induced by p -- over a universe of discourse U. The manner in which the concept of a possibility distribution may be employed to characterize the meaning of p is described in 2.

*Research supported by the National Science Foundation Grant MCS77-07568.

The principle of truth-qualification in fuzzy logic serves to provide a mechanism for the computation of the possibility distribution induced by the proposition "p is τ" from the knowledge of the possibility distributions induced by p and τ. By employing this principle, the self-referential definition (2) may be translated into a fixed-point equation which upon solution yields the possibility distribution of p for a given τ. As shown in 2, this solution is not, in general, an admissible proposition in two-valued logic. Furthermore, for certain τ, the solution does not exist, leading in the special case of (1) to Liar's paradox.

The transformation of a self-referential definition of the form (2) into a fixed-point equation whose solution is the possibility distribution of p has the effect of clarifying the basic issues arising in Liar's paradox and, perhaps, supplies its resolution. The basic ideas of the method by which (2) is transformed into a fixed-point equation are described in the following section.

2. Possibility distributions, truth-qualification principle and Liar's paradox

Our analysis of Liar's paradox is based in an essential way on the concept of a possibility distribution.[2] Informally, if X is a variable taking values in a universe of discourse U, then by a possibility distribution, Π_X, which is associated with X, is meant a fuzzy subset of U which plays the role of an elastic constraint on the values that may be assumed by X. Thus, if u is a point in U and $\mu_X(u)$ is the grade of membership of u in Π_X, then the possibility that X may take the value u is a number in the interval [0,1], denoted by $\pi_X(u)$, which is numerically equal to $\mu_X(u)$. The function π_X: $U \rightarrow [0,1]$ is termed the possibility distribution function, and a variable which is associated with a possibility distribution is called a fuzzy variable. Thus, if X is a fuzzy variable, we have, by definition,

$$\text{(3)} \qquad \mathrm{Poss}\{X = u\} = \pi_X(u)$$

where π_X is the possibility distribution function which characterizes Π_X.

The elastic constraint on the values of X may be physical or epistemic in nature. For example, if X represents the number of tennis balls that may be squeezed into a metal box, then Π_X is determined by physical constraints. On the other hand, if X is characterized by the proposition "X is small," where, SMALL, the denotation of small, is a fuzzy subset of the interval $[0,\infty)$, then Π_X is an epistemic possibility distribution such that $\pi_X(u)$ -- the degree of possibility or, simply, the possibility that $X = u$ -- is equal to $\mu_{SMALL}(u)$, the grade of membership of u in SMALL. More generally, if p is a proposition of the form

$$p \triangleq X \text{ is } F$$

where X takes values in U and F is a fuzzy subset of U, then we write

$$\text{(4)} \qquad X \text{ is } F \rightarrow \Pi_X = F$$

where the arrow stands for "translates into" and the right-hand member of (4) constitutes a possibility assignment equation. Equation (4) implies that Π_X is induced by the proposition "X is F" and that

$$\text{(5)} \qquad \pi_X(u) \triangleq \mathrm{Poss}\{X = u\} = \mu_F(u)$$

where $\mu_F\colon U \rightarrow [0,1]$ is the membership function which characterizes F.

In our analysis of Liar's paradox, we shall be concerned with propositions of the general form $p \triangleq N$ is F, where F is a fuzzy subset of the cartesian product $U_1 \times \cdots \times U_n$ of a collection of universes of discourse $U_1, \ldots, U_n$, and N is the name of an object, a variable or a proposition. In this case, the translation of p assumes the more general form

$$\text{(6)} \qquad N \text{ is } F \rightarrow \Pi_{(X_1,\ldots,X_n)} = F$$

where $X = (X_1, \ldots, X_n)$ is an n-ary variable which is implicit or explicit in N, with X_i taking values in U_i, $i = 1,\ldots,n$. To illustrate:

$$\text{(7)} \qquad \text{Naomi is young} \rightarrow \Pi_{\text{Age(Naomi)}} = \text{YOUNG}$$

where the variable Age(Naomi) is implicit in the left-hand member of (7) and YOUNG is a fuzzy subset of the interval $[0,100]$. Similarly,

$$\text{(8)} \qquad \text{John is big} \rightarrow \Pi_{(\text{Height(John)},\text{Weight(John)})} = \text{BIG}$$

where the variables Height(John) and Weight(John) are implicit, and BIG is a fuzzy subset of the product space $[0,200] \times [0,100]$ (with the height and weight assumed to be expressed in centimeters and kilograms, respectively). In general, then, a proposition of the form $p \triangleq N$ is F induces a possibility distribution of a variable $X = (X_1, \ldots, X_n)$ which is implicit or explicit in N, with F defining the distribution in question. In this sense, the meaning of the proposition "N is F" is defined by the possibility assignment equation (6), which is an instance of an expression in the meaning representation language PRUF.[3]

An important aspect of fuzzy logic relates to the ways in which the meaning of a proposition may be modified through the employment of (a) modifiers such as not, very, more or less, somewhat, etc.; and (b) qualifiers exemplified by true, false, quite true, very likely, quite possible, etc. In particular, in the case of modifiers, the pertinent rule may be stated as follows:

If m is a modifier and the translation of $p \triangleq N$ is F is of the form

$$N \text{ is } F \rightarrow \Pi_X = F \tag{9}$$

then the translation of the modified proposition $p^\dagger = N$ is mF is given by

$$N \text{ is } mF \rightarrow \Pi_X = F^\dagger \tag{10}$$

where $F^\dagger$ is a modification of F. In particular,

(a) if $m \triangleq$ not then

$$F^\dagger = F' = \text{complement of } F\ , \tag{11}$$

(12) i.e.,

$$\mu_{F'}(u) = 1 - \mu_F(u)\ , \quad u \in U \tag{12}$$

(b) if $m \triangleq$ very then[4]

$$F^\dagger = F^2\ , \tag{13}$$

(14) i.e.,

$$\mu_{F^\dagger}(u) = (\mu_F(u))^2\ , \quad u \in U \tag{14}$$

and (c) if $m =$ more or less, then

$$F^\dagger = \sqrt{F}\ , \tag{15}$$

(16) i.e.,

$$\mu_{F^\dagger}(u) = \sqrt{\mu_F(u)}\ , \quad u \in U\ . \tag{16}$$

The main point at issue in the case of Liar's paradox is the manner in which the meaning of a proposition is affected by truth-qualification. In this connection, let τ denote a linguistic truth-value, e.g., true, false, very true, not quite true, more or less true, etc., with the understanding that (a) the denotation of τ is a possibility distribution Π_τ over the unit interval $[0,1]$, and (b) once Π_{true} is specified, the denotation of τ may be computed in terms of the denotation of true through the application of a semantic rule [16]. For example, if the denotation of true is Π_{true}, then the denotation of not very true is expressed by

$$\Pi_{\text{not very true}} = (\Pi^2_{true})' \tag{17}$$

Similarly, the denotation of false, which is the antonym of true, is defined by

$$\pi_{false}(v) = \pi_{true}(1-v)\ , \quad 0 \leq v \leq 1 \tag{18}$$

where π_{false} and π_{true} are the possibility distribution functions of false and true, respectively.

Within the conceptual framework of fuzzy logic, the notion of truth-value serves, in the main, to provide a measure of the compatibility of possibility distributions. More specifically, if (p,r) is an ordered pair of propositions such that p and r induce the possibility distributions Π^p and Π^r, respectively, then the truth-value of p relative to the reference proposition r is defined as the compatibility of Π^p with Π^r, which in turn is defined by the equation

$$\text{Comp}(\Pi^p/\Pi^r) = \pi_p(\Pi^r) \tag{19}$$

where π_p is the possibility distribution function characterizing Π^p and the right-hand member of (19) expresses a possibility distribution whose possibility distribution function is given by [15]

$$\pi_\tau(v) = \mathrm{Sup}_u\ \pi_r(u)\ , \quad u \in U \tag{20}$$

subject to

$$v = \pi_p(u)\ , \quad v \in [0,1]\ .$$

The content of the definitions expressed by (19) and (20) may be stated more transparently in the form of an assertion which for convenience will be referred to as the <u>truth-qualification</u> <u>principle</u>. More specifically, let π_τ denote the possibility distribution function of a truth-value τ, and let Π^p be the possibility distribution induced by a proposition p over a universe of discourse U.[5] Then the truth-qualification principle asserts that:

(a) The possibility distribution, Π^q, induced by the truth-qualified proposition q,

$$q \triangleq p \text{ is } \tau\ , \tag{21}$$

is given by

$$\pi_q(u) = \pi_\tau(\pi_p(u))\ , \quad u \in U \tag{22}$$

where π_p and π_q are the possibility distribution functions of Π^p and Π^q, respectively.

(b) Proposition q is semantically equivalent[6] to the reference proposition r, that is,

(23) $$p \text{ is } \tau \leftrightarrow r$$

where r is the proposition with respect to which the truth-value of p is τ.

As a simple illustration of (22) and (23), consider the propositions:

$$p \triangleq \text{Susan is young}$$

$$p \triangleq \text{Susan is young is very true}$$

where young and true are defined by

(24) $$\pi_{young}(u) = (1+(\frac{u}{25})^2)^{-1}\ , \quad u \geq 0$$

(25) $$\pi_{true}(v) = (1+(\frac{1-v}{0.3})^2)^{-1}\ , \quad 0 \leq v \leq 1$$

Then by (14)

(26) $$\pi_{very\ true}(v) = (\pi_{true}(v))^2$$

and by (22)

(27) $$\pi_q(u) = (1+(\frac{1-(1+(\frac{u}{25})^2)^{-1}}{0.3})^2)^{-2}$$

which may be roughly approximated as

(28) $$\pi_q(u) \cong (\pi_{young}(u))^2$$

Thus, the proposition "Susan is young" has the truth-value very true with respect to the reference proposition r whose possibility distribution function is expressed by (27) and which is approximately semantically equivalent to "Susan is very young."

To apply the truth-qualification principle to Liar's paradox, consider a proposition p which is defined self-referentially as

(29) $$p \triangleq p \text{ is } \tau$$

with the understanding that the denotation of the truth-value τ is a possibility distribution over the unit interval, and that p induces a possibility distribution Π^p over a universe of discourse U.

On applying (22) to (29), we find that the possibility distribution functions associated with p and τ must satisfy the identity

(30) $$\pi_p(u) = \pi_\tau(\pi_p(u)) \ , \quad u \in U$$

which implies that π_p is a fixed point of the mapping π_τ: $[0,1] \rightarrow [0,1]$.

From (30) it follows at once that when

(31) $$\pi_\tau(v) = v \ , \quad v \in [0,1]$$

we have, for all p,

(32) $$p \text{ is } \tau \leftrightarrow p$$

The possibility distribution described by (31) defines a unitary truth-value which is denoted as u-true. Then,

(33) $$p \text{ is u-true} \leftrightarrow p$$

which in two-valued logic corresponds to

(34) $$p \text{ is true} \leftrightarrow p$$

The antonym of u-true is u-false, which is defined by

(35) $$\pi_{\text{u-false}}(v) = 1 - v \ , \quad v \in [0,1] \ .$$

We are now ready to raise the question "What is the proposition which is defined self-referentially by

(36) $$p \triangleq p \text{ is u-false ?}$$ "

On applying (30) to (36), we have

(37) $$\pi_p(u) = 1 - \pi_p(u) \ , \quad u \in U$$

which implies that

(38) $$\pi_p(u) = 0.5 \ , \quad u \in U \ .$$

Thus, the proposition which satisfies the self-referential definition of Liar's paradox is characterized by a uniform possibility distribution which is expressed by (38). It should be noted that p is not a proposition in two-valued logic.

In a similar vein, we may consider propositions which are defined self-referentially by strengthened or weakened forms of (36), e.g.,

(39) (a) $$p \triangleq p \text{ is very u-false}$$

(40) (b) $$q \triangleq q \text{ is more or less u-false}$$

In this case, on making use of (14), (16) and (30), we deduce

(41) $$\pi_p(u) = (1 - \pi_p(u))^2$$

and

(42) $$\pi_q(u) = \sqrt{1 - \pi_q(u)}$$

which lead, respectively, to the solutions

(43) $$\pi_p(u) = \frac{3-\sqrt{5}}{2}, \quad u \in U$$
$$\cong 0.38$$

and

(44) $$\pi_q(u) = \frac{-1+\sqrt{5}}{2}, \quad u \in U$$
$$\cong 0.62$$

More generally, if in (29) we set

$$\tau \triangleq \text{false}$$

where <u>false</u> is interpreted as a specified possibility distribution over the unit interval, then (29) becomes

(45) $$p \triangleq p \text{ is false}$$

and the corresponding fixed-point equation reads

(46) $$\pi_p(u) = \pi_{false}(\pi_p(u)), \quad u \in U,$$

where π_{false} is the possibility distribution function which characterizes <u>false</u>.

Sufficient conditions for (45) to have a non-null solution are: (i) $\pi_{false}(0) > 0$ and (ii) π_{false} is continuous. Furthermore, if π_{false} is monotone non-increasing -- which is a property that the denotation of <u>false</u> would normally be expected to have -- the solution of (45) is unique. In general, this unique solution does not define an admissible proposition in two-valued logic.

It is easy to construct a truth-value, τ, for which the fixed-point equation (30) does not have a solution other than the null solution

(47) $$\pi_p(u) \equiv 0 \ , \quad u \in U$$

For example,

(48) $$\begin{aligned} \pi_\tau(v) &= v^2 \quad , \quad 0 \le v \le 0.5 \\ &= (1-v)^2 \ , \quad 0.5 < v < 1 \end{aligned}$$

or, more compactly,

(49) $$\pi_\tau(v) = \mathrm{Min}(v^2,(1-v)^2) \ , \quad 0 \le v \le 1$$

which represents the linguistic truth-value

(50) $$\tau = \text{very u-true and very u-false}$$

In this case, the only solution of (30) is the null solution (47). Furthermore, (30) has no solution when π_τ is discontinuous at, say, $v = \beta$, $0 < \beta < 1$, and

$$\begin{aligned} \pi_\tau(v) &> v \ , \quad 0 \le v < \beta \\ \pi_\tau(v) &< v \ , \quad \beta < v \le 1 \end{aligned}$$

In such cases, then, it is the non-existence of a solution of the fixed-point equation (30) that leads to paradoxes of the Liar and strengthened Liar types.

In summary, the application of truth-qualification principle to a self-referential definition of the form

$$p \triangleq p \text{ is } \tau$$

where τ is a truth-value whose denotation is a possibility distribution over $[0,1]$, leads to the result that π_p, the possibility distribution function which is induced by p, is a solution of the fixed-point equation

(52) $$\pi_p(u) = \pi_\tau(\pi_p(u)) \ , \quad u \in U$$

In general, the solution of this equation is a uniform possibility distribution characterized by a possibility distribution function of the form

$$\pi_p(u) = \alpha \,, \quad u \in U$$

where α is a constant in the interval $[0,1]$ which is determined by τ. For some τ, however, (52) does not have a solution, in which case p does not exist, leading to the Liar and related paradoxes.

Notes

[1]More generally, the denotation of a truth-value in fuzzy logic may be a fuzzy subset of the set of truth-values of a multi-valued logic which serves as a base logic for the fuzzy logic [16].

[2]In contrast to the concept of possibility in modal logic and possible world semantics [6],[10], the possibilities associated with a possibility distribution take values in the interval [0,1] or, more generally, in a partially ordered set. The theory of possibility which is based on the concept of a possibility distribution parallels the theory of probability but, unlike the latter, is not rooted in repeated experimentation or subjective perception of likelihood. A preliminary exposition of possibility theory may be found in [18].

[3]PRUF is a relation-manipulating language which is based on the theory of fuzzy sets and, more particularly, the theory of possibility [17]. An expression in PRUF is, in general, a procedure which computes a possibility distribution or a fuzzy relation. One of the important uses of PRUF relates to the precisiation of meaning of utterances in a natural language. As a language, PRUF is considerably more expressive than first-order predicate calculus and, in particular, allows the use of fuzzy quantifiers exemplified by <u>many</u>, <u>most</u>, <u>few</u>, <u>several</u>, etc.; fuzzy truth-values, e.g., <u>very</u> <u>true</u>, <u>more or less</u> <u>true</u>, <u>quite</u> <u>false</u>, etc.; fuzzy probabilities, e.g., <u>likely</u>, <u>unlikely</u>, <u>very</u> <u>unlikely</u>, etc.; and fuzzy possibilities, e.g., <u>quite</u> <u>possible</u>, <u>almost</u> <u>impossible</u>, etc.

[4]The expressions for $F^{\dagger}$ corresponding to m = <u>very</u> and m = <u>more or less</u> should be regarded as default definitions, i.e. standardized definitions which, when necessary, may be replaced by other more elaborate or

context-dependent characterizations of $F^{\dagger}$ as a function of F.

[5]In the special case where τ is a numerical truth-value, say $\tau = \alpha$, $\alpha \in [0,1]$, the possibility distribution of τ is expressed as $\pi_{\tau}(v) = 1$ for $v = \alpha$, $\pi_{\tau}(v) = 0$ for $v \neq \alpha$. In this case, it is not merely possible but certain that α is the value of τ.

[6]Semantic equivalence of q and r, denoted as $q \leftrightarrow r$, implies and is implied by the equality $\Pi^q = \Pi^r$.

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