CHARACTERIZATION OF FINITE FUZZY MEASURES USING MARKOFF-KERNELS

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Abstract

Generalizing the definitions of L.A. Zadeh [4] and R. Lowen and E.P. Klement [3] a larger class of finite fuzzy measures is defined. It is shown that these fuzzy measures can be characterized in a unique way by a finite (classical) measure and a Markoff-kernel.

Key words: fuzzy sets, fuzzy measures, probability theory

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1. INTRODUCTION

Fuzzy probability was originally introduced by L.A. Zadeh [4] in 1968. He started with a classical probability space (X,A,P) and for each fuzzy event μ, that is a measurable function $\mu: X \to [0,1]$, he defined the probability of μ by

$$m(\mu) = \int \mu dP \ . \qquad (1)$$

More recently, the author [2] studied fuzzy σ-algebras. The most important among them are the so-called generated fuzzy σ-algebras which consist of all fuzzy sets being measurable functions with respect to some classical σ-algebra.

Together with R. Lowen [3] he gave an axiomatic definition of fuzzy probability measures and showed that in the case of a generated fuzzy σ-algebra such a fuzzy probability measure is an integral in the sense of (1) if and only if some condition (J) is fulfilled which guarantees a kind of differentiability of the measure.

In this paper we study now a much larger class of finite fuzzy measures m (not only probability measures) and show that they can be characterized in a unique way by

$$m(\mu) = \int K(x,[0,\mu(x)[)dP(x) \ ,$$

where P is some finite measure and K denotes a Markoff-kernel.

2. BASIC DEFINITIONS AND NOTATIONS

$(X,\mathcal{A})$ will denote a measurable space, that is a non-empty set X equipped with a σ-algebra $\mathcal{A}$ of subsets of X. $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$, $\mathcal{B}\cap[0,1]$ and $\mathcal{B}\cap[0,1[$ are the σ-algebras of Borel subsets of $[0,1]$ and $[0,1[$, respectively.

According to [2] we write $\sigma = \xi(\mathcal{A})$ for the fuzzy σ-algebra generated by $\mathcal{A}$, that is the family of fuzzy sets $\mu: X \to [0,1]$ where μ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}\cap[0,1]$. (In this paper we restrict ourselves to the case of generated fuzzy σ-algebras.)

A _fuzzy probability measure_ was defined in [3] to be a map $m: \sigma \to [0,1]$ fulfilling these axioms:

$$\forall\alpha \text{ constant: } \quad m(\alpha) = \alpha \tag{2}$$

$$\forall\mu \in \sigma: \quad m(1-\mu) = 1 - m(\mu) \tag{3}$$

$$\forall\mu,\nu \in \sigma: \quad m(\mu \vee \nu) + m(\mu \wedge \nu) = m(\mu) + m(\nu) \tag{4}$$

$$\forall(\mu_n)_{n\in\mathbb{N}} \subset \sigma,\ \mu \in \sigma: \quad (\mu_n)_{n\in\mathbb{N}} \uparrow \mu \ \Rightarrow\ (m(\mu_n))_{n\in\mathbb{N}} \uparrow m(\mu) \tag{5}$$

3. CHARACTERIZATION OF FUZZY PROBABILITY MEASURES

It was shown in [3] that a fuzzy probability measure is an integral, i.e. there exists a probability measure P on $(X,\mathcal{A})$ such that

$$\forall\mu\in\sigma: \quad m(\mu) = \int \mu \, dP$$

if and only if this condition (J) is fulfilled: for each $A \in \mathcal{A}$ there exists a number $u(A) \in [0,1]$ such that

(i) $\forall(\mu,\alpha) \in \sigma \times [0,1]$:

$$\mu^{-1}(]\alpha,1]) = A \ \Rightarrow\ \lim_{\beta\downarrow\alpha} \frac{m(\mu\wedge\beta) - m(\mu\wedge\alpha)}{\beta - \alpha} = u(A)$$

(ii) $u(A) + u(A^c) = 1$

(Note that this condition is sufficient only if the fuzzy σ-algebra is generated.)

Now let us consider counterexample 1 in [3]: In this example we have $X = [0,1]$, $\mathcal{A} = \mathcal{B} \cap [0,1]$ and $\sigma = \xi(\mathcal{B})$. P_0 and P_1 denote the probability measures concentrated in 0 and 1, respectively, i.e.

$$P_0(\{0\}) = P_1(\{1\}) = 1 \; .$$

The fuzzy probability measure m is defined by

$$m(\mu) = \int [(\tfrac{1}{4} \wedge \mu) + (\tfrac{3}{4} \vee \mu)] dP_0 + \int \tfrac{3}{4} \wedge (\tfrac{1}{4} \vee \mu) dP_1 - 1 \; .$$

m does not fulfill condition (J) because for

$$\mu = 1_{\{0\}}$$

we have

$$\mu^{-1}(]\tfrac{1}{2},1]) = \mu^{-1}(]\tfrac{7}{8},1]) = \{0\} \; ,$$

but

$$\lim_{\beta \downarrow \frac{1}{2}} \frac{m(\mu \wedge \beta) - m(\mu \wedge \frac{1}{2})}{\beta - \frac{1}{2}} = 0$$

and

$$\lim_{\beta \downarrow \frac{7}{8}} \frac{m(\mu \wedge \beta) - m(\mu \wedge \frac{7}{8})}{\beta - \frac{7}{8}} = 1 \; .$$

But it turns out that, if we choose the probability measure P which is uniquely determined by

$$P(\{0\}) = P(\{1\}) = \frac{1}{2}$$

and the function $K: X \times [0,1] \longrightarrow \mathbb{R}$ specified by

$$K(0,\alpha) = \begin{cases} 2\alpha & \text{if } \alpha \le \frac{1}{4} \\ \frac{1}{2} & \text{if } \frac{1}{4} \le \alpha \le \frac{3}{4} \\ 2\alpha - 1 & \text{if } \alpha \ge \frac{3}{4} \end{cases}$$

$$K(1,\alpha) = \begin{cases} 0 & \text{if } \alpha \le \frac{1}{4} \\ 2\alpha - \frac{1}{2} & \text{if } \frac{1}{4} \le \alpha \le \frac{3}{4} \\ 1 & \text{if } \alpha \ge \frac{3}{4} \end{cases}$$

and

$$K(x,\alpha) = \alpha \quad \text{if} \quad x \in]0,1[\quad \text{and} \quad \alpha \in [0,1] \; ,$$

we get the following characterization of m:

$$\forall \mu \in \sigma: \quad m(\mu) = \int K(x,\mu(x))dP(x) \; .$$

(Note that, because of $]0,1[$ being a P-null-set, in the case of $x \in]0,1[$ for $K(x,\cdot)$ each measurable function can be chosen without any change in the result.)

4. MARKOFF-KERNELS

Examining the functions $K(0,\cdot)$ and $K(1,\cdot)$ (which are the only significant ones) we realize that they are just probability distribution functions on $[0,1]$. Since a probability distribution function determines a probability measure, this observation leads us to the study of kernels, especially of Markoff-kernels, which are a powerful instrument in probability theory to describe conditional distributions, Markoff-processes, etc.

A kernel (from (X,A) to $([0,1[, \mathcal{B}\cap[0,1[)$) is a function

$$K: X \times \mathcal{B}\cap[0,1[\longrightarrow \mathbb{R}$$

such that these conditions are fulfilled:

$$\forall B \in \mathcal{B}\cap[0,1[: \quad K(\cdot,B): X \longrightarrow \mathbb{R} \quad \text{is } A\text{-}\mathcal{B}\text{-measurable} \qquad (6)$$
$$x \longrightarrow K(x,B)$$

$$\forall x \in X: \quad K(x,\cdot): \mathcal{B}\cap[0,1[\longrightarrow \mathbb{R} \quad \text{is a measure} \qquad (7)$$
$$B \longrightarrow K(x,B)$$

A kernel is called a Markoff-kernel iff

$$\forall x \in X: \quad K(x,[0,1[) = 1 \ , \qquad (8)$$

that means that $K(x,\cdot)$ is a probability measure for each $x \in X$. For more details about kernels we refer to [1].

5. FINITE FUZZY MEASURES

Now let P be a finite measure on (X,A) and K a Markoff-kernel from (X,A) to $([0,1[, \mathcal{B}\cap[0,1[)$.

Lemma. The function

$$m: \sigma \longrightarrow \mathbb{R}$$
$$\mu \longrightarrow \int K(x,[0,\mu(x)[)dP(x)$$

fulfills these properties

$$m(0) = 0 \qquad (9)$$

$$\forall \mu,\nu \in \sigma: \quad m(\mu \vee \nu) + m(\mu \wedge \nu) = m(\mu) + m(\nu) \qquad (10)$$

$$\forall (\mu_n)_{n\in\mathbb{N}} \subset \sigma,\ \mu \in \sigma: \quad (\mu_n)_{n\in\mathbb{N}} \uparrow \mu \ \Rightarrow \ (m(\mu_n))_{n\in\mathbb{N}} \uparrow m(\mu) \qquad (11)$$

Proof. First of all we denote that the function

$$K(\cdot,[0,\mu(\cdot)[): X \longrightarrow \mathbb{R}$$
$$x \longrightarrow K(x,[0,\mu(x)[)$$

is measurable for each $\mu \in \sigma$ because of the measurability of both $K(\cdot,B)$ and μ. Hence the integral

$$\int K(x,[0,\mu(x)[)dP(x)$$

always exists and m is well-defined. (9) is obviously fulfilled because of

$$K(x,\phi) = 0 \ .$$

To show (10) it is sufficient to know that for any $A, B \in \mathcal{B} \cap [0,1[$ and for each $x \in X$

$$K(x,A \cup B) + K(x,A \cap B) = K(x,A) + K(x,B)$$

holds, which is a consequence of the additivity of the measure $K(x,\cdot)$. The proof of (11) follows immediately by the continuity from below of the probability measure $K(x,\cdot)$ and by Levi's theorem of monotone convergence. ■

It is obvious that, in general, m does not fulfill properties (2) and (3), even if P is a probability measure. For example it is sufficient to consider this special Markoff-kernel

$$K(x,B) = \begin{cases} 1 & \text{if} \quad 0 \in B \\ 0 & \text{if} \quad 0 \notin B \end{cases} \quad (x \in X, \ B \in \mathcal{B} \cap [0,1[)$$

and an arbitrary probability measure P. For the constant fuzzy set

$$\mu = \frac{1}{2}$$

we get

$$m(\mu) = 1 \ ,$$

which violates both (2) and (3).

Conversely, it is straightforward that each fuzzy probability measure fulfills conditions (9)-(11).

So we can give this

Definition. A map $m: \sigma \to \mathbb{R}$ is called a finite fuzzy measure if and only if it fulfills (9)-(11).

6. CHARACTERIZATION OF FINITE FUZZY MEASURES

The following theorem establishes the main result of this paper: each finite fuzzy measure can be characterized by a finite measure and a Markoff-kernel.

Theorem. Let m be a finite fuzzy measure. Then there exists one and only one finite measure P on (X,A) and a P-almost everywhere uniquely determined Markoff-kernel K such that

$$\forall \mu \in \sigma: \quad m(\mu) = \int K(x,[0,\mu(x)[)dP(x) \quad . \tag{12}$$

Proof. (1) First we show that for each $\alpha \in \mathbb{Q} \cap [0,1]$

$$P_\alpha: A \longrightarrow \mathbb{R}$$
$$A \longrightarrow m(\alpha \wedge 1_A)$$

is a finite measure on (X,A): For each $\alpha \in \mathbb{Q} \cap [0,1]$ we obviously have

$$P_\alpha(\phi) = m(0) = 0 \ ,$$
$$P_\alpha(X) = m(\alpha) < \infty \ ,$$
$$P_\alpha(A) \geq 0 \ .$$

To prove the σ-additivity of P_α let $(A_n)_{n \in \mathbb{N}} \subset A$ be a sequence of pairwise disjoint sets. Using (10) and (11) we get

$$\begin{aligned} P_\alpha(\bigcup_{n\in\mathbb{N}} A_n) &= m(\sup_{k\in\mathbb{N}} (\alpha \wedge 1_{\bigcup_{m=1}^{k} A_n})) \\ &= \sup_{k\in\mathbb{N}} m(\alpha \wedge 1_{\bigcup_{n=1}^{k} A_n}) \\ &= \sup_{k\in\mathbb{N}} \sum_{n=1}^{k} m(\alpha \wedge 1_{A_n}) \\ &= \sum_{n=1}^{\infty} P_\alpha(A_n) \ . \end{aligned}$$

(2) Now we put

$$P = P_1$$

and show that for each $\alpha \in \mathbb{Q} \cap [0,1]$, P_α is absolutely continuous with respect to P. In order to do that we choose an $\alpha \in \mathbb{Q} \cap [0,1]$ and an $A \in \mathcal{A}$ and assume $P(A) = 0$. Then $P_\alpha(A) = 0$ follows by

$$0 \leq P_\alpha(A) = m(\alpha \wedge 1_A) \leq m(1_A) = P(A) = 0 \ .$$

(3) This allows us to apply Radon-Nikodym's theorem telling that for each $\alpha \in \mathbb{Q} \cap [0,1]$ there exists an $\mathcal{A}$-$\mathcal{B}$-measurable function $f_\alpha: X \to \mathbb{R}$ such that

$$\forall A \in \mathcal{A}: \quad P_\alpha(A) = \int_A f_\alpha dP \ . \tag{13}$$

Now we remember the following property of the Lebesgue-integral:

$$(\forall A \in \mathcal{A} \quad \int_A f dP = \int_A g dP) \Rightarrow f = g \quad P\text{-a.e.}$$

Using it leads us to these results:

$$f_0 = 0 \quad P\text{-a.e.},$$

$$f_1 = 1 \quad P\text{-a.e.},$$

$$\forall \alpha \in \mathbb{Q} \cap [0,1]: \quad f_\alpha = \sup_{\beta \in \mathbb{Q} \cap [0,\alpha[} f_\beta \quad P\text{-a.e.}$$

For our construction of the Markoff-kernel K we must have that these equalities hold everywhere. That can be easily done by changing (if necessary) the values of the functions f_α in a P-null-set to get the desired overall equalities. Of course, for these modified functions (13) still holds.

(4) Now we are able to define for each $\alpha \in [0,1]$

$$g_\alpha = \sup_{\beta \in \mathbb{Q} \cap [0,\alpha]} f_\beta .$$

Note that each g_α is the supremum of a countable family of measurable functions and hence itself measurable. We also have for each $\alpha \in [0,1]$ and each $A \in \mathcal{A}$

$$P_\alpha(A) = \int_A g_\alpha dP$$

because of (11) and Levi's theorem. Furthermore, for each $x \in X$

$$\begin{aligned} h_x\colon [0,1] &\to \mathbb{R} \\ \alpha &\to g_\alpha(x) \end{aligned}$$

is a probability distribution function which determines in a unique way a probability measure Q_x on $([0,1[, \mathcal{B} \cap [0,1[)$ fulfilling

$$Q_x([\alpha,\beta[) = h_x(\beta) - h_x(\alpha) \qquad (\alpha,\beta \in [0,1],\ \alpha < \beta) .$$

(5) Putting $\begin{aligned} K\colon X \times \mathcal{B} \cap [0,1[&\to \mathbb{R} \\ (x,B) &\to Q_x(B) \end{aligned}$ it is trivial that $K(x,\cdot)$ is a probability measure for each $x \in X$.

In order to show that $K(\cdot,B)$ is measurable for each $B \in \mathcal{B} \cap [0,1[$ we first prove that

$$\mathcal{D} = \{C \mid C \in \mathcal{B} \cap [0,1[,\ K(\cdot,C) \text{ is } \mathcal{A}\text{-}\mathcal{B}\text{-measurable}\}$$

is a Dynkin-system on $[0,1[$: $[0,1[$ belongs to $\mathcal{D}$ because of the measurability of

$$K(x,[0,1[) = Q_x([0,1[) = 1 \qquad (x \in X) .$$

Given $C, D \in \mathcal{D}$ such that $C \subset D$ we have

$$K(x,D\setminus C) = Q_x(D\setminus C) = Q_x(D)\setminus Q_x(C) = K(x,D) - K(x,C) \qquad (x \in X)$$

which implies that $K(\cdot,D\setminus C)$ is measurable, too.

Finally, if $(C_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of $\mathcal{D}$, we get

$$K(x, \bigcup_{n\in\mathbb{N}} C_n) = Q_x(\bigcup_{n\in\mathbb{N}} C_n) = \sum_{n=1}^{\infty} Q_x(C_n) = \sum_{n=1}^{\infty} K(x,C_n) \qquad (x \in X)$$

and hence the measurability of $K(\cdot, \bigcup_{n\in\mathbb{N}} C_n)$.

Because of the measurability of

$$K(x,[\alpha,\beta[) = Q_x([\alpha,\beta[) = h_x(\beta) - h_x(\alpha) = g_\beta(x) - g_\alpha(x)$$

for any $\alpha, \beta \in [0,1]$, $\alpha < \beta$ and any $x \in X$ it follows that the Dynkin-system $\mathcal{D}$ contains

$$\{[\alpha,\beta[\mid \alpha,\beta \in [0,1],\ \alpha < \beta\} ,$$

which is a $\cap$-stable generator of the σ-algebra $\mathcal{B} \cap [0,1[$. On the other hand, $\mathcal{D}$ is a subset of $\mathcal{B} \cap [0,1[$. Now a well-known classical result establishes the equality of $\mathcal{D}$ and $\mathcal{B} \cap [0,1[$. Hence K is a Markoff-kernel.

(6) Next we show that property (12) is fulfilled: if $\mu \in \sigma$ is a step function, i.e.

$$\mu = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

(A_i pairwise disjoint), we get

$$\begin{aligned} m(\mu) &= \sum_{i=1}^{n} m(\alpha_i \wedge 1_{A_i}) = \sum_{i=1}^{n} \int_{A_i} g_{\alpha_i} dP \\ &= \sum_{i=1}^{n} \int_{A_i} K(x,[0,\alpha_i[)dP(x) = \int_{A_i} \sum_{i=1}^{n} K(x,[0,\alpha_i[)\cdot 1_{A_i}(x)dP(x) \\ &= \int K(x,[0,\mu(x)[)dP(x) . \end{aligned}$$

For an arbitrary $\mu \in \sigma$ there exists always an increasing sequence $(s_n)_{n\in\mathbb{N}}$ of step functions such that

$$\mu = \sup_{n\in\mathbb{N}} s_n .$$

Then we have

$$m(\mu) = \sup_{n\in\mathbb{N}} m(s_n) = \sup_{n\in\mathbb{N}} \int K(x,[0,s_n(x)[)dP(x) = \int K(x,[0,\mu(x)[)dP(x) .$$

(7) The uniqueness of the measure P follows by

$$P(A) \doteq m(1_A) \qquad (A \in A) ,$$

the P-almost everywhere uniqueness of K follows directly from Radon-Nikodym's theorem. ■

An immediate consequence of this theorem is that each fuzzy probability measure defined on a generated fuzzy σ-algebra can be characterized by a probability measure and a Markoff-kernel, regardless whether it fulfills condition (J) or not.

Finally we note that it had not led to a larger class of fuzzy measures if we had admit general kernels instead of Markoff-kernels, as long as the result was still a finite measure, i.e. the function $K(\cdot,[0,1[)$ was integrable with respect to P. It is easily seen that in this case properties (9)-(11) are fulfilled, too. But we would lose the uniqueness of the measure P (and hence the P-almost everywhere uniqueness of the kernel K) in our theorem.

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