A STRUCTURE PRESERVING MODEL FOR
POWER SYSTEM STABILITY ANALYSIS

by

A. R. Bergen and D. J. Hill

Memorandum No. UCB/ERL M79/44

15 August 1979

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A STRUCTURE PRESERVING MODEL FOR POWER SYSTEM STABILITY ANALYSIS

A.R. Bergen, Member, IEEE

D.J. Hill, Member, IEEE

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract - A new model for the study of power system stability via Lyapunov functions is proposed. The key feature of the model is an assumption of frequency-dependent load power, rather than the usual impedance loads which are subsequently absorbed into a reduced network. The original network topology is explicitly represented. This approach has the important advantage of rigorously accounting for real power loads in the Lyapunov functions. This compares favorably with existing methods involving approximations to allow for the significant transfer conductances in reduced network models. The preservation of network topology can be exploited in stability analysis, with the concepts of critical and vulnerable cutsets playing central roles in dynamic and transient stability evaluation respectively. Of fundamental importance is the feature that the Lyapunov functions give a true representation of the spatial distribution of stored energy in the system.

I. INTRODUCTION

The analysis of power system transient stability using Lyapunov function techniques has recently achieved a status as a viable tool for on-line security assessment. Particularly promising results are reported by Gupta and El-Abiad [1], Ribbens-Pavalla, et. al. [2], and Athay, et. al. [3]. The first on-line application to a real operating power system is discussed by Saito et. al. [4]. This follows efforts beginning around 1970 to apply Lyapunov methods to realistic multimachine power systems [5-8] and some twenty years of interest in such an approach -- see surveys by Ribbens-Pavella [9] and Fouad [10]. A major difficulty which remains to be overcome rigorously is that associated with allowing for significant transfer conductances. This is essentially an issue of modeling the loads in the network. The present paper offers a new model which can bypass this difficulty while maintaining the features related to the success obtained in previous work.

While many of the assumptions made to arrive at the usual classical model for transient stability analysis are reasonable, that of ignoring transfer conductances is usually quite crude [10]. This emanates from modeling the loads as impedances (with a substantial resistive component). These are then absorbed into the bus admittance matrix for a reduced network based on generator buses. Thus, although the original transmission network is very reasonably modeled as lossless, the reduced network certainly cannot be in general. Consequently, a path-independent potential function is not readily available for construct- Lyapunov functions. Attempts to develop general Lyapunov functions have met very limited success, especially when it is considered that ultimately these functions should replace those based on assuming the conductances are zero. Pai and Murthy [11] have a Lyapunov function for the two machine case, but a generalization has inherent difficulties [12]. Jocic et. al. [13] report an approach based on large-scale systems theory, but a clear improvement in practice is not achieved [14]. The inclusion of transfer conductances is sometimes handled by some approximation, either in the system description [6,15] or in evaluating the 'Lyapunov function' (or transient energy function) [3,7,16].

A further disadvantage of forming a reduced network (by suppressing load buses) is that the original network topology is lost. This can mask the role of structural aspects in stability assessment.

In this paper, a new model is presented which does not rely on a reduced network. This follows from the reasonable assumption, for bulk power supply systems, that each load on the transmission network can be represented as a frequency-dependent power load (assuming constant bus voltages). Taking this relationship to be linear leads to a very simple dynamic model which includes the state variables of the classical model plus extra variables associated with the loads. Since the loads are not incorporated into the transmission system, it can be quite accurately modeled as one with zero transfer conductances. Thus, it is a simple matter to generate Lyapunov functions. Further, the original network topology is preserved and the model can be regarded as having structural integrity. In exploiting this feature, it is natural that circuit theory ideas play an important role. Although used in other areas of power system analysis [17], circuit theory has not played a significant role in techniques for stability assessment. Tavora and Smith [18,19] have used it in a limited way to gain insight into power system equilibria, while Jenkins and Liu [20] have formulated a network flow model and used graph theoretic ideas to develop stability results. The model used here is presented in two forms: a network form in terms of circuit matrices and an aggregate view, which is an adaptation of a similar presentation for classical models given by Bergen and Gross [21,22].

This formulation proves to be a convenient basis for consideration of the dependence of stability properties on network topology and system loads. For dynamic stability, the linearized dynamical equations are studied. Adapting results in [18,19,23], a result is given for testing stable equilibrium points in terms of so-called critical cutsets. For transient stability, reference is made to the abovementioned work on stability assessment using Lyapunov functions (and transient energy functions). Implicit in this is the importance of cutsets along which the system tends to separate. The notion of a vulnerable cutset is formulated and some indication given of how to use it in the preliminary stages of transient stability assessment. Taking a transient energy type Lyapunov function for the aggregate system, it is readily seen that this is the sum of kinetic energies associated with the generator rotors and the sum of potential energies associated with all the lines. Thus the Lyapunov function can truly represent the spatial disribution of stored energy in the physical power system. This leads to the concept of a topological

Lyapunov function.

The structure of the paper is as follows. Section II gives a description of the new model. In Section III, a discussion is given on the system equilibrium points and a test provided for stable equilibria. The concept of a topological Lyapunov function is the subject of Section IV. In Section V, this is considered, along with the idea of vulnerable cutsets, in transient stability analysis. Section VI gives some conclusions and the Appendix summarize results from circuit theory.

II. MULTIMACHINE POWER SYSTEM MODEL

In this section, a model of a multimachine power system is developed. Its novelty lies in not taking the usual step of assuming impedance loads, which are absorbed into the transmission network. Otherwise, we make the same assumptions that go with the classical model -- see [9,10,24] for instance.

Our starting point for the model is the network of buses connected by transmission lines, which is the one described by load flow equations. The system shown in Figure 1a will be used in the sequel for illustrative purposes. It has four buses, two of which have generators attached. In general, suppose there are m generators and n_0 buses in the physical system, with n_0-m buses having loads and no generation. It is convenient to introduce fictitious buses representing the internal generation voltages. These are connected to the generator buses via reactances accounting for transient reactances and connecting lines. These reactances can be regarded as 'transmission lines' and henceforth are referred to as the generator bus lines. Thus in the augmented network there is a total of $n=m+n_0$ buses. For convenience, we number the fictitious generator buses $1,\ldots,m$, the corresponding physical buses $m+1,\ldots,2m$ and the remaining load buses $2m+1,\ldots,n$. Suppose that within the transmission network there are ℓ_0 lines. Then ℓ_0 must satisfy $\ell_0 \le \frac{1}{2} n_0(n_0-1)$ and the total number of 'lines' in the augmented network is $\ell = m+\ell_0$. We number the transmission network lines $1,\ldots,\ell_0$ and the generator bus lines $\ell_0+1,\ldots,\ell$ connected to buses $1,\ldots,m$ respectively. The nth bus will be used as a reference. For the four bus example, Figure 1b shows the augmented network. At this stage, it is useful to recognize that the network is analogous to a nonlinear resistive network with real power corresponding to current and the angle difference across a line corresponding to branch voltage. Assuming a lossless transmission network and $\sum_{i=1}^{n} P_i = 0$, where P_i is the injected power at bus i, Kirchhoff's laws hold in the obvious sense. For the four bus example, the analogous circuit is shown in Figure 1c. The nonlinear resistance characteristic for each branch is given by the familiar power-angle relationship for a line. We assume that the graph for the network is connected and planar and the branches are oriented according to associated reference directions. We will make use of certain concepts and results from circuit theory. The Appendix summarizes some essential facts and further details are available in references [25-27].

Now the key assumption of dynamic loads is introduced. Let P_{D_i} be the real power drawn by the load at bus i. In general P_{D_i} is a nonlinear function of voltage and frequency. For constant voltages and small frequency variations around the operating point $P^0_{D_i}$, it is reasonable to assume

$$P_{D_i} = P^0_{D_i} + D_i\dot{\delta}_i \qquad i = m+1,\ldots,n \tag{1}$$

where $D_i > 0$. Note that as $D_i \to 0^+$ we obtain a constant load model. This load frequency dependence is usually assumed in modeling the power-frequency control system, but has not been used in modeling for transient stability. Using (1) we are led to

$$M_i\ddot{\delta}_i + D_i\dot{\delta}_i + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}\sin(\delta_i-\delta_j) = P^0_{M_i} - P^0_{D_i} \triangleq P^0_i \qquad i=1,\ldots,n \tag{2}$$

where

$M_i > 0 \quad i=1,\ldots,m$ (generator inertia constants)

$M_i = 0 \quad i=m+1,\ldots,n$

$D_i > 0 \quad i=1,\ldots,m$ (steam and mechanical damping of generator)

$D_i > 0 \quad i=m+1,\ldots,n$ (frequency coefficient of load)

$P^0_{D_i} = 0 \quad i=1,\ldots,m$

$P^0_{M_i} = 0 \quad i=m+1,\ldots,n$

Equation (2) looks similar to the usual classical swing equation model used in previous studies of transient stability via Lyapunov methods. However, there are important differences. Along with the mechanical input

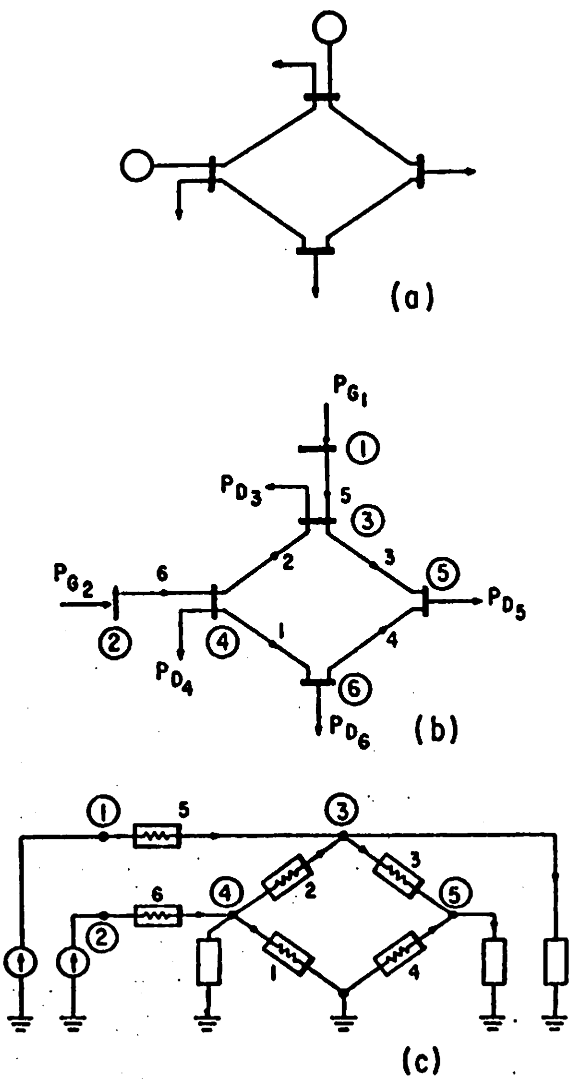

Figure 1 a) A four bus power network

b) Augmented network with generator bus lines

c) Analogous nonlinear resistive circuit powers $P^0_{M_i}$, the loads $P^0_{D_i}$ are shown explicitly. Consequently, the network topology is preserved just as in the case of the load flow model.

It will be convenient to make the assumption that $\sum_{i=1}^{n} P_i^0 = 0$, but in practice this may not be reasonable for the period following a fault. A resolution of this is achieved by adopting the idea suggested by Willems [28]. Adding equations (2) gives

$$\sum_{i=1}^{m} M_i\dot{\omega}_i + \sum_{i=1}^{n} D_i\omega_i = \sum_{i=1}^{n} P_i^0 \tag{3}$$

The required equilibrium is given by $\delta_i - \delta_j = c_{ij}$, where c_{ij} is a constant, for all i,j. Thus all ω_i approach the same constant speed ω^0. From (3), we have

$$\omega^0 = \sum_{i=1}^{n} P_i^0 \Big/ \sum_{i=1}^{n} D_i$$

(Recall that $D_i > 0$ for all i.) Then consider the transformation

$$\omega_i' = \omega_i - \omega^0$$

$$P_i' = P_i^0 - D_i\omega^0$$

It is easy to check that $\sum_{i=1}^{n} P_i' = 0$ and the equilibrium angular velocity is $\omega'^0 = 0$. Henceforth, we assume that this change of reference has been carried out if appropriate and drop the prime superscripts.

More convenient forms of the model can be derived as state-space descriptions and some flexibility is achieved by using various notions from circuit theory. The following is largely an extension of the development in [21] to the present situation. The m generators require a state-space dimension of 2m-1 with nonuniform damping [9,21] so, on including the loads, equation (2) defines a state-space of dimension n+m-1. The state variables can be chosen as the m velocities $\omega_i \triangleq \dot{\delta}_i$ and n-1 internodal angles $\alpha_i = \delta_i - \delta_n$. However, other choices of angles for the state are useful. Define $\underline{\delta} = [\delta_1 \ldots \delta_n]^T$, $\underline{\omega} = [\omega_1 \ldots \omega_n]^T$ and $\underline{\alpha} = [\alpha_1 \ldots \alpha_{n-1}]^T$. Also, define a vector of line angle differences $\underline{\sigma} = [\sigma_1 \ldots \sigma_\ell]^T$ where $\sigma_k = \delta_i - \delta_j$ for the kth line joining buses i and j. The vectors $\underline{\sigma}$ and $\underline{\alpha}$ are related to $\underline{\delta}$ via transformations $\underline{\sigma} = \underline{L}\,\underline{\delta}$ and $\underline{\alpha} = \underline{T}\,\underline{\delta}$. Matrix $\underline{T}$ is given by

$$\underline{T} = [\underline{I}_{n-1} \,\vdots\, -\underline{e}]$$

where $\underline{I}_{n-1}$ is the (n-1) identity matrix and $\underline{e}$ the (n-1) vector with unity entries. Now we introduce the reduced incidence matrix

$$\underline{A} = \left[\begin{array}{c|c} \underline{0} & \underline{I}_m \\ \hline \underline{A}_t & -\underline{I}_m \\ & \underline{0} \end{array}\right] \tag{4}$$

where $\underline{A}_t$ is the reduced incidence matrix of the transmission network. Then, we have

$$\underline{\sigma} = \underline{A}^T\underline{\alpha}$$
$$= \underline{A}^T\underline{T}\,\underline{\delta} \tag{5}$$

Now partition $\underline{T}$ according to

$$\underline{T} = \left[\begin{array}{c|c} \underline{I}_m & \underline{0} \quad -\underline{e} \\ \hline \underline{0} & \underline{T}_t \end{array}\right] \triangleq \left[\underline{T}_1 \,\middle|\, \underline{T}_2\right] \tag{6}$$

Hence

$$\underline{L} = \underline{A}^T\underline{T} = \left[\begin{array}{c|c} \underline{0} & \underline{A}_t^T\underline{T}_t \\ \hline \underline{I}_m & -\underline{I}_m \quad \underline{0} \end{array}\right]$$

As an alternative to $\underline{\alpha}$ we may wish to use a set of n-1 tree branch angle differences θ_i. With the chosen numbering system, we further number cotree branches first and then the tree branches. Writing $\underline{\sigma} = [\underline{\sigma}_c^T \underline{\theta}^T]^T$ and defining transformation $\underline{\theta} = \underline{K}\,\underline{\delta}$, we see that $\underline{K}$ can be obtained as an appropriate submatrix of $\underline{L}$. Note that $\underline{\sigma} = \underline{Q}^T\underline{\theta}$ where $\underline{Q}$ is the fundamental cutset matrix and so alternatively $\underline{L} = \underline{Q}^T\underline{K}$. These matrix relationships can be explored further for their own sake, but we only study them further as required in the sequel.

With the resistive circuit analogy in mind, we define the constitutive relationship for branch k by $p_k = g_k(\sigma_k)$ where p_k is the power flow in the branch. We have

$$g_k(\sigma_k) = b_k \sin \sigma_k \tag{7}$$

where $b_k = b_{ij}$ and it is assumed always that branch k connects buses i and j. In vector form, write $\underline{p} = \underline{g}(\underline{\sigma})$. Now $P_n^0 = -\sum_{i=1}^{n-1} P_i^0$, so there are n-1 independent excess node powers. Let $\underline{P}^0 = [P_1^0 \ldots P_{n-1}^0]^T$. Then via nodal analysis (A-3), the load flow can be written

$$\underline{P} = \underline{A}\,\underline{g}(\underline{\sigma})$$
$$= \underline{A}\,\underline{g}(\underline{A}^T\underline{\alpha}) \triangleq \underline{f}(\underline{\alpha}) \tag{8}$$

Note that

$$f_i(\underline{\alpha}) = \sum_{\substack{k=1 \\ k\neq i}}^{n-1} b_{ik} \sin(\alpha_i - \alpha_k) + b_{in} \sin \alpha_i, \quad i=1,\ldots n-1 \tag{9}$$

Alternatively, in terms of tree branch angle differences, we get from (A-5)

$$\underline{P} = \underline{A}\,\underline{g}(\underline{Q}^T\underline{\theta}) \triangleq \underline{f}_\theta(\underline{\theta}) \tag{10}$$

Now define

$$\underline{M} = \text{diag}\{M_i\}$$

$$\underline{D} = \text{diag}\{D_i\}$$

Then it is straightforward to show that (2) can be replaced by [21]

$$\underline{M}\,\dot{\underline{\omega}} + \underline{D}\,\underline{\omega} + \underline{T}^T\,[\underline{f}(\alpha) - \underline{P}^0] = \underline{0} \tag{11}$$

With appropriate partitioning of $\underline{M},\underline{D}$ and using (6), (11) can be rewritten as

$$\underline{M}_1\dot{\underline{\omega}}_1 + \underline{D}_1\underline{\omega}_1 + \underline{T}_1^T\,[\underline{f}(\underline{\alpha}) - \underline{P}^0] = \underline{0} \tag{12a}$$

$$\underline{D}_2\underline{\omega}_2 + \underline{T}_2^T\,[\underline{f}(\underline{\alpha}) - \underline{P}^0] = \underline{0} \tag{12b}$$

where subscripts 1 and 2 refer to the generators and loads respectively.

We now proceed to develop a third model description as so-called normal form or state-space form. Firstly, we have

$$\dot{\underline{\alpha}} = \underline{T}\,\underline{\omega} = \underline{T}_1\underline{\omega}_1 + \underline{T}_2\underline{\omega}_2 \qquad (13)$$

Using (12b) to eliminate $\underline{\omega}_2$ in (13) along with (12a) gives

$$\dot{\underline{\alpha}} = \underline{T}_1\underline{\omega}_1 - \underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\ [\underline{f}(\alpha) - \underline{P}^0] \qquad (14a)$$

$$\dot{\underline{\omega}}_1 = -\underline{M}_1^{-1}\underline{D}_1\underline{\omega}_1 - \underline{M}_1^{-1}\underline{T}_1^T\ [\underline{f}(\alpha) - \underline{P}^0] \qquad (14b)$$

Equations (14) define the system trajectories in a state-space of dimension m+n-1.

Equations (2), (12) and (14) give three alternative mathematical representations of the model. Equations (12) and (14) give aggregate representations, but substituting equation (8) provides the structural information in terms of circuit matrices. Sometimes it is convenient to have (12) or (14) in terms of tree branch angles $\underline{\theta}$ or write (12) in terms of branch angles $\underline{\sigma}$. Using the above transformations, these alternative representations can be developed as required.

The assumption that all $D_i > 0$ is certainly reasonable, but a comment is in order on the case where some D_i are assumed to be zero. Further, the above has implicitly assumed generator damping to be nonuniform. Setting $\underline{D}_1 = \underline{0}$ or with uniform damping the obvious extension of the classical case applies. That is, the state-space dimension is reduced by one. Setting some of the load $D_i \to 0^+$, however, gives a model in terms of differential and algebraic equations. Mathematically this is relatively more difficult to accomodate. In view of the presence always of some frequency dependence in the load, this will not be investigated further.

III. EQUILIBRIUM POINTS AND LOCAL STABILITY

Before considering the global stability properties of the system described by equations (14), attention should be given to the equilibrium points and their stability.

In the previous section, we saw that without loss of generality the equilibria correspond to $(\underline{\alpha},\underline{\omega}) = (\underline{\alpha}^e,\underline{0})$ where $\underline{\alpha}^e$ is constant. From (14), we have

$$\underline{N}[\underline{f}(\underline{\alpha}^e) - \underline{P}^0] = \underline{0} \qquad (15)$$

where

$$\underline{N} = \underline{T}_1^T + \underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T = \begin{bmatrix} \underline{I}_m & \underline{0} \\ \underline{0} & \tilde{\underline{D}}_2^{-1} \end{bmatrix} + \frac{1}{D_n}\,\underline{e}><\underline{e}$$

with $\tilde{\underline{D}}_2 = \mathrm{diag}(D_{m+1},\ldots,D_{n-1})$

Thus $\underline{N}$ has rank n-1. Then using (15) the equilibria are given by $\underline{\omega} = \underline{0}$ and the solutions of

$$\underline{f}(\underline{\alpha}^e) = \underline{P}^0 \qquad (16)$$

We call the function $\underline{f}(\cdot)$ the flow function [18]. Due to the periodic dependence of $\underline{f}(\underline{\alpha})$ on $\underline{\alpha}$, the domain of the flow function is the n-1 dimensional torus. That is we write $\underline{f}: T^{n-1} \to \mathbb{R}^{n-1}$ where

$$T^{n-1} = \{\underline{\alpha} \bmod 2\pi \ : \ \underline{\alpha} \in \mathbb{R}^{n-1}\}$$

To study properties of the solutions of (16), we note that $\underline{f}(\cdot)$ is continuously differentiable and identify its Jacobian matrix denoted by $\underline{F}(\underline{\alpha})$. The (i,j)th term of $\underline{F}(\underline{\alpha})$ is given by

$$\frac{\partial f_i(\underline{\alpha})}{\partial \alpha_j} = \begin{cases} b_{in}\cos\alpha_i + \sum\limits_{\substack{k=1\\k\neq i}}^{n-1} b_{ik}\cos(\alpha_i-\alpha_k), & i=j \\ -b_{ij}\cos(\alpha_i-\alpha_j), & i\neq j \end{cases}$$

Since $\underline{F}(\underline{\alpha})$ has full normal rank, (16) has a finite number of isolated solutions in T^{n-1} [29]. Unfortunately, there appear to be no useful results on answering questions related to the exact number of solutions for a given $\underline{P}^0$ unique stable solutions etc. Tavora and Smith [18] have given some useful insights, by way of examples, into how the number of solutions depends on network topology, line power transfer coefficients b_{ij}, and $\underline{P}^0$.

The starting point for stability analysis of (14) is a solution of (16) about which the system is locally stable. The study of stable equilibria seems largely to rely on the intuitive idea that if all lines satisfy $|\sigma_i^0| < \pi/2$, then the equilibrium is stable. From a combination of ideas in [19,23], we can actually state a precise version. In view of structural integrity of the model, the test takes some significance in its being based on making tests on cutsets. Further, the techniques employed prepare the way for the study of transient stability in later sections. For the study of local stability, we firstly linearize equations (14) about the equilibrium point $(\underline{\alpha}^0,\underline{0})$ to obtain differential equations in variables $\underline{\Delta\alpha} = \underline{\alpha} - \underline{\alpha}^0$ and $\underline{\Delta\omega}_1 = \omega_1 - \underline{\omega}_1^0 = \underline{\omega}_1$. This gives

$$\begin{bmatrix} \dot{\underline{\Delta\alpha}} \\ \dot{\underline{\omega}}_1 \end{bmatrix} = \begin{bmatrix} -\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0) & \underline{T}_1 \\ -\underline{M}_1^{-1}\underline{T}_1^T\underline{F}(\underline{\alpha}^0) & -\underline{M}^{-1}\underline{D}_1 \end{bmatrix} \begin{bmatrix} \underline{\Delta\alpha} \\ \underline{\omega}_1 \end{bmatrix} \qquad (17)$$

Study of (17) could proceed by eigenvalue techniques or Lyapunov methods. The latter turns out to give a simple answer and an appropriate Lyapunov function is a quadratic counterpart to the one to be used for transient stability [23]. It is convenient to define the polytope

$$\Lambda^{\ell} = \{\underline{\sigma} \in \mathbb{R}^{\ell} \ : \ |\sigma_i| \le \pi/2, \ \ i=1,\ldots,\ell\}$$

We observe that for $\underline{\sigma}^0 \in \Lambda^{\ell}$, then $\underline{F}(\underline{\alpha}^0)$ is nonnegative definite; this follows from Gershgorin's Theorem [30] since $\underline{F}(\underline{\alpha}^0)$ is diagonally dominant with positive diagonal elements. Motivated by stored energy, consider a possible Lyapunov function as

$$V(\underline{\Delta\alpha},\underline{\omega}_1) = \frac{1}{2}\,\underline{\omega}_1^T\underline{M}_1\underline{\omega}_1 + \frac{1}{2}\,\underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha}$$

Differentiating V along the solutions of (17) gives

$$\begin{aligned} \dot{V}(\underline{\Delta\alpha},\underline{\omega}_1) &= \underline{\omega}_1^T\underline{M}_1(-\underline{M}_1^{-1}\underline{T}_1^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} - \underline{M}_1^{-1}\underline{D}_1\underline{\omega}_1) \\ &\quad + \underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)(-\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} + \underline{T}_1\underline{\omega}_1) \\ &= -\underline{\omega}_1^T\underline{D}_1\underline{\omega}_1 - \underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} \end{aligned} \qquad (18)$$

Now $\dot{V} = 0$ implies that $\underline{\omega}_1 = \underline{0}$ and

$$\underline{T}_2^T \underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} \equiv \underline{0} \tag{19a}$$

From (17), $\underline{\omega}_1 \equiv \underline{0}$ gives

$$\underline{T}_1^T \underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} \equiv \underline{0} \tag{19b}$$

Equations (19) imply

$$\underline{T}^T \underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} \equiv \underline{0} \tag{20}$$

Then, if $\underline{F}(\underline{\alpha}^0)$ is positive definite, V is positive definite and, since $\underline{T}^T$ is a full rank matrix, (20) gives that $\dot{V} \equiv 0$ implies $(\underline{\Delta\alpha},\underline{\omega}_1) = (\underline{0},\underline{0})$. From standard Lyapunov stability theory [31], we then have that the equilibrium point $(\underline{\alpha}^0,\underline{0})$ is asymptotically stable. However, so far it has only been demonstrated that $\underline{F}(\underline{\alpha}^0)$ is nonnegative definite on polytope Λ^ℓ. To obtain the final statement of stability conditions, we use a result given by Tavora and Smith [19]. We will refer to lines with zero synchronizing coefficients, i.e., for which $\cos\sigma_i = 0$, as <u>zero-valued</u>. A cutset of zero-valued lines is called a <u>critical cutset</u>. Then, from [19], we get that $\det \underline{F}(\underline{\alpha}^0) = 0$ in Λ^ℓ if and only if the system has a critical cutset. Hence, the equilibrium point $(\underline{\alpha}^0,\underline{0})$ is asymptotically stable if $\underline{\sigma}^0 \in \Lambda^\ell$ and there are not critical cutsets. The absence of critical cutsets is ensured by having a tree of lines which are not zero-valued. We can now summarize the result as it pertains to system (14) as follows.

<u>Theorem 1</u>. Consider an equilibrium point for the power system satisfying (16). Suppose that $\underline{\sigma}^0 \in \Lambda^\ell$ and the generator bus lines are not zero-valued. Then the equilibrium point is asymptotically stable if the transmission line network has no critical cutsets.

For a normal operating condition, of course, these conditions are easily met. However, after a fault or during abnormal loading conditions the system could be operating close to the boundary of polytope Λ^ℓ. Actually, in [19] the region of stable equilibrium points is claimed to be bigger than Λ^ℓ and given by the so-called principal region. However, in general, this principal region would not be easy to calculate and it appears that Λ^ℓ is a close approximation to it.

IV. TOPOLOGICAL LYAPUNOV FUNCTION

Under normal operating conditions, the system will be in or near an equilibrium state satisfying the stability conditions of Theorem 1. A fault can alter $\underline{P}^0$, the transmission topology, or the coefficients b_{ij} giving new post-fault equilibrium states (if $\underline{P}^0$ is feasible, i.e., if $\underline{P}^0$ lies in the range of $f(\cdot)$). Whether the system settles to the post-fault s.e.p. is studied via transient stability analysis using equations (14) as the basic model. We use a Lyapunov function which is motivated by stored energy of the aggregate system. This, of course, has been the basic Lyapunov function going back to early work. However, with the present new model and using some circuit theory ideas -- see Appendix A -- additional insights into stability assessment are possible.

Suppose that $(\underline{\alpha}^0,\underline{0})$ is a stable post-fault equilibrium point. We define the Lyapunov function $V: \mathbb{R}^{n-1} \times \mathbb{R}^m \to \mathbb{R}$ by

$$V(\underline{\alpha},\underline{\omega}_1) = \frac{1}{2}\,\underline{\omega}_1^T \underline{M}_1 \underline{\omega}_1 + W(\underline{\alpha},\underline{\alpha}^0) \tag{21}$$

where

$$W(\underline{\alpha},\underline{\alpha}^0) = \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{f}(\underline{\xi}) - \underline{f}(\underline{\alpha}^0)]^T d\underline{\xi}$$

In this form, it is a direct generalization of the Lyapunov function used by Bergen and Gross [21,22] and represents the sum of aggregate kinetic energy and potential energy. The integral defining the potential function $W(\underline{\alpha},\underline{\alpha}^0)$ is evaluated over an arbitrary path between $\underline{\alpha}^0$ and $\underline{\alpha}$. Since $\underline{F}(\underline{\alpha})$ is symmetric, the integral is path independent and V is well-defined. It is interesting to note the following.

<u>Theorem 2</u>. The function V given by (21) can also be written as

$$V(\underline{\alpha},\underline{\omega}_1) = \frac{1}{2}\sum_{k=1}^{m} M_k\omega_k^2 + \sum_{k=1}^{\ell} b_k h(\sigma_k,\sigma_k^0) \tag{22}$$

where

$$h(\sigma_k,\sigma_k^0) = \int_{\sigma_k^0}^{\sigma_k} (\sin u - \sin\sigma_k^0)\,du$$

Proof: From equation (8), we have

$$\underline{f}(\underline{\xi}) = \underline{A}\,\underline{g}(\underline{A}^T\underline{\xi})$$

Then the potential function is given by

$$\begin{aligned} W(\underline{\alpha},\underline{\alpha}^0) &= \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{f}(\underline{\xi}) - \underline{f}(\underline{\alpha}^0)]^T d\underline{\xi} \\ &= \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{g}(\underline{A}^T\underline{\xi}) - \underline{g}(\underline{A}^T\underline{\alpha}^0)]^T \underline{A}^T d\underline{\xi} \\ &= \int_{\underline{\sigma}^0}^{\underline{\sigma}} [\underline{g}(\underline{u}) - \underline{g}(\underline{\sigma}^0)]^T d\underline{u} \end{aligned}$$

on setting $\underline{u} = \underline{A}^T\underline{\xi}$ and using transformation to branch angles. Using (7),

$$W(\underline{\alpha},\underline{\alpha}^0) = \sum_{k=1}^{\ell} b_k \int_{\sigma_k^0}^{\sigma_k} (\sin u - \sin\sigma_k^0)\,du \tag{23}$$

∎

Thus the total potential energy is seen to be the sum of the potential energies of the individual branches. What is interesting here is that just as the kinetic energy may be identified with individual generators, the potential energy may be identified with individual transmission lines (including generator transient reactances). Thus the Lyapunov function truly reflects the spatial distribution of stored energy in the physical system since the original topology has been preserved in the model. Hence we refer to the function (21) or (22) used in connection with model (14) as a <u>topological Lyapunov function</u>.

To actually show that V given by (21) is a Lyapunov function involves a simple modification of the steps used for the quadratic energy function in the previous section. Firstly, we determine a region where W is positive definite. Consider the function $h(\cdot,\sigma_k^0)$ and suppose $\underline{\sigma}^0 \in \Lambda^\ell$. Then $h(\cdot,\sigma_k^0)$ is a positive definite and strictly monotone increasing function over the interval $(\sigma_k^\ell,\sigma_k^u)$ with $\sigma^\ell \triangleq -\pi-\sigma_k^0$ and $\sigma^u = \pi - \sigma_k^0$. Now define the polytope

$$\Gamma^{\ell}(\underline{\sigma}^0) = \{\underline{\sigma} \in \mathbb{R}^{\ell} : \sigma_i \subset (\sigma_k^{\ell}, \sigma_i^{u}) \quad i = 1,\ldots,\ell\}$$

We denote the closure and boundary of $\Gamma^{\ell}(\underline{\sigma}^0)$ by $\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ and $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ respectively. Obviously, from (23), $W(\cdot,\underline{\sigma}^0)$ is positive definite over the polytope Γ^{ℓ} (where $\underline{\sigma} = \underline{A}^T\underline{\alpha}$ is assumed throughout). The above mentioned monotonicity property implies that all u.e.p.'s must lie on or outside of $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$. Now differentiating V along the trajectories of (14) gives

$$\dot{V}(\underline{\alpha},\underline{\omega}_1) = -\underline{\omega}_1^t D_1 \underline{\omega}_1 - [\underline{f}(\underline{\alpha})-\underline{f}(\underline{\alpha}^0)]^T \underline{T}_2 D_2^{-1} \underline{T}_2^T [\underline{f}(\underline{\alpha})-\underline{f}(\underline{\alpha}^0)] \tag{24}$$

Thus, since $\underline{D}_1 > \underline{0}$, $\underline{D}_2 > \underline{0}$, $\dot{V}$ is at least negative semidefinite. Corresponding to (20), we have $\dot{V} \equiv 0$ implying

$$\underline{T}^T[\underline{f}(\underline{\alpha}) - \underline{f}(\underline{\alpha}^0)] \equiv \underline{0} \tag{25}$$

Hence, since $\underline{T}$ is full rank, (25) implies $\underline{f}(\underline{\alpha}) - \underline{f}(\underline{\alpha}^0) \equiv \underline{0}$ and $\dot{V} \equiv 0$ only at equilibrium points. In the usual way, well-known stability results [31] determine a region of asymptotic stability defined by

$$\Omega_{\ell} = \{(\underline{\alpha},\underline{\omega}_1) : V(\underline{\alpha},\underline{\omega}_1) < V_{\ell}(\underline{\alpha}^0)\} \tag{26}$$

where V_{ℓ} is chosen so that Ω_{ℓ} excludes all the u.e.p.'s. In particular Ω_{ℓ} excludes $(\underline{\alpha}^*,\underline{0})$, the u.e.p. of lowest potential energy.

It is interesting also to note that substituting (12b) into (24) gives

$$\dot{V}(\underline{\alpha},\underline{\omega}_1) = -\underline{\omega}^T \underline{D}\,\underline{\omega} \tag{27}$$

Equation (27) shows that all the D_i act similarly to account for dissipation of energy, and the simple positivity of the coefficients insures that $\dot{V} \le 0$. Thus the precise values of the D_i, which vary and are difficult to measure, are not needed.

V. VULNERABLE CUTSETS AND TRANSIENT STABILITY ASSESSMENT

The major part of the effort to make Lyapunov methods work for transient stability assessment in realistic power systems has been directed to efficient algorithms for estimating the region of stability in the state-space. In this section, we look briefly at how the techniques can be interpreted, and possibly improved upon, with the new model. A complete presentation is beyond the scope of this paper.

Most methods for finding the extent of stability rely on calculating (or approximating) the u.e.p. $(\underline{\alpha}^*,\underline{0})$ with lowest potential $V_{\ell}(\underline{\alpha}^0) = W(\underline{\alpha}^*,\underline{\alpha}^0)$ [1,2,6,18,32]. Other work is not explicitly concerned with calculating u.e.p.'s. Bergen and Gross [33] and Pai and Narayana [34] present minimization procedures on the polytope $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ (or its equivalent in $\underline{\alpha}$ space) for estimating a close lower bound for V_{ℓ}. The novel feature of the procedure in [33] is its simple graphical calculations. Thus it is more in the spirit of the equal area criterion for two-machine systems. All of the abovementioned work is motivated by the need to avoid the prohibitive computational task of calculating all the u.e.p.'s and then testing each one to find $W(\underline{\alpha}^*, \underline{\alpha}^0)$. In looking for fundamental aspects of this problem, we are led to the role of system structure in the solution techniques. Ribbens-Pavella et al. [2] take the attitude that the most likely consequence of instability is for one generator to lose synchronism. This reduces the problem to testing 2(n-1) u.e.p.'s. In other results [1,32], the loss of groups of machines is explicitly allowed for. Physical reasoning reduces the number of possibilities for the system to split up. For instance, Gupta and El-Abiad [1] restrict attention to cutsets containing the line on which the fault occurred. For present purposes, it is sufficient merely to note that the transient stability problem seems related in a fundamental way to a ranking of the network cutsets in terms of what will be referred to here as <u>vulnerability</u>. The structural integrity of the present model adds to the meaningfulness of such a concept.

In the special case of $\underline{P}^0 = \underline{0}$ there is a simple connection between u.e.p.'s and power flows on transmission network cutsets. In particular, the u.e.p. of lowest potential may easily be identified and calculated by examining an index of vulnerability for all the cutsets. In the case $\underline{P}^0 = \underline{0}$, the solution $\underline{\sigma}^0 = \underline{0}$ is the s.e.p. and by (16) the (neighboring) u.e.p.s. have the property $\sigma_i = 0, \pm\pi$. We will refer to lines with $|\sigma_i| = \pi$ as <u>saturated</u> lines. Thus, corresponding to every u.e.p. is a set of saturated branches. A further result is stated in the following proposition.

<u>Proposition</u>. Assume that $\underline{P}^0 = 0$. Then a subset of the saturated branches corresponding to an u.e.p. form a cutset.

Proof: For a three bus triangular mesh structure the result is trivial since either all branches are zero or two are saturated and one zero. Since the system graph is planar, we can consider it as an interconnection of triangular meshes and single branches (by introducing internal zero branches if necessary).

Since we have an u.e.p., at least one branch must be saturated. Now, using KVL and the result for a single mesh, one can argue that the result holds in general. Starting from a saturated branch, we can build up a line of saturated branches through meshes with saturated branches in common. This line can terminate by having the only adjoining mesh at the zero branch or if the line rejoins itself. In either case, a cutset of saturated branches has been generated. ∎

It is easy to see that an u.e.p. can correspond to a number of saturated cutsets. For instance, each generator bus line in Figure 1b could give a separate saturated cutset at an u.e.p.

Continuing then with the simple special case of $\underline{P}^0 = \underline{0}$. Let $(\underline{\alpha}^e,\underline{0})$ be an u.e.p. of interest. Then, from Theorem 2, we have

$$W(\underline{\alpha}^e,\underline{0}) = \sum_{k=1}^{\ell} b_k h(\alpha_k^e, 0)$$

Now
$$h(\alpha_k^e, 0) = \begin{cases} 2, & \sigma_k^e = \pm\pi \\ 0, & \sigma_k^e = 0 \end{cases}$$

Thus

$$W(\underline{\alpha}^e,\underline{0}) = 2\sum_{k=k_1}^{k_s} b_k \tag{28}$$

where the summation is over the s saturated lines numbered $k_1,\ldots,k_s$. Then we have exactly, in view of the proposition, that the u.e.p. $(\alpha^*,\underline{0})$ and most vulnerable cutset are provided by minimizing the sum in (28) over all cutsets. (If there is more than one saturated cutset corresponding to an u.e.p., obviously a more vulnerable cutset can be found by setting some branch angles to zero.) It is convenient to introduce some notation. Let C_i denote the ith cutset and we write $C_i = \{i_1,\ldots,i_q\}$ where i_j identifies the jth branch in the ith cutset. Then we have

$$\mathcal{V}_i \triangleq 2 \sum_{C_i} b_k$$

as an index of vulnerability for ith cutset. (A larger $\mathcal{V}_i$ corresponds to a less vulnerable cutset.)

The situation where $\underline{P}^0 = \underline{0}$ is certainly not realistic in practice, except insofar as it approximates very low power levels. However, the idea of ranking the vulnerability of cutsets has been illustrated with a simple exact answer. Now, in general where $\underline{P}^0 \neq \underline{0}$, we will call branches saturated at an u.e.p. $\underline{\alpha}^e$ when $\pi/2 \leq |\sigma_i^e| \leq 3\pi/2$. However, exact calculation of the u.e.p.'s is to be avoided, so an index of vulnerability depending on this is not acceptable. This difficulty can be overcome by adopting some ideas used by Prabhakara and El-Abiad [32] for estimating all the u.e.p.'s. A measure of the system vulnerability at a cutset can be obtained by picturing a separation of the system into two parts along the cutset. This is illustrated in Figure 2. It is convenient to consider the polytope $\Gamma^\ell(\underline{\sigma}^0)$ corre-

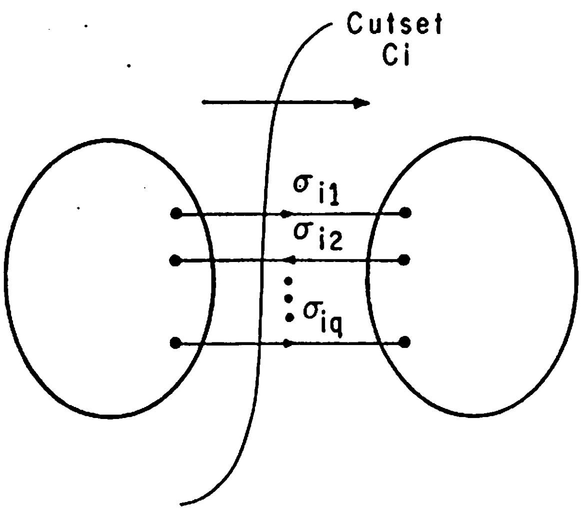

Figure 2 System separation on a cutset

sponding to s.e.p. $(\underline{\alpha}^0, \underline{0})$ where $\underline{\sigma}^0 = \underline{A}^T\underline{\alpha}^0$. By posing the hypothetical situation of the separation occurring with $\underline{\sigma}$ on $\partial\bar{\Gamma}^\ell(\underline{\sigma}^0)$, an index of vulnerability becomes obvious. Assigning a reference direction for the cutset C_i, we denote the set of positively oriented branches in C_i by C_i^+ and the remaining branches in C_i by C_i^-. Then a positive shift of line angles from $\underline{\sigma}^0$ (relative to the cutset reference) on to $\partial\bar{\Gamma}^\ell(\underline{\sigma}^0)$ selects the 'corner point' $\underline{\sigma}^+$ defined by

$$\sigma_k^+ = \begin{cases} \sigma_k^u, & k \in C_i^+ \\ \sigma_k^\ell, & k \in C_i^- \\ \sigma_k^0, & k \notin C_i \end{cases}$$

Similarly, a 'corner point' $\underline{\sigma}^-$ for negative shift of line angles can be defined with obvious modification. We have from Theorem 2

$$W(\underline{\sigma}^+, \underline{\sigma}^0) = \sum_{C_i} b_k h(\sigma_k^+, \sigma_k^0) \tag{29}$$

We propose that $W(\underline{\sigma}^+, \underline{\sigma}^0)$ and its negative separation counterpart $W(\underline{\sigma}^-, \underline{\sigma}^0)$ represent the index of vulnerability for cutset C_i. Introducing the coefficients $\mu_k^\ell \triangleq h(\sigma_k^\ell, \sigma_k^0)$ and $\mu_k^u \triangleq h(\sigma_k^u, \sigma_k^0)$ for all the lines, this motivates definition of <u>cutset vulnerability indices</u> by

$$v_i^+ = \sum_{C_i^+} b_k \mu_k^u + \sum_{C_i^-} b_k \mu_k^\ell \tag{30a}$$

$$v_i^- = \sum_{C_i^+} b_k \mu_k^\ell + \sum_{C_i^-} b_k \mu_k^u \tag{30b}$$

An overall index for the cutset is

$$\mathcal{V}_i = \min\,(v_i^+, v_i^-)$$

and for the system is

$$\mathcal{V} = \min_i \mathcal{V}_i$$

Evaluating $\mathcal{V}_i$ for each cutset gives a ranking according to vulnerability. Note that calculation of the coefficients in (30) is simply done via

$$\mu_k^u = 2[\cos\sigma_k^0 + (\sigma_k^0 - \pi/2)\sin\sigma_k^0] \tag{31a}$$

$$\mu_k^\ell = 2[\cos\sigma_k^0 + (\sigma_k^0 + \pi/2)\sin\sigma_k^0] \tag{31b}$$

For $\underline{P}^0 = \underline{0}$, we have $\mu_k^u = \mu_k^\ell = 2$ and $\mathcal{V}_i = v_i^+ = v_i^-$.

Having set up the index $\mathcal{V}_i$, how, and within what limitations, can we depend on it? Of course, in general, we cannot expect $\mathcal{V}$ to be an accurate estimate of V_ℓ. The main utility seems to lie in providing a preliminary identification of weak cutsets. Then, using this information along with other information like fault position, we can concentrate on finding the corresponding u.e.p.'s and an accurate estimate of V_ℓ. It is interesting, however, to observe that the method used by Prabhakara and El-Abiad [32] appears very accurate at least for low power levels. We can then anticipate that, for this case, $\mathcal{V}$ will indeed be a useful estimate for V_ℓ. As power levels increase, the u.e.p.'s are less related to hypothetical separation situations and there is a greater need for follow-up calculations to calculate V_ℓ.

As a simple illustration of the use of vulnerability indices, the following example is considered.

<u>Example</u>. For the network illustrated in Figure 1, we use the values for power transfer coefficients b_k and powers P_i^0 from an example in [18] (with some additions to allow for generator lines). The powers are given by

P_{G_1}	= 2.0	P_{D_3}	= 1.0
P_{G_2}	= 2.0	P_{D_4}	= 0.8
		P_{D_5}	= 1.2
		P_{D_6}	= 1.0

Firstly, we note that for $\underline{P}^0 = \underline{0}$ cutset {2,4} is most vulnerable and expect this to be the case for very low power levels. At the powers given above, the relevant coefficients (31) for each line are tabulated in Table 1. The corresponding cutset vulnerability indices are tabulated in Table 2 and they reveal that cutset {1,2} is most vulnerable. The three cutsets {1,3}, {2,4} (most vulnerable at very low powers) and {2,3} form an almost equally vulnerable group with the remainder having decreasing vulnerability. We have from [18] that the exact value of V_ℓ (found by a lowest saddle point search) is 1.63 corresponding to cutset {1,2} being saturated. Thus the vulnerability indices have identified the weakest cutset. Note that, in the case considered, the power levels are an appreciable proportion of the line capacities. In fact, at the exact u.e.p. corresponding to cutset {1,2} line 2 has $p_2 = b_2 = 0.5$. Thus we do not expect the overall vulnerability index $\mathcal{V}$ to give a close estimate of V_ℓ. However, from Table 2, we do have $\mathcal{V} = 1.89$ which is an acceptable course estimate.

TABLE 1

Calculation of Branch Vulnerability Coefficients

Line	b_k	σ_k^0 (radians)	μ_k^u	μ_k^ℓ
1	2.0	0.597	0.559	4.091
2	0.5	0.152	1.548	2.498
3	2.0	0.569	0.605	3.991
4	1.0	0.124	1.627	2.404
5	5.0	0.412	0.905	3.421
6	6.0	0.340	1.064	3.161

TABLE 2

Calculation of Cutset Vulnerability Indices

Cutset	{2,3}	{1,2}	{3,4}	{1,4}	{1,3}	{2,4}	{5}	{6}
v_i^+	8.755	1.893	2.837	3.522	2.329	2.401	4.524	6.384
v_i^-	2.460	9.431	10.385	9.810	16.163	3.653	17.103	18.968

VI. CONCLUSIONS

A new model for the study of power system stability has been discussed. The significant feature of this model is its structural integrity which goes hand-in-hand with an explicit presence of the system loads in the network. This avoids the difficult problem of how to account for transfer conductances in reduced network models. To give a conceptual view of how this model relates to stability analysis, the concepts of a topological Lyapunov function and vulnerable cutsets have been introduced. In view of the relationship with successful techniques for the classical model, the ranking of cutsets using vulnerability indices could prove to be a very useful preliminary step in transient stability assessment.

VII. ACKNOWLEDGEMENT

This research was supported by the Department of Energy, Electric Energy System Division, Contract EC-77-S-01-5105.

VIII. REFERENCES

[1] C.L. Gupta and A.H. El-Abiad, "Determination of the Closest Unstable Equilibrium State for Liapunov Methods in Transient Stability Studies," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 1699-1712, September/October 1976.

[2] M. Ribbens-Pavella, et al., "Transient Stability Analysis by Scalar Liapunov Functions: Recent Improvements and Practical Results," Coll. des Public de la Faculté des Sc. Appliquées, Univ. of Liege, No. 67, 1977.

[3] T. Athay, R. Podmore and S. Virmani, "A Practical Method for the Direct Analysis of Transient Stability," IEEE Trans. Power Apparatus and Systems, vol. PAS-98, pp. 573-584, March/April 1979.

[4] O. Saito et al., "Security Monitoring Systems Including Fast Transient Stability Studies," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 1789-1805, September/October 1975.

[5] A.H. El-Abiad and K. Nagappan, "Transient Stability Regions of Multimachine Power Systems," IEEE Trans. Power Apparatus and Systems, vol. PAS-85, pp. 169-179, February 1966.

[6] M. Ribbens-Pavella, "Transient Stability of Multi-machine Power Systems by Liapunov's Direct 71 CP17-PWR, IEEE Winter Power Meeting, New York, January 1971.

[7] G.A. Lüders, "Transient Stability of Multimachine Power Systems via the Direct Method of Lyapunov," IEEE Trans. Power Apparatus and Systems, vol. PAS-90, pp. 23-32, January/February 1971.

[8] H.F. Williams, S.A. Louie, and G.W. Bills, "Feasibility of Liapunov Functions for the Stability Analysis of Electric Power Systems Having up to 60 Generators," IEEE Trans. Power Apparatus and Systems, vol. PAS-91, pp. 1145-1153, May/June 1972.

[9] M. Ribbens-Pavella, "Critical Survey of Transient Stability Studies of Multimachine Power Systems by Liapunov's Direct Method," Proc. 9th Allerton Conf. Circuits and System Theory, Univ. of Illinois, Monticello, Illinois, pp. 751-767, 1971.

[10] A.A. Fouad, "Stability Theory - Criteria for Transient Stability," Proc. Eng. Foundation Conf. on Systems Eng. for Power: Status and Prospects, Henniker, New Hampshire, pp. 421-450, 1975.

[11] M.A. Pai and P.G. Murthy, "On Lyapunov Functions for Power Systems with Transfer Conductances," IEEE Trans. Automatic Control, vol. AC-18, pp. 181-183, April 1973.

[12] V.E. Henner, "Comments on 'On Lyapunov Functions for Power Systems with Transfer Conductances'" IEEE Trans. Automatic Control, vol. AC-19, pp. 621-622, October 1974.

[13] L.B. Jocić, M. Ribbens-Pavella, and D.D. Siljak, "Multimachine Power Systems: Stability, Decomposition, and Aggregation," IEEE Trans. Automatic Control, vol. AC-23, pp. 325-332, April 1978.

[14] M. Ribbens-Pavella, Lu.T. Grujić, and J. Sabatel, "Scalar vs Vector Liapunov Functions for Transient Stability Analysis of Large-Scale Power Systems," MECO, Athens, June 1978.

[15] M.A. Pai and S.D. Varwandkar, "On the Inclusion of Transfer Conductances in Lyapunov Functions for Multimachine Power Systems," IEEE Trans. Automatic Control, vol. AC-22, pp. 983-985, 1977.

[16] A.A. Fouad and R.L. Lugtu, "Transient Stability Analysis of Power Systems Using Liapunov's Second Method," C72145-6, IEEE Conference Paper, 1972.

[17] G.W. Stagg and A.H. El-Abiad, Computer Methods in Power System Analysis, New York: McGraw-Hill, 1958.

[18] C.J. Tavora and O.J.M. Smith, "Equilibrium Analysis of Power Systems," IEEE Trans. Power apparaand Systems, vol. PAS-91, pp. 1131-1137, May/June 1971.

[19] C.J. Tavora and O.J.M. Smith, "Stability Analysis of Power Systems," Report No. ERL-70-5, College of Engineering, University of California, Berkeley, August 1970.

[20] L. Jenkins and R-W. Liu, "Stability of Flows of a Dynamic Flow Network," IEEE Trans. Circuits

and Systems, vol. CAS-23, pp. 826-829 December 1976

[21] A.R. Bergen and G. Gross, "On Multimachine Power System Representations," Report No. ERL-M392, College of Engineering, University of California, Berkeley, July 1972.

[22] A.R. Bergen and G. Gross, "Computation of Regions of Transient Stability of Multimachine Power Systems," IEEE Trans. Automatic Control, vol. AC-19, pp. 142-143, April 1974.

[23] P.J. Moylan and D.J. Hill, "Stability Tests for Multimachine Power Systems," Proc. IFAC Symposium, Melbourne, pp. 292-296, February 1977.

[24] P.M. Anderson and A.A. Fouad, Power System Control and Stability, The Iowa State University Press, Ames, Iowa, 1977.

[25] C.A. Desoer and E.S. Kuh, Basic Circuit Theory, New York: McGraw-Hill, 1969.

[26] T.E. Stern, Theory of Nonlinear Networks and Systems, An Introduction, Reading, Mass.: Addison-Wesley, 1965.

[27] L.O. Chua and P.M. Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, New Jersey: Prentice-Hall, 1975.

[28] M. Ribbens-Pavella, Comments on J. L. Willems "Direct Methods for Transient Stability Studies in Power System Analysis," and reply by author, IEEE Trans. Automatic Control, vol, AC-17, pp. 415-417, June 1972.

[29] J.M. Ortega and W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, New York: Academic Press, 1970.

[30] A.S. Householder, The Theory of Matrices in Numerical Analysis, Blaidsell, 1964.

[31] W. Hahn, Stability of Motion, New York: Springer-Verlag, 1967.

[32] F.S. Prabhakara and A.H. El-Abiad, "A Simplified Determination of Transient Stability Regions for Lyapunov Methods," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 672-689, March/April 1975.

[33] A.R. Bergen and G. Gross, "The Projective Equal Area Criterion," C 75 037-7, IEEE Winter Power Meeting, New York, January 1975.

[34] M.A. Pai and C.L. Narayana, "Finite Regions of Attraction for Multinonlinear Systems and its Application to the Power System Stability Problem," IEEE Trans. Automatic Control, AC-21, pp. 716-721, October 1976.

APPENDIX

Some simple results in the analysis of nonlinear resistive circuits are presented. Familiarity with basic concepts is assumed. More complete details may be found in [25-27].

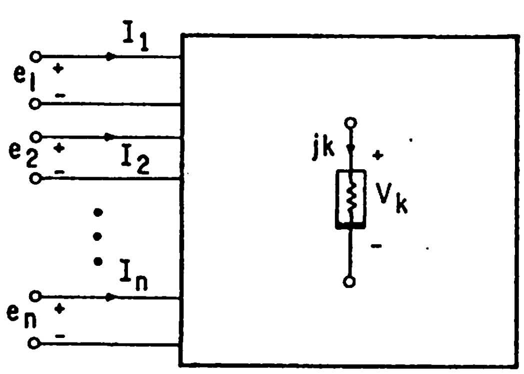

Figure 3 Nonlinear n-port resistive network

Consider the n-port representation given in Figure 3. This refers to an interconnection of ℓ nonlinear resistors at $N = n+1$ nodes. The Nth node is taken as a reference. The interconnections are described by an oriented graph which is assumed to be connected. Associated reference directions are used. The resistors are described by $j_k = g_k(v_k)$, where j_k and v_k denote the kth branch current and voltage respectively. Each node, other than the reference, has an injected current I_i, $i = 1,\ldots,n$.

Standard circuit matrices are the reduced incidence matrix $\underline{A}$ and the fundamental cutset matrix $\underline{Q}$. These $n\times\ell$ matrices have full row rank and describe the interconnections of the circuit graph. Matrix $\underline{A}$ is based on specifying branches incident at the nodes whereas $\underline{Q}$ specifies the branches in fundamental cutsets. Then Kirchhoff's laws have the convenient form for the above n-port

$$\underline{A}\underline{j} = \underline{I} \qquad \text{KCL} \tag{A-1}$$

$$\underline{v} = \underline{A}^T\underline{e} \qquad \text{KVL} \tag{A-2}$$

where $\underline{j}$, $\underline{v}$, $\underline{e}$ and $\underline{I}$ are vectors of branch currents, branch voltages, node-to-datum voltages (here serving as port voltages also) and injected currents respectively. Combining (A-1), (A-2) and the branch relationships gives

$$\underline{I} = \underline{A}\ \underline{g}(\underline{A}^T\underline{e}) \tag{A-3}$$

This specifies the aggregate n-port description in terms of circuit structure and branch resistance characteristics. An alternative description is obtained by using matrix $\underline{Q}$ to relate all branch voltages to just n tree branch voltages z_i

$$\underline{v} = \underline{Q}^T\underline{z} \qquad \text{KVL} \tag{A-4}$$

Then, we have

$$\underline{I} = \underline{A}\ \underline{g}(\underline{Q}^T\underline{z}) \tag{A-5}$$

A STRUCTURE PRESERVING MODEL FOR POWER SYSTEM STABILITY ANALYSIS

by

A. R. Bergen and D. J. Hill

Memorandum No. UCB/ERL M79/44

15 August 1979

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A STRUCTURE PRESERVING MODEL FOR
POWER SYSTEM STABILITY ANALYSIS

A.R. Bergen, Member, IEEE

D.J. Hill, Member, IEEE

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract - A new model for the study of power system stability via Lyapunov functions is proposed. The key feature of the model is an assumption of frequency-dependent load power, rather than the usual impedance loads which are subsequently absorbed into a reduced network. The original network topology is explicitly represented. This approach has the important advantage of rigorously accounting for real power loads in the Lyapunov functions. This compares favorably with existing methods involving approximations to allow for the significant transfer conductances in reduced network models. The preservation of network topology can be exploited in stability analysis, with the concepts of critical and vulnerable cutsets playing central roles in dynamic and transient stability evaluation respectively. Of fundamental importance is the feature that the Lyapunov functions give a true representation of the spatial distribution of stored energy in the system.

I. INTRODUCTION

The analysis of power system transient stability using Lyapunov function techniques has recently achieved a status as a viable tool for on-line security assessment. Particularly promising results are reported by Gupta and El-Abiad [1], Ribbens-Pavalla, et. al. [2], and Athay, et. al. [3]. The first on-line application to a real operating power system is discussed by Saito et. al. [4]. This follows efforts beginning around 1970 to apply Lyapunov methods to realistic multimachine power systems [5-8] and some twenty years of interest in such an approach -- see surveys by Ribbens-Pavella [9] and Fouad [10]. A major difficulty which remains to be overcome rigorously is that associated with allowing for significant transfer conductances. This is essentially an issue of modeling the loads in the network. The present paper offers a new model which can bypass this difficulty while maintaining the features related to the success obtained in previous work.

While many of the assumptions made to arrive at the usual classical model for transient stability analysis are reasonable, that of ignoring transfer conductances is usually quite crude [10]. This emanates from modeling the loads as impedances (with a substantial resistive component). These are then absorbed into the bus admittance matrix for a reduced network based on generator buses. Thus, although the original transmission network is very reasonably modeled as lossless, the reduced network certainly cannot be in general. Consequently, a path-independent potential function is not readily available for constructing Lyapunov functions. Attempts to develop general Lyapunov functions have met very limited success, especially when it is considered that ultimately these functions should replace those based on assuming the conductances are zero. Pai and Murthy [11] have a Lyapunov function for the two machine case, but a generalization has inherent difficulties [12]. Jocic et. al. [13] report an approach based on large-scale systems theory, but a clear improvement in practice is not achieved [14]. The inclusion of transfer conductances is sometimes handled by some approximation, either in the system description [6,15] or in evaluating the 'Lyapunov function' (or transient energy function) [3,7,16].

A further disadvantage of forming a reduced network (by suppressing load buses) is that the original network topology is lost. This can mask the role of structural aspects in stability assessment.

In this paper, a new model is presented which does not rely on a reduced network. This follows from the reasonable assumption, for bulk power supply systems, that each load on the transmission network can be represented as a frequency-dependent power load (assuming constant bus voltages). Taking this relationship to be linear leads to a very simple dynamic model which includes the state variables of the classical model plus extra variables associated with the loads. Since the loads are not incorporated into the transmission system, it can be quite accurately modeled as one with zero transfer conductances. Thus, it is a simple matter to generate Lyapunov functions. Further, the original network topology is preserved and the model can be regarded as having structural integrity. In exploiting this feature, it is natural that circuit theory ideas play an important role. Although used in other areas of power system analysis [17], circuit theory has not played a significant role in techniques for stability assessment. Tavora and Smith [18,19] have used it in a limited way to gain insight into power system equilibria, while Jenkins and Liu [20] have formulated a network flow model and used graph theoretic ideas to develop stability results. The model used here is presented in two forms: a network form in terms of circuit matrices and an aggregate view, which is an adaptation of a similar presentation for classical models given by Bergen and Gross [21,22].

This formulation proves to be a convenient basis for consideration of the dependence of stability properties on network topology and system loads. For dynamic stability, the linearized dynamical equations are studied. Adapting results in [18,19,23], a result is given for testing stable equilibrium points in terms of so-called critical cutsets. For transient stability, reference is made to the abovementioned work on stability assessment using Lyapunov functions (and transient energy functions). Implicit in this is the importance of cutsets along which the system tends to separate. The notion of a vulnerable cutset is formulated and some indication given of how to use it in the preliminary stages of transient stability assessment. Taking a transient energy type Lyapunov function for the aggregate system, it is readily seen that this is the sum of kinetic energies associated with the generator rotors and the sum of potential energies associated with all the lines. Thus the Lyapunov function can truly represent the spatial disribution of stored energy in the physical power system. This leads to the concept of a topological

Lyapunov function.

The structure of the paper is as follows. Section II gives a description of the new model. In Section III, a discussion is given on the system equilibrium points and a test provided for stable equilibria. The concept of a topological Lyapunov function is the subject of Section IV. In Section V, this is considered, along with the idea of vulnerable cutsets, in transient stability analysis. Section VI gives some conclusions and the Appendix summarize results from circuit theory.

II. MULTIMACHINE POWER SYSTEM MODEL

In this section, a model of a multimachine power system is developed. Its novelty lies in not taking the usual step of assuming impedance loads, which are absorbed into the transmission network. Otherwise, we make the same assumptions that go with the classical model -- see [9,10,24] for instance.

Our starting point for the model is the network of buses connected by transmission lines, which is the one described by load flow equations. The system shown in Figure 1a will be used in the sequel for illustrative purposes. It has four buses, two of which have generators attached. In general, suppose there are m generators and n_0 buses in the physical system, with n_0-m buses having loads and no generation. It is convenient to introduce fictitious buses representing the internal generation voltages. These are

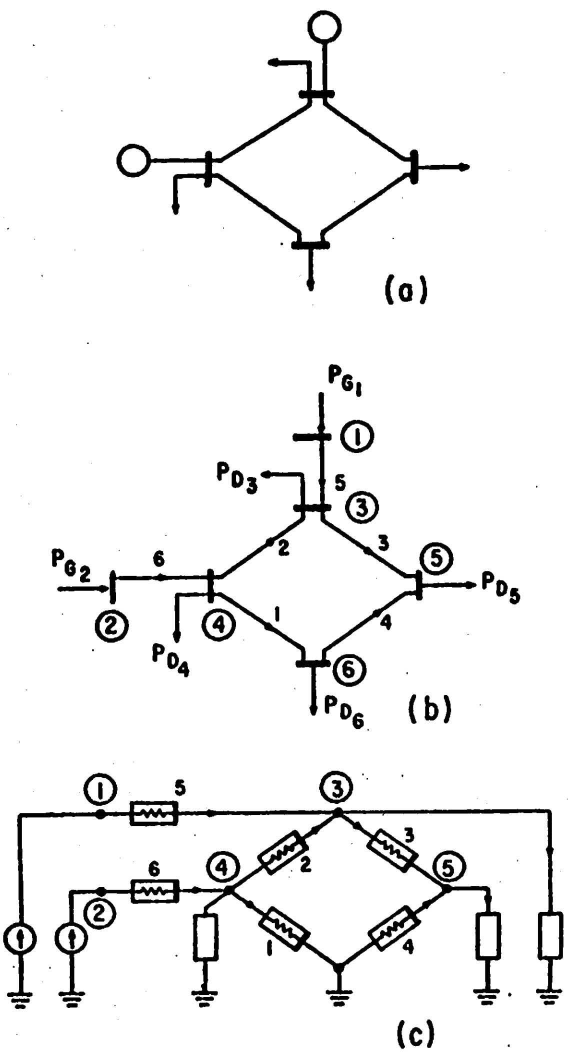

Figure 1 a) A four bus power network

b) Augmented network with generator bus lines

c) Analogous nonlinear resistive circuit

connected to the generator buses via reactances accounting for transient reactances and connecting lines. These reactances can be regarded as 'transmission lines' and henceforth are referred to as the generator bus lines. Thus in the augmented network there is a total of $n=m+n_0$ buses. For convenience, we number the fictitious generator buses $1,\ldots,m$, the corresponding physical buses $m+1,\ldots,2m$ and the remaining load buses $2m+1,\ldots,n$. Suppose that within the transmission network there are ℓ_0 lines. Then ℓ_0 must satisfy $\ell_0 \leq \frac{1}{2} n_0(n_0-1)$ and the total number of 'lines' in the augmented network is $\ell = m+\ell_0$. We number the transmission network lines $1,\ldots,\ell_0$ and the generator bus lines $\ell_0+1,\ldots,\ell$ connected to buses $1,\ldots,m$ respectively. The nth bus will be used as a reference. For the four bus example, Figure 1b shows the augmented network. At this stage, it is useful to recognize that the network is analogous to a nonlinear resistive network with real power corresponding to current and the angle difference across a line corresponding to branch voltage. Assuming a lossless transmission network and $\sum_{i=1}^{n} P_i = 0$, where P_i is the injected power at bus i, Kirchhoff's laws hold in the obvious sense. For the four bus example, the analogous circuit is shown in Figure 1c. The nonlinear resistance characteristic for each branch is given by the familiar power-angle relationship for a line. We assume that the graph for the network is connected and planar and the branches are oriented according to associated reference directions. We will make use of certain concepts and results from circuit theory. The Appendix summarizes some essential facts and further details are available in references [25-27].

Now the key assumption of dynamic loads is introduced. Let P_{D_i} be the real power drawn by the load at bus i. In general P_{D_i} is a nonlinear function of voltage and frequency. For constant voltages and small frequency variations around the operating point $P^0_{D_i}$, it is reasonable to assume

$$P_{D_i} = P^0_{D_i} + D_i\dot{\delta}_i \qquad i = m+1,\ldots,n \tag{1}$$

where $D_i > 0$. Note that as $D_i \to 0^+$ we obtain a constant load model. This load frequency dependence is usually assumed in modeling the power-frequency control system, but has not been used in modeling for transient stability. Using (1) we are led to

$$M_i\ddot{\delta}_i + D_i\dot{\delta}_i + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}\sin(\delta_i-\delta_j) = P^0_{M_i} - P^0_{D_i} \triangleq P^0_i \qquad i=1,\ldots,n \tag{2}$$

where

$M_i > 0 \quad i=1,\ldots,m$ (generator inertia constants)

$M_i = 0 \quad i=m+1,\ldots,n$

$D_i > 0 \quad i=1,\ldots,m$ (steam and mechanical damping of generator)

$D_i > 0 \quad i=m+1,\ldots,n$ (frequency coefficient of load)

$P^0_{D_i} = 0 \quad i=1,\ldots,m$

$P^0_{M_i} = 0 \quad i=m+1,\ldots,n$

Equation (2) looks similar to the usual classical swing equation model used in previous studies of transient stability via Lyapunov methods. However, there are important differences. Along with the mechanical input powers $P^0_{M_i}$, the loads $P^0_{D_i}$ are shown explicitly. Consequently, the network topology is preserved just as in the case of the load flow model.

It will be convenient to make the assumption that $\sum_{i=1}^{n} P^0_i = 0$, but in practice this may not be reasonable for the period following a fault. A resolution of this is achieved by adopting the idea suggested by Willems [28]. Adding equations (2) gives

$$\sum_{i=1}^{m} M_i \dot{\omega}_i + \sum_{i=1}^{n} D_i \omega_i = \sum_{i=1}^{n} P^0_i \tag{3}$$

The required equilibrium is given by $\delta_i - \delta_j = c_{ij}$, where c_{ij} is a constant, for all i,j. Thus all ω_i approach the same constant speed ω^0. From (3), we have

$$\omega^0 = \sum_{i=1}^{n} P^0_i \Big/ \sum_{i=1}^{n} D_i$$

(Recall that $D_i > 0$ for all i.) Then consider the transformation

$$\omega'_i = \omega_i - \omega^0$$
$$P'_i = P^0_i - D_i \omega^0$$

It is easy to check that $\sum_{i=1}^{n} P'_i = 0$ and the equilibrium angular velocity is $\omega'^0 = 0$. Henceforth, we assume that this change of reference has been carried out if appropriate and drop the prime superscripts.

More convenient forms of the model can be derived as state-space descriptions and some flexibility is achieved by using various notions from circuit theory. The following is largely an extension of the development in [21] to the present situation. The m generators require a state-space dimension of 2m-1 with nonuniform damping [9,21] so, on including the loads, equation (2) defines a state-space of dimension n+m-1. The state variables can be chosen as the m velocities $\omega_i \triangleq \dot{\delta}_i$ and n-1 internodal angles $\alpha_i = \delta_i - \delta_n$. However, other choices of angles for the state are useful. Define $\underline{\delta} = [\delta_1 \dots \delta_n]^T$, $\underline{\omega} = [\omega_1 \dots \omega_n]^T$ and $\underline{\alpha} = [\alpha_1 \dots \alpha_{n-1}]^T$. Also, define a vector of line angle differences $\underline{\sigma} = [\sigma_1 \dots \sigma_\ell]^T$ where $\sigma_k = \delta_i - \delta_j$ for the kth line joining buses i and j. The vectors $\underline{\sigma}$ and $\underline{\alpha}$ are related to $\underline{\delta}$ via transformations $\underline{\sigma} = \underline{L}\,\underline{\delta}$ and $\underline{\alpha} = \underline{T}\,\underline{\delta}$. Matrix $\underline{T}$ is given by

$$\underline{T} = [\underline{I}_{n-1} \vdots -\underline{e}]$$

where $\underline{I}_{n-1}$ is the (n-1) identity matrix and $\underline{e}$ the (n-1) vector with unity entries. Now we introduce the reduced incidence matrix

$$\underline{A} = \left[\begin{array}{c|c} \underline{0} & \underline{I}_m \\ \hline \underline{A}_t & \begin{array}{c} -\underline{I}_m \\ \underline{0} \end{array} \end{array}\right] \tag{4}$$

where $\underline{A}_t$ is the reduced incidence matrix of the transmission network. Then, we have

$$\underline{\sigma} = \underline{A}^T \underline{\alpha}$$
$$= \underline{A}^T \underline{T}\,\underline{\delta} \tag{5}$$

Now partition $\underline{T}$ according to

$$\underline{T} = \left[\begin{array}{c|cc} \underline{I}_m & \underline{0} & -\underline{e} \\ \hline \underline{0} & \multicolumn{2}{c}{\underline{T}_t} \end{array}\right] \triangleq \left[\underline{T}_1 \;\vdots\; \underline{T}_2\right] \tag{6}$$

Hence

$$\underline{L} = \underline{A}^T \underline{T} = \left[\begin{array}{c|cc} \underline{0} & \multicolumn{2}{c}{\underline{A}_t^T \underline{T}_t} \\ \hline \underline{I}_m & -\underline{I}_m & \underline{0} \end{array}\right]$$

As an alternative to $\underline{\alpha}$ we may wish to use a set of n-1 tree branch angle differences θ_i. With the chosen numbering system, we further number cotree branches first and then the tree branches. Writing $\underline{\sigma} = [\underline{\sigma}_c^T \underline{\theta}^T]^T$ and defining transformation $\underline{\theta} = \underline{K}\,\underline{\delta}$, we see that $\underline{K}$ can be obtained as an appropriate submatrix of $\underline{L}$. Note that $\underline{\sigma} = \underline{Q}^T \underline{\theta}$ where $\underline{Q}$ is the fundamental cutset matrix and so alternatively $\underline{L} = \underline{Q}^T \underline{K}$. These matrix relationships can be explored further for their own sake, but we only study them further as required in the sequel.

With the resistive circuit analogy in mind, we define the constitutive relationship for branch k by $p_k = g_k(\sigma_k)$ where p_k is the power flow in the branch. We have

$$g_k(\sigma_k) = b_k \sin \sigma_k \tag{7}$$

where $b_k = b_{ij}$ and it is assumed always that branch k connects buses i and j. In vector form, write $\underline{p} = \underline{g}(\underline{\sigma})$. Now $P^0_n = -\sum_{i=1}^{n-1} P^0_i$, so there are n-1 independent excess node powers. Let $\underline{P}^0 = [P^0_1 \dots P^0_{n-1}]^T$. Then via nodal analysis (A-3), the load flow can be written

$$\underline{P} = \underline{A}\,\underline{g}(\underline{\sigma})$$
$$= \underline{A}\,\underline{g}(\underline{A}^T \underline{\alpha}) \triangleq \underline{f}(\underline{\alpha}) \tag{8}$$

Note that

$$f_i(\underline{\alpha}) = \sum_{\substack{k=1 \\ k \neq i}}^{n-1} b_{ik} \sin(\alpha_i - \alpha_k) + b_{in} \sin \alpha_i, \quad i=1,\dots n-1 \tag{9}$$

Alternatively, in terms of tree branch angle differences, we get from (A-5)

$$\underline{P} = \underline{A}\,\underline{g}(\underline{Q}^T \underline{\theta}) \triangleq \underline{f}_\theta(\underline{\theta}) \tag{10}$$

Now define

$$\underline{M} = \mathrm{diag}\{M_i\}$$
$$\underline{D} = \mathrm{diag}\{D_i\}$$

Then it is straightforward to show that (2) can be replaced by [21]

$$\underline{M}\,\dot{\underline{\omega}} + \underline{D}\,\underline{\omega} + \underline{T}^T [\underline{f}(\alpha) - \underline{P}^0] = \underline{0} \tag{11}$$

With appropriate partitioning of $\underline{M}, \underline{D}$ and using (6), (11) can be rewritten as

$$\underline{M}_1 \dot{\underline{\omega}}_1 + \underline{D}_1 \underline{\omega}_1 + \underline{T}_1^T [\underline{f}(\underline{\alpha}) - \underline{P}^0] = \underline{0} \tag{12a}$$

$$\underline{D}_2 \underline{\omega}_2 + \underline{T}_2^T [\underline{f}(\underline{\alpha}) - \underline{P}^0] = \underline{0} \tag{12b}$$

where subscripts 1 and 2 refer to the generators and loads respectively.

We now proceed to develop a third model description as so-called normal form or state-space form. Firstly, we have

$$\dot{\underline{\alpha}} = \underline{T}\,\underline{\omega} = \underline{T}_1\underline{\omega}_1 + \underline{T}_2\underline{\omega}_2 \tag{13}$$

Using (12b) to eliminate $\underline{\omega}_2$ in (13) along with (12a) gives

$$\dot{\underline{\alpha}} = \underline{T}_1\underline{\omega}_1 - \underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T \quad [\underline{f}(\underline{\alpha}) - \underline{P}^0] \tag{14a}$$

$$\dot{\underline{\omega}}_1 = -\underline{M}_1^{-1}\underline{D}_1\underline{\omega}_1 - \underline{M}_1^{-1}\underline{T}_1^T \quad [\underline{f}(\underline{\alpha}) - \underline{P}^0] \tag{14b}$$

Equations (14) define the system trajectories in a state-space of dimension m+n-1.

Equations (2), (12) and (14) give three alternative mathematical representations of the model. Equations (12) and (14) give aggregate representations, but substituting equation (8) provides the structural information in terms of circuit matrices. Sometimes it is convenient to have (12) or (14) in terms of tree branch angles $\underline{\theta}$ or write (12) in terms of branch angles $\underline{\sigma}$. Using the above transformations, these alternative representations can be developed as required.

The assumption that all $D_i > 0$ is certainly reasonable, but a comment is in order on the case where some D_i are assumed to be zero. Further, the above has implicitly assumed generator damping to be nonuniform. Setting $\underline{D}_1 = \underline{0}$ or with uniform damping the obvious extension of the classical case applies. That is, the state-space dimension is reduced by one. Setting some of the load $D_i \to 0^+$, however, gives a model in terms of differential and algebraic equations. Mathematically this is relatively more difficult to accomodate. In view of the presence always of some frequency dependence in the load, this will not be investigated further.

III. EQUILIBRIUM POINTS AND LOCAL STABILITY

Before considering the global stability properties of the system described by equations (14), attention should be given to the equilibrium points and their stability.

In the previous section, we saw that without loss of generality the equilibria correspond to $(\underline{\alpha},\underline{\omega})=(\underline{\alpha}^e,\underline{0})$ where $\underline{\alpha}^e$ is constant. From (14), we have

$$\underline{N}[\underline{f}(\underline{\alpha}^e) - \underline{P}^0] = \underline{0} \tag{15}$$

where

$$\underline{N} = \underline{T}_1^T + \underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T = \begin{bmatrix} \underline{I}_m & \underline{0} \\ \underline{0} & \tilde{\underline{D}}_2^{-1} \end{bmatrix} + \frac{1}{D_n}\,\underline{e} >< \underline{e}$$

with $\tilde{\underline{D}}_2 = \mathrm{diag}\{D_{m+1},\ldots,D_{n-1}\}$

Thus $\underline{N}$ has rank n-1. Then using (15) the equilibria are given by $\underline{\omega} = \underline{0}$ and the solutions of

$$\underline{f}(\underline{\alpha}^e) = \underline{P}^0 \tag{16}$$

We call the function $\underline{f}(\cdot)$ the <u>flow function</u> [18]. Due to the periodic dependence of $\underline{f}(\underline{\alpha})$ on $\underline{\alpha}$, the domain of the flow function is the n-1 dimensional torus. That is we write $\underline{f}: T^{n-1} \to \mathbb{R}^{n-1}$ where

$$T^{n-1} = \{\underline{\alpha} \bmod 2\pi \,:\, \underline{\alpha} \in \mathbb{R}^{n-1}\}$$

To study properties of the solutions of (16), we note that $\underline{f}(\cdot)$ is continuously differentiable and identify its Jacobian matrix denoted by $\underline{F}(\underline{\alpha})$. The (i,j)th term of $\underline{F}(\underline{\alpha})$ is given by

$$\frac{\partial f_i(\underline{\alpha})}{\partial \alpha_j} = \begin{cases} b_{in}\cos\alpha_i + \sum\limits_{\substack{k=1\\k\neq i}}^{n-1} b_{ik}\cos(\alpha_i-\alpha_k), & i = j \\ -b_{ij}\cos(\alpha_i-\alpha_j), & i \neq j \end{cases}$$

Since $\underline{F}(\underline{\alpha})$ has full normal rank, (16) has a finite number of isolated solutions in T^{n-1} [29]. Unfortunately, there appear to be no useful results on answering questions related to the exact number of solutions for a given $\underline{P}^0$ unique stable solutions etc. Tavora and Smith [18] have given some useful insights, by way of examples, into how the number of solutions depends on network topology, line power transfer coefficients b_{ij}, and $\underline{P}^0$.

The starting point for stability analysis of (14) is a solution of (16) about which the system is locally stable. The study of stable equilibria seems largely to rely on the intuitive idea that if all lines satisfy $|\sigma_i^0| < \pi/2$, then the equilibrium is stable. From a combination of ideas in [19,23], we can actually state a precise version. In view of structural integrity of the model, the test takes some significance in its being based on making tests on cutsets. Further, the techniques employed prepare the way for the study of transient stability in later sections. For the study of local stability, we firstly linearize equations (14) about the equilibrium point $(\underline{\alpha}^0,\underline{0})$ to obtain differential equations in variables $\underline{\Delta\alpha} = \underline{\alpha} - \underline{\alpha}^0$ and $\underline{\Delta\omega}_1 = \omega_1 - \underline{\omega}_1^0 = \underline{\omega}_1$. This gives

$$\begin{bmatrix} \dot{\underline{\Delta\alpha}} \\ \dot{\underline{\omega}}_1 \end{bmatrix} = \begin{bmatrix} -\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0) & \underline{T}_1 \\ -\underline{M}_1^{-1}\underline{T}_1^T\underline{F}(\underline{\alpha}^0) & -\underline{M}^{-1}\underline{D}_1 \end{bmatrix} \begin{bmatrix} \underline{\Delta\alpha} \\ \underline{\omega}_1 \end{bmatrix} \tag{17}$$

Study of (17) could proceed by eigenvalue techniques or Lyapunov methods. The latter turns out to give a simple answer and an appropriate Lyapunov function is a quadratic counterpart to the one to be used for transient stability [23]. It is convenient to define the polytope

$$\Lambda^\ell = \{\underline{\sigma} \in \mathbb{R}^\ell \,:\, |\sigma_i| \leq \pi/2, \quad i=1,\ldots,\ell\}$$

We observe that for $\underline{\sigma}^0 \in \Lambda^\ell$, then $\underline{F}(\underline{\alpha}^0)$ is nonnegative definite; this follows from Gershgorin's Theorem [30] since $\underline{F}(\underline{\alpha}^0)$ is diagonally dominant with positive diagonal elements. Motivated by stored energy, consider a possible Lyapunov function as

$$V(\underline{\Delta\alpha},\underline{\omega}_1) = \frac{1}{2}\,\underline{\omega}_1^T\underline{M}_1\underline{\omega}_1 + \frac{1}{2}\,\underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha}$$

Differentiating V along the solutions of (17) gives

$$\begin{aligned} \dot{V}(\underline{\Delta\alpha},\underline{\omega}_1) &= \underline{\omega}_1^T\underline{M}_1(-\underline{M}_1^{-1}\underline{T}_1^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} - \underline{M}_1^{-1}\underline{D}_1\underline{\omega}_1) \\ &\quad + \underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)(-\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} + \underline{T}_1\underline{\omega}_1) \\ &= -\underline{\omega}_1^T\underline{D}_1\underline{\omega}_1 - \underline{\Delta\alpha}^T\underline{F}(\underline{\alpha}^0)\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T\underline{F}(\underline{\alpha}^0)\underline{\Delta\alpha} \end{aligned} \tag{18}$$

Now $\dot{V} \equiv 0$ implies that $\underline{\omega}_1 \equiv \underline{0}$ and

$$\underline{T}_2^T \underline{F}(\underline{\alpha}^0)\Delta\underline{\alpha} \equiv \underline{0} \tag{19a}$$

From (17), $\underline{\omega}_1 \equiv \underline{0}$ gives

$$\underline{T}_1^T \underline{F}(\underline{\alpha}^0)\Delta\underline{\alpha} \equiv \underline{0} \tag{19b}$$

Equations (19) imply

$$\underline{T}^T \underline{F}(\underline{\alpha}^0)\Delta\underline{\alpha} \equiv \underline{0} \tag{20}$$

Then, if $\underline{F}(\underline{\alpha}^0)$ is positive definite, V is positive definite and, since $\underline{T}^T$ is a full rank matrix, (20) gives that $\dot{V} \equiv 0$ implies $(\Delta\underline{\alpha}, \underline{\omega}_1) = (\underline{0}, \underline{0})$. From standard Lyapunov stability theory [31], we then have that the equilibrium point $(\underline{\alpha}^0, \underline{0})$ is asymptotically stable. However, so far it has only been demonstrated that $\underline{F}(\underline{\alpha}^0)$ is nonnegative definite on polytope Λ^ℓ. To obtain the final statement of stability conditions, we use a result given by Tavora and Smith [19]. We will refer to lines with zero synchronizing coefficients, i.e., for which $\cos \sigma_i = 0$, as zero-valued. A cutset of zero-valued lines is called a critical cutset. Then, from [19], we get that $\det \underline{F}(\underline{\alpha}^0) = 0$ in Λ^ℓ if and only if the system has a critical cutset. Hence, the equilibrium point $(\underline{\alpha}^0, \underline{0})$ is asymptotically stable if $\underline{\sigma}^0 \in \Lambda^\ell$ and there are not critical cutsets. The absence of critical cutsets is ensured by having a tree of lines which are not zero-valued. We can now summarize the result as it pertains to system (14) as follows.

Theorem 1. Consider an equilibrium point for the power system satisfying (16). Suppose that $\underline{\sigma}^0 \in \Lambda^\ell$ and the generator bus lines are not zero-valued. Then the equilibrium point is asymptotically stable if the transmission line network has no critical cutsets.

For a normal operating condition, of course, these conditions are easily met. However, after a fault or during abnormal loading conditions the system could be operating close to the boundary of polytope Λ^ℓ. Actually, in [19] the region of stable equilibrium points is claimed to be bigger than Λ^ℓ and given by the so-called principal region. However, in general, this principal region would not be easy to calculate and it appears that Λ^ℓ is a close approximation to it.

IV. TOPOLOGICAL LYAPUNOV FUNCTION

Under normal operating conditions, the system will be in or near an equilibrium state satisfying the stability conditions of Theorem 1. A fault can alter $\underline{P}^0$, the transmission topology, or the coefficients b_{ij} giving new post-fault equilibrium states (if $\underline{P}^0$ is feasible, i.e., if $\underline{P}^0$ lies in the range of $f(\cdot)$). Whether the system settles to the post-fault s.e.p. is studied via transient stability analysis using equations (14) as the basic model. We use a Lyapunov function which is motivated by stored energy of the aggregate system. This, of course, has been the basic Lyapunov function going back to early work. However, with the present new model and using some circuit theory ideas — see Appendix A — additional insights into stability assessment are possible.

Suppose that $(\underline{\alpha}^0, \underline{0})$ is a stable post-fault equilibrium point. We define the Lyapunov function $V: \mathbb{R}^{n-1} \times \mathbb{R}^m \to \mathbb{R}$ by

$$V(\underline{\alpha}, \underline{\omega}_1) = \frac{1}{2}\, \underline{\omega}_1^T \underline{M}_1 \underline{\omega}_1 + W(\underline{\alpha}, \underline{\alpha}^0) \tag{21}$$

where

$$W(\underline{\alpha}, \underline{\alpha}^0) = \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{f}(\underline{\xi}) - \underline{f}(\underline{\alpha}^0)]^T d\underline{\xi}$$

In this form, it is a direct generalization of the Lyapunov function used by Bergen and Gross [21,22] and represents the sum of aggregate kinetic energy and potential energy. The integral defining the potential function $W(\underline{\alpha}, \underline{\alpha}^0)$ is evaluated over an arbitrary path between $\underline{\alpha}^0$ and $\underline{\alpha}$. Since $\underline{F}(\underline{\alpha})$ is symmetric, the integral is path independent and V is well-defined. It is interesting to note the following.

Theorem 2. The function V given by (21) can also be written as

$$V(\underline{\alpha}, \underline{\omega}_1) = \frac{1}{2} \sum_{k=1}^{m} M_k \omega_k^2 + \sum_{k=1}^{\ell} b_k h(\sigma_k, \sigma_k^0) \tag{22}$$

where

$$h(\sigma_k, \sigma_k^0) = \int_{\sigma_k^0}^{\sigma_k} (\sin u - \sin \sigma_k^0)\, du$$

Proof: From equation (8), we have

$$\underline{f}(\underline{\xi}) = \underline{A}\, \underline{g}(\underline{A}^T \underline{\xi})$$

Then the potential function is given by

$$W(\underline{\alpha}, \underline{\alpha}^0) = \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{f}(\underline{\xi}) - \underline{f}(\underline{\alpha}^0)]^T d\underline{\xi}$$

$$= \int_{\underline{\alpha}^0}^{\underline{\alpha}} [\underline{g}(\underline{A}^T\underline{\xi}) - \underline{g}(\underline{A}^T\underline{\alpha}^0)]^T \underline{A}^T d\underline{\xi}$$

$$= \int_{\underline{\sigma}^0}^{\underline{\sigma}} [\underline{g}(\underline{u}) - \underline{g}(\underline{\sigma}^0)]^T d\underline{u}$$

on setting $\underline{u} = \underline{A}^T\underline{\xi}$ and using transformation to branch angles. Using (7),

$$W(\underline{\alpha}, \underline{\alpha}^0) = \sum_{k=1}^{\ell} b_k \int_{\sigma_k^0}^{\sigma_k} (\sin u - \sin \sigma_k^0)\, du \tag{23}$$

∎

Thus the total potential energy is seen to be the sum of the potential energies of the individual branches. What is interesting here is that just as the kinetic energy may be identified with individual generators, the potential energy may be identified with individual transmission lines (including generator transient reactances). Thus the Lyapunov function truly reflects the spatial distribution of stored energy in the physical system since the original topology has been preserved in the model. Hence we refer to the function (21) or (22) used in connection with model (14) as a topological Lyapunov function.

To actually show that V given by (21) is a Lyapunov function involves a simple modification of the steps used for the quadratic energy function in the previous section. Firstly, we determine a region where W is positive definite. Consider the function $h(\cdot, \sigma_k^0)$ and suppose $\underline{\sigma}^0 \in \Lambda^\ell$. Then $h(\cdot, \sigma_k^0)$ is a positive definite and strictly monotone increasing function over the interval $(\sigma_k^\ell, \sigma_k^u)$ with $\sigma^\ell \triangleq -\pi - \sigma_k^0$ and $\sigma^u = \pi - \sigma_k^0$. Now define the polytope

$$\Gamma^{\ell}(\underline{\sigma}^0) = \{\underline{\sigma} \in \mathbb{R}^{\ell} : \sigma_i \subset (\sigma_k^{\ell}, \sigma_i^u) \quad i = 1,\ldots,\ell\}$$

We denote the closure and boundary of $\Gamma^{\ell}(\underline{\sigma}^0)$ by $\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ and $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ respectively. Obviously, from (23), $W(\cdot,\underline{\alpha}^0)$ is positive definite over the polytope Γ^{ℓ} (where $\underline{\sigma} = \underline{A}^T\underline{\alpha}$ is assumed throughout). The above mentioned monotonicity property implies that all u.e.p.'s must lie on or outside of $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$. Now differentiating V along the trajectories of (14) gives

$$\dot{V}(\underline{\alpha},\underline{\omega}_1) = -\underline{\omega}_1^t\underline{D}_1\underline{\omega}_1 - [\underline{f}(\underline{\alpha})-\underline{f}(\underline{\alpha}^0)]^T\underline{T}_2\underline{D}_2^{-1}\underline{T}_2^T[\underline{f}(\underline{\alpha})-\underline{f}(\underline{\alpha}^0)] \tag{24}$$

Thus, since $\underline{D}_1 > \underline{0}$, $\underline{D}_2 > \underline{0}$, $\dot{V}$ is at least negative semidefinite. Corresponding to (20), we have $\dot{V} \equiv 0$ implying

$$\underline{T}^T[\underline{f}(\underline{\alpha}) - \underline{f}(\underline{\alpha}^0)] \equiv \underline{0} \tag{25}$$

Hence, since $\underline{T}$ is full rank, (25) implies $\underline{f}(\underline{\alpha}) - \underline{f}(\underline{\alpha}^0) \equiv \underline{0}$ and $\dot{V} \equiv 0$ only at equilibrium points. In the usual way, well-known stability results [31] determine a region of asymptotic stability defined by

$$\Omega_{\ell} = \{(\underline{\alpha},\underline{\omega}_1) : V(\underline{\alpha},\underline{\omega}_1) < V_{\ell}(\underline{\alpha}^0)\} \tag{26}$$

where V_{ℓ} is chosen so that Ω_{ℓ} excludes all the u.e.p.'s. In particular Ω_{ℓ} excludes $(\underline{\alpha}^*,\underline{0})$, the u.e.p. of lowest potential energy.

It is interesting also to note that substituting (12b) into (24) gives

$$\dot{V}(\underline{\alpha},\underline{\omega}_1) = -\underline{\omega}^T\underline{D}\,\underline{\omega} \tag{27}$$

Equation (27) shows that all the D_i act similarly to account for dissipation of energy, and the simple positivity of the coefficients insures that $\dot{V} \leq 0$. Thus the precise values of the D_i, which vary and are difficult to measure, are not needed.

V. VULNERABLE CUTSETS AND TRANSIENT STABILITY ASSESSMENT

The major part of the effort to make Lyapunov methods work for transient stability assessment in realistic power systems has been directed to efficient algorithms for estimating the region of stability in the state-space. In this section, we look briefly at how the techniques can be interpreted, and possibly improved upon, with the new model. A complete presentation is beyond the scope of this paper.

Most methods for finding the extent of stability rely on calculating (or approximating) the u.e.p. $(\underline{\alpha}^*,\underline{0})$ with lowest potential $V_{\ell}(\underline{\alpha}^0) = W(\underline{\alpha}^*,\underline{\alpha}^0)$ [1,2,6,18,32]. Other work is not explicitly concerned with calculating u.e.p.'s. Bergen and Gross [33] and Pai and Narayana [34] present minimization procedures on the polytope $\partial\bar{\Gamma}^{\ell}(\underline{\sigma}^0)$ (or its equivalent in $\underline{\alpha}$ space) for estimating a close lower bound for V_{ℓ}. The novel feature of the procedure in [33] is its simple graphical calculations. Thus it is more in the spirit of the equal area criterion for two-machine systems. All of the abovementioned work is motivated by the need to avoid the prohibitive computational task of calculating all the u.e.p.'s and then testing each one to find $W(\underline{\alpha}^*, \underline{\alpha}^0)$. In looking for fundamental aspects of this problem, we are led to the role of system structure in the solution techniques. Ribbens-Pavella et al. [2] take the attitude that the most likely consequence of instability is for one generator to lose synchronism. This reduces the problem to testing 2(n-1) u.e.p.'s. In other results [1,32], the loss of groups of machines is explicitly allowed for. Physical reasoning reduces the number of possibilities for the system to split up. For instance, Gupta and El-Abiad [1] restrict attention to cutsets containing the line on which the fault occurred. For present purposes, it is sufficient merely to note that the transient stability problem seems related in a fundamental way to a ranking of the network cutsets in terms of what will be referred to here as <u>vulnerability</u>. The structural integrity of the present model adds to the meaningfulness of such a concept.

In the special case of $\underline{P}^0 = \underline{0}$ there is a simple connection between u.e.p.'s and power flows on transmission network cutsets. In particular, the u.e.p. of lowest potential may easily be identified and calculated by examining an index of vulnerability for all the cutsets. In the case $\underline{P}^0 = \underline{0}$, the solution $\underline{\sigma}^0 = \underline{0}$ is the s.e.p. and by (16) the (neighboring) u.e.p.s. have the property $\sigma_i = 0, \pm\pi$. We will refer to lines with $|\sigma_i| = \pi$ as <u>saturated</u> lines. Thus, corresponding to every u.e.p. is a set of saturated branches. A further result is stated in the following proposition.

<u>Proposition</u>. Assume that $\underline{P}^0 = 0$. Then a subset of the saturated branches corresponding to an u.e.p. form a cutset.

Proof: For a three bus triangular mesh structure the result is trivial since either all branches are zero or two are saturated and one zero. Since the system graph is planar, we can consider it as an interconnection of triangular meshes and single branches (by introducing internal zero branches if necessary).

Since we have an u.e.p., at least one branch must be saturated. Now, using KVL and the result for a single mesh, one can argue that the result holds in general. Starting from a saturated branch, we can build up a line of saturated branches through meshes with saturated branches in common. This line can terminate by having the only adjoining mesh at the zero branch or if the line rejoins itself. In either case, a cutset of saturated branches has been generated. ∎

It is easy to see that an u.e.p. can correspond to a number of saturated cutsets. For instance, each generator bus line in Figure 1b could give a separate saturated cutset at an u.e.p.

Continuing then with the simple special case of $\underline{P}^0 = \underline{0}$. Let $(\underline{\alpha}^e,\underline{0})$ be an u.e.p. of interest. Then, from Theorem 2, we have

$$W(\underline{\alpha}^e,\underline{0}) = \sum_{k=1}^{\ell} b_k h(\alpha_k^e, 0)$$

Now $h(\alpha_k^e,0) = \begin{cases} 2, & \sigma_k^e = \pm\pi \\ 0, & \sigma_k^e = 0 \end{cases}$

Thus

$$W(\underline{\alpha}^e,\underline{0}) = 2\sum_{k=k_1}^{k_s} b_k \tag{28}$$

where the summation is over the s saturated lines numbered $k_1,\ldots,k_s$. Then we have exactly, in view of the proposition, that the u.e.p. $(\alpha^*,\underline{0})$ and most vulnerable cutset are provided by minimizing the sum in (28) over all cutsets. (If there is more than one saturated cutset corresponding to an u.e.p., obviously a more vulnerable cutset can be found by setting some branch angles to zero.) It is convenient to introduce some notation. Let C_i denote the ith cutset and we write $C_i = \{i_1,\ldots,i_q\}$ where i_j identifies the jth branch in the ith cutset. Then we have

$$\mathcal{V}_i \triangleq 2 \sum_{C_i} b_k$$

as an index of vulnerability for ith cutset. (A larger $\mathcal{V}_i$ corresponds to a less vulnerable cutset.)

The situation where $\underline{P}^0 = \underline{0}$ is certainly not realistic in practice, except insofar as it approximates very low power levels. However, the idea of ranking the vulnerability of cutsets has been illustrated with a simple exact answer. Now, in general where $\underline{P}^0 \neq \underline{0}$, we will call branches saturated at an u.e.p. $\underline{\alpha}^e$ when $\pi/2 \leq |\sigma_i^e| \leq 3\pi/2$. However, exact calculation of the u.e.p.'s is to be avoided, so an index of vulnerability depending on this is not acceptable. This difficulty can be overcome by adopting some ideas used by Prabhakara and El-Abiad [32] for estimating all the u.e.p.'s. A measure of the system vulnerability at a cutset can be obtained by picturing a separation of the system into two parts along the cutset. This is illustrated in Figure 2. It is convenient to consider the polytope $\Gamma^\ell(\underline{\sigma}^0)$ corre-

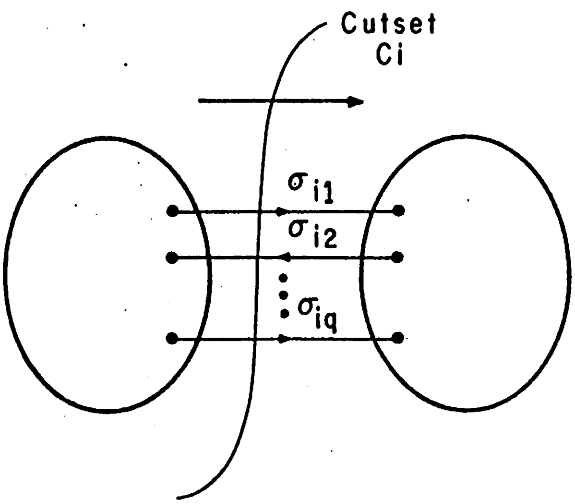

Figure 2 System separation on a cutset

sponding to s.e.p. $(\underline{\alpha}^0, \underline{0})$ where $\underline{\sigma}^0 = \underline{A}^T\underline{\alpha}^0$. By posing the hypothetical situation of the separation occurring with $\underline{\sigma}$ on $\partial\bar{\Gamma}^\ell(\underline{\sigma}^0)$, an index of vulnerability becomes obvious. Assigning a reference direction for the cutset C_i, we denote the set of positively oriented branches in C_i by C_i^+ and the remaining branches in C_i by C_i^-. Then a positive shift of line angles from $\underline{\sigma}^0$ (relative to the cutset reference) on to $\partial\bar{\Gamma}^\ell(\underline{\sigma}^0)$ selects the 'corner point' $\underline{\sigma}^+$ defined by

$$\sigma_k^+ = \begin{cases} \sigma_k^u, & k \in C_i^+ \\ \sigma_k^\ell, & k \in C_i^- \\ \sigma_k^0, & k \notin C_i \end{cases}$$

Similarly, a 'corner point' $\underline{\sigma}^-$ for negative shift of line angles can be defined with obvious modification. We have from Theorem 2

$$W(\underline{\sigma}^+, \underline{\sigma}^0) = \sum_{C_i} b_k h(\sigma_k^+, \sigma_k^0) \tag{29}$$

We propose that $W(\underline{\sigma}^+, \underline{\sigma}^0)$ and its negative separation counterpart $W(\underline{\sigma}^-, \underline{\sigma}^0)$ represent the index of vulnerability for cutset C_i. Introducing the coefficients $\mu_k^\ell \triangleq h(\sigma_k^\ell, \sigma_k^0)$ and $\mu_k^u \triangleq h(\sigma_k^u, \sigma_k^0)$ for all the lines, this motivates definition of <u>cutset vulnerability indices</u> by

$$v_i^+ = \sum_{C_i^+} b_k \mu_k^u + \sum_{C_i^-} b_k \mu_k^\ell \tag{30a}$$

$$v_i^- = \sum_{C_i^+} b_k \mu_k^\ell + \sum_{C_i^-} b_k \mu_k^u \tag{30b}$$

An overall index for the cutset is

$$\mathcal{V}_i = \min\{v_i^+, v_i^-\}$$

and for the system is

$$\mathcal{V} = \min_i \mathcal{V}_i$$

Evaluating $\mathcal{V}_i$ for each cutset gives a ranking according to vulnerability. Note that calculation of the coefficients in (30) is simply done via

$$\mu_k^u = 2[\cos \sigma_k^0 + (\sigma_k^0 - \pi/2)\sin \sigma_k^0] \tag{31a}$$

$$\mu_k^\ell = 2[\cos \sigma_k^0 + (\sigma_k^0 + \pi/2)\sin \sigma_k^0] \tag{31b}$$

For $\underline{P}^0 = \underline{0}$, we have $\mu_k^u = \mu_k^\ell = 2$ and $\mathcal{V}_i = v_i^+ = v_i^-$.

Having set up the index $\mathcal{V}_i$, how, and within what limitations, can we depend on it? Of course, in general, we cannot expect $\mathcal{V}$ to be an accurate estimate of V_ℓ. The main utility seems to lie in providing a preliminary identification of weak cutsets. Then, using this information along with other information like fault position, we can concentrate on finding the corresponding u.e.p.'s and an accurate estimate of V_ℓ. It is interesting, however, to observe that the method used by Prabhakara and El-Abiad [32] appears very accurate at least for low power levels. We can then anticipate that, for this case, $\mathcal{V}$ will indeed be a useful estimate for V_ℓ. As power levels increase, the u.e.p.'s are less related to hypothetical separation situations and there is a greater need for follow-up calculations to calculate V_ℓ.

As a simple illustration of the use of vulnerability indices, the following example is considered.

<u>Example</u>. For the network illustrated in Figure 1, we use the values for power transfer coefficients b_k and powers P_i^0 from an example in [18] (with some additions to allow for generator lines). The powers are given by

$P_{G_1} = 2.0$	$P_{D_3} = 1.0$
$P_{G_2} = 2.0$	$P_{D_4} = 0.8$
	$P_{D_5} = 1.2$
	$P_{D_6} = 1.0$

Firstly, we note that for $\underline{P}^0 = \underline{0}$ cutset {2,4} is most vulnerable and expect this to be the case for very low power levels. At the powers given above, the relevant coefficients (31) for each line are tabulated in Table 1. The corresponding cutset vulnerability indices are tabulated in Table 2 and they reveal that cutset {1,2} is most vulnerable. The three cutsets {1,3}, {2,4} (most vulnerable at very low powers) and {2,3} form an almost equally vulnerable group with the remainder having decreasing vulnerability. We have from [18] that the exact value of V_ℓ (found by a lowest saddle point search) is 1.63 corresponding to cutset {1,2} being saturated. Thus the vulnerability indices have identified the weakest cutset. Note that, in the case considered, the power levels are an appreciable proportion of the line capacities. In fact, at the exact u.e.p. corresponding to cutset {1,2} line 2 has $p_2 = b_2 = 0.5$. Thus we do not expect the overall vulnerability index $\mathcal{V}$ to give a close estimate of V_ℓ. However, from Table 2, we do have $\mathcal{V} = 1.89$ which is an acceptable course estimate.

TABLE 1

Calculation of Branch Vulnerability Coefficients

Line	b_k	σ_k^0 (radians)	μ_k^u	μ_k^ℓ
1	2.0	0.597	0.559	4.091
2	0.5	0.152	1.548	2.498
3	2.0	0.569	0.605	3.991
4	1.0	0.124	1.627	2.404
5	5.0	0.412	0.905	3.421
6	6.0	0.340	1.064	3.161

TABLE 2

Calculation of Cutset Vulnerability Indices

Cutset	{2,3}	{1,2}	{3,4}	{1,4}	{1,3}	{2,4}	{5}	{6}
v_i^+	8.755	1.893	2.837	3.522	2.329	2.401	4.524	6.384
v_i^-	2.460	9.431	10.385	9.810	16.163	3.653	17.103	18.968

VI. CONCLUSIONS

A new model for the study of power system stability has been discussed. The significant feature of this model is its structural integrity which goes hand-in-hand with an explicit presence of the system loads in the network. This avoids the difficult problem of how to account for transfer conductances in reduced network models. To give a conceptual view of how this model relates to stability analysis, the concepts of a topological Lyapunov function and vulnerable cutsets have been introduced. In view of the relationship with successful techniques for the classical model, the ranking of cutsets using vulnerability indices could prove to be a very useful preliminary step in transient stability assessment.

VII. ACKNOWLEDGEMENT

This research was supported by the Department of Energy, Electric Energy System Division, Contract EC-77-S-01-5105.

VIII. REFERENCES

[1] C.L. Gupta and A.H. El-Abiad, "Determination of the Closest Unstable Equilibrium State for Liapunov Methods in Transient Stability Studies," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 1699-1712, September/October 1976.

[2] M. Ribbens-Pavella, et al., "Transient Stability Analysis by Scalar Liapunov Functions: Recent Improvements and Practical Results," Coll. des Public de la Faculté des Sc. Appliquées, Univ. of Liege, No. 67, 1977.

[3] T. Athay, R. Podmore and S. Virmani, "A Practical Method for the Direct Analysis of Transient Stability," IEEE Trans. Power Apparatus and Systems, vol. PAS-98, pp. 573-584, March/April 1979.

[4] O. Saito et al., "Security Monitoring Systems Including Fast Transient Stability Studies," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 1789-1805, September/October 1975.

[5] A.H. El-Abiad and K. Nagappan, "Transient Stability Regions of Multimachine Power Systems," IEEE Trans. Power Apparatus and Systems, vol. PAS-85, pp. 169-179, February 1966.

[6] M. Ribbens-Pavella, "Transient Stability of Multi-machine Power Systems by Liapunov's Direct 71 CP17-PWR, IEEE Winter Power Meeting, New York, January 1971.

[7] G.A. Lüders, "Transient Stability of Multimachine Power Systems via the Direct Method of Lyapunov," IEEE Trans. Power Apparatus and Systems, vol. PAS-90, pp. 23-32, January/February 1971.

[8] H.F. Williams, S.A. Louie, and G.W. Bills, "Feasibility of Liapunov Functions for the Stability Analysis of Electric Power Systems Having up to 60 Generators," IEEE Trans. Power Apparatus and Systems, vol. PAS-91, pp. 1145-1153, May/June 1972.

[9] M. Ribbens-Pavella, "Critical Survey of Transient Stability Studies of Multimachine Power Systems by Liapunov's Direct Method," Proc. 9th Allerton Conf. Circuits and System Theory, Univ. of Illinois, Monticello, Illinois, pp. 751-767, 1971.

[10] A.A. Fouad, "Stability Theory - Criteria for Transient Stability," Proc. Eng. Foundation Conf. on Systems Eng. for Power: Status and Prospects, Henniker, New Hampshire, pp. 421-450, 1975.

[11] M.A. Pai and P.G. Murthy, "On Lyapunov Functions for Power Systems with Transfer Conductances," IEEE Trans. Automatic Control, vol. AC-18, pp. 181-183, April 1973.

[12] V.E. Henner, "Comments on 'On Lyapunov Functions for Power Systems with Transfer Conductances'" IEEE Trans. Automatic Control, vol. AC-19, pp. 621-622, October 1974.

[13] L.B. Jocić, M. Ribbens-Pavella, and D.D. Siljak, "Multimachine Power Systems: Stability, Decomposition, and Aggregation," IEEE Trans. Automatic Control, vol. AC-23, pp. 325-332, April 1978.

[14] M. Ribbens-Pavella, Lu.T. Grujić, and J. Sabatel, "Scalar vs Vector Liapunov Functions for Transient Stability Analysis of Large-Scale Power Systems," MECO, Athens, June 1978.

[15] M.A. Pai and S.D. Varwandkar, "On the Inclusion of Transfer Conductances in Lyapunov Functions for Multimachine Power Systems," IEEE Trans. Automatic Control, vol. AC-22, pp. 983-985, 1977.

[16] A.A. Fouad and R.L. Lugtu, "Transient Stability Analysis of Power Systems Using Liapunov's Second Method," C72145-6, IEEE Conference Paper, 1972.

[17] G.W. Stagg and A.H. El-Abiad, Computer Methods in Power System Analysis, New York: McGraw-Hill, 1958.

[18] C.J. Tavora and O.J.M. Smith, "Equilibrium Analysis of Power Systems," IEEE Trans. Power apparaand Systems, vol. PAS-91, pp. 1131-1137, May/June 1971.

[19] C.J. Tavora and O.J.M. Smith, "Stability Analysis of Power Systems," Report No. ERL-70-5, College of Engineering, University of California, Berkeley, August 1970.

[20] L. Jenkins and R-W. Liu, "Stability of Flows of a Dynamic Flow Network," IEEE Trans. Circuits

and Systems, vol. CAS-23, pp. 826-829 December 1976
[21] A.R. Bergen and G. Gross, "On Multimachine Power System Representations," Report No. ERL-M392, College of Engineering, University of California, Berkeley, July 1972.
[22] A.R. Bergen and G. Gross, "Computation of Regions of Transient Stability of Multimachine Power Systems," IEEE Trans. Automatic Control, vol. AC-19, pp. 142-143, April 1974.
[23] P.J. Moylan and D.J. Hill, "Stability Tests for Multimachine Power Systems," Proc. IFAC Symposium, Melbourne, pp. 292-296, February 1977.
[24] P.M. Anderson and A.A. Fouad, Power System Control and Stability, The Iowa State University Press, Ames, Iowa, 1977.
[25] C.A. Desoer and E.S. Kuh, Basic Circuit Theory, New York: McGraw-Hill, 1969.
[26] T.E. Stern, Theory of Nonlinear Networks and Systems, An Introduction, Reading, Mass.: Addison-Wesley, 1965.
[27] L.O. Chua and P.M. Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, New Jersey: Prentice-Hall, 1975.
[28] M. Ribbens-Pavella, Comments on J. L. Willems "Direct Methods for Transient Stability Studies in Power System Analysis," and reply by author, IEEE Trans. Automatic Control, vol, AC-17, pp. 415-417, June 1972.
[29] J.M. Ortega and W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, New York: Academic Press, 1970.
[30] A.S. Householder, The Theory of Matrices in Numerical Analysis, Blaidsell, 1964.
[31] W. Hahn, Stability of Motion, New York: Springer-Verlag, 1967.
[32] F.S. Prabhakara and A.H. El-Abiad, "A Simplified Determination of Transient Stability Regions for Lyapunov Methods," IEEE Trans. Power Apparatus and Systems, vol. PAS-94, pp. 672-689, March/April 1975.
[33] A.R. Bergen and G. Gross, "The Projective Equal Area Criterion," C 75 037-7, IEEE Winter Power Meeting, New York, January 1975.
[34] M.A. Pai and C.L. Narayana, "Finite Regions of Attraction for Multinonlinear Systems and its Application to the Power System Stability Problem," IEEE Trans. Automatic Control, AC-21, pp. 716-721, October 1976.

APPENDIX

Some simple results in the analysis of nonlinear resistive circuits are presented. Familiarity with basic concepts is assumed. More complete details may be found in [25-27].

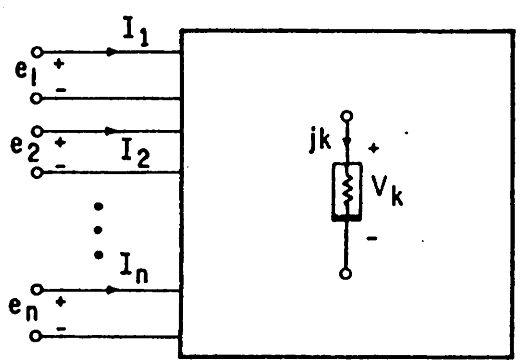

Figure 3 Nonlinear n-port resistive network

Consider the n-port representation given in Figure 3. This refers to an interconnection of ℓ nonlinear resistors at $N = n+1$ nodes. The Nth node is taken as a reference. The interconnections are described by an oriented graph which is assumed to be connected. Associated reference directions are used. The resistors are described by $j_k = g_k(v_k)$, where j_k and v_k denote the kth branch current and voltage respectively. Each node, other than the reference, has an injected current I_i, $i = 1,\ldots,n$.

Standard circuit matrices are the reduced incidence matrix $\underline{A}$ and the fundamental cutset matrix $\underline{Q}$. These $n\times\ell$ matrices have full row rank and describe the interconnections of the circuit graph. Matrix $\underline{A}$ is based on specifying branches incident at the nodes whereas $\underline{Q}$ specifies the branches in fundamental cutsets. Then Kirchhoff's laws have the convenient form for the above n-port

$$\underline{A}\underline{j} = \underline{I} \qquad \text{KCL} \qquad \text{(A-1)}$$

$$\underline{v} = \underline{A}^T\underline{e} \qquad \text{KVL} \qquad \text{(A-2)}$$

where $\underline{j}$, $\underline{v}$, $\underline{e}$ and $\underline{I}$ are vectors of branch currents, branch voltages, node-to-datum voltages (here serving as port voltages also) and injected currents respectively. Combining (A-1), (A-2) and the branch relationships gives

$$\underline{I} = \underline{A}\,\underline{g}(\underline{A}^T\underline{e}) \qquad \text{(A-3)}$$

This specifies the aggregate n-port description in terms of circuit structure and branch resistance characteristics. An alternative description is obtained by using matrix $\underline{Q}$ to relate all branch voltages to just n tree branch voltages z_i

$$\underline{v} = \underline{Q}^T\underline{z} \qquad \text{KVL} \qquad \text{(A-4)}$$

Then, we have

$$\underline{I} = \underline{A}\,\underline{g}(\underline{Q}^T\underline{z}) \qquad \text{(A-5)}$$