THE WORKING SET SIZE DISTRIBUTION
OF THE MARKOV PROGRAM BEHAVIOR MODEL

by

Makoto Kobayashi

Memorandum No. UCB/ERL M79/46

11 July 1979

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

THE WORKING SET SIZE DISTRIBUTION
OF THE MARKOV PROGRAM BEHAVIOR MODEL*

Makoto Kobayashi

Computer Science Division
Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California
Berkeley, California 94720
U.S.A.

Summary

The distribution of the working set size for the page reference strings generated by the Independent Reference Model (IRM), Easton's model, and the Markov model is obtained analytically. Simpler equations to compute the average working set size for these models are also given. Numerical examples show that the IRM and the Markov model produce quite different averages and distributions of the working set size. Examples also suggest that the working set size of the Markov model is not normally distributed.

Keywords: Paging, Program Behavior, Working Set, Markov Chain, Performance Evaluation.

*The research reported here has been partially supported by the National Science Foundation under grant MCS78-24618.

1. Introduction

In order to evaluate the performance of a computer system we have to specify its hardware organization, its workload, and the systems software which determines how hardware and software resources should be used to process the workload. The workload can be characterized by the CPU demands, I/O demands, and memory demands of its components (jobs, job steps, processes, transactions, and so on). Characterization of the memory demands is particularly important in virtual memory systems for the purpose of both performance evaluation and performance improvement.

For the analytical performance evaluation of virtual memory systems, many models of dynamic address reference behavior of programs, i.e., mathematical description of how a program references its address space over time, have been proposed and actually applied [1]. One of the significant and important characteristics of address reference behavior (called program behavior for short in the sequel) is locality of reference, namely the property that a program exhibits of referencing only subsets of its entire address space during relatively long intervals of its execution time [2]. As program behavior models which are simple enough to be mathematically tractable, yet reflect locality of reference reasonably well, the Markov model and the least recently used (LRU) stack models have been commonly used. The independent reference model (IRM) has also been popular for its extreme simplicity, although it does not exhibit locality of reference. Several operating systems for today's virtual memory computers use the working set memory management algorithm [3,4,5]. The working set at any given time is roughly defined as a set of pages referenced in the recent past [6,7] and the working set memory management algorithm is the one which holds the working set of a program in main memory while the program is running. Hence, the measurement of the working set size, which is the number of pages in the working set, of actual programs is of great interest and importance [8,9]. Similarly, the derivation of the distribution of the working set size for program behavior models is the first step towards analytical performance evaluation of virtual storage systems using the working set memory management algorithm.

The distribution of the working set size for the simple LRU stack model can be computed either recursively [17] or explicitly [1,10]. The exact distribution has also been known for the IRM [11] but not for the Markov model. The principal purpose of this paper is to present a derivation of the exact distribution of the working set size for the Markov model. In the next section page reference strings, the working set, and three different Markov models, i.e., the IRM, Easton's model, and the first-order Markov model, will be formally introduced. Then, the average working set size of these three models will be computed in the following section, which leads to the section presenting the exact distribution of the working set size for the three models. Numerical examples of the average and the distribution of the working set size will be given in the last section.

2. The IRM, Easton's, and the Markov Model

Let us assume that a program consists of n pages whose page indices are denoted by $1,\ldots,n$ and let N be the set of the page indices, i.e, $N = \{1,2,\ldots,n\}$. Let $\ldots,r_{t-1},r_t,r_{t+1},\ldots$ be an infinite page reference string of the program, where $r_t \in N$ denotes the index of the page referenced at discrete virtual time t. The working set at time t with window size T, $W(t,T)$, is defined as the set of distinct pages referenced in the interval $[t-T+1,t]$. The working set size $w(t,T)$ is defined as the number of pages in $W(t,T)$ [6]. The average working set size $\bar{w}(T)$ is the average of $W(t,T)$ over t.

The independent reference model (IRM) assumes that page i, $i = 1,\ldots,n$, is referenced with a fixed probability b_i, $b_i > 0$, $i = 1,\ldots,n$, $\sum_{i=1}^{n} b_i = 1$, independently of the pages referenced in the past [12]. That is, the IRM is a zeroth-order Markov chain whose states correspond to page indices. Let us define a page reference probability vector b such that

$$b = (b_1,\ldots,b_n) \;, \quad b_i > 0,\; i = 1,\ldots,n,\; \sum_{i=1}^{n} b_i = 1 \;.$$

The IRM tends to produce larger average working set size than the measured size for small values of the window [12]. One solution to this problem is to use a more sophisticated method of estimating the page reference probabilities [13]. Another solution is to use higher order, i.e., first-order Markov model [14,15]. The first-order Markov chain model will be simply called the Markov model in the sequel. Let $P = (p_{ij})$ be the transition probability matrix of the chain, where $p_{ij} = \Pr\{r_t = j \mid r_{t-1} = i\}$, $\sum_{j=1}^{n} p_{ij} = 1$, $i = 1,\ldots,n$. In the subsequent analysis, we assume that this Markov chain is irreducible and ergodic. Therefore, it has a limiting probability vector $\lambda = (\lambda_1,\ldots,\lambda_n)$, $\lambda_i > 0$, $i = 1,\ldots,n$, $\sum_{i=1}^{n} \lambda_i = 1$, where $\lambda_i = \Pr\{r_t = i\}$. λ is the eigenvector of the transition probability matrix P, i.e., satisfies the eigen equation $\lambda P = \lambda$. Let P_i, $i = 1,\ldots,n$, be the column vectors of the matrix P. Thus, we may write $P = (P_1,\ldots,P_n)$.

The Markov model has certainly more modeling power than the IRM. However, the number of parameters of the Markov model is n^2 instead of n for the IRM. This prohibits practical application of the Markov model to programs consisting of numerous pages, e.g., database systems. Observing consecutive references to the same page, a special first order Markov model requiring only $n+1$ parameters has been proposed [16]. We will call this model Easton's model after its inventor's name. The transition probabilities of Easton's model are given by

$$p_{ii} = r + (1-r)\lambda_i \ ,$$
$$p_{ij} = (1-r)\lambda_j \ , \quad i \neq j \ ,$$

where $0 \leq r < 1$, $\sum_{i=1}^{n} \lambda_i = 1$, and $\lambda_i > 0$ for $i = 1,\ldots,n$. Note that $\lambda = (\lambda_1,\ldots,\lambda_n)$ is the limiting probability for Easton's model. Note also that the model degenerates to the IRM when $r = 0$.

The average working set size of these three models will be computed first in the next section, since it gives the basic ideas on how to compute the distribution of their working set sizes.

3. Average Working Set Sizes

We shall consider a segment of the page reference string of length T generated either by the IRM, by the Markov model, or by Easton's model. The string will be represented by $r_1, r_2, \ldots, r_T$, $T > 0$. The working set, the working set size, and the average working set size will be denoted as $W(T)$, $w(T)$, and $\bar{w}(T)$, respectively. Since page reference strings generated by the three models are stationary, $W(T) = W(t,T)$, $w(T) = w(t,T)$, and $\bar{w}(T) = \bar{w}(t,T)$ for any given time t.

Let us first compute the average working set size for the IRM, $\bar{w}_{IRM}(T)$. Since $\Pr\{i \notin W(T)\} = (1-b_i)^T$, $1 \leq i \leq n$,

$$\begin{aligned} \bar{w}_{IRM}(T) &= \sum_{i=1}^{n} \Pr\{i \in W(T)\} \\ &= \sum_{i=1}^{n} [1 - \Pr\{i \notin W(T)\}] \qquad (1) \end{aligned}$$

$$= \sum_{i=1}^{n} [1 - (1-b_i)^T]$$

$$= n - \sum_{i=1}^{n} (1-b_i)^T \ .$$

The average working set size for the Markov model, $\bar{w}_{Markov}(T)$, can be derived as follows. Let $x^t(i) = (x_1^t(i),\dots,x_n^t(i))$ be a probability vector such that

$$x_j^t(i) = \Pr\{r_t = j | r_1 \neq i,\dots,r_{t-1} \neq i\} \ , \quad \text{when} \quad j \neq i \ ,$$
$$= 0 \ , \qquad \text{when} \quad j = i \ ,$$

where $t = 2,3,\dots,T$. Conditioning on r_{t-1}, $x_j^t(i)$ can be recursively expressed as

$$x_j^t(i) = \sum_{k=1}^{n} x_k^{t-1}(i) p_{kj} \ , \quad i = 1,2,\dots,n \ .$$

These equations can be rewritten in a vector form as

$$x^t(i) = x^{t-1}(i) Q_i \ , \tag{2}$$

where the matrix Q_i is defined by

$$Q_i = (P_1,\dots,P_{i-1},0,P_{i+1},\dots,P_n) \ .$$

That is, Q_i is obtained from the transition probability matrix P by replacing its i-th column vector P_i with a zero vector. Using equation (2) recursively, we have

$$x^t(i) = x^1(i) Q_i^{t-1} \ , \quad t = 2,3,\dots,T \ , \tag{3}$$

where Q_i^{t-1} is the (t-1)-st power of Q_i. Conditioning on r_0, the page referenced immediately before r_1, and due to the stationarity of the

Markov chain, $x_j^1(i)$ is given by

$$x_j^1(i) = \sum_{\substack{k=1\\k\neq i}}^{n} \Pr\{r_0 = k\}p_{kj} = \sum_{\substack{k=1\\k\neq i}}^{n} \lambda_k p_{kj} \quad \text{when} \quad j \neq i \,,$$
$$= 0 \qquad\qquad\qquad\qquad \text{when} \quad j = i \,,$$

where $\lambda = (\lambda_1,\ldots,\lambda_n)$ is the limiting probability vector of the Markov chain. The above equation can be expressed in vector form as $x^1(i) = \lambda Q_i$. Substituting $x^1(i)$ into expression (3), we have

$$x^t(i) = \lambda Q_i^t \,, \quad t = 1,2,\ldots,T \,. \tag{4}$$

According to the definition of $x^t(i)$, the probability of not referencing page i in $[1,T]$ is given by

$$\begin{aligned}\Pr\{i \notin W(T)\} &= \Pr\{r_1 \neq i,\ldots,r_T \neq i\}\\ &= \sum_{j=1}^{n} x_j^T(i)\\ &= \|x^T(i)\|\\ &= \|\lambda Q_i^T\| \,,\end{aligned} \tag{5}$$

where $\|y\|$ is the norm of vector $y = (y_1,\ldots,y_n)$, that is $\|y\| = \sum_{k=1}^{n} y_k$.

Therefore, the average working set size with window size T, $\bar{w}_{Markov}(T)$, is computed by

$$\begin{aligned}\bar{w}_{Markov}(T) &= \sum_{i=1}^{n} \Pr\{i \in W(T)\}\\ &= \sum_{i=1}^{n} [1 - \Pr\{i \notin W(T)\}]\\ &= \sum_{i=1}^{n} (1 - \|\lambda Q_i^T\|)\\ &= n - \sum_{i=1}^{n} \|\lambda Q_i^T\| \,.\end{aligned} \tag{6}$$

In particular, when $T = 1$, using the above equation, $\lambda P = \lambda$, and $\|\lambda\| = 1$,

$$
\begin{aligned}
\bar{w}_{Markov}(1) &= n - \sum_{i=1}^{n} \|\lambda Q_i\| \\
&= n - \|\lambda \sum_{i=1}^{n} Q_i\| \\
&= n - (n-1)\|\lambda P\| \\
&= 1 .
\end{aligned}
$$

Since the IRM is a special case of the Markov model, $\bar{w}_{Markov}(T)$ in equation (6) should reduce to $\bar{w}_{IRM}(T)$ in equation (1) when

$$
P = \begin{pmatrix} b_1, \ldots, b_n \\ \vdots \qquad \vdots \\ b_1, \ldots, b_n \end{pmatrix}, \quad b_i > 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} b_i = 1 .
$$

Noticing that b is the eigenvector of P, a little computation reveals

$$
\lambda Q_i^T = (1-b_i)^{T-1}(b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n) .
$$

Hence,

$$
\begin{aligned}
\bar{w}_{Markov}(T) &= n - \sum_{i=1}^{n} \|\lambda Q_i^T\| \\
&= n - (1-b_i)^T ,
\end{aligned}
$$

which agrees with the expression of $\bar{w}_{IRM}(T)$ derived above (equation (1)).

The average working set size for Easton's model $\bar{w}_{Easton}(T)$ can be computed either by recognizing that this model is a particular case of the Markov model or by an independent approach. Let us first compute it according to the latter [16]. If we assume $r_{t-1} \neq i$, $2 \leq t \leq T$, then

$$
\begin{aligned}
Pr\{r_t \neq i \mid r_{t-1} \neq i\} &= 1 - Pr\{r_t = i \mid r_{t-1} \neq i\} \\
&= 1 - \lambda_i(1-r) . \qquad (7)
\end{aligned}
$$

Since λ is the eigenvector of the transition probability matrix, we have

$$\Pr\{r_1 \neq i\} = 1 - \lambda_i \quad . \tag{8}$$

Applying equation (7) recursively with equation (8) yields

$$\Pr\{i \notin W(T)\} = (1-\lambda_i)[1 - \lambda_i(1-r)]^{T-1} \; . \tag{9}$$

Hence,

$$\begin{aligned} \bar{w}_{\text{Easton}}(T) &= \sum_{i=1}^{n} \Pr\{i \in W(T)\} \\ &= \sum_{i=1}^{n} [1 - \Pr\{i \notin W(T)\}] \\ &= n - \sum_{i=1}^{n} (1-\lambda_i)[1 - \lambda_i(1-r)]^{T-1} \; . \end{aligned} \tag{10}$$

Now, let us derive $\bar{w}_{\text{Easton}}(T)$ from $\bar{w}_{\text{Markov}}(T)$. The transition probability matrix P is given by

$$P = \begin{pmatrix} r+(1-r)\lambda_1 & (1-r)\lambda_2 & \cdots & (1-r)\lambda_n \\ (1-r)\lambda_1 & r+(1-r)\lambda_2 & \cdots & (1-r)\lambda_n \\ \vdots & \vdots & & \vdots \\ (1-r)\lambda_1 & (1-r)\lambda_2 & \cdots & r+(1-r)\lambda_n \end{pmatrix} .$$

After a little computation we find for $t = 1,2,\ldots,T$,

$$\lambda Q_i^t = [1 - (1-r)\lambda_i]^{t-1}(\lambda_1,\ldots,\lambda_{i-1},0,\lambda_{i+1},\ldots,\lambda_n) \; .$$

Therefore, from equation (5) we have

$$\begin{aligned} \Pr\{i \notin W(T)\} &= \|\lambda Q_i^T\| \\ &= (1-\lambda_i)[1 - (1-r)\lambda_i]^{T-1} \; , \end{aligned}$$

which agrees with equation (9). Thus, even $\bar{w}_{\text{Easton}}(T)$ can actually be derived from $\bar{w}_{\text{Markov}}(T)$.

4. Distribution of the Working Set Size

The distribution of the working set size for the IRM was computed by H. Vantilborgh [11]. We will briefly review his approach; then a new approach which can be extended to the Markov model will be introduced.

Following Vantilborgh's notations, we define the probability mass function of the working set size as

$$p(m;n,T) = \Pr\{w(T)=m\} , \quad 1 \le m \le n, \quad 1 \le m \le T .$$

Because of the independence of page references, $p(m;n,T)$ has a multinomial distribution

$$p(m;n,T) = \sum_{\substack{s_1,\ldots,s_m > 0 \\ s_1+\cdots+s_m = T}} \left(\frac{T!}{s_1!\cdots s_m!} \sum_{\substack{j_1,\ldots,j_m \\ x\neq y \Rightarrow j_x \neq j_y}}^{n} {}_{1} \; b_{j_1}^{s_1}\cdots b_{j_m}^{s_m}\right) .$$

Using the inversion formula for binomial coefficients, we have

$$p(m;n,T) = \sum_{\ell=0}^{m} (-1)^{m-1} \binom{n-\ell}{n-m} D(\ell;n,T) ,$$

where $D(\ell;n,T)$ is defined as

$$D(\ell;n,T) = \begin{cases} \displaystyle\sum_{1\le i_1 \le \cdots \le i_\ell \le n} (b_{i_1}+\cdots+b_{i_\ell})^T , & 1 \le \ell \le n , \\ 0 , & \text{otherwise.} \end{cases}$$

Since page references are not independent in the Markov model, Vantilborgh's approach cannot be applied to the Markov model.

The basic idea of the new solution is to first compute $\Pr\{W(T) \subseteq S_m\}$, where S_m is a subset of $N = \{1,\ldots,n\}$ such that $|S_m| = m$, $m = 1,\ldots,n$. Then $\Pr\{W(T) = S_m\}$ is computed by the recursive formula

$$\Pr\{W(T)=S_m\} = \Pr\{W(T)\subseteq S_m\} - \sum_{\ell=1}^{m-1} \sum_{S_\ell \subset S_m} \Pr\{W(T)=S_\ell\} . \tag{11}$$

Although $P(m;n,T)$ can be computed using this equation after all $\Pr\{W(T)=S_\ell\}$, $1 \le \ell \le m-1$, have been obtained, this approach is not practically applicable since 2^n-1 words of memory are required to store all $\Pr\{W(T)\subseteq S_m\}$. The number n of pages in database systems, for example, is very large. Even for $n = 100$, 2^n-1 is approximately 1.27×10^{30}. This limitation due to memory requirements can fortunately be eliminated. Summing up both left and right hand sides of equation (11) over all S_m yields

$$P(m;n,T) = \sum_{S_m} \Pr\{W(T)\subseteq S_m\} - \sum_{S_m} \sum_{\ell=1}^{m-1} \sum_{S_\ell \subset S_m} \Pr\{W(T)=S_\ell\} . \tag{12}$$

Due to the symmetry of the page indices, the second term of the right hand side of equation (12) can be rewritten as $\sum_{i=1}^{m-1} a_i \sum_{S_i} \Pr\{W(T)=S_i\} = \sum_{i=1}^{m-1} a_i P(i;n,T)$, where the a_i's are coefficients to be derived. Since there are $\binom{n}{m}$ distinct sets of the S_m type (i.e., of size m), and each S_m has $\binom{m}{i}$ terms of $\Pr\{W(T)=S_i\}$, for a given value of i, $1 \le i \le m$, there are $\binom{n}{m}\binom{m}{i}$ terms of $\Pr\{W(T)=S_i\}$ for a particular value of i in the term $\sum_{S_m} \sum_{\ell=1}^{m-1} \sum_{S_\ell \subset S_m} \Pr\{W(T)=S_\ell\}$. On the other hand, $\sum_{S_i} \Pr\{W(T)=S_i\}$ has $\binom{n}{i}$ terms. Hence, the coefficient of $\sum_{S_i} \Pr\{W(T)=S_i\}$ is given by

$$a_i = \frac{\binom{n}{m}\binom{m}{i}}{\binom{n}{i}} = \binom{n-i}{n-m} .$$

Consequently, equation (12) becomes

$$P(m;n,T) = \sum_{S_m} \Pr\{W(T)\subseteq S_m\} - \sum_{i=1}^{m-1} \binom{n-i}{n-m} P(i;,n,T) , \quad m = 2,\ldots,n . \tag{13}$$

Recursive equation (13) requires only n words of memory to store $\sum_{S_m} Pr\{W(T) \subseteq m\}$ or $P(m;n,T)$ instead of the 2^n-1 words needed by the original equation.

Applying the inclusion-exclusion principle, $Pr\{W(T)=S_m\}$ may also be computed in terms of $Pr\{W(T) \subseteq S_\ell\}$, $\ell = 1,\ldots,m$, as

$$Pr\{W(T)=S_m\} = Pr\{W(T) \subseteq S_m\} + \sum_{\ell=1}^{m-1} (-1)^\ell \sum_{S_{m-\ell} \subset S_m} Pr\{W(T) \subseteq S_{m-\ell}\} . \quad (14)$$

Then, equation (13) will be replaced with

$$P(m;n,T) = \sum_{S_m} Pr\{W(T) \subseteq S_m\} + \sum_{\ell=1}^{m-1} (-1)^\ell \sum_{S_{m-\ell}} \binom{n-m+\ell}{\ell} Pr\{W(T) \subseteq S_{m-\ell}\} . \quad (15)$$

$Pr\{W(T) \subseteq S_m\}$ for any given $S_m \subseteq N$ for the IRM is readily computed by

$$Pr\{W(T) \subseteq S_m\} = \Big(\sum_{i \in S_m} b_i\Big)^T . \quad (16)$$

The computational complexity of this new solution presented by equations (13) or (14) and (16) is comparable to that of the Vantilborgh's solution. However, $Pr\{W(T) \subseteq S_m\}$ can be computed for the Markov model, and hence the working set size distribution of the Markov model can be obtained, as will be shown next.

In order to compute $Pr\{W(T) \subseteq S_m\}$ for the Markov model, let us observe equation (5), i.e., $Pr\{i \notin W(T)\} = \|\lambda Q_i^T\|$. Since the event $i \notin W(T)$ is equivalent to the event $W(T) \subseteq N-\{i\}$, equation (5) provides an expression for $Pr\{W(T) \subseteq S_m\}$ in a special case, i.e., when $S_m = N-\{i\}$. Generalizing this idea, we define a vector $x^t(S_m) = (x_1^t(S_m),\ldots,x_n^t(S_m))$ at time t, $t = 2,3,\ldots,T$, such that

$$x_j^t(S_m) = \begin{cases} Pr\{r_t = j \mid r_1 \in S_m,\ldots,r_{t-1} \in S_m\} & \text{when } j \in S_m \\ 0 & \text{when } j \notin S_m . \end{cases}$$

Conditioning on r_{t-1}, $x_j^t(S_m)$ may be expressed as a linear function of $x^{t-1}(S_m)$, that is,

$$x_j^t(S_m) = \sum_{k=1}^{n} x_k^{t-1}(S_m)p_{kj}$$
$$= (x^{t-1}(S_m), {}^tP_j) \ , \qquad (17)$$

where $(x^{t-1}(S_m), {}^tP_j)$ is the inner product of vectors $x^{t-1}(S_m)$ and ${}^tP_j$, and ${}^tP_j$ is the transpose of P_j, the j-th column vector of the transition probability matrix. Equation (17) can be rewritten in matrix form as

$$x^t(S_m) = x^{t-1}(S_m)Q_{S_m} \ , \quad t = 2,\ldots,T \ , \qquad (18)$$

where matrix Q_{S_m} is defined as

$$Q_{S_m} = (\ldots,P_{i_1},\ldots,0,\ldots,P_{i_m},\ldots,0)$$
$$i_j \in S_m \ , \quad j = 1,\ldots,m \ .$$

Thus, Q_{S_m} is obtained from P by replacing its k-th column vector with a zero column vector for all $k \in N - S_m$. $x^1(S_m)$ is computed in terms of the limiting page reference probabilities λ and the transition probability matrix P by conditioning on r_0 as follows:

$$x_j^1(S_m) = \begin{cases} \sum\limits_{\substack{k=1 \\ k\in S_m}}^{n} \Pr\{r_0 = k\}p_{kj} = \sum\limits_{\substack{k=1 \\ k\notin S_m}}^{n} \lambda_k p_{kj} \ , & \text{when} \quad j \in S_m \ , \\ 0 \ , & \text{when} \quad j \notin S_m \ , \end{cases}$$

or in vector form

$$x^1(S_m) = \lambda Q_{S_m} \ . \qquad (19)$$

Repeating equation (18) recursively starting from $t = T$ down to $t = 2$ and using equation (19), we get

$$x^T(S_m) = \lambda Q_{S_m}^T \ . \tag{20}$$

According to the definition of $x^t(S_m)$, $\Pr\{W(T) \subseteq S_m\}$ is given by

$$\begin{aligned} \Pr\{W(T) \subseteq S_m\} &= \| x^T(S_m) \| \\ &= \| \lambda Q_{S_m}^T \| \ . \end{aligned} \tag{21}$$

In particular, when $T = 1$ and $m = 1$,

$$\begin{aligned} \Pr\{W(1) = 1\} &= \| \lambda \sum_{S_1} Q_{S_1} \| \\ &= \| \lambda P \| \\ &= \| \lambda \| \\ &= 1 \ . \end{aligned}$$

For the IRM, it turns out that

$$\lambda Q_{S_m}^T = \Big(\sum_{i \in S_m} b_i \Big)^{T-1} (0, \ldots, b_{i_1}, \ldots, 0, \ldots, b_{i_m}, \ldots, 0)$$
$$i_\ell \in S_m, \quad \ell = 1, \ldots, m.$$

Hence, from equation (21),

$$\begin{aligned} \Pr\{W(T) \subseteq S_m\} &= \| \lambda Q_{S_m}^T \| \\ &= \Big(\sum_{i \in S_m} b_i \Big)^T , \end{aligned}$$

which agrees with equation (16).

$\Pr\{W(T) \subseteq S_m\}$ of Easton's model can also be computed either independently or by using the solution (21) for the Markov model. The direct solution will now be described. Since the probability of referencing page i consecutively is $r + (1-r)\lambda_i$,

$$\Pr\{r_t \in S_m \mid r_{t-1} \in S_m\} = r + (1-r) \sum_{i \in S_m} \lambda_i \ , \quad t = 2,\ldots,T \ ,$$

and for $t = 1$,

$$\Pr\{r_1 \in S_m\} = \sum_{i \in S_m} \lambda_i \ .$$

From these two equations, we have

$$\Pr\{W(T) \subseteq S_m\} = \Pr\{r_1 \in S_m\} \prod_{t=2}^{T} \Pr\{r_t \in S_m \mid r_{t-1} \in S_m\}$$

$$= \left(\sum_{i \in S_m} \lambda_i\right)\left[r + (1-r) \sum_{i \in S_m} \lambda_i\right]^{T-1} \ . \tag{22}$$

This equation can also be obtained from equation (21). However, we will not present the procedure in detail; we shall only mention that

$$\lambda Q_{S_m}^T = \left[r + (1-r) \sum_{i \in S_m} \lambda_i\right]^{T-1} (0,\ldots,\lambda_{i_1},\ldots,0,\ldots,\lambda_{i_m},\ldots,0) \ ,$$
$$i_\ell \in S_m, \quad \ell = 1,\ldots,m \ .$$

In order to compute the working set size mass function $p(m;n,T)$ in a systematic, effective fashion, the following algorithm can be followed:

<u>Step 1</u>: Solve the eigen equation $\lambda P = \lambda$ for the limiting page reference probabilities λ.

<u>Step 2</u>: Compute $q(m;n,T) = \sum_{S_m} \| \lambda Q_{S_m}^T \|$.

<u>Step 3</u>: Compute $p(m;n,T) = q(m;n,T) - \sum_{i=1}^{m-1} \binom{n-i}{n-m} p(i;n,T)$ recursively from $m = 2$ up to $m = n$ with $p(1;n,T) = q(1;n,T)$. $p(m;n,T)$ can also be computed by $p(m,n,T) = q(m;n,T)$ $+ \sum_{\ell=1}^{m-1} (-1)^\ell \binom{n-m+\ell}{\ell} q(m-\ell;n,T)$.

The computational complexity of $q(m;n,T)$ is on the order of $Tn^3 2^n$, since the summation of $\| \lambda Q_{S_m}^T \|$ has to be taken over all 2^n-1 subsets of N and the computation of $Q_{S_m}^T$ requires about Tn^3 multiplications. However, $Q_{S_m}^T$ can be computed easier by noticing that if a matrix A is expressed as PBP^{-1}, where matrix B is a diagonal matrix, then $A^T = PB^TP^{-1}$ and B^T is a diagonal matrix whose diagonal element is given by the T-th power of corresponding element of B. Therefore, the computational complexity of $q(m;n,T)$ can be reduced to $n^3 2^n$. Hence, the computational complexity of the algorithm for the Markov model is on the other of $n^3 2^n$. Since $q(m;n,T) = \sum_{S_m} (\sum_{i \in S_m} b_i)^T$ for the IRM and $q(m;n,T) = \sum_{S_m} (\sum_{i \in S_m} \lambda_i)[r + (1-r) \sum_{i \in S_m} \lambda_i]^{T-1}$ for Easton's model, the computational complexity of the algorithm for these models is on the order of 2^n.

5. Numerical Examples

As numerical examples of the distributions of the working set size of the IRM, Easton's model, and the Markov model, page transition matrices were obtained from instruction trace data. The programs used in this paper are a FORTRAN execution (FORTRANGO) and a COBOL execution (COBOLGO). The program size of FORTRANGO is about 28K bytes and about half a million instructions were executed and traced. COBOLGO has a size of 2K bytes and 3.6 million instructions were executed and traced. Since instruction references were available at hand, they were used in the examples to construct page transition matrices with a page size of 4096 bytes. The distributions and the averages of the working set size for window sizes up to 2048 have been computed using the algorithm presented in the previous section.

The value of the parameter r of Easton's model has been estimated by two methods as follows. The first, rather crude, method is to sum up both sides of the expression $p_{ii} = r + (1-r)\lambda_i$ over $i = 1,\ldots,n$. This yields

$$\sum_{i=1}^{n} p_{ii} = (n-1)r + 1 ,$$

and

$$r = \frac{\sum_{i=1}^{n} p_{ii} - 1}{n - 1} .$$

The second method is to take the weighted average of r for each page. Let $r(i)$ be the estimated value of r for page i, which is directly derived from $p_{ii} = r + (1-r)\lambda_i$ as $r(i) = \frac{p_{ii}-\lambda_i}{1-\lambda_i}$. Since page i is referenced with probability λ_i, the weighted average is given by

$$\begin{aligned} r &= \sum_{i=1}^{n} \lambda_i r(i) \\ &= \sum_{i=1}^{n} \frac{\lambda_i(p_{ii}-\lambda_i)}{1-\lambda_i} . \end{aligned}$$

The estimated values according to the first and the second methods are called r_1 and r_2, respectively. It should be noticed that these methods are independent of the window size T. Easton suggested another method for a given window size and a given page fault rate [16]. Since the page fault rate for Easton's model under the working set strategy is theoretically given by $\bar{M}(T) = (1-r) \sum_{i=1}^{n} \lambda_i(1-\lambda_i)[1 - \lambda_i(1-r)]^{T-1}$, we can solve this equation for r if λ_i $(i = 1,\ldots,n)$, T, and $\bar{M}(T)$ are given. However, since we are interested in estimating r using the page transition probability matrix and independently of the window size, we will not use Easton's method.

Table I shows the transition probability matrix P, the limiting page reference probability vector λ, and the estimates of r for Easton's model of COBOLGO. The measured and the estimated average working set sizes as a function of the window size are shown in Fig. 1. Since the estimated values of r_1 and r_2 and hence the corresponding estimated working set

Table I

Transition Probability Matrix and Limiting Page Reference Probabilities

COBOLGO

$$P = \begin{pmatrix} .97120 & .20710\times10^{-1} & .63098\times10^{-2} & .17849\times10^{-2} & .63091\times10^{-6} & 0. \\ .25655\times10^{-1} & .96641 & 0. & 0. & .79396\times10^{-4} & 0. \\ .37401\times10^{-1} & 0. & .95077 & .11829\times10^{-1} & 0. & 0. \\ .28279\times10^{-1} & 0. & .31618\times10^{-1} & .91833 & 0. & .28269\times10^{-1} \\ 0. & .40046\times10^{-1} & 0. & 0. & .95995 & 0. \\ 0. & 0. & 0. & .23333\times10^{-1} & 0. & .97667 \end{pmatrix}$$

$$\lambda = (.43944,\ .35475,\ .74135\times10^{-1},\ .27366\times10^{-1},\ .70341\times10^{-1},\ .33602\times10^{-1})$$

$$r_1 = .94727 \qquad r_2 = .94865$$

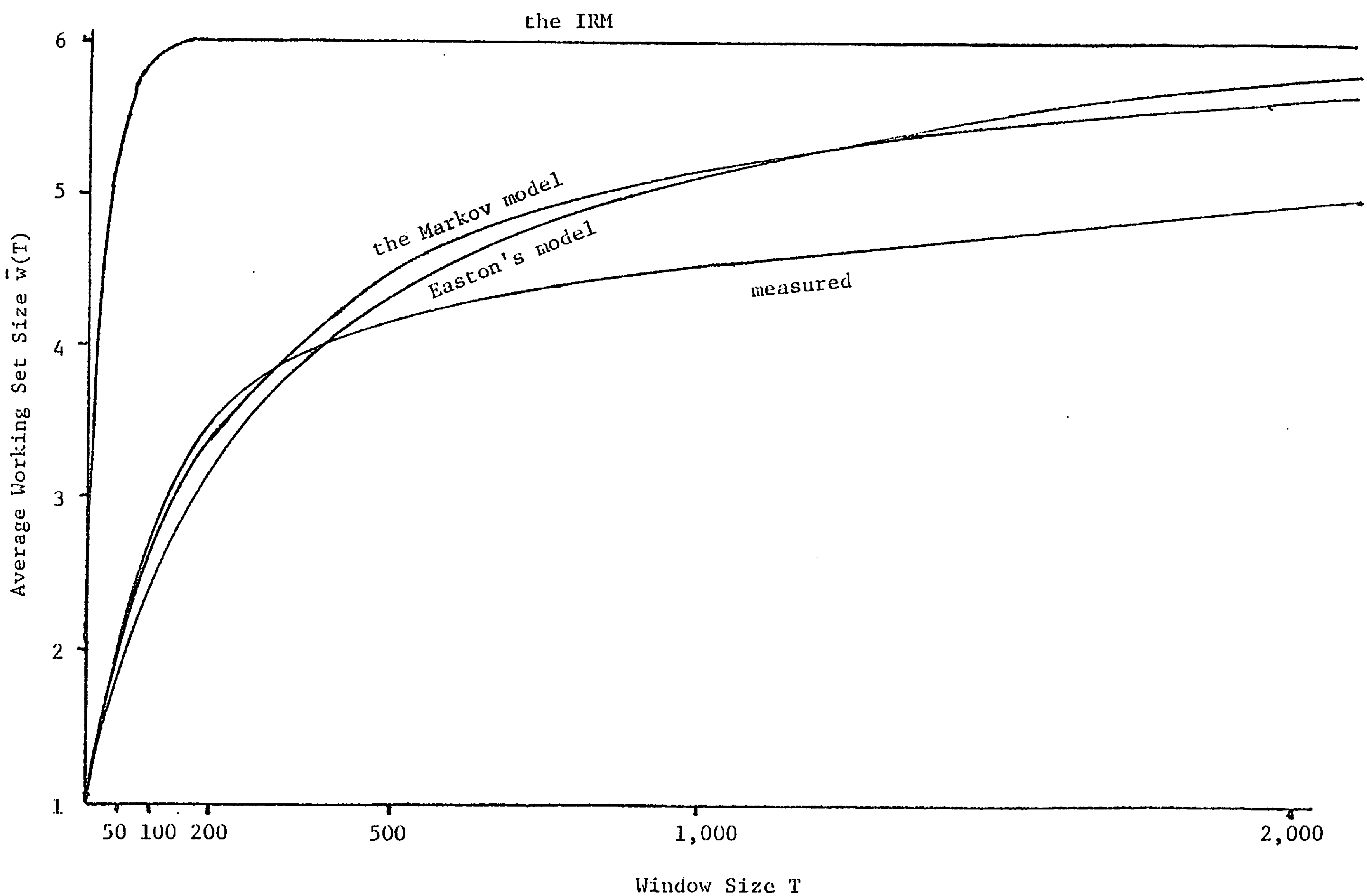

Figure 1. Average Working Set Size of COBOLGO

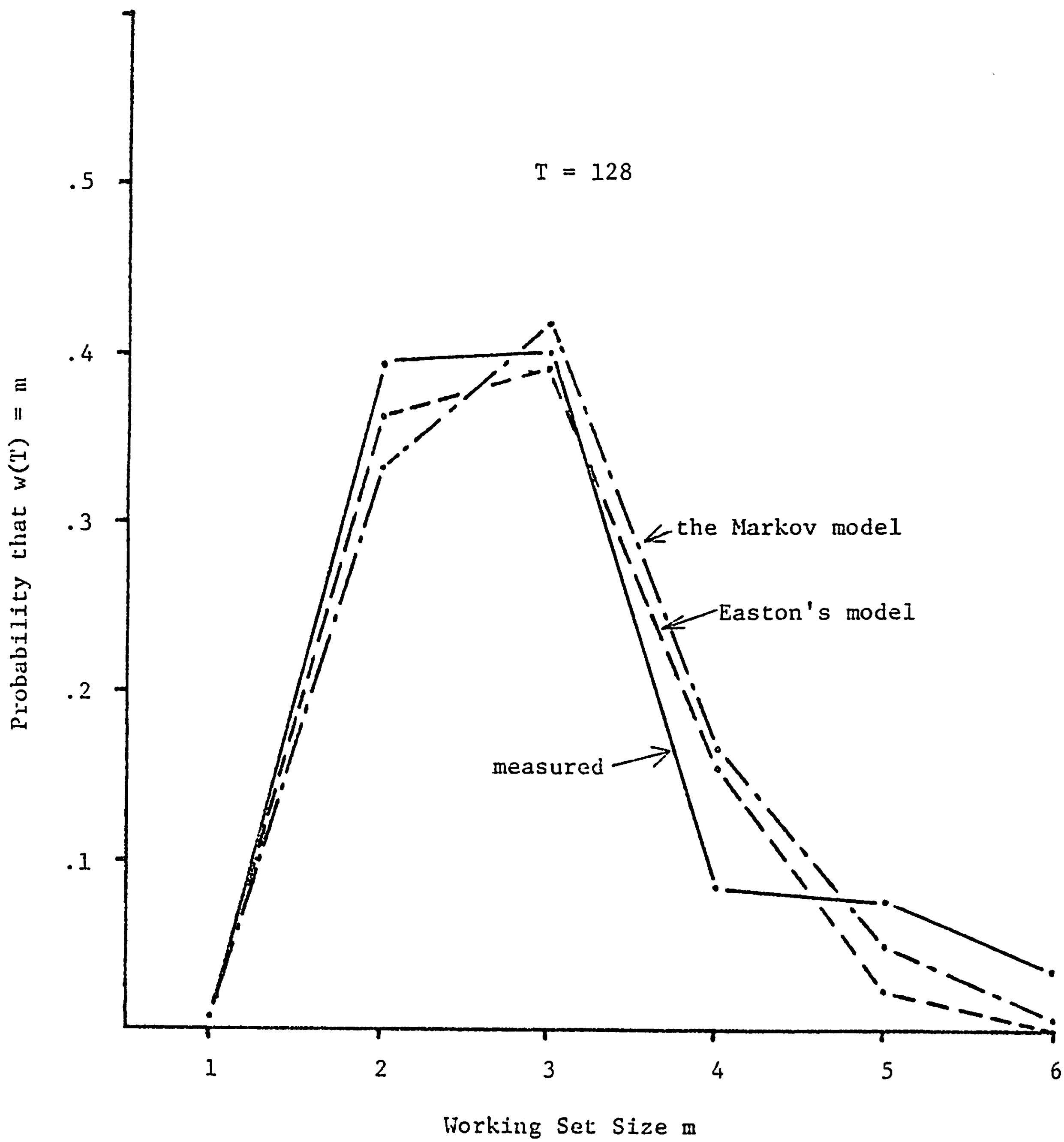

Figure 2. Working Set Size Distribution of COBOLGO (1)

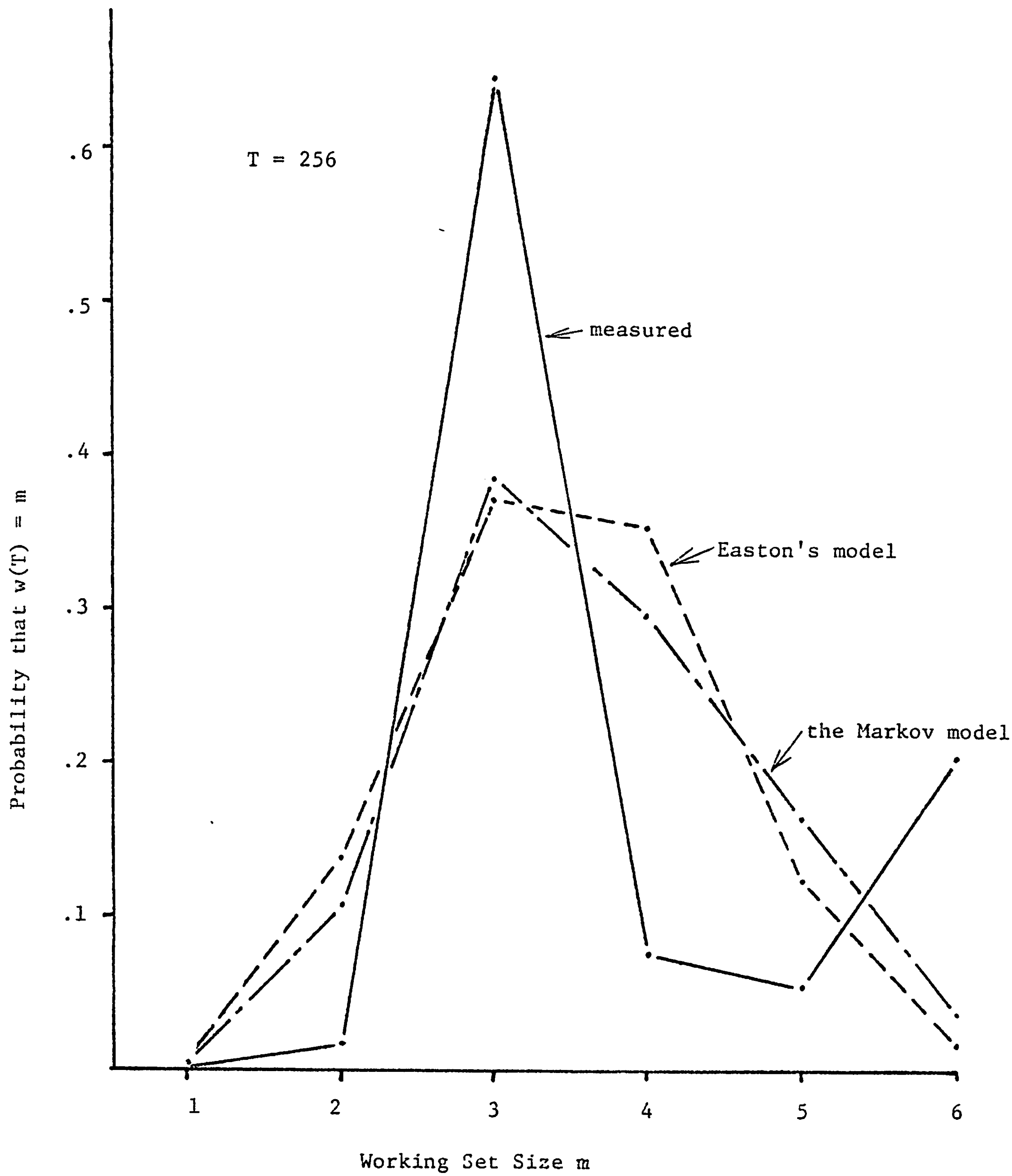

Figure 2. Working Set Size Distribution of COBOLGO (2)

sizes are very close to each other, the average working set sizes estimated by Easton's model with r_1 and r_2 are indistinguishable in the diagram. In Fig. 1 we can see that the IRM grossly overestimates the average working set size. Easton's model is as good as the Markov model, although both of them overestimate the average working set. The measured and estimated distributions of the working set size for window sizes of 128 and 256 instructions are shown in Fig. 2. Both Easton's model and the Markov model provide good approximations to the distribution.

The page transition matrix, the limiting page reference probabilities of FORTRANGO, and the values of r_1 and r_2 for Easton's model are shown in Table II. The measured and the estimated average working set sizes are shown in Fig. 3. The IRM is quite inaccurate again. Easton's model using r_1 overestimates the average working set size, while with r_2 it is accurate when the window size is small and underestimates the average working set size when the window is large. On the other hand, the Markov model underestimates the average working set size when the window size is small and is better behaved when it is large. Fig. 4 shows the distributions of the working set sizes measured and estimated by the Markov and Easton's (with r_2) models for window sizes of 256 and 512 instructions. Since the window sizes are small, Easton's model with r_2 gives a better estimate than the Markov model, as can be expected from Fig. 3. Note that the measured distribution for $T = 256$ is bimodal. Bimodal distributions were found for other programs which are not discussed in this paper.

6. Conclusions

The exact distribution of the working set size for page reference strings generated by the zeroth- and the first-order Markov chain models,

Table II

Page Transition Matrix and Limiting Page Reference Probabilities

FORTRANGO

$$P = \begin{pmatrix} 74826 & 1834 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 584 & 78861 & 2247 & 4 & 567 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1248 & 996 & 45968 & 3 & 744 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 14 & 0 & 1492 & 25 & 0 & 14 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 744 & 32 & 33497 & 855 & 0 & 0 & 0 & 0 & 738 & 0 \\ 7 & 558 & 0 & 2 & 285 & 76426 & 8 & 15 & 0 & 0 & 0 & 612 \\ 0 & 0 & 0 & 14 & 3 & 6 & 2017 & 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14 & 1 & 687 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 529 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 108 & 0 & 0 \\ 0 & 0 & 0 & 0 & 738 & 0 & 13 & 0 & 0 & 0 & 152185 & 0 \\ 0 & 0 & 0 & 0 & 0 & 612 & 0 & 0 & 0 & 0 & 0 & 918 \end{pmatrix}$$

$$= (.1592 \times 10^{-2},\ .1709 \times 10^{-2},\ .1017 \times 10^{-2},\ .3280 \times 10^{-4},\ .7456 \times 10^{-3},\ .1620 \times 10^{-4},$$
$$.4302 \times 10^{-4},\ .1476 \times 10^{-4},\ .1116 \times 10^{-4},\ .2417 \times 10^{-5},\ .3180 \times 10^{-2},\ .3181 \times 10^{-4})$$

$r_1 = .92862 \qquad r_2 = .96716$

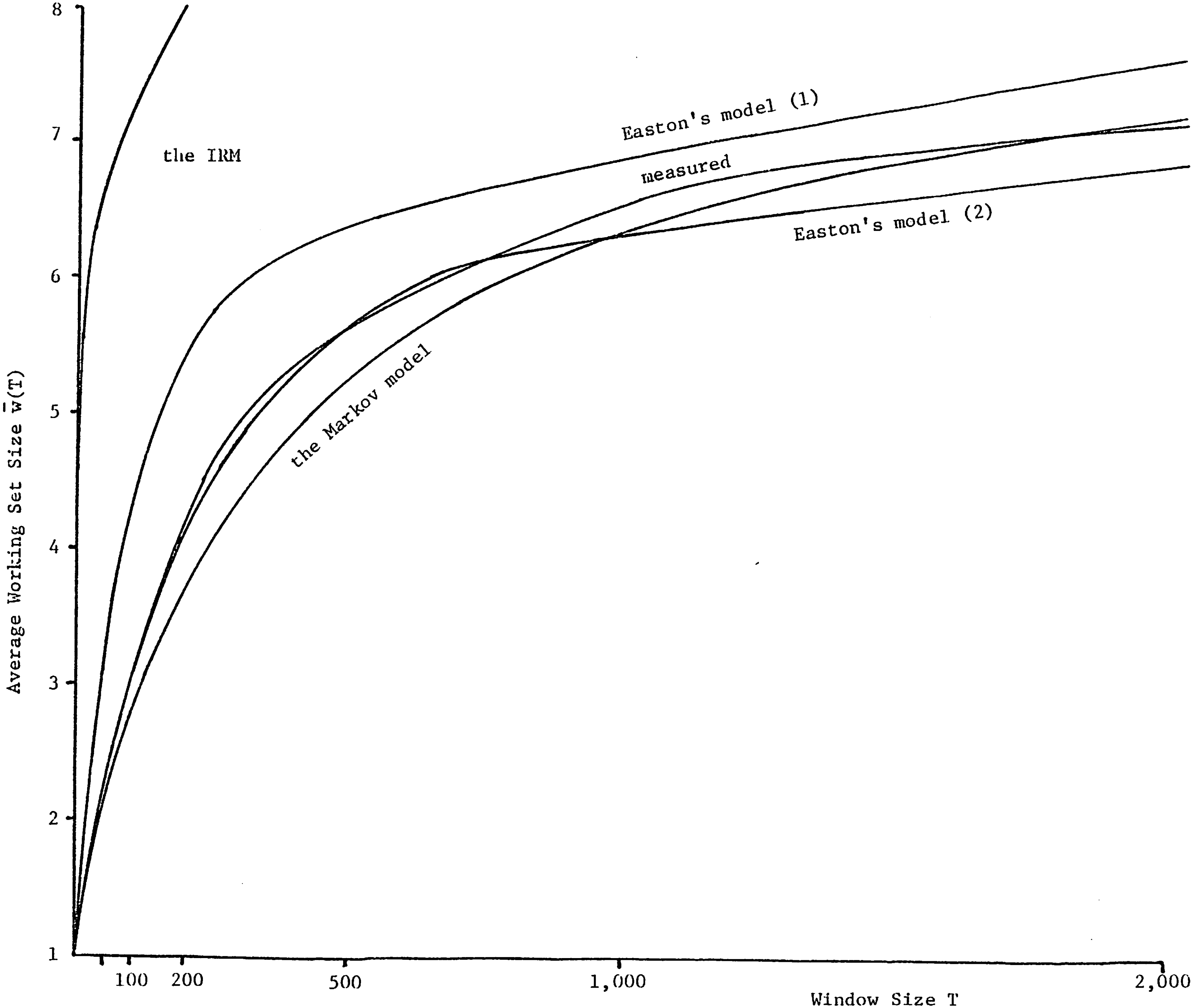

Figure 3. Average Working Set Size of FORTRANGO

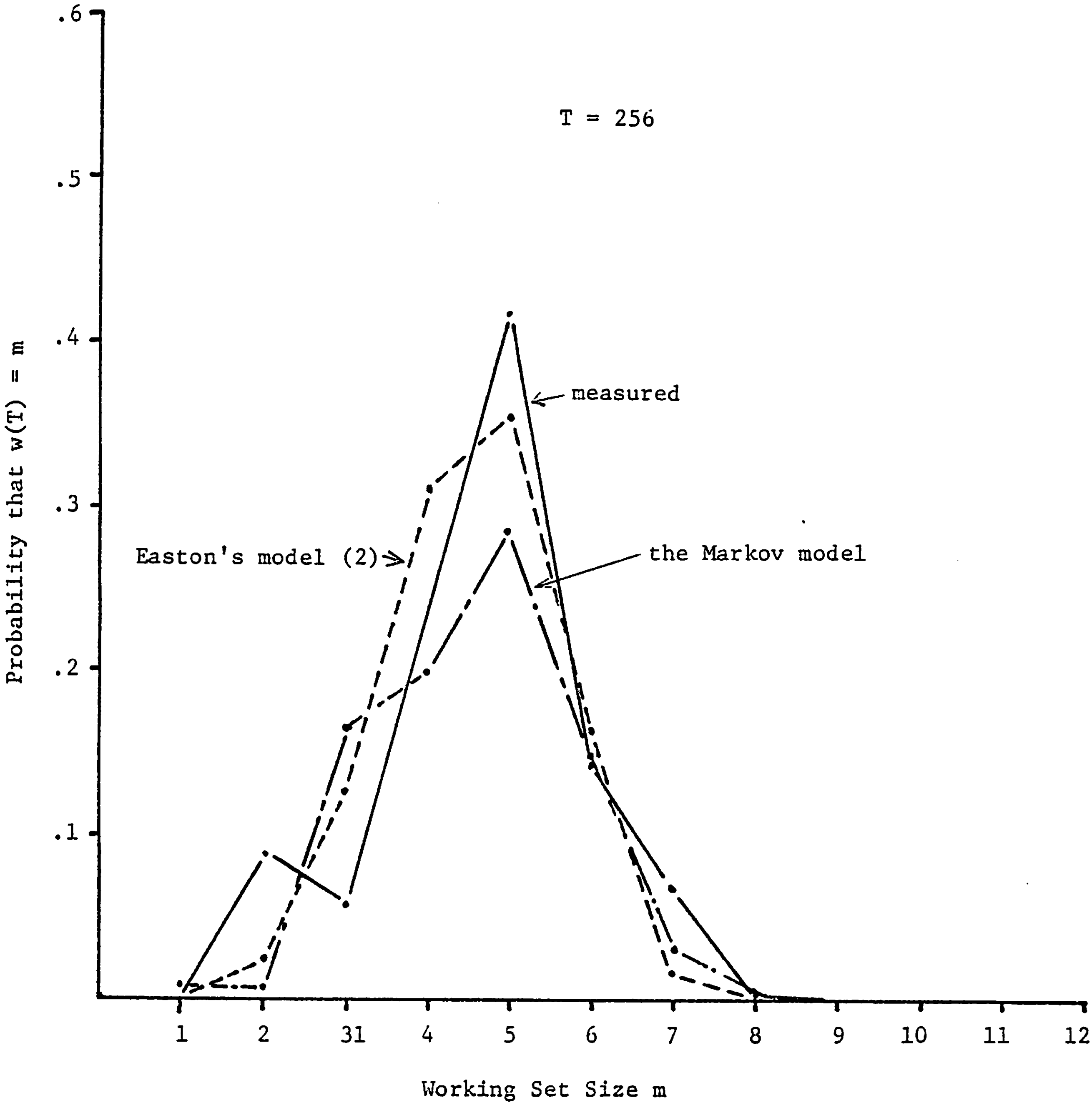

Figure 4. Working Set Size Distribution of FORTRANGO (1)

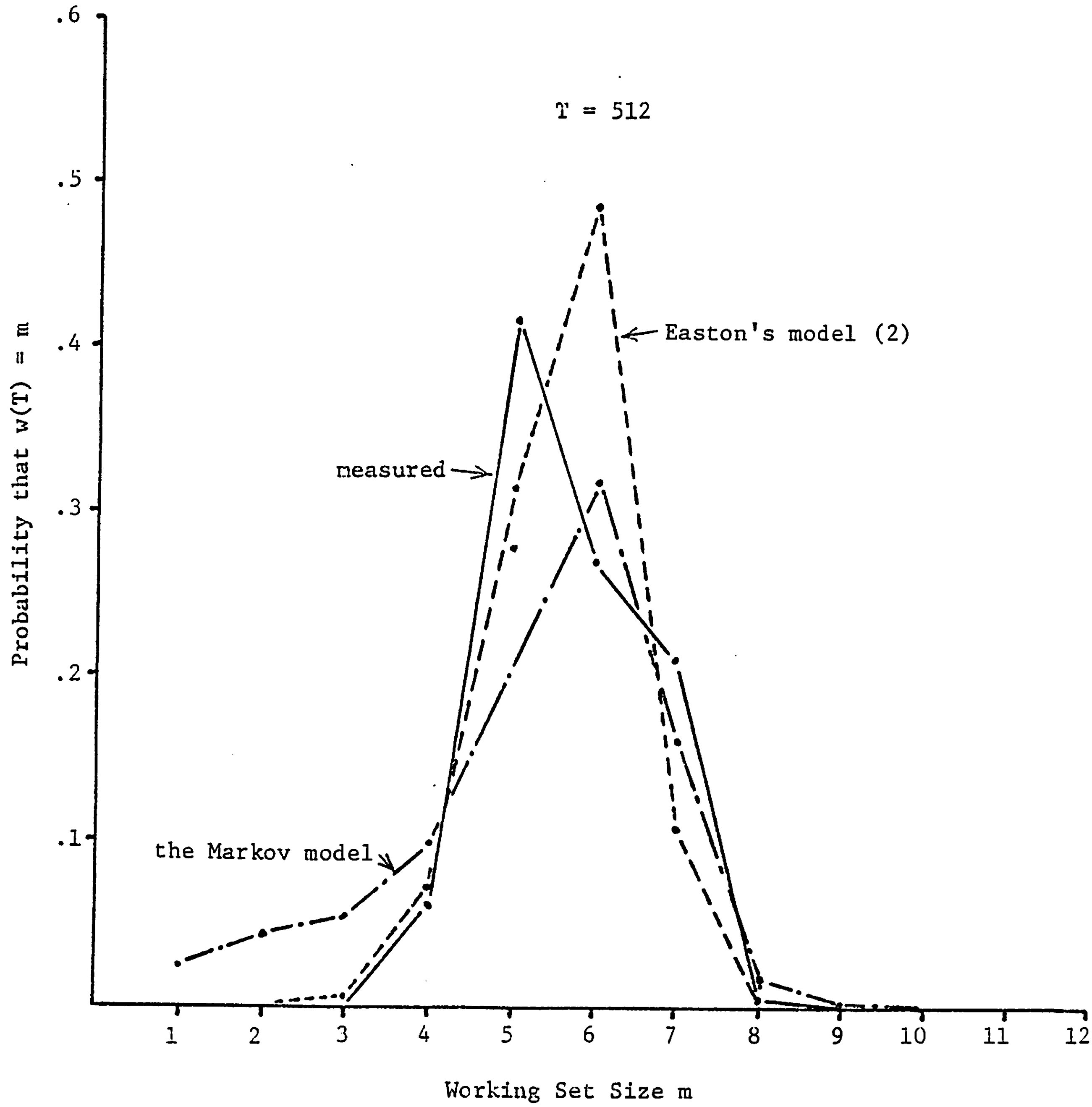

Figure 4. Working Set Size Distribution of FORTRANGO (2)

that is, the Independent Reference Model (IRM), Easton's model, and the so-called Markov model, has been obtained. A practical algorithm to compute the distribution has been given. The algorithm requires $o(n^2)$ words of storage and $o(n^3 2^n)$ CPU time for the Markov model, and $o(n)$ words of storage and $o(2^n)$ CPU time for the IRM and Easton's model, where n is the total number of pages of a program. A simple equation to compute the average working set size for the Markov model has also been given.

Numerical examples of the use of the algorithm to compute the average and the distribution of working set sizes have been given. The examples have been based on transition probabilities matrices and the limiting page reference probabilities obtained from actual trace data of programs. The examples show that the IRM and the Markov model have quite different averages and distributions of the working set size if the page reference probabilities of the IRM are estimated simply by the limiting page reference probabilities of the Markov model and not by a sophisticated method [13]. Easton's model gives almost as good an estimate as the Markov model in terms of both the average and the distribution of working set sizes when the parameter r of Easton's model is computed appropriately. Finally, working set size distributions were observed to be bimodal for several programs.

Acknowledgments

The author wishes to thank the members of the PROGRES project in the Department of Electrical Engineering and Computer Sciences at the University of California at Berkeley for their valuable comments made when the contents of this paper were presented at regular meetings of the project. His special thanks go to Domenico Ferrari for his careful reading of the manuscript. He is also very grateful to Amdahl Corporation for permitting him to use the trace data on which the examples of this paper are based.

References

[1] J.R. Spirn, Program Behavior: Models and Measurements, Elsevier North-Holland, Inc., 1977.

[2] P.J. Denning, "On modeling program behavior," SJCC 1972 Conference Proceedings, pp. 937-944.

[3] J.B. Morris, "Demand paging through utilization of working sets on the MANIAC II," Comm. ACM, Vol. 15, No. 10, October 1972, pp. 867-872.

[4] M.H. Fogel, "The VMOS paging algorithms: a practical implementation of the working set model," ACM Operating Systems Review, Vol. 8, No. 1, 1974, pp. 8-17.

[5] M.A. Auslander, J.F. Jaffe, A.L. Scherr and J.P. Birch, "Functional structure of IBM virtual storage operating systems," IBM Syst. J., No. 4, 1974, pp. 368-411.

[6] P.J. Denning, "The working set model of program behavior," Comm. ACM, Vol. 11, No. 5, May 1968, pp. 323-333.

[7] P.J. Denning and S.C. Schwartz, "Properties of the working set model," Comm. ACM, Vol. 15, No. 3, March 1972, pp. 191-198.

[8] J. Rodriguez-Rosell, "Empirical working set behavior," Comm. ACM, Vol. 16, No. 9, September 1973, pp. 556-560.

[9] T. Masuda, N. Takahashi and Y. Yoshizawa, "Comparison of swapping algorithms and analysis of program behavior in paging machine," Information Processing, Japan, Vol. 13, No. 2, February 1972, pp. 81-88.

[10] J. Lenfant, "Comparison of the working sets and bounded locality intervals of a program," Rapport de recherche n° 41, Université de Rennes.

[11] H. Vantilborgh, "On the working set size and its normal approximation," BIT, No. 14, 1974, pp. 240-251.

[12] J.R. Spirn and P.J. Denning, "Exepriments with program locality," Proc. FJCC 1972, pp. 611-622.

[13] F. Baskell and A. Rafii, "The AØ inversion model of program paging behavior," SLAC PUB-1826, STAN-CS-76-579, October 1976.

[14] R.P. Bogott and M.A. Franklin, "Evaluation of Markov program models in virtual memory systems," Software-Practice and Experience, Vol. 5, 1975, pp. 337-346.

[15] M.A. Franklin and R.K. Gupta, "Computation of page fault probability from program transition diagram," Comm. ACM, Vol. 17, No. 4, April 1974, pp. 186-191.

[16] M.C. Easton, "Model for interactive data base reference string," _IBM Res. Develop._, November 1975, pp. 550-556.

[17] P.J. Denning, J.E. Savage and J.R. Spirn, "Models for locality in program behavior," Princeton University, Department of Electrical Engineering and Computer Science Report, April 1972.