COMPLETE STABILITY OF NON-RECIPROCAL NONLINEAR NETWORKS

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ABSTRACT

This paper reports a recent breakthrough in research on deriving complete stability criteria for non-reciprocal network having multiple equilibrium points. Although the proof is quite mathematical in nature, the method itself is circuit-theoretic and is applicable to a large class of non-reciprocal network.

INTRODUCTION

An autonomous system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is said to be "completely stable" if all solutions converge to some equilibrium points in the state space. This property is extremely important in practice because it excludes not only oscillations, but also other more exotic modes, such as almost-periodic oscillations. In the special case where the system has only one equilibrium point, the concept of complete stability [1,2] reduces to that of "global asymptotic stability." Much research has been directed at deriving conditions for identifying completely stable nonlinear network [3-7]. The results obtained so far, however, have been restricted to either "reciprocal" networks [3,5,6] (or "eventually reciprocal [4] networks), or networks which have only one equilibrium point [7]. Unfortunately, these results are not applicable to many non-reciprocal networks of practical interest, such as switching networks which invariably have multiple equilibrium points. This "non-reciprocity" barrier has not been overcome inspite of much research efforts over the past decade because of the formidable problem of constructing global Lyapunov functions for such networks. In this paper, we report a breakthrough in this non-reciprocity barrier which we believe would have far reaching significance for research in this area.

DEFINITIONS AND BASIC PROPERTIES [8]

Consider the system described by an ordinary differential equation

$$\dot{x} = f(x),\ x \in \Sigma \subseteq \mathbb{R}^n, \quad \Sigma \text{ is open} \tag{1}$$

where $f : \Sigma \to \mathbb{R}^n$ is assumed to be continuously differentiable (unless otherwise specified). A particular point x is called an equilibrium point of the system if $f(x) = 0$. We will be dealing only with systems having isolated and a finite number of equilibrium points.

A function $x : \mathbb{R}_+ \to \Sigma$ is called a solution of the system if $\frac{d}{dt}\ x(t) = f(x(t))$ for all $t \in \mathbb{R}_+$ (where $\mathbb{R}_+$ = set of all nonnegative real numbers). A solution is bounded if its range is bounded. A point p is called an ω-limit point of a solution $x(\cdot)$ if there is an unbounded increasing sequence $\{t_k\}$ such that $\lim_{k\to\infty} x(t_k) = p$. The set of all ω-limit points of a solution is the ω-limit set of that solution. A subset $M \subseteq \Sigma$ is called positively invariant if every solution $x(\cdot)$ with $x(0) \in M$ satisfies $x(t) \in M$ for all $t > 0$.

Let ω be a map from Σ to the collection of all subsets of Σ which sends a point p to the ω-limit sets of all solutions $x(\cdot)$ with $x(0) = p$. Since f is continuously differentiable, there is one and only one solution with p as the initial point. A fundamental result which will be invoked quite often in the proofs is that if the solution starting at p is bounded, then $\omega(p)$ is nonempty, compact and connected. Also well known, is that $\omega(p)$ contains entire solutions, not necessarily $x(\cdot)$, and $\omega(p)$ is invariant in the sense that all solutions originating in $\omega(p)$ remain within $\omega(p)$.

Also useful in this paper is the notion of a general solution which is a function $\phi : \mathbb{R} \times \Sigma \to \Sigma$ with the property that (for our purpose) $\phi(\cdot,p)$ is a solution for all $p \in \Sigma$. A theorem about the general solution states that if f is r times continuously differentiable so is ϕ.

GENERAL THEOREMS ON COMPLETE STABILITY

A subset K of the space Σ is called a complete set of the system if K is positively invariant and contains its ω-limit sets i.e.

$$K \supset \omega(K) \triangleq \bigcup_{p\in K} \omega(p)$$

The following theorem is a more general version of an earlier result [6].

Theorem 1. Let $K \subseteq \Sigma$ be a complete set and $V : K \to \mathbb{R}$ be a continuously differentiable function on K with

a) $(\nabla V(x))^T f(x) \le 0$ for all $x \in K$,

b) $(\nabla V(x))^T f(x) = 0$ if and only if $f(x) = 0$,

then all bounded solutions in K converge to equilibrium points in K.

Corollary 1. If in Theorem 1, $K = \Sigma$ then, the system is completely stable in the sense that all of its solutions that are bounded converge to some equilibrium points. □

Our next theorem allows us to transform the extremely difficult problem of global stability analysis into several albeit much easier stability analysis within appropriately subdivided complete subsets of the state space Σ.

Theorem 2. System (1) is completely stable if: 1) there is a finite collection $\{(K_\alpha, V_\alpha) | \alpha \in J\}$ where each K_α is a complete set and its associated V_α satisfies on K_α, the hypotheses of Theorem 1, and 2) there is a continuously differentiable function $V_0 : \Sigma \to \mathbb{R}$ such that for $x \in \Sigma - K_J (K_J \triangleq \bigcup_{\alpha \in J} K_\alpha)$, we have

a) $(\nabla V_0(x))^T f(x) \leq 0$

b) $(\nabla V_0(x))^T f(x) = 0$ if $f(x) = 0$. □

Intuitively, we can interpret each K_α as a "reduced" state space and V_α as a Lyapunov function on it. These reduced systems are completely stable in view of Theorem 1. Now, the existance of V_0 implies that each bounded solution of the full system is attracted to some K_α and, hence, eventually to some equilibrium point. We can think of V_0 as a "global" Lyapunov function outside of K_J. Since the behavior of V_0 within each $K_\alpha \subset K_J$ is irrelevant, it is usually much easier to construct V_0 once a suitable set of K_α has been identified. The above interpretation allows us to think of each "complete set" K_α as a magnified "super" stable equilibrium point. Roughly speaking, we can establish complete stability of (1) in two steps: First, identify a suitable set of "super" stable equilibrium points. Second, show that each bounded solution tends to one of these "super" equilibrium points. We remark that sets with properties similar to K_α are sometimes called "regions of attractions" in the literature. Here, we are more precise since we also specify the mechanism of attraction.

APPLICATIONS TO NONRECIPROCAL AUTONOMOUS NONLINEAR NETWORKS

Consider now the very general class of autonomous nonlinear networks (see Fig. 1) whose state equations are described in [9]. These state equations are made up of the following 3 components: 1) Constitutive relation of the "non-reciprocal" resistive n-ports: $y = g(x)$. 2) Constitutive relations of the "reciprocal" capacitors and inductors: $x = h(z)$ where h is an injective continuously differentiable state function and H is its potential function, $\nabla H(z) = h(z)$. 3) Port interconnections: $\dot{z} = -y$

The networks in this class are described by systems of the form

$$\dot{z} = -g \circ h(z) \triangleq f(z) \tag{2}$$

Associated with the network, x_0 will be called an operating point if $x_0 = h(z_0)$ where z_0 is an equilibrium point of the system (2).

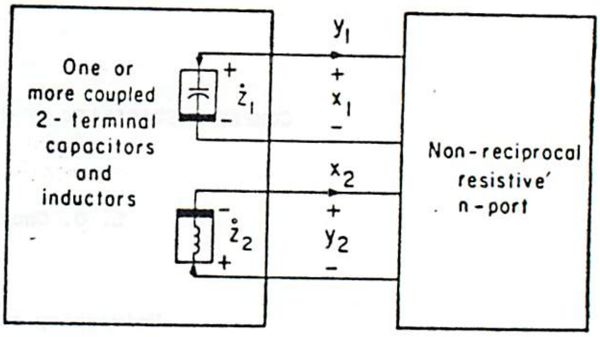

Fig. 1. A general autonomous network.

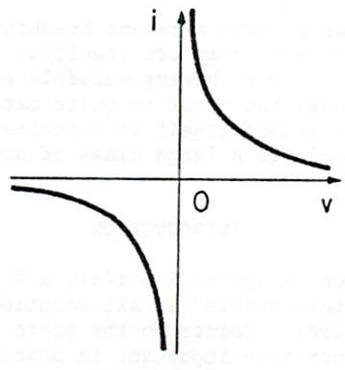

Fig. 2. A strictly passive one-port resistor which is not relatively passive.

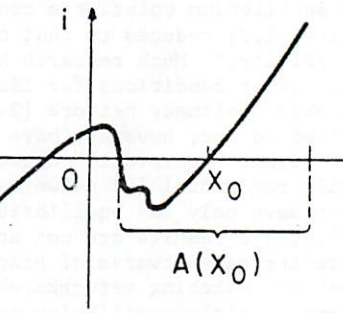

Fig. 3. An active one-port resistor which is strictly passive relative to x_0 in $A(x_0)$.

Definition. The resistive n-port is strictly passive relative to an operating point in a set $A(x_0)$ containing x_0 if

$$\langle x - x_0, g(x) \rangle > 0 \quad \text{for all } x \in A(x_0) - x_0$$

Remarks: 1. This definition is different from the classical definition of "passivity" and "local passivity" as illustrated in Figs. 2 and 3, respectively.

2. This definition would still be meaningful even if $x_0 \notin A(x_0)$. However, such an extention is not intuitively appealing as will be clear later. It is also possible to relax the definition where x_0 need not be an operating point. However, there doesn't seem to be any value in such a more

general setting either.

3. Classical thinking would tend to take $A(x_0)$ to be a sphere around x_0. That is all right except that our dynamics occurs in z-space (charges and fluxes) rather than x-space (voltages and currents). Note that a sphere around an equilibrium point in z-space when mapped into the x-space is not necessarily a sphere around the corresponding operating point.

4. In this paper, it is convenient to think of "energy" and "potential" as concepts pertaining to the dynamical variables in the z-space, and to think of "power" and "dissipation" as concepts pertaining to the network variables in the x-space. These two independent concepts can be represented graphically as an "energy profile" and a "power profile," respectively. To each operating point, we can draw a corresponding power profile. The port interconnection "matches" the energy profile with each of the "power profile." We will now show in our next theorem, (the main result of this paper) that if the matching is right, there can be no oscillation.

Theorem 3. The network described by (2) is completely stable if:

1. there is a set S of operating points such that for each $x_0 \in S$, g is strictly passive relative to x_0 with

$$A(x_0) = \{x = h(z) \mid [H(z)-H(z_0)] - \langle x_0, z-z_0\rangle < a(x_0)\}$$

for some $a(x_0)$ and z_0 such that $x_0 = h(z_0)$.

2. $$A(s) \triangleq \bigcup_{x_0 \in S} A(x_0) \supset \{x \mid \langle x, g(x)\rangle \leq 0 \text{ and } g(x) \neq 0\}. \quad \square$$

This theorem gives a criteria for matching the "energy" to the "power" profiles so that the network is completely stable. To give an interpretation of condition 1, consider the following Taylor expansion:

$$H(z) = H(z_0) + \langle \nabla H(z_0), z-z_0\rangle + \frac{1}{2}\langle z-z_0, \frac{\partial^2 H}{\partial z^2}(z_0)(z-z_0)\rangle + O(\|z-z_0\|^3).$$

Since $\nabla H(z_0) = h(z_0) = x_0$, and $\frac{\partial^2 H}{\partial z^2}(z_0) = \frac{\partial h}{\partial z}(z_0)$, we can write

$$[H(z)-H(z_0)] - \langle x_0, z-z_0\rangle = \frac{1}{2}\langle z-z_0, \frac{\partial h}{\partial z}(z_0)(z-z_0)\rangle + O(\|z-z_0\|^3)$$

and interpret this quantity as an approximation of the incremental potential function around the equilibrium point z_0.

V. AN ILLUSTRATIVE EXAMPLE

Consider a non-reciprocal autonomous nonlinear network described by the following component equations:

1) $$g_1(x,y) = x(x^2-1) + \frac{3}{2}xy^2$$
$$g_2(x,y) = y + y^3 + \frac{1}{2}x^2y$$

2) $$(x,y) = h(z_1,z_2) = (z_1,z_2)$$

3) $$\dot{z}_1 = -g_1(x,y)$$
$$\dot{z}_2 = -g_2(x,y)$$

The associated state equations are given by

$$\dot{x} = -[x(x^2-1) + \frac{3}{2}xy^2]$$

$$\dot{y} = -[y + y^3 + \frac{1}{2}x^2y]$$

and the operating points are (-1,0), (0,0), (1,0). Let $S = \{(-1,0),(1,0)\}$.

Relative to (1,0): $$(x-1)\,x(x^2-1) + \frac{3}{2}(x-1)xy^2 + y^2 + y^4 + \frac{1}{2}x^2y^2$$
$$= x(x-1)^2(x+1) + y^2(2x^2-\frac{3}{2}x+1) + y^4.$$
$$> 0 \quad \text{for} \quad x > 0$$

Suppose we choose

$a((1,0)) = \frac{1}{2}$, then

$$[H(z)-H(z_0)] - \langle x_0, z-z_0\rangle = \frac{1}{2}\langle z-z_0, z-z_0\rangle = \frac{1}{2}[(x-1)^2+y^2] < \frac{1}{2}$$

implies that $(x-1)^2 + y^2 < 1$. Consequently, A((1,0)) is an open disc of radius 1 centered at (1,0).

Relative to (-1,0): $$(x+1)x(x^2-1) + \frac{3}{2}(x+1)xy^2 + y^2 + y^4$$
$$> 0 \quad \text{for} \quad x < 0.$$

Choosing again $a((-1,0)) = \frac{1}{2}$, we obtain

$$[H(z)-H(z_0)] - \langle x_0, z-z_0\rangle = \frac{1}{2}\langle z-z_0, z-z_0\rangle = \frac{1}{2}[(x+1)^2+y^2] < \frac{1}{2}$$

and $(x+1)^2 + y^2 < 1$. Consequently, A((-1,0)) is an open disc of radius 1 centered at (-1,0).

Now to check condition 2, we calculate

$$\langle (x,y),g(x,y)\rangle = x^2(x^2-1)+\frac{3}{2}x^2y^2+y^2+y^4+\frac{1}{2}x^2y^2$$
$$+(x^2+y^2)^2-(x^2-y^2)$$

Observe that this expression is negative inside the shaded area shown in Fig. 4, which in turn is a subset of A(S). Since all hypotheses of Theorem 3 are satisfied, it follows that the network is completely stable.

PROOFS OF THEOREMS

1. Proof of Theorem 1. Let $x(t)$ be a bounded solution in K. Then $\omega(x(0))$ is compact and therefore $V(\omega(x(0)))$ is bounded. Moreover,

$$\lim_{t\to\infty} x(t) \in \omega(x(0))$$

$$\Rightarrow \lim_{t\to\infty} V(x(t)) = V(\lim_{t\to\infty} x(t)) \in V(\omega(x(0)))$$

Hence, for any unbounded increasing sequence $\{t_k\}$, $t_k \in [0,\infty)$, $\{V(x(t_k))\}$ is a bounded sequence with a finite limit. It remains for us to show that this sequence actually converges. Since $x(t_k) \in K$ $\forall t_k \in [0,\infty)$,

$$\dot{V}(x(t)) = \langle \nabla V(x(t)),f(x(t))\rangle \le 0 \quad \forall t \in [0,\infty)$$

Therefore, $\{V(x(t))\}$ is a decreasing sequence of real numbers. Hence it converges to a unique limit V_∞.

Next we show that V is constant on $\omega(x(0))$. Let $x \in \omega(x(0))$, then there is a sequence $\{t_k\} \underset{k\to\infty}{\to} \infty$ such that $\{x(t_k)\} \underset{k\to\infty}{\to} x$. then

$$V(x) = V(\lim_{k\to\infty} x(t_k)) = \lim_{k\to\infty} V(x(t_k)) = V_\infty .$$

Finally, because $\omega(x(0))$ is invariant, there is a solution $\phi(t,x) \subset \omega(x(0))$ $\forall t$. So along this solution,

$\dot{V}(\phi(t,x)) \equiv 0$, in particular $\dot{V}(\phi(0,x)) = \dot{V}(x) = 0$

Consequently, $\dot{V}(\omega(x(0))) = 0$ and it follows from the hypothesis that $\omega(x(0))$ consists of equilibrium points only.

2. Proof of Theorem 2. By Theorem 1, every solution that starts in K_α, converges to equilibrium points in K_α, $\alpha \in J$. Since the solution through each point is unique, every solution whose path intersects K_J must also converge to equilibrium points.

Let $X(t)$ be a bounded solution that is contained in $\Sigma-K_J$ for all finite time. If $\omega(x(0)) \cap K_J \neq \phi$, then either $(x(0))$ consists of equilibrium points and we are done, or it doesn't. In the latter case, there is a solution that is non constant which starts in $\omega(x(0)) \cap K_J$ and is contained in $\omega(x(0))$. But such solution converges to equilibrium points in K_J.

Finally we are left with the case that $\omega(x(0)) \subset \Sigma-K_J$. But then $\{x(t)\} \cup \omega(x(0))$ is a complete set and V_0 is defined on $\Sigma-K_J$ so it is defined on $\{x(t)\} \cup \omega(x(0))$. It follows from Theorem 1 that $x(t)$ converges to an equilibrium point in $\Sigma-K_J$.

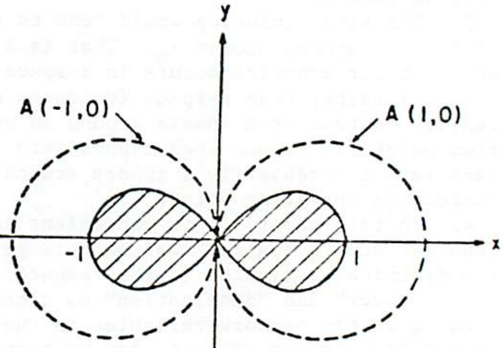

Fig. 4. The two complete sets A((1,0)) and A((-1,0)). The point (0,0) is not in A(S).

3. Proof of Theorem 3. Let $\hat{A}(z_0) \triangleq \{z | h(z) \in A(x_0)\}$. We will show that $A(z_0)$ is a complete set for all z_0 such that $h(z_0) \in S$.

For $z \in \hat{A}(z_0)$ define

$$V_{z_0}(z) \triangleq [H(z)-H(z_0)] - \langle h(z_0),z-z_0\rangle$$
$$< a(x_0).$$

Pick a point $p \in \hat{A}(z_0) - z_0$, then $V_{z_0}(p) < a(x_0)$. Let $z(t)$ be a solution starting at p. Since

$$\dot{V}_{z_0}(z(t))_{t=0} = \dot{V}_{z_0}(p) = \langle \nabla V_{z_0}(p),\dot{z}(0)\rangle$$
$$= \langle h(z(0))-h(z_0),\dot{z}(0)\rangle$$
$$= \langle h(p)-x_0,-g(h(p)\rangle.$$

and $p \in \hat{A}(z_0)$ means $h(p) \in A(x_0)$, therefore

$$\dot{V}_{z_0}(p) < 0$$

in view of the "relative passivity" hypothesis. Hence for some small time $\varepsilon > 0$,

$$V_{z_0}(z(\tau)) < V_{z_0}(p).$$

Let t_1 be the earliest time such that $V_{z_0}(z(t_1)) = V_{z_0}(p)$. By the mean-value theorem, there exists a $\tau \in (0,t_1)$ such that

$$V_{z_0}(z(t_1)) - V_{z_0}(p) = \dot{V}_{z_0}(z(\tau))\cdot t_1 = 0.$$

But this is impossible because by our choice of t_1,

$V_{z_0}(z(\tau)) < V_{z_0}(p) < a(x_0)$ so that $z(\tau) \in \hat{A}(z_0)$

and therefore $\dot{V}_{z_0}(z(\tau)) < 0$.

Consequently, for all $t \in (0,\infty)$,

$V_{z_0}(z(t)) < V_{z_0}(p) < a(x_0)$, i.e. $z(t) \in \hat{A}(z_0)$.

Hence, $\lim_{t\to\infty} V_{z_0}(z(t)) < a(x_0)$ and $\omega(p) \subset \hat{A}(z_0)$.

Then $\{\hat{A}(z_0) | h(z_0) \in S\}$ is a collection of complete sets and $\{V_{z_0} | h(z_0) \in S\}$ satisfies the hypotheses of Theorem 2.

Finally, let $V_0(z) = H(z)$, then

$$\dot{V}_0(z) = \langle \nabla H(z), \dot{z} \rangle = -\langle h(z), g(h(z)) \rangle$$
$$= -\langle x, g(x) \rangle, \text{ where } x = h(z),$$

which by hypothesis is negative if $x \notin A(S)$ and $g(x) \neq 0$. Hence, $\dot{V}_0(z) < 0$ if $z \notin \bigcup_{h(z_0)\in S} \hat{A}(z_0)$ and $g \circ h(z) \neq 0$.

Since all hypotheses of Theorem 2 are satisfied the network is completely stable. □

CONCLUSION

Although networks containing "locally active" and "non-reciprocal" elements are quite susceptible to oscillation, Theorem 3 provides us with an invaluable tool for uncovering a subclass of such networks where oscillation is impossible. Intuitively speaking, if one measures power with respect to ground, a network belonging to this class may appear "active" in some regions of the state space. Such regions may correspond to the "charging" of capacitors, etc. Now if one searches for a "target" point where the active charging network is "aiming" for, then around that point, a "non-oscillatory" network should appear passive. In another words, the difference in energy levels between the instantaneous value and that of the target diminishes.

It might be conjectured that a substantial subclass of all "switching circuits" operates in this manner: they all have stable targets by design, and the switching mechanism is active. However, since each target is a stable state, these must exist a region which is relatively passive with respect to each target.

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REFERENCES

[1] Lasalle, J.P. and S. Lefschetz, Stability by Lyapunov's Direct Method, Academic Press, New York, 1961.

[2] Hahn, W., Stability of Motion, Springer-Verlag, New York, 1967.

[3] Brayton, R.K. and J. K. Moser, "A Theory of Nonlinear Networks I and II," Quarterly of Appl. Math., vol. 22, pp. 1-33, April 1964; and pp. 81-104, July 1964.

[4] Moser, J.K., "On Nonoscillating Networks," Quarterly of Appl. Math., vol. 25, no. 1, pp. 1-9, April 1967.

[5] Brayton, R.K., "Nonlinear Reciprocal Networks," in Mathematical Aspect of Electrical Network Analysis, SIAM-AMS Proc., vol. 3, American Math. Soc., 1971, pp. 1-15.

[6] Chua, L.O. and N.N. Wang, "Complete Stability of Autonomous Reciprocal Nonlinear Networks," Int. J. of Circuit Theory and Applications, vol. 6, no. 3, pp. 211-241, July, 1978.

[7] Sandberg, I.W., "Some Theorems on the Dynamic Response of Nonlinear Transistor Networks," Bell System Technical Journal, vol. 48, pp. 35-54, Jan. 1969.

[8] Hale, J.K., Ordinary Differential Equations, Wiley-Interscience, New York, 1969.

[9] Chua, L.O. and D.N. Green, "A Qualitative Analysis of the Behavior of Dynamic Nonlinear Networks: Stability of Autonomous Networks," IEEE TRans. on Circuits and Systems, vol. CAS-23, no. 6, June 1976, pp. 355-379.