MARTINGALES PARAMETERIZED BY SETS

AND MULTIPLE ITO INTEGRALS

by

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SUMMARY

Martingales parameterized by certain families of convex subsets of $\mathbb{R}^n$, termed set martingales, are studied. The collection of subsets is partially ordered by set inclusion and an increasing family of σ-fields is naturally generated by an independent, random measure. It is shown that square integrable set martingales may be represented as a sum of certain stochastic integrals with respect to the random measure. The stochastic integrals are named multiple Ito integrals since they generalize both the multiple Wiener integral introduced by K. Ito and the stochastic integral of K. Ito. Some properties of multiple Ito integrals are found.

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1. Introduction

Inspired by work of Kakutani [6], K. Ito [4] introduced the isometric multiple Wiener integral. K. Ito [3] also introduced random integrals in the theory of stochastic integration. It is thus fitting to name the multiple stochastic integrals with random integrands originally presented by Wong and Zakai [7] multiple Ito integrals. The purpose of this paper is to identify and study a broad class of multiple Ito integrals.

Let $\mathcal{A}$ be a collection of subsets of $E = \mathbb{R}^n$. Suppose that $(\Omega,\mathcal{F},P)$ is a probability space and that $\{\mathcal{F}_A : A \in \mathcal{A}\}$ is a collection of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_A \subset \mathcal{F}_B$ whenever $A \subset B$ and $\mathcal{F} = \mathcal{F}_E$. A collection of integrable random variables $\{X_A : A \in \mathcal{A}\}$ is defined to be a set martingale relative to $\{\mathcal{F}_A : A \in \mathcal{A}\}$ if $E[X_A|\mathcal{F}_B] = X_B$ a.s. whenever $A \supset B$.

In this and the following section, it is assumed that the σ-fields $\mathcal{F}_A$ are generated by a Gaussian white noise as follows. Let $\{W(B) : B \in \mathcal{B}(E)\}$ be a centered Gaussian random measure (i.e. a process parameterized by $\mathcal{B}(E)$, the Borel subsets of E) with $E[W(A)W(B)] = \mu(A \cap B)$, where μ denotes Lebesgue measure on E. It is assumed that $\mathcal{F}_A = \sigma(W(B) : B \subset A) \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of P-null sets. When n=2 and $\mathcal{A}$ consists of sets of the form $[0,z_1] \times [0,z_2]$ for $z_1, z_2 \geq 0$, the framework of Wong and Zakai [7] is recovered. The case when $\mathcal{F}_A$ is generated by a differential process [5], [2] (or "general independent white noise"), which includes both Gaussian white noise and poisson point processes as special cases, will be studied in Section 3.

A certain class of parameter sets $\mathcal{A}$ is studied in this paper.

This includes the case when $\mathcal{A}$ is the collection of all closed convex sets and the case when $\mathcal{A}$ is the collection of all closed rectangles (where a rectangle in E is any set $A \subset E$ such that $\prod_{i=1}^{n} (a_i,b_i) \subset A \subset \prod_{i=1}^{n} [a_i,b_i]$ for $-\infty \le a_i \le b_i \le +\infty$).

Given a collection of sets $\mathcal{A}$, points $s_1,\ldots,s_k$ in E are said to be unordered if each of the points lies outside of some $A \subset \mathcal{A}$ which contains the other k-1 points. The multiple Ito integral of order k involves stochastic integration over the unordered portion of E^k. The multiple Wiener integral defined by Ito [4] is the special case when $A = \mathcal{B}(E)$ and unordered means distinct.

It is shown that multiple Wiener integrals may be "collapsed" into multiple Ito integrals. This is used to establish the completeness of multiple Ito integrals in the space of square integrable set parameter martingales. This fact generalizes the representation theorem of Wong and Zakai [7] which, in turn, has its roots in the work of Ito [4] and Kakutani [6].

In the remainder of this section, some facts regarding multiple Wiener integrals will be reviewed. The multiple Ito integral introduced in the following section will have similar properties.

The multiple Wiener integral of order k as defined in [4] is a map $f \to I_k(f)$ of $L^2(E^k) \to L^2(\Omega)$ which is characterized by the properties.

(i) $I_k(h) = \prod_{i=1}^{k} W(A_i)$ if $h = 1_{A_1 \times \ldots \times A_k}$ for disjoint rectangles $A_1,\ldots,A_k$.

(ii) $I_k(f+g) = I_k(f) + I_k(g)$.

(iii) If $f_j \to f$ in $L^2(E^k)$, then $I_k(f_j) \to I_k(f)$ in $L^2(\Omega)$.

By convention, if $k = 0$ and $h \in L^2(E^0) \cong \mathbb{R}$, define $I_0(h) = h$.

For $f: E^k \to \mathbb{R}$ let $\tilde{f}$ defined by

$$\tilde{f}(t_1,\ldots,t_k) = \frac{1}{k!} \sum_{\pi} f(t_{\pi_1},\ldots,t_{\pi_k})$$

denote the symmetrization of f. For $f \in L^2(E^k)$, $\tilde{f}$ is the projection of f onto the subspace $L^2_S(E^k)$ of $L^2(E^k)$ spanned by symmetric functions. Note that $\|\tilde{f}\| \leq \|f\|$ by the Schwartz inequality.

The suggestive notation

$$\int_{\hat{E}^k} f(s_1,\ldots,s_k) W(ds_1)\ldots W(ds_k),$$

where $\hat{E}^k = \{(s_1,\ldots,s_k) \in E^k : s_i \neq s_j \text{ if } i \neq j\}$ will also be used for $I_k(f)$. The multiple Wiener integral also has the following properties (see [2],[4]):

(iv) For $f \in L^2(E^k)$ and $g \in L^2(E^{k'})$, $I_k(f) = I_k(\tilde{f})$ and

$$E[I_k(f) I_{k'}(g)] = 1_{\{k=k'\}} k! \ \langle \tilde{f}, \tilde{g} \rangle_{L^2(E^k)}$$

(v) For $\phi \in L^2(E)$ and $\lambda \in \mathbb{C}$,

$$\exp(\lambda \int_E \phi_s W(ds) - \frac{1}{2} \lambda^2 \int_E \phi_s^2 ds) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_{E^k} \phi_{s_1} \cdots \phi_{s_k} W(ds_1) \ldots W(ds_k) \quad (1.1)$$

(vi) The Wiener integrals span $L^2(\mathbb{R})$. Thus (using $\oplus$ to denote orthogonal sum),

$$L^2(\Omega) = \bigoplus_{k=0}^{\infty} \{I_k(f) : f \in L^2(E^k)\} \cong \bigoplus_{k=0}^{\infty} L^2_S(E^k).$$

2. Multiple Ito Integrals and Representation of Set Martingales

In this section we will define a class of multiple stochastic integrals analogous to the integrals $\theta \circ W$ and $W \circ r \circ W$ introduced by Wong and Zakai [7]. Such integrals will be called multiple Ito integrals since, as in the one parameter case, they generalize (multiple) Wiener integrals in that random integrands are allowed. Also the indefinite integrals will be defined so that the resulting integrals will be set martingales relative to a collection $\mathcal{A}$ of subsets of $E = \mathbb{R}^n$.

It will be assumed that $\mathcal{A}$ is of the following form. Let $\{\theta_\alpha\}$ be a subset of the unit sphere in $E = \mathbb{R}^n$ and let p denote its (possibly infinite) cardinality. Let $\mathcal{A} = \mathcal{A}_{\{\theta_\alpha\}}$ be the collection of all closed convex subsets A in E such that $\{\theta_\alpha\}$ contains an outword normal to A at each point in the boundary of A. The special case $p = 2n$ and $\{\theta_1,\ldots,\theta_p\} = \{(0,\ldots,0,\pm 1,0,\ldots,0)\}$ corresponds to the collection of all closed rectangles. The special case when $\{\theta_\alpha\}$ is equal to the unit sphere in $E = \mathbb{R}^n$ corresponds to the collection of all convex sets.

Each $A \subset \mathcal{A}_{\{\theta_\alpha\}}$ has the representation

$$A_h = \{x \in E : x\cdot\theta_\alpha \le h_\alpha \quad \forall\alpha\} \tag{2.1}$$

where $h = (h_\alpha)_{\alpha\in p} \in (\mathbb{R}\cup\{+\infty\})^p$. Let $|x-y|$ be absolute value for $x,y \in \mathbb{R} \cup \{+\infty\}$ with the conventions $(+\infty) - (+\infty) = 0$ and $|\pm\infty| = +\infty$. A metric is defined for $A,B \in \mathcal{A}_{\{\theta_k\}}$ by

$$d(A,B) = \min(1, \inf \sum_\alpha |k_\alpha - h_\alpha|) \tag{2.2}$$

where the infimum is over $k = (k_\alpha)$, $h = (h_\alpha) \in (\mathbb{R} \cup \{+\infty\})^p$ such that $A = A_k$ and $B = B_k$.

As in Section 1, let $\{W(B) : B \in \mathcal{B}(E)\}$ be a centered Gaussian random measure with $E[W(a)W(B)] = \mu(A\cap B)$, defined on a probability space $(\Omega,\mathcal{F},P)$. Let $\mathcal{F}_A = \sigma(W(B) : B \subset A) \vee \mathcal{N}$ for $A \in \mathcal{B}(E)$, where $\mathcal{N}$ denotes the collection of P-null sets. Formally, $W(A) = \int_A \eta_s ds$ for a white Gaussian noise η and

$\mathcal{F}_A = \sigma(\eta_s : s \in A)$.

Given subsets $T_1,\ldots,T_k$ of $E = \mathbb{R}^n$, define $R_{T_1,\ldots,T_k}$ to be the intersection of all $A \in \mathcal{A}$ such that $A \cap T_i \neq \phi$ for each i. Let $R_{s_1,\ldots,s_k} = R_{\{s_1\},\ldots,\{s_k\}}$ for $s_1,\ldots,s_k \in E^k$. A set of points $s_1,\ldots,s_\ell$ is called unordered if for $1 \le j \le \ell$, s_j is not contained in $R_{s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_\ell}$. Note that a set of unordered points contains at most p points. A collection of subsets $A_1,\ldots,A_\ell$ will be called unordered if $s_1,\ldots,s_\ell$ are unordered whenever $s_i \in A_i; i = 1,\ldots,\ell$. Given a subset D of E^ℓ, $\hat{D}$ will denote the set of $(s_1,\ldots,s_\ell) \in \hat{D}$ such that $s_1,\ldots,s_\ell$ are unordered.

Let $L_a^2(\widehat{E^k}\times\Omega)$ denote the set of adapted $\mu^k \times P$ square integrable functions on $E^k \times \Omega$ — i.e. $f \in L_a^2(\widehat{E^k}\times\Omega)$ if $f:\widehat{E^k} \times \Omega \to \mathbb{R}$ and

1) f is $\mathcal{B}(\widehat{E^k}) \times \mathcal{F}$ measurable.

2) f is $\mu^k \times P$ square integrable.

3) $f(s,\cdot)$ is $\mathcal{F}(R_{s_1,\ldots,s_k})$ measurable for each $s = (s_1,\ldots,s_k) \in \widehat{E^k}$

As usual, two functions in $L_a^2(\widehat{E^k}\times\Omega)$ are identified if they are equal $\mu^k\times P$ a.e., so that $L_a^2(\widehat{E^k}\times\Omega)$ becomes a Hilbert space with $\|f\|$ denoting the norm of f.

For $f \in L_a^2(\widehat{E^k}\times\Omega)$, let $\tilde{f}$ denote the symmetrization of f:

$$\tilde{f}(t_1,\ldots,t_k) = \frac{1}{k!}\sum_{\pi} f(t_{\pi_1},\ldots,t_{\pi_k})$$

Note that $\|\tilde{f}\| \le \|f\|$ by the Schwartz inequality. Let $L_{a,S}^2(\widehat{E^k}\times\Omega)$ denote the collection of symmetric functions in $L_a^2(\widehat{E^k}\times\Omega)$; then $\tilde{f}$ is the projection of f onto $L_{a,S}^2(\widehat{E^k}\times\Omega)$.

An elementary function $f \in L_a^2(\widehat{E^k}\times\Omega)$ is a finite linear combination of functions of the form $1_{E_1\times\cdots\times E_k}(s)Z(\omega)$ where $E_1,\ldots,E_k$ is an unordered collection of bounded rectangles and Z is a bounded, $\mathcal{F}(R_{E_1,\ldots,E_k})$ measurable random variable. The collection of elementary functions is dense in $L_a^2(\widehat{E^k}\times\Omega)$ as shown in the appendix.

If $f \in L_a^2(\widehat{E^k} \times \Omega)$ is elementary then f can be expressed as

$$f(t_1,\ldots,t_k,\omega) = Z_{i_1,\ldots,i_k}(\omega) \text{ for } (t_1,\ldots,t_k) \in T_{i_1} \times \ldots T_{i_k}$$
$$= 0 \quad \text{otherwise}$$

where $T_1,\ldots,T_m$ are disjoint rectangles, $Z_{i_1,\ldots,i_k}$ is zero unless $T_{i_1},\ldots,T_{i_k}$ are unordered rectangles and then $Z_{i_1,\ldots,i_k}$ is a bounded, $\mathcal{F}(R_{T_{i_1},\ldots,T_{i_k}})$ measurable random variable. For such f, the (indefinite) multiple Ito integral of order k, denoted $f \circ W^k$, is defined by

$$f \circ W_A^k = \sum Z_{i_1,\ldots,i_k} W(T_{i_1} \cap A) \ldots W(T_n \cap A)$$

for each $A \in \mathcal{A}$. It is not hard to see that $\tilde{f} \circ W_A^k = f \circ W_A^k$, that $(f+g) \circ W_A^k = f \circ W_A^k + g \circ W_A^k$ if $g \in L_a^2(\widehat{E^k} \times \Omega)$ is also elementary, and that for any $A \subset \mathcal{A}^o$,

$$E[(f \circ W_A^k)^2] = \|\tilde{f} 1_{A^k}\| \leq \|f 1_{A^k}\|$$

Thus, for fixed $A \subset \mathcal{A}$, the multiple Ito integral $f \circ W_A^k$ may be extended to all $f \in L_a^2(\widehat{E^k} \times \Omega)$ by the requirements that $f \circ W_A + g \circ W_A = (f+g) \circ W_A$ a.s. and $f_n \circ W_A^k \to f \circ W$ in $L^2(\Omega)$ whenever $\|f_k - f\| \to 0$.

It is easily checked that $f \circ W^k$ is a set parametered martingale relative to $\mathcal{A}$ if f is an elementary function. Since conditional expectations commute with limits in $L^2(\Omega)$, it follows that $f \circ W^k$ is a set martingale relative to $\mathcal{A}$ for any $f \in L_a^2(\widehat{E^k} \times \Omega)$.

If p is finite it will be shown next (Proposition 2.3 below) that there is a sample continuous modification of the multiple Ito integral $\{f \circ W_A^k : A \in \mathcal{A}\}$. The topology on $\mathcal{A}$ is induced by the metric defined in (2.2). The map $h \to A_h$ from $\mathbb{R}^p$ to $\mathcal{A}$ defined by (2.1) is continuous in this topology.

Lemma 2.1. Let $B \in \mathcal{B}(E)$ be a bounded subset of E. Then there is a sample continuous modification of $\{W(A \cap B): A \in \mathcal{A}\}$ if $\rho < +\infty$.

Proof. If each random variable $W(A \cap B)$ for $A \in \mathcal{A}$ is redefined on a P-null set, then the Gaussian random process $\{X_h = W(A_h \cap B), h \in \mathbb{R}^p\}$ can be made sample continuous. This is a consequence of that fact that $E[X_k X_{k'}] \leq C_F |k-k'|$ for all $k, k' \in F$, where F is any bounded subset of $\mathbb{R}^{p+2n}$ and C_F is a constant depending on F. By the definition of the metric on $\mathcal{A}$, the process $\{W(A \cap B): A \in \mathcal{A}\}$ is sample continuous for the same modification. ◻

Lemma 2.2. Suppose $p < +\infty$. If M is a separable square integrable martingale relative to $\mathcal{A}$, then

$$E[\sup_{A \in \mathcal{A}} |M(A)|^2] \leq 4^p E[|M(E)|^2].$$

This inequality may be proved by repeated application of Doob's maximal inequality for 1-parameter martingales and positive submartingales [1].

Proposition 2.3. If $p < +\infty$ and $f \in L_a^2(\hat{E}^k \times \Omega)$, there is a sample continuous modification of $\{f \circ W_A^k : A \in \mathcal{A}\}$.

Proof. By Lemma 3.1, the proposition is true if f is an elementary function. By Lemma 3.2, a continuous modification of $f \circ W^k$ in the general case is obtained as the a.s. uniform (in $A \in \mathcal{A}$) limit of the sample continuous integrals of elementary functions. ◻

It is convenient to extend the multiple Ito integral $f \circ W_A$ to $f \circ W_B$ for any Borel set $B \subset E$ as follows. If $f \in L_a^2(\hat{E}^k \times \Omega)$ and if B is a rectangle, then

$$f \circ W_B^k = (f 1_{B^k}) \circ W^k \quad \text{a.s.} \tag{2.3}$$

Indeed, (2.3) is true when f is an elementary function and hence for all f by approximation. Now, if B is any Borel subset of E then we can define a random variable $f\circ W_B^k$ by (2.3) since the right hand side is still well defined. $f\circ W_B^k$ will always be F_{B*} measurable, where B* is the intersection of all $A \in A$ such that $B \subset A$. $f\circ W_B^k$ need not be F_B measurable.

A suggestive alternative notation for $f\circ W_B^k$ is

$$f\circ W_B^k = \int_{\hat{B}^k} f(s_1,\ldots,s_k,\omega)W(ds_1)\ \ldots\ W(ds_k).$$

This emphasises that the multiple Ito integral permits random integrands and integration is restricted to unordered k-tuples of points in E.

<u>Theorem 2.4</u>. (Properties of Multiple Ito Integral)

a) (Linearity) $(f+g)\circ W_B^k = f\circ W_B^k + g\circ W_B^k$ a.s. whenever $f,g \in L_a^2(\hat{E}^k\times\Omega)$ and $B \in \mathcal{B}(E)$.

b) (Orthogonality and Isometric Properties) If $f \in L_a^2(\hat{E}^k\times\Omega)$, $g \in L_a^2(\hat{E}^{k'}\times\Omega)$, and $B \in \mathcal{B}(E)$, then

$$E[(f\circ W_B^k)(g\circ W_B^{k'})] = 1_{\{k=k'\}}\ \langle \tilde{f}1_{B^k},\tilde{g}\rangle_{L_a^2(\hat{E}^k\times\Omega)}$$

c) (Uniqueness of Representation) For $f,f' \in L_a^2(\hat{E}^k\times\Omega)$ and $B \in \mathcal{B}(E)$, $f\circ W_B^k = f'\circ W_B^k$ a.s. if and only if $\tilde{f}1_{B^k} = \tilde{f}'1_{B^k}$ a.e. $(\mu^k\times P)$.

d) (Projection Property) Given $A,B \in \mathcal{B}(E)$, $1 \le k \le p$, and $f \in L_a^2(\hat{E}^k\times\Omega)$, there exists an element $E[f|F_B] \in L_a^2(\hat{E}^k\times\Omega)$ characterized by the fact that $E[f|F_B](s,\cdot) = E[f(s,\cdot)|F_B]$ for all $s \in \hat{E}^k$. The multiple Ito integral satisfies

$$E[f\circ W_A^k|F_B] = (E[f|F_B])\circ W_{A\cap B} \tag{2.4}$$

e) (Elementary Exponential Representation) Suppose $p < +\infty$. For all $\phi \in L^2(E)$, all complex λ, and all $A \in \mathcal{B}(E)$,

$$L_A^{(\lambda)} = \exp(\lambda\int_A \phi_s W(ds) - \frac{\lambda^2}{2}\int_A \phi_s^2 ds)$$

$$= 1 + \sum_{k=1}^{p} \frac{\lambda^k}{k!} (L^{(\lambda)}_{R_{s_1,\ldots,s_k}} \phi_{s_1}\cdots\phi_{s_k}) \circ W_A^k \qquad (2.5)$$

f) (Relation to Multiple Wiener Integral) Let $h \in L^2(E^m)$. Then the multiple Wiener integral $I_m(h)$ has the representation

$$I_m(h) = E[I_m(h)] + \sum_{k=1}^{\min(m,p)} h_k \circ W_E^k \qquad (2.6)$$

where $h_k \in L_a^2(\widehat{E^k}\times\Omega)$ for $k \leq \min(m,p)$ satisfies

$$h_k(s_1,\ldots,s_k,\omega) = I_{m-k}(\tilde{h}(s_1,\ldots,s_k,\cdot)1_{\widehat{R^{m-k}_{s_1,\ldots,s_k}}}(\cdot))(\omega) \quad \text{a.e.} \qquad (2.7)$$

g) (Completeness or Martingale Representation) Every square integrable set martingale M relative to $\{F_A : A \in \mathcal{A}\}$ has a (sample continuous, if $p < +\infty$) modification with the representation

$$M_A = E[M_A] + \sum_{k=1}^{p} \alpha_k \circ W_A^k \text{ for } A \in \mathcal{A}. \qquad (2.8)$$

The sum converges in $L^2(\Omega)$ for each $A \in \mathcal{A}$ if p is infinite.

<u>Proof</u>. (a) and (b) are easily verified if f and g are elementary functions, and the assertions extend to the general case by an obvious limiting argument. (c) follows directly from (b). To prove (d), assume first that f is an elementary function. Then f is a finite linear combination of functions of the form $\theta(s,\omega) = Z(\omega)\, 1_{A_1\times\ldots\times A_k}(s)$ where $A_1,\ldots,A_k$ is an unordered collection of bounded rectangles and Z is a bounded, $F(R_{A_1,\ldots,A_k})$ measurable random variable. Now

$$E[\theta \circ W_A^k | F_B] = E[Z \prod_{i=1}^{k} W(A \cap A_i) | F_B]$$

$$= E[Z|F_B] \prod_{i=1}^{k} W(A \cap B \cap A_i)$$

$$= (E[\theta|F_B]) \circ W^k_{A \cap B}$$

since a version of $E[\theta|F_B]$ is given by

$$E[\theta|F_B](s,\omega) = E[Z|F_B](\omega)\, 1_{A_1 \times \ldots \times A_k}(s).$$

Thus (d) is true for elementary functions f by linearity. The case of general f follows from (b) and the fact that the map $f \to E[f|F_B]$ is norm decreasing in $L_a^2(\widehat{E^k} \times \Omega)$.

By replacing ϕ by $\phi 1_A$, it suffices to prove (e) for the case A = E. Then (e) may be proved as in [8] by a two step procedure. First, the differential formula for one parameter processes is applied to $L_A^{(\lambda)}$ in each one of the p directions $\theta_1, \ldots, \theta_p$ in E. This yields a representation of $L_A^{(\lambda)}$ as a sum of iterated integrals of order up to p. The second part of the proof then is to note the equivalence of the iterated integrals and multiple Ito integrals. This is accomplished by first considering elementary processes for integrands. The details are straight forward and are almost the same as in [8], and are hence omitted.

To prove (f), first suppose that h has the form $h = 1_{A_1 \times \ldots \times A_m}$ where $A_1, \ldots, A_m$ are disjoint closed, bounded rectangles such that $A_{i_1}, \ldots, A_{i_\ell}$ are unordered and $A_{i_{\ell+1}}, \ldots, A_{i_m} \subset R_{A_{i_1}, \ldots, A_{i_\ell}}$ for some permutation $i_1, \ldots, i_m$ of $1, \ldots, m$. Then

$$I_m(h) = \prod_{k=1}^{m} W(A_k)$$

$$= \left(\prod_{k=\ell+1}^{m} W(A_{i_k})\right) \prod_{k=1}^{\ell} W(A_{i_k})$$

$$= h_\ell \circ W_E^\ell$$

where

$$h_\ell(s_1,\ldots,s_\ell,\omega) = \Big(\prod_{k=\ell+1}^{m} W(A_{i_k})(\omega)\Big) 1_{A_{i_1}\times\cdots\times A_{i_\ell}} . \tag{2.9}$$

Thus, $I_m(h)$ has the representation (2.6) with $h_k = 0$ if $k \geq m$, and (2.7) follows from (2.9). Since linear combinations of such functions h are dense in $L^2(E^m)$, (f) is proved for general h by approximation using the isometric properties of the multiple Wiener and Ito integrals.

To prove (g), note first that by (f), multiple Wiener integrals can be expressed as sums of multiple Ito integrals (evaluated at E). Hence, since multiple Wiener integrals are total in $L^2(\Omega)$, so are multiple Ito integrals. Thus, the collection of random variables of the form $\sum_{k=0}^{p} \alpha_k \circ W_E^k$ is dense in $L^2(\Omega)$, and is a closed subspace of $L^2(\Omega)$, being isometric to $\bigoplus_{k=0}^{p} L^2_{a,S}$. Thus, any square integrable random variable has an integral representation $\sum_{k=0}^{p} \alpha_k \circ W_E^k$. Thus, if M is a square integrable martingale with respect to $\mathcal{A}$ then

$$M_E = \sum_{k=0}^{p} \alpha_k \circ W_E^k$$

for some $\alpha_k \in L_a(\widehat{E^k}\times\Omega)$, $k = 0,\ldots p$. Hence, a modification of M satisfies (2.8) since each side is a martingale with common final value. ¤

<u>Remark</u>. If p is finite, the spanning property (g) can also be proven by using (a), (b) and (e). Indeed, by (e), $\exp(\lambda \int_E \phi_s dW_s)$ has a representation in terms of multiple Ito integrals. Random variables of this form are total in $L^2(\Omega,P)$ by a monotone class argument and the fact that exponentials span the class of square integrable functions of finite collections of Gaussian random variables. Hence, the collection of random variables of the form $\sum_{k=0}^{p} \alpha_k \circ W_E^k$ is dense in $L^2(\Omega)$. The proof is completed as before.

The idea for this proof is essentially due to Yor [8]. It has the advantage that completeness is proved from scratch, while the proof we gave depends on the completeness of multiple Wiener integrals. However, we have not established (e) in case p is not finite, so that Yor's proof cannot (yet) be used in this case.

<u>Conjecture</u>. I conjecture that (e) of Theorem 2.1 is also valid for $p = \infty$. One proof might be based on iterated integrals as in the case $p < \infty$, using (b) and (g) to control the limit. A perhaps more general approach would be to use property (f) in conjunction with the exponential formula (1.1) for multiple Wiener integral.

<u>Remark</u>. In all cases, Ito integrals are characterized by the fact that random integrands are allowed so that integration may be restricted to unordered points. A different class of integrands, the analog of one parameter predictable processes, is considered in the next section.

It is interesting to note that if A is the collection of all Borel subsets of $\mathbb{R}^n$ and if the definitions in this section are used, then "unordered" is the same as "disjoint." Then the resulting multiple Ito integral is just the multiple Wiener integral of Section 1.

3. <u>σ-Fields Generated by General Stationary White Noise</u>

The multiple Ito integral and representation theorems are given in this section in case the σ-fields are generated by a general stationary white noise. Suppose that there is a stationary independent Borel random measure $\{M(A):A \in \mathcal{B}(E)\}$ defined on the probability space (Ω,F,P). (By independent, we mean that $M(A)$ is independent of $M(B)$ if $A \cap B = \phi$.) Assume that $P[|M(A)| > \varepsilon] \to 0$ if $\mu(A) \to 0$ for any $\varepsilon > 0$. For each $A \in \mathcal{B}(E)$, let $F_A = \sigma(M(B):B \subset A) \vee N$, where N is the collection of P-null sets.

Suppose $A = A_{\{\theta_\alpha\}}$ as in Section 2. A multiple Ito integral representation of all square integrable set martingales relative to $\{F_A : A \in A\}$ will be obtained. The relevant stochastic integrals involve the Levy representative of M. Some facts about multiple Wiener integrals (which correspond to A = all Borel sets) will be proved first and then multiple Ito integrals are considered.

Our assumptions on M imply that

$$E[\exp(iuM(A))] = \exp(\mu(A)\psi(u))$$

where

$$\psi(u) = iub - \frac{1}{2} u^2 \Pi(\{0\}) + \int_{0<|\lambda|\leq 1} (e^{iu\lambda} - 1 - iu\lambda)\Pi(d\lambda) + \int_{|\lambda|>1} (e^{iu\lambda} - 1)\Pi(d\lambda)$$

for some $b \in \mathbb{R}$ and σ-finite Borel measure Π on $\mathbb{R}$ with $\int_{|\lambda|>0} \frac{|\lambda|^2}{1+|\lambda|^2} \Pi(d\lambda) < +\infty$. Furthermore, M has the representation

$$M(A) = b\mu(M) + W(\{t : (t,0) \in A\}) + \int_{A\times(\mathbb{R}-\{0\})} \lambda[q(dt,d\lambda) + 1_{|\lambda|>1} dt\Pi(d\lambda)] \quad (3.1)$$

where W is a centered Gaussian independent random measure parameterized by $\mathcal{B}(E)$ with $E[W(A)W(B)] = \Pi(\{0\})\mu(A\cap B)$ and q is a compensated σ-finite poisson point process (viewed as a random measure) on $E \times (\mathbb{R}-\{0\})$ with intensity measure $dt\Pi(d\lambda)$. W and q are independent random processes. The integral in (3.1) is improper at $A\times\{0\}$ and converges in probability.

Define an independent Borel random measure Y on $E \times \mathbb{R}$ by

$$Y(dt,d\lambda) = q(dt,d\lambda) + W(dt)\mathcal{E}(d\lambda).$$

Let $\underline{\underline{E}} = E \times \mathbb{R}$ and let $\underline{\underline{\mu}}$ denote the σ-finite measure $\underline{\underline{\mu}}(dt,d\lambda) = dt \times \Pi(d\lambda)$ on $\underline{\underline{E}}$. Hence, $E[Y(dt,d\lambda)^2] = \underline{\underline{\mu}}(dt,d\lambda)$. For functions $f_1,\ldots,f_k$ on some set S, define the tensor product $f_1 \otimes \ldots \otimes f_k$ to be the function on S^k such that

$$f_1 \otimes \ldots \otimes f_k(s_1,\ldots,s_k) = f_1(s_1)\ldots f_k(s_k).$$

The multiple Wiener integral $I_k : L^2(\underline{\underline{E}}^k,\underline{\underline{\mu}}^k) \to L^2(\Omega)$ is defined by the following three properties [5], [2]:

i) $I_1(1_{A_1} \otimes \ldots \otimes 1_{A_k}) = \prod_{i=1}^{k} Y(A_i)$ whenever $A_1,\ldots,A_k \in \mathcal{B}(E)$ are disjoint and $\underline{\underline{\mu}}(A_i) < +\infty$, $i = 1,\ldots,k$.

ii) $I_k(f+g) = I_k(f) + I_k(g)$

iii) $I_k(f_n) \to I_k(f)$ in probability if $\|f-f_n\| \to 0$

The alternative notation

$$I_k(f) = \int_{\underline{\underline{E}}^k,(s_i,\lambda_i)\text{distinct}} f(s_1,\lambda_1,\ldots,s_k,\lambda_k)Y(ds_1,d\lambda_1)\ldots Y(ds_k,d\lambda_k)$$

is suggestive.

Proposition 3.1. (Additional Properties of General Multiple Wiener Integral)

iv) (Isometric Properties) For $f \in L^2(\underline{\underline{E}}^k,\underline{\underline{\mu}}^k)$, $g \in L^2(\underline{\underline{E}}^{k'},\underline{\underline{\mu}}^{k'})$,

$$E[I_k(f)I_{k'}(g)] = 1_{\{k=k'\}}\; k!\langle \tilde{f},\tilde{g}\rangle_{L^2(\underline{\underline{E}}^k,\underline{\underline{\mu}}^k)}$$

v) (Product Decomposition) Let $f \in L^2(\underline{\underline{E}}^k,\underline{\underline{\mu}}^k)$ $g \in L^2(\underline{\underline{E}}^\ell,\underline{\underline{\mu}}^\ell)$.

Suppose f and g have totally disjoint supports in the sense that there exist $A,B \in \mathcal{B}(\underline{\underline{E}})$ with $A \cap B = \phi$ such that $f = f1_{A^k}$ and $g = g1_{B^\ell}$. Then

$$I_{k+\ell}(f \otimes g) = I_k(f)I_\ell(g) \tag{3.2}$$

vi) (Exponential Formula). Let $\alpha : \underline{\underline{E}} \to \mathbb{R}$ be Borel measurable. Define

$$f(s,\lambda) = \begin{cases} \alpha(s,0) & \text{if } \lambda = 0 \\ e^{\alpha(s,\lambda)} - 1 & \text{otherwise} \end{cases} \tag{3.3}$$

$$h(s,\lambda) = \begin{cases} \frac{1}{2}\,\alpha(s,0)^2 & \text{if } \lambda = 0 \\ e^{\alpha(s,\lambda)} - 1 - \alpha(s,\lambda) & \text{otherwise.} \end{cases} \tag{3.4}$$

Suppose that $\alpha, f \in L^2(\underline{\underline{E}},\underline{\underline{\mu}})$. Then

$$L^{(\alpha)} = \exp\left(\int_{E\times\mathbb{R}} \alpha(s,\lambda)Y(ds,d\lambda) - \int_{E\times\mathbb{R}} h(s,\lambda)ds\Pi(d\lambda)\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} I_k(f^{\otimes k}) \qquad (3.5)$$

If the condition $\alpha \in L^2(\underline{\underline{E}},\underline{\underline{\mu}})$ is removed, then

$$L^{(\alpha)} = \exp\Big(\int_{E\times\mathbb{R}} \alpha(s,\lambda)[Y(ds,d\lambda) + 1_{\{\alpha(s,\lambda)>1,\lambda\neq 0\}}ds\Pi(d\lambda)]$$

$$+ \int_{\{\alpha(s,\lambda)\leq 1,\lambda\neq 0\}} (e^{\alpha(s,\lambda)}-1-\alpha(s,\lambda))ds\Pi(d\lambda)$$

$$+ \int_{\{\alpha(s,\lambda)>1,\lambda\neq 0\}} (e^{\alpha(s,\lambda)}-1)ds\Pi(d\lambda) \qquad (3.6)$$

$$+ \frac{1}{2}\Pi(\{0\}) \int_E \alpha(s,0)^2 ds\Big)$$

is still well defined and is equal to the right side of (3.5).

vii) (Completeness of multiple Wiener integrals)

$$L^2(\Omega) = \bigoplus_{k=0}^{\infty} \{I_k(f): f \in L^2(\underline{\underline{E}},\underline{\underline{\mu}})\}$$

$$= \bigoplus_{k=0}^{\infty} L_S^2(\underline{\underline{E}}^k,\underline{\underline{\mu}}^k)$$

Proof.

(iv) follows by approximation by elementary functions. (v) is proved by approximating f and g by elementary functions f_j, g_j with totally disjoint supports.

(vi) is true if $\alpha = \alpha_1$ where $\alpha_1(s,\lambda) = 0$ if $\lambda \neq 0$, for then (3.5) and (3.6) specialize to the exponential formula (1.1) for Gaussian white noise. (vi) is also true if $\alpha = \alpha_2$ where α_2 is bounded and $\alpha_2(s,\lambda) = \alpha_2(s,\lambda)1_A$ for some $A \in \mathcal{B}(\underline{\underline{E}})$ with $A \subset E \times (\mathbb{R}-\{0\})$ and $\underline{\underline{\mu}}(A) < +\infty$.

Indeed, in this case the quantities in (3.5) and (3.6) may be interpreted as Stieltjes integrals (defined for each fixed ω) with respect to the compensated Poisson point process $Y|_A$ of finite total intensity measure. This reduces (3.5) and (3.6) to algebraic facts which may be easily proven by induction on the (a.s. finite) number of point masses of Y contained in A.

Now, for $\alpha(s,\lambda) = \alpha_1(s,\lambda) + \alpha_2(s,\lambda)$, property (v) yields

$$
\begin{aligned}
L^{(\alpha)} &= L^{(\alpha_1)} L^{(\alpha_2)} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{k!} \frac{1}{\ell!} I_k(\alpha_1^{\otimes k}) \, I(\alpha_2^{\otimes \ell}) \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{k!} \frac{1}{\ell!} I_{k+\ell}(\alpha_1^{\otimes k} \otimes \alpha_2^{\otimes \ell}) \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \left\{ \sum_{k=0}^{j} \binom{j}{k} I_j(\alpha_1^{\otimes k} \otimes \alpha_2^{\otimes (j-k)}) \right\} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} I_j((\alpha_1+\alpha_2)^{\otimes j}) = \sum_{j=0}^{\infty} \frac{1}{j!} I_j(\alpha^{\otimes j}).
\end{aligned}
$$

(3.5) and (3.6) then follow for general α by an easy approximation argument.

The completeness of the multiple Wiener integrals follows from (3.5) and the fact that random variables of the form

$$\exp\left(\int_{E\times\mathbb{R}} \alpha(s,\lambda)Y(ds,d\lambda)\right),$$

with $\alpha = \alpha_1+\alpha_2$ as in the proof of (vi), are total in $L^2(\Omega)$. ¤

<u>Remark</u>. Proposition 3.1 and its proof easily generalize to the case when E is an arbitrary separable measure space with σ-finite, non-atomic measure μ. Many properties of multiple Wiener integrals follow from the exponential formula (3.3), which we have not seen elsewhere.

The multiple Ito integral with respect to Y will now be defined.

Let $A = A_{\{\theta_\alpha\}}$ be a collection of subsets of $E = \mathbb{R}^n$ as in Section 2.

Let, $\underline{\underline{A}} = A \times \{\mathbb{R}\} = \{A \times \mathbb{R} : A \in A\}$, and $\underline{\underline{F}}_A = \sigma(Y(B) : B \in \mathcal{B}(E),\ \underline{\underline{\mu}}(B) < +\infty,$

$B \subset A)$ for $A \subset \mathcal{B}(\underline{\underline{E}})$. Note that $F_A = \underline{\underline{F}}_{A \times \mathbb{R}}$ for $A \in \mathcal{B}(E)$. Hence, under the correspondence $A \leftrightarrow A \times \mathbb{R}$, set martingales relative to $\{F_A : A \in A\}$ may be identified with set martingales relative to $\{\underline{\underline{F}}_A : A \in \underline{\underline{A}}\}$. Define $R_{T_1,\ldots T_k}$, $R_{s_1,\ldots s_k}$, "unordered," and $\hat{D}$ as in

Section III.2. These definitions are relative to A or $\underline{\underline{A}}$.

Fix a positive integer k. A function defined on $\hat{\underline{\underline{E}}}^k \times \Omega$ is elementary if it is a finite linear combination of functions of the form

$$Z1_{A_1 \times \ldots \times A_k} \tag{3.7}$$

where $A_1,\ldots,A_k \subset \underline{\underline{E}}$ are unordered, bounded rectangles with $\underline{\underline{\mu}}(A_i) < +\infty$ and $A_i \cap R_{A_1,\ldots,A_k} = \phi$ for $i = 1,\ldots,k$, and Z is a bounded, $F_{R_{A_1,\ldots,A_k}}$ measurable random variable. Let $\mathcal{P}$ be the σ-algebra of subsets of $\hat{\underline{\underline{E}}}^k \times \Omega$ generated by the elementary functions, and define $L^2_k(Y)$ $= L^2(\hat{\underline{\underline{E}}}^k \times \Omega, \mathcal{P}, \underline{\underline{\mu}}^k \times P)$ to be the Hilbert space of $\mathcal{P}$-measurable, $\mu^k \times P$ square integrable functions on $\hat{\underline{\underline{E}}}^k \times \Omega$. By partitioning rectangles into unions of smaller rectangles and adding like terms, it can always be assumed that there is at most one non-zero term in the finite sum of random variables defining an elementary function at each point of $\hat{E}^k$. This fact makes it clear that the collection of elementary functions is an algebra closed under pointwise minimums and maximums. By the Stone-Weierstrass theorem and a monotone class argument, it follows that the collection of elementary functions is dense in $L^2_k(Y)$.

If $\theta \in L_k^2(Y)$ is of the form (3.7), define the multiple Ito integral $\theta \circ Y_A^k$ for $A \in \underline{\underline{A}}$ by

$$\theta \circ Y_A^k = Z(\omega) \prod_{i=1}^{k} Y(A \cap A_i) \tag{3.8}$$

Extend the definition to elementary functions $f \in L_k^2(Y)$ by linearity. For elementary f,g, $(f+g)\circ Y_A^k = f\circ Y_A^k + g\circ Y_A^k$, and

$$E[(f\circ Y_A^k)^2]^{1/2} = \|\tilde{f}\| \leq \|f\|.$$

Also, $f\circ Y_A$ is a set parameter martingale relative to $\underline{\underline{A}}$ and $\{\underline{\underline{F}}_A : A \in \underline{\underline{A}}\}$. By invoking the requirement that $f_i\circ Y_A^k \to f\circ Y_A^k$ in $L^2(\Omega)$ whenever $\|f_i - f\| \to 0$ for $f_i, f \in L_k^2(Y)$ the multiple Ito integral $I_k(f)$ is defined for all $f \in L_k^2(Y)$.

If p is finite, there exists a version of the multiple Ito integral with nice sample path properties. Note that $\underline{\underline{A}} = A_{\{\theta_\alpha \times \{0\}\}}$ and the metrics on A and $\underline{\underline{A}}$ defined by (2.2) agree under the correspondence $A \leftrightarrow A \times \mathbb{R}$. A function $\eta : A \to \mathbb{R}$ (or equivalently $\eta : \underline{\underline{A}} \to \mathbb{R}$) is outer continuous (or continuous from the outside) if $\lim_{\substack{B\to A\\ B\supseteq A}} \eta_B = \eta_A$. η has inner limits (or has limits from the inside) if $\lim_{\substack{B\to A\\ B\subseteq A}} \eta_A$ exists for each $A \in A$.

Let $B \in \mathcal{B}(\underline{\underline{E}})$ be a bounded subset of E with $\mu(B) < +\infty$.

<u>Lemma 3.2</u>. There is a modification of $\{Y(A\cap B) : A \in \underline{\underline{A}}\}$ that is outer continuous and has inner limits with probability one.

<u>Proof</u>. Let $G_i = \{(t,\lambda) \in \underline{\underline{E}} = E \times \mathbb{R} : |t| \leq i,\ (|\lambda| > \frac{1}{i} \text{ or } \lambda = 0)\}$ Then $\underline{\underline{\mu}}(G_i) < +\infty$ and $\bigcup_{i=1}^{\infty} G_i = \underline{\underline{E}}$. Let $Y^i(A) = Y(A\cap G_i \cap B)$. For each i, Y^i is the sum of an independent Gaussian measure and a compensated Poisson point process of finite total intensity, so we may choose an outer

continuous with inner limits modification of $Y^i|_{\underline{\underline{A}}}$ (Use Lemma 2.1 for the Gaussian part.) $\{Y(A\cap B):A \in \underline{\underline{A}}\}$ and Y^i are set martingales relative to $\{F_A:A \in A\}$, and

$$E[(Y^i(A)-Y(A\cap B))^2] = \underline{\underline{\mu}}(A\cup G_i) \to 0 \quad \text{as } i \to +\infty$$

for each $A \in \underline{\underline{A}}$. Hence, by Lemma 2.2 and a diagonal subsequence argument, there is a subsequence $i_1, i_2, \ldots$ of positive integers such that, for each i, $Y^{i_k}(A)$ converges, a.s. uniformly for $A \subset G_i$, to a modification of $\{Y(A\cap B):A \in \underline{\underline{A}}\}$. This provides the desired modification of $\{Y(A\cap B):A \in \underline{\underline{A}}\}$. ∎

<u>Proposition 3.3</u>. Let $p < +\infty$. For $f \in L^2_k(Y)$, there exists a modification of the multiple Ito integral $f\circ Y^k$ such that $f\circ Y^k|_{\underline{\underline{A}}}$ is outer continuous with left limits.

<u>Proof</u>. If f is an elementary function, then the multiple Ito integral $f\circ Y^k$ is outer continuous with left limits by Proposition 3.2. In general, a modification of $\{f\circ Y^k_A:A \in \underline{\underline{A}}\}$ is the a.s. uniform limit of the outer continuous inner limited multiple Ito integrals of a sequence of elementary functions by Proposition 2.1. This modification of $f\circ Y^k$ is outer continuous and has inner limits. ∎

For $f \in L_k(Y)$ and any $B \in B(\underline{\underline{E}})$, define $f\circ Y^k_B$ by $f\circ Y^k_B = (f1_{B^k})\circ Y^k_E$. This agrees with the previous definition if $B \in \underline{\underline{A}}$ as is easily proved for elementary functions and then for general $f \in L^2_k(Y)$ by approximation.

<u>Theorem 3.4</u>. (Properties of Multiple Ito Integral — General Independent Noise)

a) For $f, f' \in L^2_k(Y)$, $g \in L^2_{k'}(Y)$ and $B \in B(\underline{\underline{E}})$,

$$(f+f')\circ Y^k_B = f\circ Y^k_B + f'\circ Y^k_B$$

$$E[(f\circ Y^k_B)(g\circ Y^{k'}_B)] = 1_{\{k=k'\}}\langle \tilde{f}1_{B^k}, \tilde{g}\rangle$$

$$f\circ Y^k_B = f'\circ Y^k_B \text{ a.s.} \Leftrightarrow \tilde{f}1_{B^k} = \tilde{f}'1_{B^k} \text{ a.e. } \underline{\underline{\mu}} \times P.$$

b) (Projection Property) For $A, B \in \mathcal{B}(E)$ and $f \in L_k^2(Y)$,

$$E[f \circ Y_A^k | \underline{\underline{F}}_B] = (E[f|F_B]) \circ W_{A \cap B}^k \tag{3.9}$$

(In (3.9), $E[f|F_B](s,\cdot) = E[f(s,\cdot)|F_B]$ a.e. for each $s \in \underline{\underline{E}}^k$ and a version $E[f|F_B] \in L_k^2(Y)$ is chosen)

c) (Elementary Exponential Representation) Suppose $p < +\infty$. Let $\alpha \in L^2(\underline{\underline{E}}, \underline{\underline{\mu}})$ and define f,h by (3.3), (3.4). Suppose $\alpha, f \in L^2(\underline{\underline{E}}, \underline{\underline{\mu}})$. Then for each $A \in \mathcal{B}(E)$,

$$L_A^{(\alpha)} \triangleq \exp\left(\int_A \alpha(t,\lambda) Y(dt,d\lambda) - \int_A h(t,\lambda) \underline{\underline{\mu}}(dt,d\lambda)\right)$$

$$= 1 + \sum_{k=1}^{p} \frac{1}{k!} \left(L_{R_{s_1,\ldots,s_k}}^{(\alpha)} f^{\otimes k}\right) \circ Y_A^k$$

d) (Relation to Multiple Wiener Integral) Let $h \in L^2(\underline{\underline{E}}^m, \underline{\underline{\mu}})$. Then the multiple Wiener integral $I_m(h)$ has the representation

$$I_m(h) = E[I_m(h)] + \sum_{k=1}^{\min(m,p)} h_k \circ Y_E^k$$

where $h_k \in L_k^2(Y)$ for $k \leq \min(m,p)$ satisfies

$$h_k(s_1,\ldots,s_k,\omega) = I_{m-k}(\tilde{h}(s_1,\ldots,s_k,\cdot) 1_{\widehat{R^{m-k}_{s_1,\ldots,s_k}}}(\cdot))(\omega).$$

for a.e. ω and $s_i = (t_i,\lambda_i) \in \underline{\underline{E}}$ for $i = 1,\ldots k$.

e) (Completeness or Martingale Representation) Every square integrable set martingale N (relative to $\underline{\underline{A}}$) has a (outer continuous with inner limits, if $p < +\infty$) modification with the representation

$$N_A = \sum_{k=0}^{p} \alpha_k \circ Y_A^k, \quad A \in \underline{\underline{A}}$$

The sum converges in $L^2(\Omega)$ in case p is infinite.

Remark. The proof of Theorem 2.4 easily extends to prove Theorem 3.4. The properties a), b), d) and e) of Theorem 3.4 and their proofs are very much independent of what type of independent random measure Y is. This is only true for e) when given the fact that multiple Wiener integrals with respect to Y span $L^2(\Omega)$.

APPENDIX

The notation of Section 2 will be used in this appendix. The purpose of this appendix is to prove the following proposition.

<u>Proposition A.1</u>. The class of elementary functions is dense in $L_a^2(\hat{E}^k \times \Omega)$.

<u>Lemma A.2</u>. The class of sample-continuous, adapted functions on $\hat{E}^k \times \Omega$ is dense in $L_a^2(\hat{E}^k \times \Omega)$.

<u>Proof of Lemma</u>. Define an open set $\subset \hat{E}^k$ by

$$G = \{(t_1,\ldots,t_k) \in \hat{E}^k : R_{t_1,\ldots,t_k} \text{ has non-empty interior}\}.$$

Suppose that $f \in L_a^2(\hat{E}^k \times \Omega)$. Then $f = f_1 + f_2$ where $f_1 = f1_G$ and $f_2 = f1_{G^c}$. It will be shown that f_1 and f_2 (and hence f) may each be approximated with arbitrary precision in $L_a^2(\hat{E}^k \times \Omega)$ by sample-continuous adapted functions.

It suffices to consider the case when f_1 is supported by an open set G_0 with compact closure in G for a.e. ω. For $\varepsilon > 0$, define

$$f_1^\varepsilon(s_1,\ldots,s_k,\omega) = \begin{cases} \dfrac{1}{\mu^k(\Lambda^\varepsilon(s_1,\ldots,s_k))} \displaystyle\int_{\Lambda^\varepsilon(s_1,\ldots,s_k)} f_1(r_1,\ldots,r_k)dr_1\ldots dr_k \\ \qquad\qquad \text{if } (s_1,\ldots,s_k) \in G_0 \\ 0 \qquad \text{otherwise} \end{cases} \tag{A.1}$$

where, for $(s_1,\ldots,s_k) \in G^0$,

$$\Lambda^\varepsilon(s_1,\ldots,s_k) = \{(t_1,\ldots,t_k) \in \hat{E}^k : t_i \in R_{s_1,\ldots,s_k}$$
$$\text{and } |t_i - s_i| \leq \varepsilon(1+\text{diam}\{s_1,\ldots,s_k\})\ \forall i\}$$

Then f_1^ε is adapted and sample-continuous on G_0.

Claim: f_1^{ε} converges to f_1 in $L_a^2(\hat{E}^k \times \Omega)$ as $\varepsilon \to 0$. To prove the claim, first note that f_1 can be well-approximated in $L^2(\hat{E}^k \times \Omega)$ by (not necessarily adapted) functions $g(s_1,\ldots,s_k,\omega)$ which are bounded, continuous and have support in G_0 for each fixed ω. (By an easy monotone class argument.) For such g, if g^{ε} is defined by the right side of (A.1) with f_1 replaced by g, then g^{ε} converges to g pointwise and hence in $L^2(\hat{E}^k \times \Omega)$ by Lebesgues' bounded convergence theorem. By Jensen's inequality and Fubini's lemma,

$$\|g^{\varepsilon}-f^{\varepsilon}\|^2 = E[\int_{G_0} (\frac{1}{\mu^k(\Lambda^{\varepsilon}(s_1,\ldots,s_k)} \int_{\Lambda^{\varepsilon}(s_1,\ldots,s_k)} f_1(r_1,\ldots,r_k) - g(r_1,\ldots,r_k)dr_1\ldots dr_k)^2 ds_1\ldots ds_k]$$

$$\leq E[\int_{G_0} \frac{1}{\mu^k(\Lambda^{\varepsilon}(s_1,\ldots,s_k)} \int_{\Lambda^{\varepsilon}(s_1,\ldots,s_k)} f_1(r_1,\ldots,r_k) - g(r_1,\ldots,r_k))^2 dr_1\ldots dr_k ds_1\ldots ds_k]$$

$$= E[\int_{G_0} (f_1(s_1,\ldots,s_k)-g(s_1,\ldots,s_k))^2 S^{\varepsilon}(s_1,\ldots,s_k)ds_1\ldots ds_k] \tag{A.2}$$

where

$$S^{\varepsilon}(s_1,\ldots,s_k) = \int_{G_0} \frac{1}{\mu^k(\Lambda^{\varepsilon}(r_1,\ldots,r_k))} 1_{\{\Lambda^{\varepsilon}(r_1,\ldots,r_k)\supset(s_1,\ldots,s_k)\}} dr_1\ldots dr_k.$$

It is not hard to see that $S^{\varepsilon}(s_1,\ldots,s_k)$ is locally bounded on $G_0 \times [0,\varepsilon_0]$ for some $\varepsilon_0 > 0$, so that $S^{\varepsilon}(s_1,\ldots,s_k) \leq K$ for all $s_1,\ldots,s_k$ and $\varepsilon \leq \varepsilon_0$ by the compactness of $\bar{G}_0$. Thus by (A.2), $\|g -f \| \leq K\|g-f\|$. Therefore

$$\|f-f^{\varepsilon}\| \leq \|f-g\| + \|g-g^{\varepsilon}\| + \|g^{\varepsilon}-f^{\varepsilon}\| \leq (1+K)\ \|f-g\| + \|g-g^{\varepsilon}\|$$

for $\varepsilon \leq \varepsilon_0$. Since $\|f-g\|$ and $\|g-g^{\varepsilon}\|$ can be made arbitrarily small, the claim is proven.

The functions f^{ε} are adapted and continuous on the open set G_0. The functions f^{ε}, and so also f, can therefore be well-approximated in $L_a^2(\hat{E}^k \times \Omega)$ by sample continuous functions on $\hat{E}^k$ of the form uf^{ε} where u is a continuous (deterministic) function on $\hat{E}^k$, $0 \leq u \leq 1$, and $u = 0$ on G_0^c.

It remains to show that f_2 may be well-approximated by sample continuous functions in $L_a^2(\hat{E}^k \times \Omega)$. Now, for $s_1,\ldots,s_k \in \hat{G}^c$, $R_{s_1,\ldots,s_k}$ has zero Lebesgue measure so that $F(R_{s_1,\ldots m s_k}) = N$, the collection of P-null sets. Thus, $f_2(s_1,\ldots,s_k,\omega) = g(s_1,\ldots,s_k)$ a.e. where $g \in L^2(\hat{E}^k)$ is defined by $g(s_1,\ldots,s_k) = E[f_2(s_1,\ldots,s_k)]$. By a monotone class argument, there is a sequence of continuous functions on E^k converging to g in $L^2(\hat{E}^k)$. Since deterministic functions are always adapted, the same sequence converges to f_2 in $L_a^2(\hat{E}^k \times \Omega)$. ¤

<u>Proof of Proposition</u>. Suppose that f is a bounded, sample-continuous, adapted function on $\hat{E}^k \times \Omega$. Assume that the support of f is contained in a fixed compact subset of $\hat{E}^k$ for each ω. By Lemma A.1 it suffices to prove that f may be approximated in $L_a^2(\hat{E}^k \times \Omega)$ by elementary functions.

Let m be a positive integer. For each n-tuple $i = (i^{(1)},\ldots,i^{(n)})$ of integers, let Δ_i denote the rectangle in $E = \mathbb{R}^n$ defined by

$$\Delta_i = (\frac{i^{(1)}-1}{m}, \frac{i^{(1)}}{m}] \times \ldots \times (\frac{i^{(n)}-1}{m}, \frac{i^{(n)}}{m}]$$

For each unordered collection $\Delta_{i_1},\ldots,\Delta_{i_k}$ of k such rectangles, choose a k-tuple

$$(s_1^{i_1,\ldots,i_k},\ldots,s_k^{i_1,\ldots,i_k}) \in R_{\Delta_{i_1},\ldots,\Delta_{i_k}} \cap (\bar{\Delta}_{i_1}\times\ldots\times\bar{\Delta}_{i_k})$$

Define

$$h_m(s_1,\ldots,s_k) = \begin{cases} f(s_1^{i_1,\ldots,i_k},\ldots,s_k^{i_1,\ldots,i_k}) & \text{if } (s_1,\ldots,s_k) \in \Delta_{i_1}\times\ldots\times\Delta_{i_k} \\ & \text{and } \Delta_{i_1},\ldots,\Delta_{i_k} \text{ are unordered for} \\ & \text{some } i_1,\ldots,i_k \\ 0 & \text{otherwise} \end{cases}$$

Then h_m is an elementary function for each m and h_m converges to f pointwise as $m \to \infty$. Furthermore, h_m is uniformly bounded and has support contained in a bounded subset of E^k, independently of m. Hence, $h_m \to f$ in $L_a^2(E^k\times\Omega)$ by Lebesgue's bounded convergence theorem. ∎

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