ASYMPTOTIC UNBOUNDED ROOT LOCI BY

THE SINGULAR VALUE DECOMPOSITION

by

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ABSTRACT

In this paper the asymptotic behavior of the closed-loop eigenvalues (root loci) of a strictly proper linear time-invariant control system as the loop gain goes to ∞ is studied. Basic properties of the singular value decomposition are introduced and then used to obtain formulae for the asymptotic values, as the loop gain goes to ∞, of the unbounded (with loop gain) root loci. The geometric interpretation of these formulae is developed and a numerically sound way of computing them is proposed. Perturbation techniques are used under mild assumptions to obtain the complete asymptotes of the unbounded root loci. Using these calculations necessary and sufficient conditions for the closed loop exponential stability of a strictly proper linear time-invariant system under arbitrarily high feedback gain are derived. This is the generalization to multi-input, multi-output of a well-known result for single input, single output systems.

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Section I. Introduction

It is well known that high loop gain in a feedback control system enhances the desirable effects of feedback, for instance, desensitization and disturbance attenuation, etc. see, for example, Desoer and Wang [13]. It is also well-known that practical control systems are driven to instability by high gain feedback. With this design philosophy in mind this paper presents a new geometric way of comprehending and a numerically stable way of computing the asymptotic behavior of unbounded root loci of a strictly proper, linear, time-invariant feedback control system shown in Figure 1 as the loop gain $\rightarrow \infty$ ($k \rightarrow \infty$).

The asymptotic behavior of unbounded root loci has been studied by Kouvaritakis and Shaked [1], Kouvaritakis [2], Kouvaritakis and Edmunds [8] and Owens [3,4,5] but the geometric interpretation of their results is not clear. Also, their assumptions are not explicit (see for e.g. [5]) and there are some formulae for the asymptotes in [3] and in [8] which we do not understand.

The present paper recognizes that the computation of the asymptotic values of the unbounded root loci is essentially a process of identifying subspaces in the input space ($\mathbb{R}^m$) and the output space ($\mathbb{R}^m$) where the effects of the $0(k)$ (order (k)), $0(\sqrt{k})$, $0(\sqrt[3]{k})$,... unbounded root loci dominate asymptotically. This suggests naturally the use of the numerically stable tool — the singular value decomposition (see for instance Golub and Reinsch [6] and Stewart [7]) for the computation. We then use standard perturbation calculations formalized for instance in Dieudonné [10] and Kato [11] to compute the asymptotes of the unbounded root loci.

As illustrations of the power of the clear geometric picture of the root locus behavior we present two applications of the calculation: (i) State feedback invariance of the asymptotic values of the root loci in the instance that $G(s)$ has a minimal realization (A,B,C).
(ii) A necessary and sufficient condition for the stability of the feedback control system of Figure 1 for arbitrarily high gain $k \geq k_o$. This is the multivariable generalization of a well known result for single input, single output systems (see, for instance, Ogata [14]). More applications are clearly possible and one of them, namely a formula for establishing the degree of a minimal realization of a rational transfer function $G(s)$ is stated without proof in the Conclusions (Sec. VIII). The organization of the paper is as follows.

<u>Section I</u>. The present Introduction.

<u>Section II</u>. Mathematical Preliminaries.

<u>Section III</u>. System Description and Assumptions.

<u>Section IV</u>. Summary of Results.

<u>Section V</u>. Asymptotic Root Loci-Asymptotic Values.

<u>Section VI</u>. Calculations of the Asymptotes of the Unbounded Root Loci.

<u>Section VII</u>. A necessary and sufficient condition for the closed loop exponential stability of a strictly proper linear time invariant system under arbitrarily high gain.

<u>Section VIII</u>. Conclusions.

<u>Appendix</u>. Proofs.

Section II. Mathematical Preliminaries

In this section we state some results and some propositions which we will use repeatedly in the paper.

II.1. The Singular Value Decomposition (S.V.D.) (see for e.g. [6])

A matrix $A \in \mathbb{C}^{m\times m}$ may be decomposed as

$$A = [U_1 \,\vdots\, U_2] \left[\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^* \\ \hline V_2^* \end{array}\right] \tag{II.1}$$

where $U = [U_1 \vdots U_2] \in \mathbb{C}^{m\times m}$ with $U_1 \in \mathbb{C}^{m\times r}$; $U_2 \in \mathbb{C}^{m\times(m-r)}$ and $V = [V_1 \vdots V_2] \in \mathbb{C}^{m\times r}$; $V_2 \in \mathbb{C}^{m\times(m-r)}$ are unitary matrices and $\Sigma_1 \in \mathbb{R}_+^{r\times r}$ is a matrix of positive real numbers and r is the rank of the matrix A. ¤

The columns of V_1, U_1 represents orthonormal bases for the range spaces of A*, A respectively and the columns of V_2, U_2 represent orthonormal basis for the null spaces of A, A* respectively. [$A^* \in \mathbb{C}^{m\times m}$ stands for the conjugate transpose of A]. The structure of the linear map A may be viewed as in Figure 1.

II.2. Restriction of a Linear Map in Domain and Range.

Definition 1. Given a linear map A from $\mathbb{C}^m$ to $\mathbb{C}^m$ and two subspaces S_1, S_2 of $\mathbb{C}^m$ define the restriction of A to S_1 in the domain and S_2 in the range to be the linear map which associates with $x \in S_1 \subset \mathbb{C}^m$ the orthogonal projection of Ax onto $S_2 \subset \mathbb{C}^m$. The notation for the restricted linear map is $A|_{S_1 \to S_2}$.

Comment. If S_1 and S_2 both have dimension m_1, $A|_{S_1\to S_2}$ which is a linear map from $S_1 \subset \mathbb{C}^m$ to $S_2 \subset \mathbb{C}^m$ can be represented by a matrix $\tilde{A} \in \mathbb{C}^{m_1\times m_1}$ which gives a description of the action of $A|_{S_1\to S_2}$ once suitable bases have been found for S_1 and S_2. This is shown in Figure 2. In the instance that the bases are orthonormal a particularly simple formula for $\tilde{A}$ accrues. This is proved in proposition II.1.

<u>Proposition II.1</u>. (Matrix Representation of $A_{S_1 \to S_2}$)

Let the columns of $P_1 \in \mathbb{C}^{m \times m_1}$ form an orthonormal basis $\mathcal{B}_1$ for a subspace $S_1 \subset \mathbb{C}^m$ and the columns of $P_2 \in \mathbb{C}^{m \times m_1}$ form an orthonormal basis $\mathcal{B}_2$ for a subspace $S_2 \subset C^m$ and the map $A : C^m \to C^m$ be linear. Then the matrix representation of $A|_{S_1 \to S_2}$ with respect to the basis $\mathcal{B}_1$ in the domain and $\mathcal{B}_2$ in the range is $P_2^* A P_1 \in \mathbb{C}^{m_1 \times m_1}$. ¤

II.3. <u>Eigenvalues of the Restriction of a Linear Map in Domain and Range</u>

<u>Definition 2</u>. If S_1, S_2 are subspaces of $\mathbb{C}^m$ of dimension m_1; $\lambda \in \mathbb{C}$ is an eigenvalue of $A|_{S_1 \to S_2}$ if $\exists x \in \mathbb{C}^m$ such that $A|_{S_1 \to S_2} x = \lambda I|_{S_1 \to S_2} x$, ($I|_{S_1 \to S_2}$ denotes the restriction of the identity map to S_1 in the domain and S_2 in the range).

<u>Proposition II.2</u>. [Polynomial equation for the eigenvalues of $A|_{S_1 \to S_2}$].

Let the notation of Proposition II.1 stand. Then, the eigenvalues of $A|_{S_1 \to S_2}$ are the zeros of the polynomial

$$\det[\lambda P_2^* P_1 - P_2^* A P_1] = 0 \tag{II.2}$$

¤

<u>Comment</u>. If $P_2^* P_1$ is of rank m_1 [i.e. there is no vector in S_1 which is orthogonal to S_2] then the polynomial of (II.2) has leading term λ^{m_1} so that there are m_1 non-zero values of λ satisfying (II.2).

We define what we mean by simple null structure of a map $A|_{S_1 \to S_2}$. Unlike the previous two definitions which were basis free we use orthonornal bases for this definition.

Definition 3. With the notation of Proposition II.1; $A|_{S_1\to S_2}$ is said to have simple null structure if $\nexists$ non zero $x \in \mathbb{C}^{m_1}$ such that

$$P_2^* A P_1 x \neq \theta_{m_1} \text{ but } (P_2^* A P_1)^2 x = \theta_{m_1}.$$

Comments. (1) The condition above means that there are no generalized eigenvectors associated with $\lambda = 0$ for the map $A|_{S_1\to S_2}$.

(2) If $A|_{S_1\to S_2}$ does not have simple null structure, then, considering the Jordan form of the matrix representation $P_2^* A P_1$ we see that there are arbitrarily small perturbations of A which will give to the perturbed A a simple null structure.

We use the definition of simple null structure to equate the number of non zero eigenvalues of $A|_{S_1\to S_2}$ to the rank of $P_2^* A P_1$ when $P_2^* P_1$ has rank m_1.

Proposition II.3. (Number of non zero eigenvalues of $A|_{S_1\to S_2}$).

Let the notation of proposition (II.1) hold. Assume that $P_2^* P_1 \in \mathbb{C}^{m_1 \times m_1}$ is non singular and that $P_2^* A P_1 \in \mathbb{C}^{m_1 \times m_1}$ has rank r. Then, if $A|_{S_1\to S_2}$ has simple null structure the number of non zero eigenvalues of $A|_{S_1\to S_2}$ is r. ¤

A similar definition can be made for simple structure associated with an eigenvalue of $A|_{S_1\to S_2}$.

Definition 4. With the notation of Proposition II.1, $A|_{S_1\to S_2}$ is said to have simple structure associated with $\lambda \in \mathbb{C}$, an eigenvalue of $A|_{S_1\to S_2}$, if the map $(\lambda I - A)|_{S_1\to S_2}$ has simple null structure.

Comment. With the notation of Proposition II.1; $A|_{S_1\to S_2}$ is said to have simple structure associated with an eigenvalue λ if $\nexists$ nonzero $x \in \mathbb{C}^{m_1}$ such that $(\lambda P_2^* P_1 - P_2^* A P_1)x \neq \theta_{m_1}$ but $(\lambda P_2^* P_1 - P_2^* A P_1)^2 x = \theta_{m_1}$.

II.4. Adjoint of the Restriction of a Linear Map in Domain and Range

Definition 5. With the notation of Definition 1, the adjoint of $\hat{A} := A|_{S_1 \to S_2}$ is defined to be the linear map $A^*|_{S_2 \to S_1} : \mathbb{C}^m \to \mathbb{C}^m$ and is denoted by $\hat{A}^*$.

The following proposition is obvious.

Proposition II.4 [Matrix representation of $\hat{A}^*$.]

With the notation of Proposition II.1 the matrix representation of $\hat{A}^*$ with respect to basis $\mathcal{B}_2$ in the domain and $\mathcal{B}_1$ in the range is $P_1^* A^* P_2 \in \mathbb{C}^{m_1 \times m_1}$. ¤

Section III. System Description and Assumptions.

The system under study is the system of Figure 3 where G(s) is the m×m transfer function matrix of a linear time-invariant, strictly proper control system with Taylor expansion about $s = \infty$ given by (III.1).

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \frac{G_3}{s^3} + \ldots \qquad \forall |s| > M \tag{III.1}$$

with $G_1, G_2, \ldots \in \mathbb{R}^{m \times m}$; k is real and positive. For instance G(s) can be a strictly proper rational transfer function matrix i.e. $G(s) \in \mathbb{R}(s)^{m \times m}$. If G(s) is the transfer function matrix of a system with state space representation (A,B,C) then the Neumann series for G(s) converges $\forall |s| > |\lambda_{max}(A)|$, the spectral radius of A, and we have

$$G(s) = \frac{CB}{s} + \frac{CAB}{s^2} + \ldots \quad \forall |s| > |\lambda_{max}(A)| \tag{III.2}$$

$G_1 = CB$, $G_2 = CAB$, etc. are referred to in the literature as the Markov parameters of the system [1].

We study the closed loop poles of the system of Figure 3 as $k \to \infty$. The motivation for this is that G(s) represents the composition of a linear time invariant plant and a linear time-invariant controller and $k \to \infty$ represents high gain feedback [with gains tending to ∞] in all control channels. The curves traced by closed-loop eigenvalues as a function of k are referred to as the multivariable root loci. As $k \to \infty$ some of the root loci tend to finite points in the complex plane located at the (McMillan) zeros of the system [see for e.g., 1,2], the others go to ∞ as $k \to \infty$ and are referred to as the unbounded root loci of the system. We classify the unbounded root loci by the velocity (with k) with which they tend to ∞.

Definition. An unbounded multivariable root locus $s_n(k)$ is said to be an n^{th} order unbounded root locus ($n=1,2,3,\ldots$) if asymptotically

$$s_n(k) = \mu_n(k)^{1/n} + O(k^0) \tag{III.3}$$

where $|\mu_n| < \infty$ and $O(k^0)$ is a term of order k^0.

We now state the assumptions and the results of the paper.

Assumption 0. (Non triviality assumption).

Let $\hat{G}_2 := G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$; $\hat{G}_3 := G_3\big|_{\mathcal{N}(G_2)\to\mathcal{N}(\hat{G}_2^*)}$; and so on.

Assume; either

a) Output nontriviality

$$\mathbb{R}^m = \mathcal{R}(G_1) \overset{\perp}{\oplus} \mathcal{R}(\hat{G}_2) \overset{\perp}{\oplus} \mathcal{R}(\hat{G}_3) + \cdots \tag{III.4}$$

or

b) Input nontriviality

$$\mathbb{R}^m = \mathcal{R}(G_1^*) \overset{\perp}{\oplus} \mathcal{R}(\hat{G}_2^*) \overset{\perp}{\oplus} \mathcal{R}(\hat{G}_3^*) + \cdots \tag{III.5}$$

Comments. (1) (III.4) guarantees that no output (or linear combination thereof) is trivial i.e., identically zero and (III.5) guarantees that no input (or linear combination thereof) is trivial i.e., the output is independent of the value of that particular linear combination of inputs.

For an example of a system violating this assumption, consider

Example 1

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad G_k = 0 \quad k \geq 3$$

The block diagram for the system is shown in Figure 4. Note that $y_2 = y_1 + y_3$ and that u_1 is useless. Also note that $\mathcal{R}(G_1) + \mathcal{R}(\hat{G}_2)$ $= \mathrm{sp}\left[\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right] \subsetneqq \mathbb{R}^3$.

(2) Only output nontriviality or input nontriviality needs to be assumed since one implies the other.

Assumption 1. (Simple Null Structure Assumption).

$$G_1, \hat{G}_2 := G_2\Big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}, \quad \hat{G}_3 := G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}, \text{ and so on}$$

have simple null structure.

Comment: (1) The assumption 1 of simple null structure is generic (i.e., given arbitrary matrices $G_1, G_2.G_3, \ldots \in \mathbb{R}^{m\times m}$ the assumption is satisfied almost surely). However, it is more than just a technical condition required for the asymptotic calculations. We will at this point give an example of systems that violate our simple null structure assumption to illustrate that the assumption makes engineering sense.

Example 2.

$$G_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad G_k = 0 \qquad k \geq 3$$

Notice that G_1 does not have simple null structure. Figure 5a is a block diagram of the system (for large k). Notice, that a more desirable control system is formed by relabelling the outputs of G(s) before closing the feedback loops as is shown in Figure 5b. (It is obvious that the wrong output was being used for control in Figure 5a). Notice that the unbounded root loci of Figure 5a are of the $O(k^{2/3})$ so that they are not of the 1st, 2nd,...,nth order defined above and those of Figure (5b) are of the 1st and 2nd order.

Assumption 2. [Non Redundancy Assumption].

$$\mathcal{N}(G_{k-1}) \neq \phi \text{ and } G_k \neq 0 \Rightarrow \hat{G}_k := G_k\Big|_{\mathcal{N}(\hat{G}_{k-1})\to\mathcal{N}(\hat{G}^*_{k-1})} \neq 0$$

$$\text{for } k = 1,2,3,\ldots$$

Comment. The non-redundancy assumption is clearly generic; also it makes engineering sense in that it rules out redundant use of integrators in design as is illustrated in the example below (due to Owens [5]).

Example 3.

$$G_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that $\mathcal{N}(G_1) \neq 0$ and $\hat{G}_2 := G_2\Big|_{\mathcal{N}(\hat{G}_1)\to\mathcal{N}(\hat{G}^*_1)} = 0$. A block diagram of the system for large k is shown in Figure 6a. We notice that the integrators associated with G_2 are redundant for large k; since they are dominated by the single integrator in G_1. Figure 6b shows the system with G_2 set to 0. There are two non-interacting control loop with 1st and 3rd order unbounded root locus behavior.

Assumption 3. (Simple Structure Assumption).

$$G_1, \hat{G}_2 := G_2\Big|_{\mathcal{N}(G_1)\to\mathcal{N}(G^*_1)}, \quad \hat{G}_3 := G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}^*_2)}, \text{ etc.}$$

have simple structure associated with all of their eigenvalues.

Comment: The simple structure assumption is needed to make asymptotic estimates of the $O(k^0)$ terms in equation (III.3) for the nth order unbounded root loci (n=1,2,3,...). It is not needed for the calculation of the asymptotic values of the unbounded root loci. This assumption is purely technical. Note that assumption 3 includes assumption 1.

Section IV. Summary of Results

Under Assumptions 0, 1 and 2, the simple null structure and non-redundancy assumptions of Section III we prove in Section V that the only unbounded root loci of the system of Figure 3 are the 1st,...,nth order unbounded root loci and asymptotic formulae for them are given by

$$s_n(k) = \sqrt[n]{-\lambda_{i,n} k} + c_{i,n} + O(k^{1/n}) \tag{IV.1}$$

where $\lambda_{i,n}$ is the eigenvalue of $\hat{G}_n := G_n \Big|_{\mathcal{N}(\hat{G}_{n-1}) \to \mathcal{N}(\hat{G}^*_{n-1})}$ for $n = 1,2,3,\ldots$

Under the simple structure assumption 3 of Section III we give formulae for the $c_{i,n}$ in Section VI. Means of computing the $\lambda_{i,n}$ and $c_{i,n}$ are also described.

Using these calculations a necessary and sufficient condition for the exponential stability of the system of Figure 3 for arbitrarily large k is derived in Section VII.

Section V. Unbounded Root Loci-Asymptotic Values

In this section we establish formulae and methods for computing the values of the unbounded root loci as $k \to \infty$.

For the values of the finite root loci at $k = \infty$ the following procedure is well-known (see for e.g., [8]): The closed loop transfer function of the system shown in Fig. 3 is $kG(s)[I+kG(s)]^{-1}$. A right coprime factorization of G(s) of the form

$$G(s) = N_r(s)D_r(s)^{-1} \tag{V.1}$$

yields that the closed loop eigenvalues of the system are the zeros of $\det[N_r(s) + \frac{D_r(s)}{k}]$. The finite zeros of this polynomial at $k = \infty$ are the zeros of

$$\det N_r(s) = 0 \tag{V.2}$$

which are the zeros of G(s).

To obtain the unbounded root loci of the system notice that for k sufficiently large the zeros of det(I+kG(s)) of unbounded magnitude (with k) are closed loop eigenvalues of the system. To obtain the first, second,...,nth order unbounded root loci of the system we derive the asymptotic values of s satisfying

$$\det(I+kG(s)) = 0 \tag{V.3}$$

as $k \to \infty$ such that $\frac{s}{k} = \lambda_1$; $\frac{s^2}{k} = \lambda_2$;...;$\frac{s^n}{k} = \lambda_n$, respectively, where the λ_i's are finite and nonzero.

V.1. Asymptotic values of the first order unbounded root loci

Using (III.1) we note that

$$\lim_{\substack{k,s\to\infty \\ k/s=1/\lambda_1}} \det(I+kG(s)) = \det\left(I+\frac{G_1}{\lambda_1}\right) \tag{V.4}$$

Then a first order root locus will exist provided $\exists \lambda_1 \neq 0$ such that $\det\left(I+\frac{G_1}{\lambda_1}\right) = 0$. Let G_1 have the S.V.D. given by (V.5)

$$G_1 = [U_1^1 \,\vdots\, U_2^1] \left[\begin{array}{c|c} \Sigma_1^1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{1*} \\ \hline V_2^{1*} \end{array}\right] = U^1 \left[\begin{array}{c|c} \Sigma_1^1 & 0 \\ \hline 0 & 0 \end{array}\right] V^{1*} \tag{V.5}$$

with $U_1^1 \in \mathbb{R}^{m\times m_1}$; $U_2^1 \in \mathbb{R}^{m\times(m-m_1)}$; $V_1^1 \in \mathbb{R}^{m\times m_1}$; $V_2^1 \in \mathbb{R}^{m\times(m-m_1)}$.

Then, we have (using III.1)

$$\lim_{\substack{k,s\to\infty \\ k/s=1/\lambda_1}} \det(I+kG(s)) = \det\left[I + k[U_1^1 \; U_2^1] \left[\begin{array}{c|c} \Sigma_1^1/\lambda_1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{1*} \\ \hline V_2^{1*} \end{array}\right]\right] = 0$$

Premultiplying by the unitary matrix U^{1*} and post multiplying by the unitary matrix V^1, we obtain

$$\det \left[\begin{array}{c|c} U_1^{1*}V_1^1 + \Sigma_1^1/\lambda_1 & U_1^{1*}V_2^1 \\ \hline U_2^{1*}V_1^1 & U_2^{1*}V_2^1 \end{array}\right] = 0 \tag{V.6}$$

Proposition V.1 now follows immediately by comparing (V.4) and (V.6).

Proposition V.1. Given the Taylor expansion (III.1) of $G(s)$ and the S.V.D. of G_1 given by (V.5) the values of λ_1 for which (V.6) holds are the negatives of the non-zero eigenvalues $\lambda_{i,1}$ of G_1. ¤

We conclude that the first order unbounded root loci of the system of Figure 1 are of the form

$$s_{i,1} = -k\lambda_{i,1} + O(k^0) \tag{V.7}$$

where each $\lambda_{i,1}$ is a non-zero eigenvalue of G_1. Under the simple null structure assumption (Assumption 1) it will be shown in the proof of Theorem 1 that the number of non-zero λ_1 satisfying (V.6) is m_1, where m_1 is the rank of G_1. We assume $m_1 < m$.

V.2. Asymptotic values of the second order unbounded root loci

Recalling (III.1) we now label $\frac{G_2}{s^2} + \frac{G_3}{s^3} + \ldots$ as $P_1(s)$ (which is $O(\frac{1}{s^2})$). Then, we have

$$\det(I+kG(s)) = \det \left[\begin{array}{c|c} U_1^{1*}(I+kP_1(s))V_1^1 + \frac{k}{s}\Sigma_1^1 & U_1^{1*}(I+kP_1(s))V_2^1 \\ \hline U_2^{1*}(I+kP_1(s))V_1^1 & U_2^{1*}(I+kP_1(s))V_2^1 \end{array}\right] \tag{V.8}$$

If $G_2 = 0$ we skip this step and proceed to the next step for the computation of the third order unbounded root loci as detailed in Section (V.3). Using the Schur formula for the determinants of partitioned matrices in (V.8) we have, either

$$\det[U_1^{1*}(I+kP_1(s)V_1^1 + \frac{k}{s}(\Sigma_1^1)] = 0 \tag{V.9}$$

or

$$\det[U_2^{1*}(I+kP_1(s))V_2^1 + U_2^{1*}(I+kP_1(s))V_1^1[U_1^{1*}(I+kP_1(s))V_1^1 + \frac{k}{s}\Sigma_1^1]^{-1}\cdot$$

$$U_1^{1*}(I+kP_1(s))V_2^1] = 0 \tag{V.10}$$

We now examine (V.9) as k,s both $\to\infty$ with $k/s^2 = 1/\lambda_2$ where λ_2 is a non-zero constant: then $P_1(s) \to G_2/\lambda_2$ and $k/s = s/\lambda_2$, thus

$$\det[U_1^{1*}(I+kP_1(s))V_1^1 + \frac{k}{s}\Sigma_1^1] \simeq \det[U_1^{1*}(I+G_2/\lambda_2)V_1^1 + s\Sigma_1^1/\lambda_2]$$

$$\simeq (\frac{s}{\lambda_2})^{m_1} \det(\Sigma_1^1)$$

Since Σ_1^1 is nonsingular by construction (see (V.5)), the left hand side of (V.9) goes to infinity along the <u>2</u>nd order root loci. In other words, the second order root loci are specified exclusively by (V.10). Now using the calculation above, the inverse in the second term of (V.10) becomes as k,s $\to\infty$ with $k/s^2 = 1/\lambda_2$,

$$\frac{\lambda_2}{s}(\Sigma_1^1)^{-1}$$

and the second term is easily seen to be $0(1/s)$; whereas the first term becomes

$$U_2^{1*}(I+G_2/\lambda_2)V_2^1$$

Hence, in the limit (V.10) becomes

$$\det[U_2^{1*}V_2^1 + \frac{1}{\lambda_2}U_2^{1*}G_2V_2^1] = 0. \tag{V.11}$$

By our assumption 2, (non redundancy), we have $G_2 \neq 0 \Rightarrow U_2^{1*}G_2V_2^1 \neq 0$ so that equation (V.11) is indeed a polynomial equation in λ_2. From Proposition (II.2) it follows that the values of λ_2 satisfying (V.11) are the negatives of the non-zero eigenvalues of $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$. Thus, the second order unbounded root loci are of the form

$$s_{i,2} = \sqrt{-k\lambda_{i,2}} + 0(k^0). \tag{V.12}$$

where each $\lambda_{i,2}$ is a non-zero eigenvalue of $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$.

This procedure is in the spirit of the singular perturbations approach (see for e.g. Desoer and Shensa [7]) in the following sense: to compute the second order unbounded root loci which tend to ∞ at a lower velocity than the first order unbounded root loci we renormalize the s variable so as to place at ∞ the first order unbounded root loci and then examine the (slower) second order unbounded root loci.

Let the S.V.D. of $U_2^{1*}G_2V_2^1 \in \mathbb{R}^{(m-m_1)\times(m-m_1)}$ be given by

$$U_2^{1*}G_2V_2^1 = [U_1^2\ U_2^2]\begin{bmatrix}\Sigma_1^2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^{2*}\\ V_2^{2*}\end{bmatrix} \tag{V.13}$$

with $U_1^2 \in \mathbb{R}^{(m-m_1)\times m_2}$; $U_2^2 \in \mathbb{R}^{(m-m_1)\times(m-m_1-m_2)}$; $V_1^2 \in \mathbb{R}^{(m-m_1)\times m_2}$; $V_2^2 \in \mathbb{R}^{(m-m_1)\times(m-m_1-m_2)}$. It is shown in the proof of Theorem 1 that under the simple null structure assumption on $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ the number of non-zero solutions to (V.12) is m_2, the rank of $U_2^{1*}G_2V_2^1$. For each non-zero solution $\lambda_{i,2}$ equation (V.12) gives two asymptotic root loci corresponding to the two branches of the square root. In order to have third order root loci we assume $m_1 + m_2 < m$.

We explain at this point what could happen in the absence of the non-redundancy assumption (Assumption 2). If we had $G_2 \neq 0$, $U_2 \neq 0$ and $U_2^{1*}G_2V_2^1 = 0$ then as $k,s \to \infty$ such that $\frac{k}{s^2} = \frac{1}{\lambda_2} < \infty$ we would obtain from equation (V.10)

$$\det(U_2^{1*}V_2^1) = 0$$

which is false as is seen in the proof of Thereom 1. Thus none of the solutions of equation (V.15) are second order unbounded root loci.

If we do take the limit of equation (V.10) as $k,s \to \infty$ such that $\frac{k}{s^3} = \frac{1}{\mu} < \infty$ we obtain

$$\det[U_2^{1*}V_2^1 + \mu(U_2^{1*}G_3V_2^1 + U_2^{1*}G_2G_1^{\dagger}G_2V_2^1)] = 0$$

where $G_1^{\dagger}$ is the pseudo inverse of G_1 given by $V_1^1\Sigma_1^{1^{-1}}U_1^1$. Thus, there are now third order unbounded root loci in the solution of equation (V.10) in the absence of Assumption 2 (non-redundancy).

Thus, assumption 2 yields us at the end of the procedure a bona fide polynomial equation in λ_2 guaranteeing the existence of second order unbounded root loci. In the absence of assumption 2 the calculation can still be made and the procedure can be suitably modified as suggested above. Notice, however, the redundancy of the integrators associated with G_2 which is reflected in the fact that there are no second order unbounded root loci.

V.3. Asymptotic values of the third order unbounded root loci

Mimicking the previous section we label $\frac{G_3}{s^3} + \frac{G_4}{s^4} + \ldots$ as $P_2(s)$ (which is $O(\frac{1}{s^2})$). Then using (V.13), the S.V.D. of $U_2^{1*}G_2V_2^1$ in equation (V.11) we have

$$\det\left[\begin{array}{c|c} U_1^{2*}U_2^{1*}(I+kP_2(s))V_2^1V_1^2 + k\Sigma_1^2/s^2 & U_1^{2*}U_2^{1*}(I+kP_2(s))V_2^1V_2^2 \\ \hline U_2^{2*}U_2^{1*}(I+kP_2(s))V_2^1V_1^2 & U_2^{2*}U_2^{1*}(I+kP_2(s))V_2^1V_2^2 \end{array}\right] \qquad (V.14)$$

Using the Schur formula for equation (V.14); and identifying the equation corresponding to the higher order (order ≥ 2) unbounded root loci and partially simplifying it as $k,s \to \infty$ such that $\frac{k}{s^i} < \infty$ for some $i = 3,4$,

$$\det[U_2^{2*}U_2^{1*}(I+kP_2(s))V_2^1V_2^2] = 0 \tag{V.15}$$

Taking limits as $k,s \to \infty$ such that $\frac{k}{s^3} = \frac{1}{\lambda_3}$ equation (V.15) yields

$$\det\left[U_2^{2*}U_2^{1*}V_1^1V_2^2 + \frac{U_2^{2*}U_2^{1*}G_3V_2^1V_2^2}{\lambda_3}\right] = 0 \tag{V.16}$$

By the non-redundancy assumption $G_3 \neq 0 \Rightarrow U_2^{2*}U_2^{1*}G_3V_2^1V_2^2 \neq 0$, so that equation (V.16) is indeed a polynomial equation in λ_3.

It follows from Proposition (II.1) that the λ_3 satisfying equation (V.16) are the negatives of the non-zero eigenvalues of $G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}$. In this step the m_1 first order unbounded root loci and the $2m_2$ second order unbounded root loci are driven to ∞ to study the third order unbounded root loci of the form

$$s_{i,3} = \sqrt[3]{-k\lambda_{i,3}} + 0(k^0).$$

whereas each $\lambda_{i,3}$ is a non-zero eigenvalue of $G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}$.

Let the S.V.D. of $U_2^{2*}U_2^{1*}G_3V_2^1V_2^2 \in \mathbb{R}^{(m-m_1-m_2)\times(m-m_1-m_2)}$ be given by

$$U_2^{2*}U_2^{1*}G_3V_2^1V_2^2 = [U_1^3 \mid U_2^3]\begin{bmatrix}\Sigma_1^3 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^{3*}\\ V_2^{3*}\end{bmatrix} \tag{V.17}$$

with $U_1^3 \in \mathbb{R}^{(m-m_1-m_2)\times m_3}$, $U_2^3 \in \mathbb{R}^{(m-m_1-m_3)\times(m-m_1-m_2-m_3)}$; $V_1^3 \in \mathbb{R}^{(m-m_1-m_2)\times m_3}$; $V_2^3 \in \mathbb{R}^{(m-m_1-m_2)\times(m-m_1-m_2-m_3)}$. Then, under the simple null structure assumption on $G_1, G_2\Big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)} =: \hat{G}_2$; $G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}$ it is shown in the proof of Theorem 1 that there are m_3 non-zero solutions to (V.16) where m_3 is the rank of $U_2^{2*}U_2^{1*}C_3V_2^1V_2^2$. Under the non-redundancy assumption $m_1 + m_2 < m, G_3 \neq 0 \Rightarrow m_3 \geq 1$.

V.4. Higher order unbounded root loci

These are computed in a manner exactly as described above for the first, second, and third order unbounded root loci. The procedure ends at the n_oth order unbounded root loci provided that $U_2^{n_o+1}$ is the zero matrix; that is $G_{n_o}\Big|_{\mathcal{N}(\hat{G}_{n_o-1})+\mathcal{N}(\hat{G}^*_{n_o-1})} =: \hat{G}_{n_o}$ has no zero eigenvalues. Assumption 0 guarantees that the procedure ends and that $m_1+m_2+\ldots+m_{n_o} = m$ so that the entire input (and output space) has been decomposed as the orthogonal direct sum of subspaces associated with the 1st,2nd,...,n_oth order root loci as will be explicated in section V.5. Then, the total number of unbounded root loci is $m_1+2m_2+3m_3+\ldots+n_o m_{n_o}$.

V.5. Interpretation of the results so far

The usefulness of the S.V.D. in extracting the equations for the values of the unbounded root loci at $k = \infty$ stems from the fact that the orthogonal bases it provides for the null space of a linear map and its adjoint give an explicit representation of the restriction of a linear map in its domain and range.

Equation (V.6) for the first order unbounded root loci is an equation for computing the negatives of the eigenvalues of G_1. Equation (V.11) for the second order unbounded root loci is an equation for computing the negatives of the eigenvalues of $\hat{G}_2 := G_2\Big|_{\mathcal{N}(G_1)+\mathcal{N}(G_1^*)}$ using orthonormal basis for the $\mathcal{N}(G_1)$ and $\mathcal{N}(G_1^*)$ furnished by the columns of $U_2^1 \in \mathbb{R}^{m\times(m-m_1)}$ and $V_2^1 \in \mathbb{R}^{m\times(m-m_1)}$ respectively. Equation (V.16) for the third order unbounded root loci is an equation for computing the negatives of the eigenvalues of $G_3\Big|_{\mathcal{N}(\hat{G}_2)+\mathcal{N}(\hat{G}_2^*)}$. Notice that the $\mathcal{N}(\hat{G}_2)$, $\mathcal{N}(\hat{G}_2^*)$ are subspaces of the $\mathcal{N}(G_1)$, $\mathcal{N}(G_1^*)$ respectively. The bases used for the representation of this restriction of G_3 are the orthonormal columns of $V_2^1V_2^2$ ($m-m_1-m_2$ of them) in the domain and the orthonormal columns of $U_2^1U_2^2$ ($m-m_1-m_2$ of them) in the range. The interpretation for the higher order unbounded root loci follows naturally.

Roughly speaking, the procedure consits of identifying in turn subspaces of the input space ($\mathbb{R}^m$) and output space ($\mathbb{R}^m$) where the effects of the first order, second order,... unbounded root loci dominate. Thus $\mathcal{R}(V_1^1)$, $\mathcal{R}(V_2^1,V_1^2)$, $\mathcal{R}(V_2^1V_2^2V_1^3)$,... are subspaces of the <u>input space</u> where the effects of the 1st order, 2nd order, 3rd order,... unbounded root loci respectively dominate and the $\mathcal{R}(U_1^1)$, $\mathcal{R}(U_2^1U_1^2)$, $\mathcal{R}(U_2^1U_2^2U_1^3)$ are subspaces of the <u>output space</u> where the 1st order, 2nd order, 3rd order,... unbounded root loci respectively dominate. The identification procedure is demonstrated pictorially for the second order and the third order unbounded root loci in Fig. 7. Assumption 0 guarantees that the <u>entire</u> input and output space can be written as the orthogonal direct sum of subspaces associated with the 1st, 2nd,...,n_oth order unbounded root loci. We now state the main result of this section.

<u>Theorem 1</u>. (Asymptotic values of the unbounded root loci)

Given a strictly proper linear time-invariant control system with Taylor series expansion about s = ∞ given by

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \ldots \qquad \text{(III.1)}$$

where $G_1,G_2,\ldots \in \mathbb{R}^{m\times m}$ satisfy:

(i) Assumption (0) - Non-triviality.

(ii) Assumption (1) - Simple Null Structure

(iii) Assumption (2) - Non-redundancy

The nth order unbounded root loci of the system of Fig. 3 consist of collections of n branches given by

$$s_{i,n} = \sqrt[n]{-k\lambda_{i,n}} + O(k^0) \qquad n = 1,2,3,\ldots,n_o$$

where $\lambda_{i,n}$ is a non-zero eigenvalue of $\hat{G}_n := G_n\Big|_{\mathcal{N}(\hat{G}_{n-1})\to\mathcal{N}(\hat{G}^*_{n-1})}$ and $\mathcal{N}(\hat{G}_{n_o}) = 0$.

The number of nth order root loci is nm_n where m_n is the rank of the matrix representation of $G_n\Big|_{\mathcal{N}(\hat{G}_{n-1})\to\mathcal{N}(G^*_{n-1})}$.

Furthermore the 1st, 2nd,...,n_oth order unbounded root loci are the only unbounded root loci of the system. ⌑

Proof: See Appendix.

V.6. <u>Ways of computing the asymptotic values of the unbounded root loci</u>

Under the assumption of simple null structure for $G_1,\hat{G}_2,\hat{G}_3$,etc... the amount of computation required to solve equations (V.6), (V.11) and (V.16) can be reduced. The motivation for doing this is that the number of operations involved in solving a generalized eigenvalue problem is the order of the cube of the size of the matrix. Clearly, if a generalized eigenvalue problem is known to have a certain number of zero solutions, computational effort is wasted in computing them. Also, the accuracy of computation is empirically observed to be higher with smaller dimensional matrices. From the proof of Theorem 1, we notice that equation (V.6) is an eigenvalue problem in $\mathbb{R}^{m\times m}$ with m_1 non-zero solutions, (V.11) is an eigenvalue problem in $\mathbb{R}^{(m-m_1)\times(m-m_1)}$ with m_2 non-zero solutions and so on. Our goal then is to replace (V.6), (V.11), (V.16) and subsequent equations for higher order unbounded root loci by eigenvalue problems in $\mathbb{R}^{m_1 \times m_1}$; $\mathbb{R}^{m_2 \times m_2}$; $\mathbb{R}^{m_3 \times m_3}$ and so on. We carry out this program in detail for the first and second order unbounded root loci. The extension to higher order unbounded root loci is essentially obvious.

Recalling the S.V.D.s of G_1 and $U_2^{1*} G_2 V_2^1$ from equation (V.5) and (V.13) respectively we state the reduced dimension versions of (V.6) and (V.11) in Proposition V.2.

Proposition (V.2) [Computation of 1st and 2nd order unbounded root loci]

With the notation and assumptions of Theorem 1 the non-zero solutions to equation (V.6) are the solutions of the generalized eigenvalue problem of (V.18) in matrices of dimension m_1

$$\det[\mu(U_1^{1*} V_1^1 - U_1^{1*} V_2^1 (U_2^{1*} V_2^1)^{-1} U_2^{1*} V_1^1) - \Sigma_1^1] = 0. \tag{V.18}$$

Similarly, the non-zero solutions to equation (V.12) are the solution of the generalized eigenvalue problem of (V.19) in matrices of dimension m_2

$$\begin{aligned} \det[\mu(U_1^{2*} (U_2^{1*} V_2^1) V_1^2 - U_1^{2*} (U_2^{1*} V_2^1) V_2^2 [U_2^{2*} (U_2^{1*} V_2^1) V_2^2]^{-1} U_2^{2*} \\ (U_2^{1*} V_2^1) V_1^2) - \Sigma_1^2] = 0. \end{aligned} \tag{V.19}$$

Proof: See Appendix

Comment: Since the matrices $\Sigma_1^1 \in \mathbb{R}^{m_1 \times m_1}$ and $\Sigma_1^2 \in \mathbb{R}^{m_2 \times m_2}$ are invertible the generalized eigenvalue problems of (V.18) and (V.19) can be restated as ordinary eigenvalue problems.

V.7. State Feedback Invariance of the Asymptotic Values of the Unbounded Root Loci System Representation (A,B,C)

Let the linear time invariant system represented by the transfer function G(s) have a minimal state space realization (A,B,C) with

$A \in \mathbb{R}^{n\times n}$; $B \in \mathbb{R}^{n\times m}$; $C \in \mathbb{R}^{m\times n}$ then the Markov parameters are $G_1 = CB$, $G_2 = CAB$, $G_3 = CA^2B$, and so on, we now move the invariance of the asymptotic values as $k \to \infty$ of the unbounded root loci under state feedback $u = -Fx$ with $F \in \mathbb{R}^{m\times n}$, under the assumptions 0,1 and 2 of non-triviality, simple null structure and non-redundancy. This property of the asymptotic values at $k = \infty$ of the unbounded root loci is reminiscent of the identical property for the McMillan zeros of the system, which is well known.

Let $G_F(s)$ denote the transfer function of the system (A,B,C) with state-feedback $u = -Fx$. Let the Markov parameters of the system with feedback be

$$G_1^F = CB;\ G_2^F = C(A+BF)B;\ G_3^F = C(A+BF)^2B$$

and so on. Then, we have

<u>Proposition V.3</u> [Asymptotic values of the root loci are invariant under constant state-feedback]

The asymptotic values of the root loci of the system with transfer function $G_F(s)$ are the same as the asymptotic values of the root loci of the system with transfer function $G(s)$ provided both $G(s)$ and $G_F(s)$ satisfy assumptions 0,1 and 2.

<u>Comments</u>. Proposition (V.3) establishes the existence of <u>n feedback invariants</u> where n is the order of the system. Since (A,B,C) is minimal, λ is a McMillan zero of $G(s)$ if and only if λ is a root of

$$\det \left[\begin{array}{c|c} \lambda I - A & B \\ \hline -C & 0 \end{array}\right] = 0 \qquad \text{(V.20)}$$

(see for instance Desoer and Schulman [12].) However (V.20) is equivalent to (V.21) below as is easily seen by elementary column operations

$$\det \left[\begin{array}{c|c} \lambda I - A - BF & B \\ \hline -C & 0 \end{array}\right] = 0 \tag{V.21}$$

The proof of Proposition (V.3) establishes the feedback invariance of the so-called "infinite zeros" (terminology of [1],[2],[3],[4],[5],[8]). It is clear that $n_z + m_1 + 2m_2 + \ldots + n_o m_{n_o} = n$ where n_z is the number of solutions, λ, of equation (V.20).

Section VI. Calculation of the Asymptotes of the Unbounded Root Loci

By the asymptotes of the unbounded root loci is meant the leading term in the series for the $O(k^0)$ terms in the asymptotic expansion of the unbounded root loci. We postulate that the nth order unbounded root loci have asymptotic series of the form given by (VI.1)

$$s_{i,n} = \sqrt[n]{-k\lambda_{i,n}} + c_{i,n} + \frac{d_{i,n}}{\sqrt[n]{k}} + \ldots \quad n = 1,2,3,\ldots,n_o \tag{VI.1}$$

with $\lambda_{i,n}$; $c_{i,n}$; $d_{i,n} \in \mathbb{C}$.

This form for the asymptotic series is justified by explicit substitution into the equation that they should satisfy and a verification that terms of different orders in k sum independently to zero for suitable choices of $\lambda_{i,n}$; $c_{i,n}$; $d_{i,n}$ etc. We present the steps in the derivation of formulae for $c_{i,n}$ for the first and second order unbounded root loci ($n = 1,2$) and then present a way of computing them. The extension to higher order unbounded root loci is essentially obvious. To make the estimates we make the <u>Simple Structure Assumption</u> (Assumption 3) in addition to the <u>Simple Null Structure</u> and <u>Non-Redundancy Assumptions</u>. The procedure followed is essentially that prescribed by Dieudonné [10] or Kato [11].

VI.1 Asymptotes for the first order unbounded root loci

The asymptotic formula for a first order unbounded root locus $s_{i,1}$ is of the form of (VI.2)

$$s_{i,1} = -\lambda_{i,1}k + c_{i,1} + \frac{d_{i,1}}{k} + \ldots \tag{VI.2}$$

To compute $c_{i,1}$ we substitute (VI.2) for s in det(I+kG(s)) and take limits as $k,s \to \infty$ and retain terms of the order of k^0 and k^{-1} as suggested by Dieudonné [10]. We then obtain

$$\det[(-\lambda_{i,1}I+G_1) + \frac{1}{k}(c_{i,1}I - \frac{1}{\lambda_{i,1}} G_2) + 0(\frac{1}{k^2})] = 0 \tag{VI.3}$$

By the simple structure assumption G_1 has no generalized eigenvectors associated with its eigenvalues. Thus only two cases can arise namely $\lambda_{i,1}$ having multiplicity 1 and $\lambda_{i,1}$ having multiplicity > 1. These are treated separately as Cases 1 and 2, respectively.

Case 1. $\lambda_{i,1}$ has multiplicity 1.

Then, $\mathcal{N}(\lambda_{i,1}I-G_1) \subset \mathbb{C}^m$ is one dimensional, say, spanned by $e_{i,o}$. Let us write the null space of the matrix in (VI.3) as the span of $e_{i,o} + \frac{e_{i,1}}{k}$ for some $e_{i,1} \in \mathbb{C}^m$. Using this in (VI.3) we obtain after some simplification

$$[-\lambda_{i,1}I+G_1 \mid e_{i,o}] \begin{bmatrix} e_{i,1} \\ c_{i,1} \end{bmatrix} = \frac{1}{\lambda_{i,1}} G_2 e_{i,o} \tag{VI.4}$$

Proposition VI.1. [Formula for $c_{i,1}$ when $\lambda_{i,1}$ has multiplicity 1]
If $\lambda_{i,1}$ has multiplicity 1 as an eigenvalue of G_1 then there do exist $e_{i,1} \in \mathbb{C}^m$ and $c_{i,1} \in \mathbb{C}$ satisfying equation (VI.4) for any $G_2 \in \mathbb{C}^{m\times m}$. Furthermore, any solution to equation (VI.4) yields the same value of $c_{i,1}$.

Proof: See Appendix. ¤

Case 2. $\lambda_{i,1}$ has multiplicity p.

Under the simple structure assumption there exist p linearliy independent eigenvectors say $e_{i,o}^{(1)},\ldots,e_{i,o}^{(p)}$ each $\in \mathbb{C}^m$. It is expected that formulae similar to (VI.4) hold for each of the p $c_{i,1}$'s associated with these p first order unbounded root loci. This is almost true and we have

Proposition VI.2 [Formula for $c_{i,1}$ when $\lambda_{i,1}$ has multiplicity p].

There exist p vectors $\{\alpha^{(k)}\}_{k=1}^{p} \subset \mathbb{C}^p$ such that the equations (VI.5) each have a solution

$$\begin{pmatrix} e_{i,1}^{(k)} \\ c_{i,1}^{(k)} \end{pmatrix} \in \mathbb{C}^{m+1} \text{ for } k = 1,\ldots,p$$

$$\left[-\lambda_{i,1}I+G_1 \;\middle|\; \sum_{\ell=1}^{p} \alpha_\ell^{(k)} c_{i,o}^{(\ell)}\right] \begin{bmatrix} e_{i,1}^{(k)} \\ \hline c_{i,1}^{(k)} \end{bmatrix} = \frac{1}{\lambda_{i,1}} \sum_{\ell=1}^{p} \alpha_\ell^{(k)} G_2 e_{i,o}^{(\ell)} \qquad \text{(VI.5)}$$

with $\underline{\alpha}^{(k)} = (\alpha_1^{(k)}\ldots\alpha_p^{(k)})^*$ $k = 1,\ldots,p$. Furthermore for each k any two solutions of (VI.5) yield the same value of $c_{i,1}^{(k)}$. ¤

Comment: We notice that (VI.5) and (VI.4) are essentially the same since any linear combination of eigenvectors associated with an eigenvalue is an eigenvector for that eigenvalue.

Proof: See Appendix. ¤

VI.2 Asymptotes for the second order unbounded root loci

The asymptotic formulae for the second order unbounded root loci are

$$s_{i,2} = \sqrt{-k\lambda_{i,2}} + c_{i,2} + \frac{d_{i,2}}{\sqrt{k}} + \ldots \qquad \text{(VI.6)}$$

Using (VI.6) in (V.8) and taking limits as $k,s \to \infty$ and retaining terms of order k^0 and $k^{-1/2}$ we obtain the finite solutions of the resulting equation [i.e. neglecting those driven to ∞] to be the solutions of

$$\det\left[(-\lambda_{i,2}U_2^{1*}V_2^1+U_2^{1*}G_2V_2^1) + \frac{1}{\sqrt{k}}\left(3\sqrt{-\lambda_{i,2}}\; c_{i,2}\; U_2^{1*}V_2^1 + \frac{c_{i,2}}{\sqrt{-\lambda_{i,2}}}\; U_2^{1*}G_2V_2^1 + \frac{U_2^{1*}G_3V_2^1}{\sqrt{-\lambda_{i,2}}}\right) + o\left(\frac{1}{k}\right)\right] = 0 \qquad \text{(VI.7)}$$

Recall that the $\lambda_{i,2}$ are solutions to the generalized eigenvalue problem of (V.10) namely

$$\det(\lambda_{i,2}U_2^{1*}V_2^1 - U_2^{1*}G_2V_2^1) = 0. \qquad \text{(V.10)}$$

Under the simple structure assumption on $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ with respect to $\lambda_{i,2}$ we treat the two cases $\lambda_{i,2}$ having multiplicity 1 and $\lambda_{i,2}$ having multiplicity $p > 1$ separately.

Case 1. $\lambda_{i,2}$ has multiplicity 1:

We assume that the null space of the matrix in (VI.7) is of the form $e_{i,o} + \frac{e_{i,1}}{\sqrt{k}} \in \mathbb{C}^{m-m_1}$ with

$$[\lambda_{i,2}U_2^{1*}V_2^1 - U_2^{1*}G_2V_2^1]e_{i,o} = \theta_{m-m_1} \qquad \text{(VI.8)}$$

we then obtain from (VI.7)

$$[-\lambda_{i,2}I + (U_2^{1*}V_2^1)^{-1}U_2^{1*}G_2V_2^1 \;\vdots\; e_{i,o}]\begin{bmatrix} e_{i,1} \\ \hline 2c_{i,2}\end{bmatrix} = (U_2^{1*}V_2^1)^{-1}\,\frac{U_2^{1*}G_3V_2^1e_{i,o}}{\sqrt{-\lambda_{i,2}}} \qquad \text{(VI.9)}$$

[The existence of $(U_2^{1*}V_2^1)^{-1}$ was established in Proposition (V.1)]. By the same proof as that of Proposition (VI.1) we can verify the existence of $\begin{bmatrix} e_{i,1} \\ c_{i,2}\end{bmatrix} \in \mathbb{C}^{m-m_1+1}$ satisfying (VI.9) and that two solutions of (VI.9) yield the same value of $c_{i,2}$.

Case 2. $\lambda_{i,2}$ has multiplicity p.

Under the simple structure assumption there are p vectors $\{e_{i,o}^{(\ell)}\}_{\ell=1}^{p}$ associated with the eigenvalue $\lambda_{i,2}$ of $G_2\big|_{\mathcal{N}(G)\to\mathcal{N}(G_1^*)}$, i.e.

$$[\lambda_{i,2}U_2^{1*}V_2^1 - U_2^{1*}G_2V_2^1]e_{i,o}^{(\ell)} = \theta_{m-m_1} \qquad \ell = 1,\ldots,p \tag{VI.10}$$

As in Proposition (VI.2) we can again affirm the existence of p vectors $\{\underline{\alpha}^{(\ell)}\}_{\ell=1}^{p} \in \mathbb{C}^{m-m_1}$ such that the equations (VI.11) have a solution $\begin{bmatrix} e_{i,1}^{(k)} \\ c_{i,2}^{(k)} \end{bmatrix} \in \mathbb{C}^{m-m_1+1}$.

$$\left[-\lambda_{i,2}I+(U_2^{1*}V_2^1)^{-1}U_2^{1*}G_2V_2^1 \;\middle|\; \sum_{\ell=1}^{p} \alpha_\ell^{(k)} e_{i,o}^{(\ell)}\right] \cdot \begin{bmatrix} e_{i,1}^{(k)} \\ 2c_{i,2}^{(k)} \end{bmatrix}$$

$$= -(U_2^{1*}V_2^1)^{-1} \frac{U_2^{1*}G_3V_2^1}{\sqrt{-\lambda_{i,2}}} \sum_{\ell=1}^{p} \alpha_\ell^{(k)} e_{i,o}^{(\ell)} \qquad k = 1,\ldots,p. \tag{VI.11}$$

Furthermore all solutions of (VI.11) yield the same value of $c_{i,2}^{(k)}$.

(VI.3) Interpretation of the Results of Sections (VI.1) and (VI.2).

Equation (VI.4) expresses the fact that the perturbation in the eigendirection at $k = \infty$ for finite k depends on the effect of G_2 in that direction. Thus, for instance if $G_2e_{i,o} = \theta_m$; $c_{i,1} = 0$. To elaborate on the picture of the previous section we imagine a further partition of the range into the direct sum of eigenspaces associated with $\lambda_{i,1}$. To calculate $c_{i,1}$ we examine the effect of G_2 on individual eigenspaces in $\mathcal{R}(G_1)$. When an eigenvalue has multiplicity p and simple structure it is necessary to pick p linearly independent directions in the subspace associated with $\lambda_{i,1}$ to compute the p $c_{i,1}$'s associated with the p first order unbounded root loci. This is achieved by the choice of the $\{\alpha^{(k)}\}_{k=1}^{p}$ described in Proposition (VI.2) and equation (VI.5).

For higher order unbounded root loci the picture is the same-one of further subdivision of $\mathcal{R}(\hat{G}_2)$, $\mathcal{R}(\hat{G}_3)$,... into the direct sum of eigenspaces on which individually we examine the effect of $G_3, G_4, \ldots,$ respectively.

(VI.4) Computational Method for Calculating the Asymptotes using the Formulae of this Section.

We will state the method only for equations (VI.4) and (VI.5) which are for the first order unbounded root loci. For higher order unbounded root loci the extension is straightforward. Of the two equations (VI.4) can easily be solved by any linear equation solving method since a solution is guaranteed. To solve (VI.5) the vectors $\{\underline{\alpha}^{(k)}\}_{k=1}^{p}$ need to be computed. But these are shown in the proof of Proposition VI.2 to be the eigenvectors of a matrix $\beta \in C^{p\times p}$: we now illustrate how to compute the matrix β. Let

$$[-\lambda_{i,1}I+G_1 \mid e_{i,o}^{(1)} \quad \cdots \mid e_{i,o}^{(p)}] = U_{G_1}[\Sigma_1 \mid 0]V_{G_1}^{*} \tag{VI.12}$$

with $U_{G_1} \in \mathbb{C}^{m_1\times m_1}$; $V_{G_1} \in \mathbb{C}^{(m_1+p)\times(m_1+p)}$ and $\Sigma_1 \in \mathbb{R}_+^{m_1\times m_1}$ (a diagonal matrix) be the S.V.D. of the matrix $[-\lambda_{i,1}I+G_1 \mid e_{i,o}^{(1)} \cdots \mid e_{i,o}^{(p)}]$. Then, to obtain β_k which is the list of coordinates of the vector $G_2e_{i,o}^{(k)}$ along $e_{i,o}^{(1)},\ldots,e_{i,o}^{(p)}$ we take a pseudo universe of the matrix of (VI.12) to obtain (VI.13)

$$\begin{bmatrix} f_{i,1}^{(k)} \\ \\ \beta_k \end{bmatrix} = V_{G_1}\begin{bmatrix} \Sigma_1^{-1} \\ \hline 0 \end{bmatrix} U_{G_1}^{*} G_2 e_{i,o}^{(k)} \tag{VI.13}$$

where $f_{i,1}^{(k)} \in \mathbb{C}^{m_1}$ and $\underline{\beta}_k \in \mathbb{C}^p$ are as defined in the proof of proposition (VI.2). By repeating (VI.13) for $k = 1,\ldots,p$ the matrix β is obtained and the $c_{i,1}^{(k)}$'s can be computed from the eigenvalues $\delta^{(k)}$ of β as

$$c_{i,1}^{(k)} = \delta^{(k)}/\lambda_{i,1} \qquad k = 1,\ldots,p. \tag{VI.14}$$

Section VII. A Necessary and Sufficient Condition for the Closed Loop Exponential Stability of a Strictly Proper Linear Time-Invariant Control System Under Arbitrarily High Gain Feedback

It is well known (see for e.g. [14]) that a proper, linear time-invariant control system is exponentially stable for sufficiently high feedback gain $k \geq k_o$ with the closed loop eigenvalues uniformly (with k) bounded away from the $j\omega$ axis for $k \geq k_o$ iff

(i) its zeros are in the open left half plane;

(ii) the pole excess of the system is no larger than 2;

(iii) if the pole excess is 2, the intercept of the asymptotes of the unbounded root loci with the real line is in the open left half plane.

Using the calculations of the previous sections a similar result for the exponential stability of linear, time-invariant, multivariable systems for sufficiently high feedback may be derived under the non-triviality, non-redundancy and simple structure assumptions. For this result conditions have to be found so as to exclude third order unbounded root loci since at least one of the three cube roots of $-k\lambda_{i,3}$ will lie in the right half plane for k sufficiently large. Also, second order unbounded root loci can be tolerated only if the eigenvalues of $\hat{G}_2$, namely $\lambda_{i,2}$, are real and positive and the $c_{i,2}$ associated with them have negative real part. The precise statement is as follows.

__Theorem 2__ [High gain stability]

Under the set up of Theorem 1 with Assumptions 0, 2 and 3 of non-triviality, non-redundancy and simple structure respectively, the closed loop system of Figure 1 is exponentially stable for all $k \geq k_o$

(where k_o is some finite constant dependent on G(s)) with all the closed loop eigenvalues uniformly (with k) bounded away from the $j\omega$ axis for $k \geq k_o$, iff

(i) the McMillan zeros of G(s) are in the open left half plane;

(ii) the non-zero eigenvalues of G_1 are in the open right half plane;

(iii) the eigenvalues of $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ are real and positive;

(iv) the $c_{i,2}$ associated with each eigenvalue of $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ have negative real part; and

(v) $\mathbb{R}^m = \mathcal{R}(G_1) + \mathcal{R}(G_2\big|_{\mathcal{N}(G_1)})$.

(Here the restriction of G_2 is only in the domain). ¤

Proof: See Appendix.

Section VIII. Conclusions.

In this paper the tool of the singular value decomposition has been used to establish a clear geometrical picture and a numerically sound procedure for computing the asymptotic behavior of the unbounded root loci; under the assumptions of simple structure, non-triviality and non-redundancy.

It has been explained how the non-redundancy assumption can be removed for the purposes of the computation — but the result is more complicated to understand geometrically. Also, the simple structure assumption is needed only for the asymptote calculations of Section VI. For the calculations of asymptotic values a simple null structure assumption (Assumption 1) suffices.

To show the benefits of this clear understanding two applications of the calculations have been demonstrated - namely the state feedback invariance of the asymptotic values of the root loci of (V.6) and the necessary and sufficient conditions for exponential stability under arbitrarily high gain $k \geq k_o$ have been demonstrated. More applications of these calculations are clearly possible - in fact one on the order of a minimal realization of a strictly proper rational transfer function is stated without proof as a Proposition here in the Conclusion to indicate the scope of the method.

Proposition VIII.1. (Degree of Minimal realization of a strictly proper rational transfer function).

Given a strictly proper rational transfer function $G(s) \in \mathbb{R}(s)^{m\times m}$ with asymptotic behavior given by

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s} + \ldots$$

with $G_1, G_2, \ldots \in \mathbb{R}^{m\times m}$ the order of a minimal realization of $G(s)$ is given by

$$n = n_z + \rho(G_1) + \rho(\hat{G}_2) + \ldots$$

where n_z = number of McMillan zeros of $G(s)$, $\rho(G_1)$ = dimension $\mathcal{R}(G_1)$, $\rho(\hat{G}_2)$ = dimension $\mathcal{R}(\hat{G}_2)$, and so on, provided $G(s)$ satisfies assumption 1 (simple null structure). ⌑

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Appendix-Proofs

Proposition II.1. (Matrix Representation of $A|_{S_1 \to S_2}$).

Proof: Note that $x \in S_1$ iff

$$x = P_1 y, \text{ for some } y \in \mathbb{C}^{m_1}$$

Here the elements of $y \in \mathbb{C}^{m_1}$ represent the coordinates of x with respect to the orthonormal basis $\mathcal{B}_1$. $A|_{S_1 \to S_2} x$ is the projection of Ax onto S_2 = span of $\mathcal{B}_2$ = span of the columns of P_2. The orthogonal projection of C^m onto S_2 is given by $P_2 P_2^*$: indeed,

$$\forall \xi \in S_2 \quad \xi = P_2 \eta \text{ for some } \eta \in \mathbb{C}^{m_1}; \text{ thus } P_2 P_2^* \xi = P_2 P_2^* P_2 \eta = P_2 \eta = \xi$$

Consider, now $\rho \in S_2^{\perp}$.

Then $P_2 P_2^* \rho = 0$, since ρ is orthogonal to each column of P_2. Now, $\forall x \in S_1$, $\exists y \in \mathbb{C}^{m_1}$ such that $x = P_1 y$ thus

$$A|_{S_1 \to S_2} x = P_2 P_2^* A x = P_2 P_2^* A P_1 y \in S_2 \tag{A.1}$$

Hence,

$$A|_{S_1 \to S_2} x = P_2 z \text{ for some } z \in \mathbb{C}^{m_1} \tag{A.2}$$

Since the columns of P_2 are orthonormal, we obtain from (A.1) and (A.2)

$$z = P_2^* A P_1 y$$

where $y \in \mathbb{C}^{m_1}$ is the coordinate list of $x \in S_1$ with respect to basis $\mathcal{B}_1$ and $z \in \mathbb{C}^{m_1}$ is the coordinate list of $A|_{S_1 \to S_2} x \in S_2$. Hence, $P_2^* A P_1$ is the matrix representation of $A|_{S_1 \to S_2}$ with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$. ¤

Proposition II.2 [Polynomial equation for the eigenvalues of $A|_{S_1 \to S_2}$].

Proof: The proof is immediate from the definition and the fact that $P_2^* P_1 \in \mathbb{C}^{m_1 \times m_1}$ and $P_2^* A P_1 \in \mathbb{C}^{m_1 \times m_1}$ are the matrix representations of

$I|_{S_1\to S_2}$ and $A|_{S_1\to S_2}$ with respect to the bases furnished by the columns of P_1 (in the domain) and the columns of P_2 (in the range). ¤

<u>Proposition II.3</u> [Number of non-zero eigenvalues of $A|_{S_1\to S_2}$]

<u>Proof</u>: Since r is the dimension of the range space of $P_2^*AP_1 \in \mathbb{C}^{m_1\times m_1}$; (m_1-r) is the dimension of the null space of $P_2^*AP_1$. Thus, there are m_1-r linearly independent vectors $\{e_i\}_{i=1}^{m_1-r}$ such that

$$P_2^*AP_1e_i = \theta_{m_1} \quad i = 1,\ldots,m_1-r.$$

All of these are eigenvectors of $P_2^*AP_1 \in \mathbb{C}^{m_1\times m_1}$ associated with the zero eigenvalue. But, by the assumption that $A|_{S_1\to S_2}$ has simple null structure (m_1-r) is precisely the number of zero eigenvalues of $P_2^*AP_1$ [there are no generalized eigenvectors associated with the zero eigenvalue]. Also, since $P_2^*P_1 \in \mathbb{C}^{m_1\ m_1}$ is non-singular, the number of eigenvalues (counting multiplicities) is m_1 as noted in the comment following Proposition II.2. Hence, the number of non-zero eigenvalues $A|_{S_1\to S_2}$ is r.

<u>Theorem 1</u>. (Asymptotic values of the unbounded root loci)

<u>Proof</u>: The proof consits of establishing the cardinality of the nth order unbounded root loci (n = 1,2,...) and establishing that the 1st, 2nd,...,n_oth order unbounded root loci are the only unbounded root loci of the system by a counting argument.

<u>Part I</u>

To show that the number of nth order unbounded root loci is nm_n where m_n is the rank of the matrix representation of $G_n|_{\mathcal{N}(\hat{G}_{n-1})\to\mathcal{N}(\hat{G}^*_{n-1})}$; we first establish a lemma.

Lemma

Let $Q \in \mathbb{C}^{n\times n}$ have simple null structure. Let the S.V.D. of Q be given by (A.3)

$$Q = [U_1^Q \mid U_2^Q] \left[\begin{array}{c|c} \Sigma_1^Q & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{Q*} \\ \hline V_2^{Q*} \end{array}\right] \tag{A.3}$$

with $U_1^Q \in \mathbb{C}^{n\times n_Q}$; $U_2^Q \in \mathbb{C}^{n\times(n-n_Q)}$; $V_1^Q \in \mathbb{C}^{n\times n_Q}$; $V_2^Q \in \mathbb{C}^{n\times(n-n_Q)}$. Then, the matrix $U_2^{Q*}V_2^Q \in \mathbb{C}^{(n-n_Q)\times(n-n_Q)}$ has rank $n-n_Q$.

Proof: Q has simple null structure $\Leftrightarrow \mathcal{R}(Q) \cap \mathcal{N}(Q) = \{\theta_n\}$. (A.4)

Now, for a proof by contradiction, assume $U_2^{Q*}V_2^Q$ not full rank; equivalently

$$\exists\, \eta \neq \theta_{n-n_Q} \in \mathbb{C}^{n-n_Q} \ni U_2^{Q*}V_2^Q\eta = \theta_{n-n_Q} \tag{A.5}$$

But

$$\xi := V_2^Q\eta \in \mathcal{N}(Q) \text{ and } \xi \neq \theta_n \tag{A.6}$$

and by (A.5),

$$U_2^{Q*}\xi = \theta_{n-n_Q} \text{ or equivalent } \xi \in \mathcal{R}(Q) \tag{A.7}$$

(A.6) and (A.7) contradict (A.4). ¤

Now, G_1 has simple null structure and has rank m_1 hence by Proposition II.3 G_1 has m_1 non-zero eigenvalues and hence there are m_1 unbounded root loci. Also, $U_2^{1*}G_2V_2^1 \in \mathbb{R}^{m-m_1\ m-m_1}$ has rank m_2 (from equation (V.13)). Hence by Proposition II.3 the number of non-zero eigenvalues to $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ is m_2 provided $U_2^{1*}V_2^1$ is non-singular. But, this follows from the preceding lemma. Since, there correspond two second order unbounded root loci to every non-zero eigenvalue of $G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$ there are $2m_2$ second order unbounded root loci.

$U_2^{2*}U_2^{1*}G_3V_2^1V_2^2 \in \mathbb{R}^{(m-m_1-m_2)\times(m-m_1-m_2)}$ has rank m_3 (from equation (V.17)). Also $U_2^{2*}U_2^{1*}V_2^1V_2^2$ is non-singular from the preceding lemma applied to $G_2\Big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$. Hence by Proposition (II.3) the number of non-zero eigenvalues of $G_3\Big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}$ is m_3 and the number of third order unbounded root loci is $3m_3$ and so on.

<u>Part II</u>. The only unbounded root loci are the 1st, 2nd,...,n_oth order unbounded root loci.

When the Schur formula is applied repeatedly as discussed in Sections (V.1) and (V.2) and the limit of the resulting product taken as k,s → ∞ such that $\frac{k}{s^{n_o}} = \frac{1}{\lambda_{n_o}} < \infty$; assumption 0 of non-triviality guarantees that the matrix I + kG(s) may be <u>entirely</u> decomposed and we obtain

$$\lim_{\substack{k,s\to\infty \\ \frac{k}{s^{n_o}}<\infty}} \frac{1}{\left(k^{1-\frac{1}{n}}\right)^{m_1}} \frac{1}{\left(k^{1-\frac{2}{n}}\right)^{m_2}} \cdots \frac{1}{\left(k^{1-\frac{n_o-1}{n_o}}\right)^{m_{n_o-1}}} \cdot \det(I+kG(s))$$

$$= \det \frac{\Sigma_1^1}{\sqrt[n_o]{\lambda_{n_o}}} \cdot \det \frac{\Sigma_1^2}{\sqrt[n_o]{\lambda_{n_o}^2}} \cdots \det(Y + \frac{X}{\lambda_{n_o}}) \qquad \text{(A.8)}$$

where Y, X are the matrix representations of $I\Big|_{\mathcal{N}(\hat{G}_{n_o-1})\to\mathcal{N}(\hat{G}^*_{n_o-1})}$ and $G_{n_o}\Big|_{\mathcal{N}(\hat{G}_{n_o-1})\to\mathcal{N}(\hat{G}^*_{n_o-1})}$. X is full rank ($m_{n_o}$) since there are no higher order unbounded root loci and Y is full rank from the argument in Part I of the proof. Thus there are m_{n_o} finite values of λ_{n_o} satisfying the equation (A.9)

$$\det(Y + \frac{X}{\lambda_{n_o}}) = 0; \qquad \text{(A.9)}$$

Thus the number of solutions in s to the equation in (A.8) are $m_1 + 2m_2 + \ldots + (n_o-1)m_{n_o-1}$ of infinite magnitude and $n_o m_{n_o}$ of finite magnitude [the n_oth roots of the solution to equation (A.9)]. Thus the number of unbounded solutions (with k) to $\det(I+kG(s)) = 0$ is $m_1 + 2m_2 + \ldots + n_o m_{n_o}$. But, from Part I of proof this is precisely the number obtained by our procedure. Thus, all the unbounded solutions to

$$\lim_{k\to\infty} \det(I+kG(s)) = 0$$

are given by the first, second,...,n_oth order unbounded root loci.

Q.E.D.

Proposition V.2 [Computation of 1st and 2nd order unbounded root loci]

Proof: The key to the proof is a lemma requiring the assumption of simple null structure.

Lemma

If a matrix $Q \in C^{n\times n}$ has simple null structure and S.V.D. given by (A.10)

$$Q = [U_1^Q \; U_2^Q] \left[\begin{array}{c|c} \Sigma_1^Q & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^{Q*} \\ \hline V_2^{Q*} \end{array}\right] \tag{A.10}$$

with $V_1^Q \in \mathbb{C}^{n\times n_Q}$; $V_2^Q \in \mathbb{C}^{n\times n-n_Q}$; $U_1^Q \in \mathbb{C}^{n\times n_Q}$; $U_2^Q \in \mathbb{C}^{n\times n-n_Q}$ where n_Q is the rank of Q; then the non-zero eigenvalues of Q are the zeros of (A.11)

$$\det[\,(U_1^{Q*}V_1^Q + U_1^{Q*}V_2^Q(U_2^{Q*}V_2^Q)^{-1}U_2^{Q*}V_1^Q) - \Sigma_1^Q] = 0 \tag{A.11}$$

Proof: From the lemma in Theorem 1 we have $U_2^{Q*}V_2^Q \in \mathbb{C}^{(n-n_Q)\times(n-n_Q)}$ to be invertible. Now, $\det(\lambda I-Q) = 0$ may be written as

$$\det\left[\begin{array}{c|c} \lambda U_1^{Q*}V_1^Q - \Sigma_1^Q & \lambda U_1^{Q*}V_2^Q \\ \hline \lambda U_2^{Q*}V_1^Q & \lambda U_2^{Q*}V_2^Q \end{array}\right] = 0; \tag{A.12}$$

and if $\lambda \neq 0$ we have by elementary column operations

$$\det\left[\begin{array}{c|c} \lambda U_1^{Q*}V_1^Q+\lambda U_1^{Q*}V_2^Q(U_2^{Q*}V_2^Q)^{-1}U_2^{Q*}V_1^Q & 0 \\ \hline \lambda U_2^{Q*}V_1^Q & \lambda U_2^{Q*}V_2^Q \end{array}\right] = 0 \tag{A.13}$$

Equation (A.11) follows readily from the upper triangular system of (A.13) and the fact that $U_2^{Q*}V_2^Q$ is non-singular. ¤

The proof of the proposition now follows in straightforward fashion by inserting the S.V.D.s of G_1 and $U_2^{1*}G_2V_2^1$ into (V.7) and (V.12) respectively.

¤

<u>Proposition V.3</u> [Feedback invariance of unbounded root loci]

<u>Proof</u>: To establish the equality between the unbounded root loci of $G(s)$ and $G_F(s)$ we establish tbe equality of the maps $G_1, G_2\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)}$, $G_3\big|_{\mathcal{N}(\hat{G}_2)\to\mathcal{N}(\hat{G}_2^*)}$, ... and $G_1^F, G_2^F\big|_{\mathcal{N}(G_1^F)\to\mathcal{N}(G_1^{F*})}$, $G_3^F\big|_{\mathcal{N}(\hat{G}_2^F)\to\mathcal{N}(\hat{G}_2^{F*})}$, ...

G_1 is clearly the same as G_1^F and so $\mathcal{N}(G_1) = \mathcal{N}(G_1^F)$ and $\mathcal{N}(G_1^*) = \mathcal{N}(G_1^{F*})$. Since $G_2^F = CAB + CBFB$ and $CBFB\big|_{\mathcal{N}(G_1)\to\mathcal{N}(G_1^*)} = 0$ [the range space of CBFB is contained in the range space of CB and hence orthogonal to $\mathcal{N}(G_1^*)$] we have $G_2^F\big|_{\mathcal{N}(G_1^F)\to\mathcal{N}(G_1^{F*})} = G_2\big|_{\mathcal{N}(\hat{G}_1)\to\mathcal{N}(\hat{G}_1^*)}$. A straightforward calculation verifies the equality of $G_3^F\big|_{\mathcal{N}(\hat{G}_2^F)\to\mathcal{N}(\hat{G}_2^{F*})}$ and $G_3\big|_{\mathcal{N}(G_2)\to\mathcal{N}(G_2^*)}$ and so on. ¤

<u>Proposition VI.1</u> [Formula for $c_{i,1}$ when $\lambda_{i,1}$ has multiplicity 1].

<u>Proof</u>: Equation (VI.4) has a solution for any $G_2 \in \mathbb{C}^{m\times m}$ if the matrix $[-\lambda_{i,1}I+G_1 \,|\, e_{i,o}]$ is full rank. Clearly $e_{i,o} \notin \mathcal{R}(\lambda_{i,1}I-G_1)$ since this would contradict Assumption 3, the simple structure of G_1 at each of its eigenvalues. Furthermore $\mathcal{R}(\lambda_{i,1}I-G_1)$ is of dimension $(m-1)$ since $\lambda_{i,1}$ has multiplicity 1 as an eigenvalue of G_1. Hence the matrix $[-\lambda_{i,1}I+G_1 \,|\, e_{i,o}]$ has rank m.

Since the $\mathcal{N}[-\lambda_{i,1}I+G_1 \mid e_{i,o}] = \mathrm{sp}\left[\begin{pmatrix} e_{i,o} \\ 0 \end{pmatrix}\right] \subset \mathbb{C}^{m+1}$ we notice that any two solutions to (VI.4) yield the same value of $c_{i,1}$. ¤

<u>Proposition VI.2</u> [Formula for $c_{i,1}$ when $\lambda_{i,1}$ has multiplicity p].

<u>Proof:</u> By the simple structure assumption and the same argument as in Proposition VI.1 above we have

$$\mathbb{C}^m = \mathcal{R}(-\lambda_{i,1}I+G_1) \oplus \mathrm{sp}(e_{i,o}^{(1)}) \oplus \cdots \oplus \mathrm{sp}(e_{i,o}^{(p)}) \tag{A.14}$$

Then, by (A.14) we may write

$$G_2 e_{i,o}^{(k)} = \sum_{\ell=1}^{p} \beta_{\ell,k} e_{i,o}^{(\ell)} + (\lambda_{i,1}I-G_1) f_{i,1}^{(k)} \quad k = 1,\ldots,p \tag{A.15}$$

for some $f_{i,1}^{(k)} \in \mathbb{C}^m$ and $\{\beta_{\ell,k}\}_{\ell=1}^{p} \subset \mathbb{C}$. Let $\beta \in \mathbb{C}^{p\times p}$ denote the matrix of $\{\beta_{\ell k}\}_{\substack{\ell=1,\ldots,p \\ k=1,\ldots,p}}$ and $\{\underline{\alpha}^{(k)}\}_{k=1}^{p}$ the eigenvectors of $\underline{\beta}(\subset\mathbb{C}^p)$ with corresponding eigenvalues $\{\delta_k\}_{k=1}^{p}$. Then, we have

$$G_2 \begin{bmatrix} | & & | \\ e_{i,o}^{(1)} & \cdots & e_{i,o}^{(p)} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_{i,o}^{(1)} & \cdots & e_{i,o}^{(p)} \\ | & & | \end{bmatrix} \beta + (-\lambda_{i,1}I+G_1) \begin{bmatrix} | & & | \\ f_{i,1}^{(1)} & \cdots & f_{i,1}^{(p)} \\ | & & | \end{bmatrix}$$

and

$$G_2 \begin{bmatrix} | & & | \\ e_{i,o}^{(1)} & \cdots & e_{i,o}^{(p)} \\ | & & | \end{bmatrix} \underline{\alpha}^{(k)} = \begin{bmatrix} | & & | \\ e_{i,o}^{(1)} & \cdots & e_{i,o}^{(p)} \\ | & & | \end{bmatrix} \delta_k \underline{\alpha}^{(k)} + (-\lambda_{i,1}I+G_1) \begin{bmatrix} | & & | \\ f_{i,1}^{(1)} & \cdots & f_{i,1}^{(p)} \\ | & & | \end{bmatrix} \underline{\alpha}^{(k)}$$

Let $\begin{bmatrix} | & & | \\ f_{i,1}^{(1)} & \cdots & f_{i,1}^{(p)} \\ | & & | \end{bmatrix} \underline{\alpha}^{(k)} = e_{i,1}^{(k)}$

we can then write (A.15) as

$$[-\lambda_{i,1}I+G_1 \mid \sum_{\ell=1}^{p} \alpha_\ell^{(k)} e_{i,o}^{(\ell)}] \begin{bmatrix} e_{i,1}^{(k)} \\ \hline c_{i,1}^{(k)} \end{bmatrix} = \frac{1}{\lambda_{i,1}} \sum_{k=1}^{p} \alpha_\ell^{(k)} G_2 e_{i,o}^{(\ell)}$$

$$\text{with } c_{i,1}^{(k)} = \delta^{(k)}/\lambda_{i,1} \tag{VI.5}$$

[In the event that $\underline{\beta}$ does not have a complete set of eigenvectors, perturb the matrix $\underline{\beta}$ by an arbitrarily small amount so that it has simple structure].

This establishes the first part of proposition (VI.2). The second part follows immediately since the null space of $[-\lambda_{i,1}I+G_1 \mid \sum_{\ell=1}^{p} \alpha_\ell^{(k)} e_{i,o}^{(\ell)}]$ is $sp\left\{\begin{pmatrix} e_{i,o}^{(\ell)} \\ 0 \end{pmatrix}\right\}_{\ell=1}^{p}$ for any $\underline{\alpha}^{(k)} \neq \theta_p$. ¤

Theorem 2 [High gain stability]

Proof: Sufficiency

Condition (i) guarantees that for k sufficient large the finite root loci lie in the open left half plane (neighborhoods of the McMillan zeros). Condition (ii) guarantees that the first order unbounded root loci lie in the open left half plane for k sufficiently large. Conditions (iii) and (iv) together guarantee that the second order unbounded root loci (if any) lie in the open left half plane since by equation (VI.6)

$$s_{i,2} = \sqrt{-k\lambda_{i,2}} + c_{i,2} + 0\left(\frac{1}{\sqrt{k}}\right) \qquad \text{(VI.6)}$$

We show that condition (v) guarantees non-existence of third order unbounded root loci by showing $U_2^{1*}G_2V_2^1$ to be full rank. But this is essentially obvious since

$$\mathbb{R}^m = \mathcal{R}(G_1) + \mathcal{R}(G_2|_{\mathcal{N}(G_1)}) \Rightarrow U_2^{1*}(\mathbb{R}^m) = U_2^{1*}\mathcal{R}(G_1) + U_2^{1*}(\mathcal{R}(G_2|_{\mathcal{N}(G_1)}))$$

$$\Rightarrow sp(U_2^{1*}) = sp(U_2^{1*}G_2V_2^1) = \mathbb{R}^{m-m_1}.$$

It remains to be shown that the closed loop eigenvalues are uniformly bounded away from the $j\omega$ axis for all $k \geq k_o$. To shown, this we use the asymptotic series of equation (VI.1). If $\{z_i\}_{i=1}^p \subset \mathbb{C}$ is the set of intercepts of the asymptotes of the second order unbounded root loci, choose

$$\varepsilon = +\frac{1}{2} \max \text{ real part}\{\{z_i\}_{i=1}^{p}, \{c_{i,2}\}_{i=1}^{m_2}\}.$$

By the hypothesis of the theorem $\varepsilon < 0$. We will establish the existence of k_o such that the closed loop eigenvalues have their real part less than or equal to ε, for all $k \geq k_o$. Clearly, $\exists k_1 < \infty$ such that the p bounded closed loop eigenvalues $\{s_{i,o}(k)\}_{i=1}^{p}$ are such that

$$|s_{i,o}(k)-z_i| < \varepsilon \qquad \forall k \geqq k_1$$
$$\forall i = 1,\ldots,p.$$

Also from the asymptotic series for the first order unbounded root loci, we have

$$s_{i,1}(k) = -k\lambda_{i,1} + c_{i,1} + 0(\tfrac{1}{k}) \quad i = 1,\ldots,m_1$$

with $\mathrm{Re}(\lambda_{i,1}) < 0$ for $i = 1,\ldots,m_1$. Hence $\exists k_2 < \infty$ such that the first order unbounded root loci have their real parts less than or equal to ε for all $k \geqq k_2$. From the asymptotic series for the second order unbounded root loci, we have

$$s_{i,2}(k) = \sqrt{-k\lambda_{i,2}} + c_{i,2} + 0(\tfrac{1}{\sqrt{k}}) \qquad i = 1,\ldots,m_2$$

with $\sqrt{-k\lambda_{i,2}}$ purely imaginary for $i = 1,\ldots,m_2$. Hence, $\exists k_3 < \infty$ such that

$$\mathrm{Re}(s_{i,2}(k)-c_{i,2}) < \varepsilon \qquad \forall k \geqq k_3.$$

We then conclude that for all $k \geqq k_o = \max(k_1,k_2,k_3)$ the closed loop eigenvalues have their real part less than or equal to ε. This completes the sufficiency.

Necessity

The necessity of conditions (ii) and (v) is obvious from the discussion so far. The hypothesis that the closed loop eigenvalues are uniformly bounded away from the $j\omega$ axis for all $k \geq k_o$ necessitates that the McMillan zeros of $G(s)$ lie in the open left half plane. Condition (iii) is necessary since one of the two square roots of $\sqrt{-\lambda_{i,2}}$ has strictly positive real part unless $\lambda_{i,2}$ is real and positive. Further, from the asymptotic series for the second order unbounded root loci,

$$s_{i,2}(k) - \sqrt{-k\lambda_{i,2}} - c_{i,2} = 0(\frac{1}{\sqrt{k}}) \quad i = 1,\dots,m_2,$$

the hypothesis that the closed loop eigenvalues are uniformly bounded away from the $j\omega$ axis for all $k \geqq k_o$ and the fact that $\sqrt{-k\lambda_{i,2}}$ is purely imaginary we obtain the necessity of condition (iv).

This completes the necessity. Q.E.D.

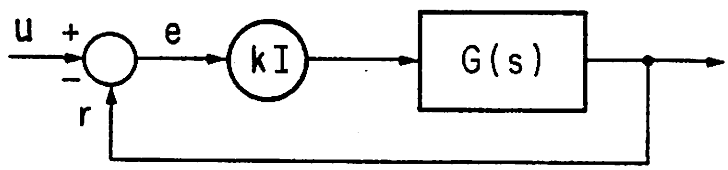

Figure 1. System configuration.

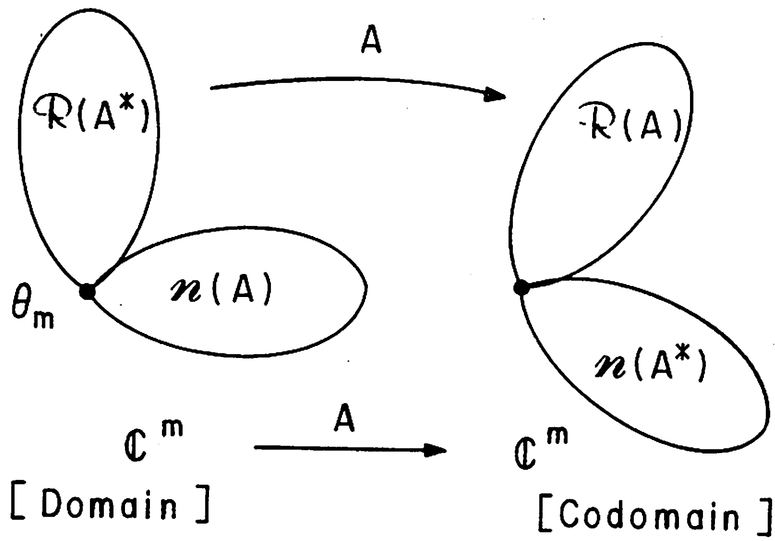

Figure 2. Structure of a linear map.

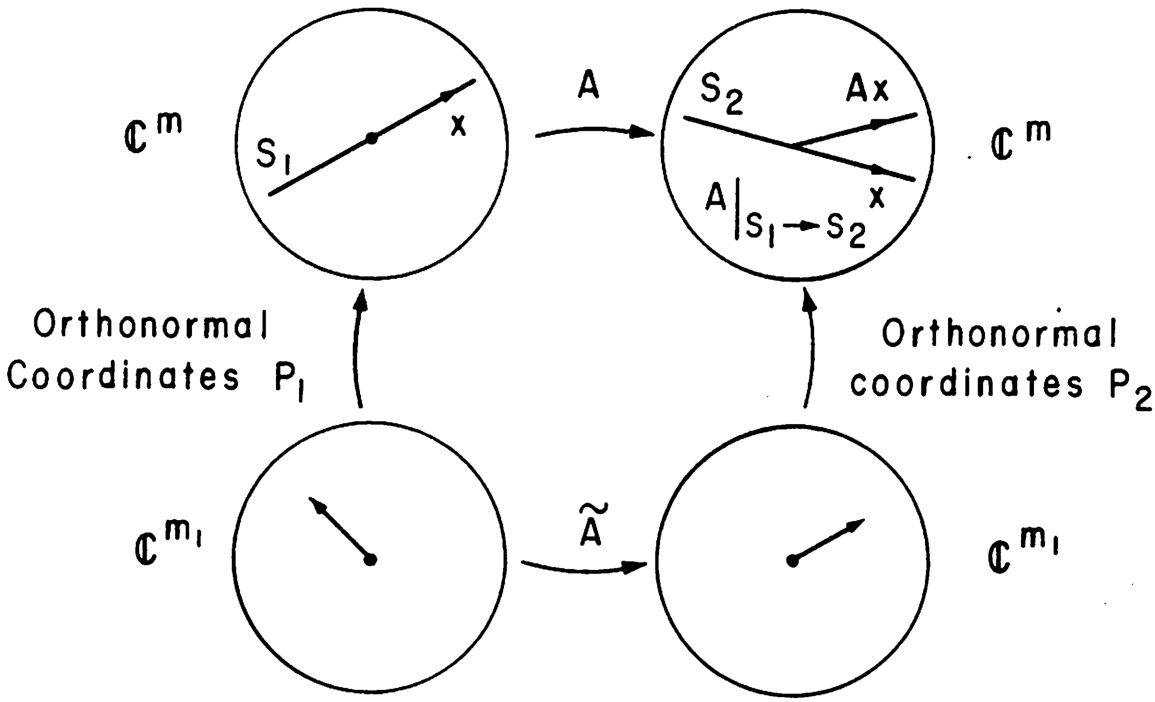

Figure 3. Depiction of an orthonormally coordinatized version of $A|_{S_1 \to S_2}$ as a map $A : \mathbb{C}^{m_1} \to \mathbb{C}^{m_1}$.

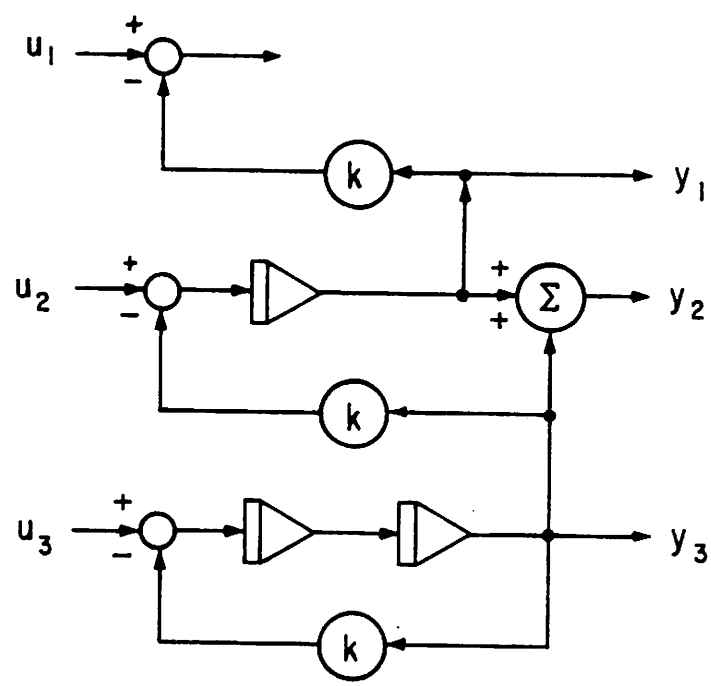

Figure 4. Block diagram for the asymptotic behavior of the system of example 2 showing the triviality of u_1 and $y_2 (=y_1+y_3)$.

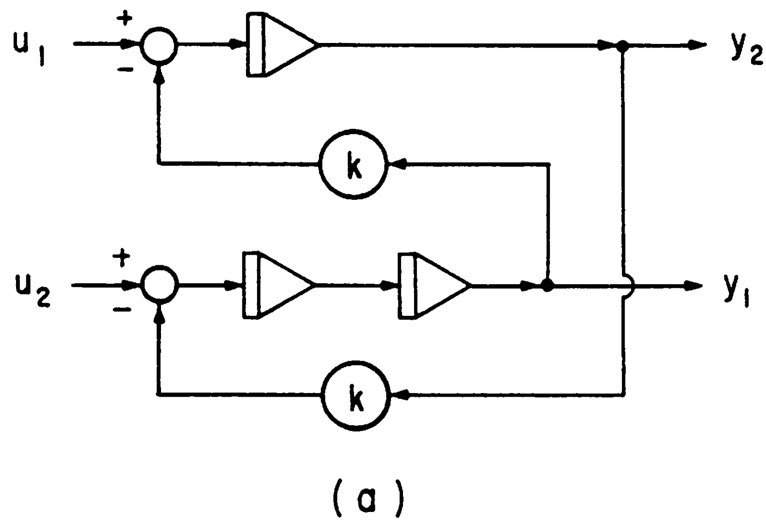

Figure 5a. Block diagram of asymptotic behavior of system violating the simple null structure assumption.

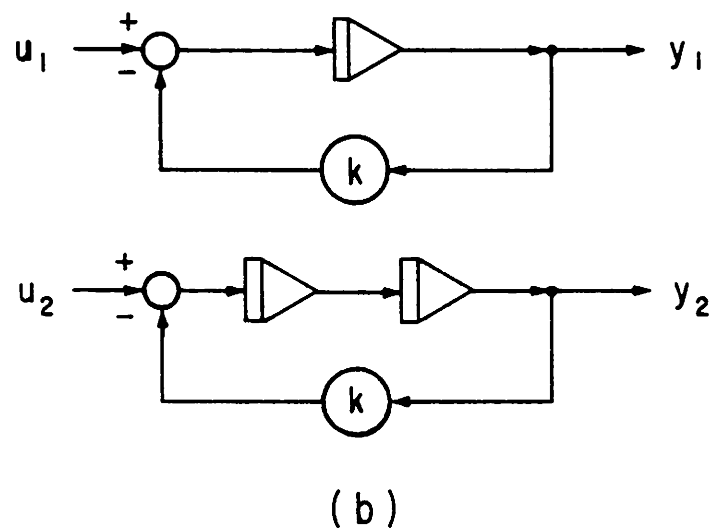

Figure 5b. The same system as in Figure 5a with the outputs relabelled before closing loops.

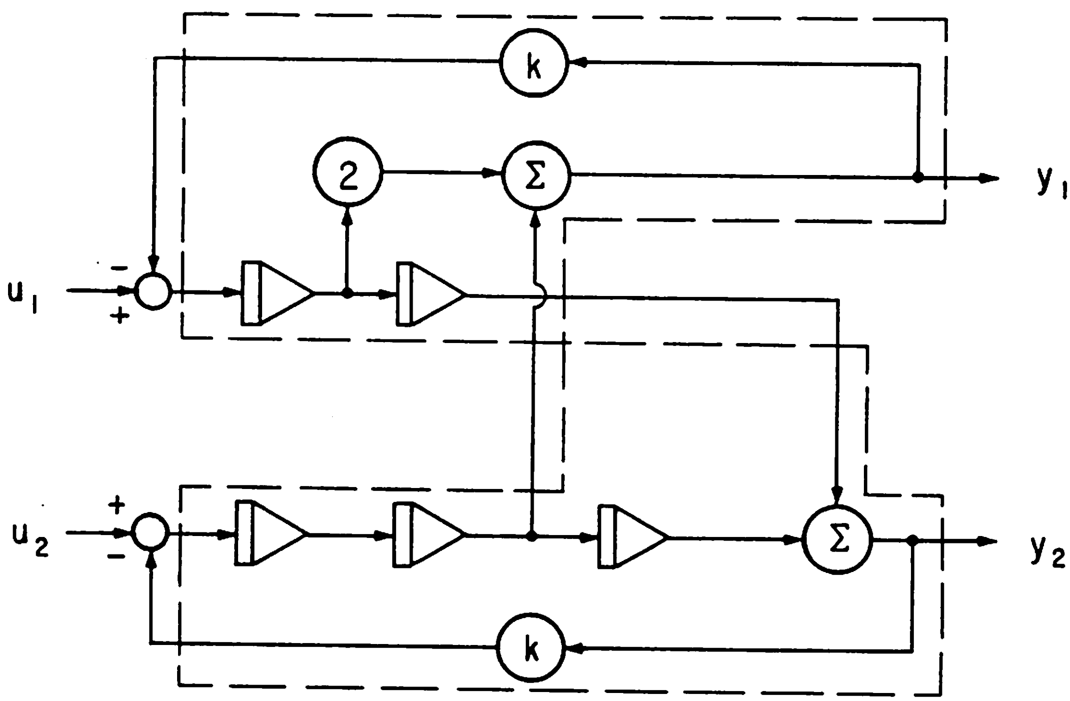

Figure 6. Asymptotic block diagram of a system violating the non-redundancy assumption. (the loop shown dotted is redundant)

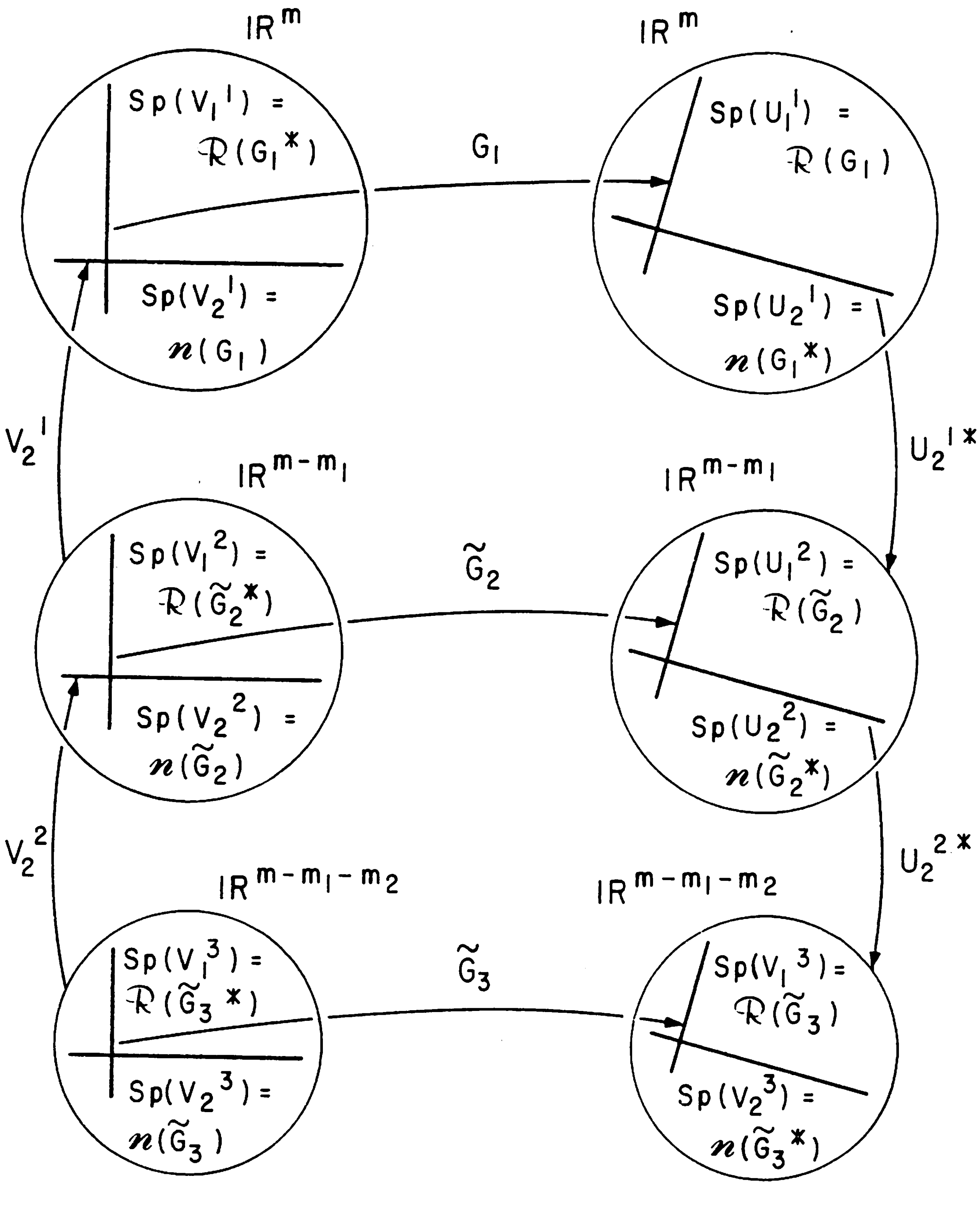

Figure 7. Identification of the subspaces associated with the second and third order unbounded root loci.