THE ROBUSTNESS OF CONTROLLABILITY AND OBSERVABILITY OF LINEAR TIME-VARYING SYSTEMS WITH APPLICATION TO THE EMERGENCY CONTROL OF POWER SYSTEMS

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Abstract

Fixed point methods from nonlinear analysis are used to establish conditions under which the uniform complete controllability of linear time-varying systems is preserved under non-linear perturbations in the state dynamics and the zero-input uniform complete observability of linear time-varying systems is preserved under non-linear perturbation in the state dynamics and output read out map. Algorithms for computing the specific input to steer the perturbed systems from a given initial state to a given final state are also presented.

As an application, a very specific emergency control of an interconnected power system is formulated as a steering problem and it is shown that this emergency control is indeed possible in finite time.

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Section I. Introduction

Controllability and observability are key issues in system theory. To be specific, consider a class of physical dynamical systems which are adequately modelled by ordinary differential equations with inputs u and a static read-out map h: more precisely,

$$\dot{x} = f(x,u,t) \tag{I.1}$$

$$y = h(x,t) \tag{I.2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $y \in \mathbb{R}^{n_o}$, $t \in \mathbb{R}_+$ and f, h C^o functions; f satisfies Lipschitz and growth conditions so that solutions exist, are unique and can be extended to all of $\mathbb{R}_+$. In optimal control, it is well known (see for e.g. [29]) that controllability has fundamental interconnections with the existence of optimal controls and their feedback synthesis. In process control, controllability and observability are crucial in the study of stabilizability of plants. On a more abstract level, it has been shown by Willems [23, 24] that if a dynamical system is completely controllable and observable in a suitably defined sense, input-output properties (notably, finite gain stability and dissipativeness) are reflected into properties of the state space description of the system (as global asymptotic stability in the sense of Lyapunov and the existence of a storage function, respectively). In the theory of diffusions arising from dynamical systems the question of the existence of a probability density for the diffusion, posed by Ito, have been answered by Elliott [9] in terms of controllability of the underlying dynamical system. Finally in what is perhaps the best known application of the concepts of complete controllability and observability we have Kalman's results (see for e.g. [4]) on the minimal realization of linear dynamical systems. The reader will notice that we have been loose with our use of controllability and observability for a variety of related but not identical notions. Precise definitions of these for our purposes are relegated to Sections III and VIII. Also, for the purposes of this paper the state space, input space and output space are all vector spaces.

In view of their obvious importance there is a rather large literature on the controllability and observability of non-linear systems. We will not be exhaustive in briefly reviewing it, but will point out what we feel to be three approaches to this issue in the literature:

(i) the differential geometric approach developed by Brockett [5], Hermann [12], Krener [14], Lobry [16], Sussmann and Jurdjevic [22]. The most comprehensive survey appears in a recent paper of Hermann and Krener [13].

(ii) the nonlinear analysis approach to null controllability (i.e. controllability to the origin) using classical Lyapunov theory and the more recently introduced theory of cone valued Lyapunov function. This approach has been developed among others by Chukwu [6] and Sinha [20].

(iii) the global analysis approach to the zero-input observability of Morse-Smale dynamical systems, due to Aeyels [1]; see also Aeyels and Elliott [2]. Some other work which does not fit under any of these headings are the paper on global (complete) observability of non-linear systems by Yamamoto and Sugiura [25] and the paper on global (complete) controllability of non-linear systems by Lukes [16].

The results as they stand in the differential geometric approach have reached final form for C^∞ systems with control u appearing linearly (i.e., with $f(x,u,t) = f_1(x) + f_2(x)u$ for suitably chosen $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{n \times n_i}$, C^∞ functions). Implicit in the results is the assumption of high differentiability. The results are of necessity local in nature and the causality of physical dynamical systems is lost in the formulation of the results. Further, the results are of an "existence " nature so that they do not explicity give controls for steering the system between prescribed states.

In the null controllability results using Lyapunov and cone valued Laypunov techniques, conditions are imposed on the non linearities of the state dynamics so as to make the domain of null controllability the entire state space. The results are rather restrictive in that they discuss only controllability to the origin.

In the global analysis [1,2] approach we have a sufficient condition for the global (complete) observability of Morse-Smale systems to be the rank condition of Kalman for complete observability of linear sytems applied to the linearized dynamics and linearized output map at each of the (finitely many) fixed points and orbits of the flow. The tools used are the properties of Morse-Smale systems and a Banach space implicit function theorem. In the paper of Yamamoto and Sugiura the contraction mapping theorem (see for e.g. Marsden [17]) is used to obtain some results for the observability of non linear systems with "small" nonlinearities. In the paper of Lukes, controllability of an autonomous dynamical system is treated as a boundary value problem and sufficient conditions for the controllability of certain perturbed, linear time-invariant systems is derived using compactness arguments (Arzela-Ascoli theorem) in a Banach space. In fact, Theorem V.1 of the present paper has also been proven by Lukes. The present proof is of course new.

We now discuss the philosophy of our approach: In the present paper, we take the engineers view of complete controllability: $\exists\ T \in \mathbb{R}_+$ such that given any t_o, initial time, and any two states x_o, the initial state and x_1, the final state, there exists a control that will steer the system from x_o at t_o to x_1 at $t_o + T$. The same view is held of zero-input observability: $\exists\ T \in \mathbb{R}_+$ such that given any t_o and the output of the system with zero-input on $[t_o, t_o + T]$ we can determine (uniquely) the state of the system at time t_o. In keeping with our view point we give, wherever we prove complete controllability, a procedure for obtaining explicitly a control law to perform any required steering and wherever we prove complete observability, a procedure for obtaining explicitly the initial state of the system. Since our results are global (complete) controllability and observability results for nonlinear systems which are in some sense close to being linear, we choose to think of our results as being robustness results for the uniform controllability and observability of linear time-varying systems (precise definitions are given in Sections III and VIII) in the presence of nonlinear perturbations of various types.

The major mathematical tool for the paper is a solvability theorem for operator equations with a quasibounded nonlinearity, due to Granas [11], which is reminiscent of the small gain theorem (see for e.g. Desoer and Vidyasagar [7]). The heart of the theroem lies in the Rothe (or equivalently the Schauder) fixed point theorems, which are essentially topoligical tools in nonlinear analysis.

We illustrate the use of our results in the derivation of control laws, during a very specific emergency, for interconnected power systems, by posing the emergency control problem as a steering problem. That this formulation is indeed the right one for emergency control has been suggested by a recent research report [10]. We mention that another application may be in economics for establishing the existence of homeostatic trajectories for certain adapting economic systems which satisfy differential equations rather than inclusions, as is suggested in a paper of Aubin and Day [3].

CONTENTS

Section II. Notation

The dynamical systems that we study are differential dynamical systems (DDS) with finite dimensional vector spaces as input, output and state space, respectively $\mathbb{R}^{n_i}$, $\mathbb{R}^{n_o}$ and $\mathbb{R}^n$ with the representation

$$\dot{x} = f(x,u,t) \tag{II.1}$$

$$y = h(x,t) \tag{II.2}$$

where $t \in \mathbb{R}_+$, f is a C^o function from $\mathbb{R}^n \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^n$ which is globally Lipschitz continuous in its first argument (to guarantee uniqueness of solution to (2.1) when the initial condition is given) and h is a C^o function from $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n_o}$. <u>Finite Dimensional Linear Dynamical Systems</u> (FDLS) with a <u>bounded realization</u> are differential dynamical systems of the form (II.3), (II.4)

$$\dot{x} = A(t)x + B(t)u \tag{II.3}$$

$$y = C(t)x \tag{II.4}$$

with $\|A(\cdot)\|$, $\|B(\cdot)\|$, $\|C(\cdot)\|$ bounded on $\mathbb{R}_+$. (II.5)

Section III. <u>Characterization of controllability for finite dimensional linear systems</u>.

The definitions and propositions of this section are well known, though not standardized. We restate them here to establish the terminology and notation. The definitions aredrawn from Silverman [19] and the proofs may be found in standard books (see for e.g. [4]).

<u>Definition III.1</u>. (Uniform complete controllability (UCC))

A differential dynamical system represented by (II.1), (II.2) is said to be <u>uniformly completely controllable</u> if $\exists T > 0$ such that $\forall t_o \in \mathbb{R}_+$ and $\forall x_o, x_1 \in \mathbb{R}^n$, $\exists$ an input $u \in L_2^{n_i}([t_o, t_o+T])$ which drives the system from $x(t_o) = x_o$ to $x(t_o+T) = x_1$. ∎

For FDLS with bounded realization a simple characterization of UCC accrues from the fact that equation (II.3) can be solved explicitly on $[t_o, t_o+T]$ given $x(t_o) = x_o$ and input $u \in L_2^{n_i}([t_o, t_o+T])$ to yield equation (III.1).

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\tau)\; B(\tau)\; u(\tau)d\tau \tag{III.1}$$

$$\forall t \in [t_o, t_o+T]$$

where $\Phi(t,t_o) \in \mathbb{R}^{nxn}$ denotes the fundamental solution of the homogeneous matrix equation

$$\dot{X}(t) = A(t)\; X(t), \quad X(t_o) = I \tag{II.2}$$

with $X(t) \in \mathbb{R}^{nxn}$.

To obtain the desired characterization define, for fixed $t_o \in \mathbb{R}_+$, the linear map $\mathcal{L}_R$ (called the <u>reachability map</u>) from $L_2^{n_i}([t_o,t_o+T])$ to $\mathbb{R}^n$ by

$$\mathcal{L}_R u = \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\; B(\tau)\; u(\tau)d\tau \tag{III.3}$$

Then at $t = t_o + T$, equation (III.1) may be rewritten as:

$$x(t_o+T) = \Phi(t_o+T,t_o)x_o + \mathcal{L}_R u \tag{III.4}$$

The adjoint map of $\mathcal{L}_R$, denoted $\mathcal{L}_R^*$, then is the linear map from $\mathbb{R}^n$ to $L_2^{n_i}([t_o,t_o+T])$ defined by

$$\mathcal{L}_R^* x = B^*(\cdot)\; \Phi^*(t_o+T,\cdot)x \tag{III.5}$$

Since the realization (II.3) is bounded (say by K) we have from the Bellman Gronwall lemma that

$$\|\phi(t,\tau)\| \leqq \exp K(t-\tau) \qquad \forall t,\tau \in \mathbb{R}_+ \tag{III.6}$$

Also,

$$\|B(\tau)\| \leqq K \qquad \forall \tau \in \mathbb{R}_+ \tag{III.7}$$

Using (III.6), (III.7) it is easy to check that $\mathcal{L}_R$ and $\mathcal{L}_R^*$ are continuous linear maps.

Theorem III.1. (Characterization of Uniform Complete controllability for FDLS)

The FDLS with bounded realization represented by (II.3) and (II.4) is uniformly completely controllable $\Leftrightarrow \forall t_o \in \mathbb{R}_+$, the reachability map $\mathcal{L}_R : L_2^{n_i}([t_o,t_o+T]) \to \mathbb{R}^n$ defined in (III.3) is onto $\Leftrightarrow \forall t_o \in \mathbb{R}_+$ the composition of the reachability map and its adjoint namely $\mathcal{L}_R\mathcal{L}_R^* : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection.
¤

Comment

The second characterization of uniform complete controllability is particularly handy since it is in terms of the rank of a linear map $\mathcal{L}_R\mathcal{L}_R^* : \mathbb{R}^n \to \mathbb{R}^n$. The properties of this linear map will be of use in the sequel and hence we define its representation explicitly.

Definition III.2. (Reachability grammian)

Given $t_o \in \mathbb{R}_+$, the matrix representation of the continuous linear map $\mathcal{L}_R\mathcal{L}_R^* : \mathbb{R}^n \to \mathbb{R}^n$ is the reachability grammian, denoted $W_R[t_o,t_o+T] \in \mathbb{R}^{n\times n}$

$$W_R[t_o,t_o + T] = \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\; B(\tau)\; B^*(\tau)\; \Phi^*(t_o+T,\tau)d\tau \qquad \text{(III.8)}$$
¤

Thus, the FDLS with bounded realization given by equations (II.3), (II.4) is uniformly completely controllable iff $W_R[t_o,t_o+T]$ is non-singular $\forall t_o \in \mathbb{R}_+$. Notice, however, that the FDLS can be uniformly completely controllable with the smallest eigenvalue of $W_R[t_o,t_o+T]$ tending to 0 as $t_o \to \infty$.

In addition to providing a test for uniform complete controllability the reachability grammian provides information about the minimum size of the L_2 norm (energy) of the input required to make the transfer from $x_o \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}^n$ in $[t_o,t_o+T]$ as stated below.

Proposition III.1. (Least L_2 norm of control)

If the FDLS with bounded realization given by equations (II.3), (II.4) is uniformly completely controllable; then, the least L_2 norm of the control required to transfer the system from x_o at t_o to x_1 at t_o+T is given by

$$[(x_1-\Phi(t_o+T,t_o)x_o)^*(W_R[t_o,t_o+T])^{-1}(x_1-\Phi(t_o+T,t_o)x_o)]^{1/2} \tag{III.9}$$

¤

Comment. From Proposition (III.1) it follows that the least L_2 norm control required to reach $x_1 \in \mathbb{R}^n$ at $t_o + T$ from the origin at t_o is given by

$$[x_1^*(W_R[t_o,t_o + T])^{-1}x_1]^{1/2}$$

Uniform complete controllability does not guarantee that this quantity is bounded $\forall x_1 \in \mathbb{R}^n$ and $\forall t_o \in \mathbb{R}_+$ as was noted after definition (II.2). To guarantee this we define a slightly stronger form of controllability.

Definition (III.3). (Strong Uniform Complete Controllability)

A FDLS with bounded realization represented by (II.3) and (II.4) is strongly, uniformly completely controllable if $\exists T > 0$, $\lambda_s > 0$ such that $\forall t_o \in \mathbb{R}_+$

$$W_R[t_o,t_o + T] \geqq \lambda_s^2 I \tag{III.10}$$

¤

Comments

(i) The boundedness of the realization guarantees that $\exists \lambda_L \in \mathbb{R}_+$ such that $\forall t_o \in \mathbb{R}_+$

$$\lambda_L^2 I \geqq \sup_{\tau\in[0,T]} W_R[t_o,t_o + \tau] \tag{III.11}$$

(ii) It is obvious that it is more costly to reach certain directions than others (the cost is the minimal L_2 norm of the input required to reach a unit norm vector in a certain direction starting from the origin). Motivated by the condition number of numerical analysis (see for example, Ortega [28]) we define the reachability condition over T seconds of a strongly uniformly controllable FDLS.

<u>Definition III.4</u>. (Reachability condition number)

Consider a strongly uniformly controllable FDLS with bounded realization, let $\tilde{\lambda}_L > 0$ and $\tilde{\lambda}_s > 0$ be defined by

$$\tilde{\lambda}_L = \sup_{t_o \in \mathbb{R}_+} \sup_{t \in [0,T]} \lambda_{max}(W_R[t_o, t_o + T])^{1/2} \tag{III.12}$$

and

$$\tilde{\lambda}_s = \inf_{t_o \in \mathbb{R}_+} \lambda_{min}(W_R[t_o, t_o + T])^{1/2} \tag{III.13}$$

then the <u>reachability condition number over T seconds</u> χ_R is defined by

$$\chi_R = \frac{\tilde{\lambda}_L}{\tilde{\lambda}_S} \tag{III.14}$$

The burden of this paper consists in demonstrating the <u>robustness</u> of strong <u>uniform complete controllability of an FDLS in the face of nonlinear perturbations in the dynamics</u> both bounded and unbounded. Our methods seem to indicate that FDLS with smaller reachability condition number are more robust than others with larger reachability condition number.

Section IV. <u>Solvability of an operator equation with a quasibounded nonlinearity in normed spaces</u>

The main mathematical tool used in the investigation of the robustness of controllability is a solvability theorem for an operator equation in normed spaces with a quasibounded nonlinearity proved in its present form by Granas [11]; see also Mawhin [18]. The heart of the theorem lies in fixed point methods in nonlinear analysis: specifically, the Rothe fixed point theorem which we state in the Appendix. For details, the reader is referred to the excellent monograph of Smart [21].

Definition IV.1. [Quasibounded maps]

Given $\mathcal{X}$ and $\mathcal{Y}$ Banach spaces with respective norms $|\cdot|_{\mathcal{X}}$ and $|\cdot|_{\mathcal{Y}}$ and F a map from $\mathcal{X}$ to $\mathcal{Y}$, F is said to be quasibounded if the number

$$\rho(F) := \inf_{0 \leq \rho < \infty} \sup_{|x|_{\mathcal{X}} \geq \rho} \frac{|F(x)|_{\mathcal{Y}}}{|x|_{\mathcal{X}}} \tag{IV.1}$$

is finite and this number is called the quasinorm of F. ¤

Comments

(i) A continuous linear map is quasibounded and its quasinorm corresponds to the usual induced norm.

(ii) If for instance for some $c_1, c_2, c_3 \in \mathbb{R}$

$$|F(x)|_{\mathcal{Y}} \leq c_1 |x|_{\mathcal{X}} + c_2 \qquad \forall x \in \{x : |x|_{\mathcal{X}} \geq c_3\} \tag{IV.2}$$

(that is, (IV.2) holds for all $x \in \mathcal{X}$ outside a ball of radius c_3) then F is quasibounded and its quasinorm is less than or equal to c_1. In particular, if $c_1 = 0$ then the quasinorm of F is zero

(iii) If F is a compact map on $\mathcal{X}$ then F is quasibounded $\Leftrightarrow |F(x)|_{\mathcal{Y}} \leq c_1 |x|_{\mathcal{X}} + c_2$ for some $c_1, c_2 \in \mathbb{R}$.

(Recall that a continuous map $F : \mathcal{X} \to \mathcal{Y}$ is said to be compact if the closure of the image of any bounded set is compact).

Theorem IV.1. (Solvability Theorem)

If $F:\mathcal{X}\to\mathcal{X}$ is a continuous, quasibounded, compact map on the Banach space $\mathcal{X}$ and if

$$\rho(F) < 1$$

then the equation

$$x + F(x) = y \tag{IV.3}$$

has at least one solution for every $y \in \mathcal{X}$.

Proof. Let y_o be an arbitrary point in $\mathcal{X}$. We shall prove that $\exists\, x_o$ such that $x_o + F(x_o) = y_o$. Let $\bar{F}:\mathcal{X}\to\mathcal{X}$ be a compact map defined by

$$\bar{F}(x) = y_o - F(x) \qquad \text{for } x \in \mathcal{X}$$

Now $\rho(F) < 1$ implies that

$$\frac{|F(x)|}{|x|} < \delta < 1 \qquad \text{for } |x| \geq r_1$$

where δ and r_1 are some constants. Choose $\varepsilon > 0$ such that $\varepsilon + \delta < 1$ and define $r := \max(r_1, |y_o|/\varepsilon)$. Now $S_r = \{x \in X : |x| = r\}$ is the (topological) boundary of the ball $B_r = \{x \in X : |x| = r\}$ and for $x \in S_r$ we have

$$\frac{|\bar{F}(x)|}{|x|} \leqq \frac{|y_o|}{|x|} + \frac{|F(x)|}{|x|}$$

hence

$$\frac{|\bar{F}(x)|}{|x|} \leqq \varepsilon + \delta \qquad \text{(by the definition of } r\text{)}$$

By the definition of ε we have

$$|\bar{F}(x)| < |x| \qquad \forall x \in S_r. \tag{IV.4}$$

From (IV.4), $\bar{F}(S_r) \subset B_r$. Since $\bar{F}$ is a compact map we have by the Rothe fixed point theorem (see Appendix) that $\bar{F}$ has at least one fixed point $x_o \in B_r$. Hence,

$$\bar{F}(x_o) = y_o - F(x_o) = x_o.$$

completing the proof. ¤

Comment.

Theorem IV.1 bears a resemblance to the well known small-gain theorem in the analysis of feedback systems (see for e.g. [7]), if the operator F were thought of as representing the plant in a unity feedback gain control system. At the cost of a topological restriction (continuous, compact) on the plant operator F (there are no topological restrictions in the small gain theorem) Theorem (IV.1) yields the existence of a bounded solution to the feedback equation (IV.3) for every $y \in \mathcal{X}$ provided the "asymptotic gain" (quasinorm) of F is less than 1.

Section V. Robustness of strong uniform complete controllability under bounded perturbations in the dynamics

In this section we consider the uniform, complete controllability of the F.D.L.S. of (II.3), (II.4) whose dynamics are perturbed by a bounded C^o function $h : \mathbb{R}^n \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^n$ which is in addition globally Lipschitz continuous in its first argument (to assure uniqueness of solution of the resulting differential equation, given the initial condition) to give the state evolution equation of (V.1)

$$\dot{x}(t) = A(t)x + B(t)u + h(x,u,t) \tag{V.1}$$

with

$$\sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^{n_i}, t\in\mathbb{R}_+} |h(x,u,t)| = K_o < \infty \qquad (V.2)$$

Such a perturbation might arise from the study of a DDS of the form (II.1) which is in addition "almost linear" in the sense of (V.1), (V.2) above. Yet another application of the study of such perturbation is illustrated in Section X for emergency control of an interconnected power system. In Section (V.1) we prove the main result of this section which is the following theorem.

Theorem V.1. (Robustness of uniform complete controllability under bounded perturbations in the dynamics)

Given that the FDLS with bounded realization of equations (II.3), (II.4) is strongly uniformly completely controllable over T seconds, the perturbed system represented by equations (V.1), (V.2) is uniformly completely controllable over T seconds. ¤

In Section (V.2) we give an algorithm for the computation of an input u to take the perturbed system from any initial state $x_o \in \mathbb{R}^n$ (at t_o) to any final state $x_1 \in \mathbb{R}^n$ (at t_o+T). The proof of the existence of accumulation points in the algorithm involves the use of the Arzela Ascoli theorem.

V.1 Proof of Theorem (V.1)

Fix $t_o \in \mathbb{R}_+$; $x_o \in \mathbb{R}^n$ the initial state; define $x_1(t)$, $x_2(t)$ to be the state of the FDLS and the perturbed FDLS respectively at time $t \in [t_o, t_o+T]$. Then, we have

$$\dot{x}_1(t) = A(t)\, x_1(t) + B(t)\, u(t), \quad x_1(t_o) = x_o \qquad (V.3)$$

and

$$\dot{x}_2(t) = A(t)\, x_2(t) + B(t)\, u(t) + h(x_2(t),u(t),t), \quad x_2(t_o) = x_o \qquad (V.4)$$

Subtracting equation (V.4) from (V.3) and defining $\Delta x := x_2 - x_1$ we have

$$\Delta\dot{x} = A(t)\Delta x + h(x_2(t),u(t),t), \quad \Delta x(t_o) = \theta_n \tag{V.5}$$

To obtain a bound on $|\Delta x(t_o+T)|$ define the continuous linear map $\mathcal{L}$ from $L_2^n([t_o,t_o+T])$ to $\mathbb{R}^n$ by

$$\mathcal{L}(v) = \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\ v(\tau)d\tau \tag{V.6}$$

with $\Phi(t_o+T,\cdot)$ as defined in equation (III.2) (continuity of $\mathcal{L}$ follows from the boundedness of the realization and equation (III.6)). Then,

$$|\Delta x(t_o+T)| \leqq |\mathcal{L}|_i\ \|v(\cdot)\|$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$

$\|\cdot\|$ stands for the usual L_2 norm on $[t_o,t_o,T]$

and $|\ |_i$ stands for the operator norm induced on a linear map from $L_2^n\ ([t_o,t_o\ +T])$ to $\mathbb{R}^n$ by the above norms,

with $v(t) := h(x_2(t),u(t),t)$.

From (V.2) we have $\forall t \in [t_o,t_o,+T]$

$$|v(t)| \leqq K_o$$

so that

$$\|v(\cdot)\| \leqq K_o T^{1/2}.$$

and

$$|\Delta x(t_o+T)| = |\mathcal{L}|_i K_o T^{1/2} \quad \forall u \in L_2^{n_i}([t_o,t_o+T]) \tag{V.7}$$

Now, think of $\Delta x(t_o+T)$ as the value of a (continuous) map $N_{x_o} : L_2^{n_i}([t_o,t_o+T])$ to $\mathbb{R}^n$ (the subscript x_o emphasizing that the map

N depends on x_o) with

$$\Delta x(t_o+T) := N_{x_o}(u) = \int_{t_o}^{t_o+T} \Phi(x_o+T,\tau)\; h(x_2(\tau),u(\tau),\tau)d\tau \tag{V.8}$$

where $x_2(\cdot)$ satisfies (V.4). Then, observe that N_{x_o} is a quasibounded nonlinear map (actually bounded) with quasinorm 0 (independent of x_o). Also, with the definition of $\mathcal{L}_R$ from Section III

$$x_1(t_o+T) = \mathcal{L}_R(u) + \Phi(t_o+T,t_o)x_o \tag{V.9}$$

and

$$x_2(t_o+T) = \mathcal{L}_R(u) + N_{x_o}(u) + \Phi(t_o+T,t_o)x_o$$

Uniform complete controllability of the FDLS guarantees that $\mathcal{L}_R$ is onto. To show the uniform complete controllability of the perturbed system we will show that $(\mathcal{L}_R+N_{x_o})$ is onto for each $x_o \in \mathbb{R}^n$. Infact we will show that the image under $\mathcal{L}_R+N_{x_o}$ of a finite dimensional subspace of $L_2^{n_i}([t_o,t_o+T])$ is $\mathbb{R}^n$ using Theorem (IV.1).

Define $\mathcal{M} := \mathcal{L}_R^*(\mathbb{R}^n)$, an n dimensional (by uniform complete controllability) subspace of $L_2^{n_i}([t_o,t_o+T])$. Clearly $\mathcal{L}_R$ is a bijection of $\mathcal{M}$ onto $\mathbb{R}^n$ and we can define the inverse of $\mathcal{L}_R$ on $\mathcal{M}$; $\mathcal{L}_R^{-1} : \mathbb{R}^n \to \mathcal{M}$ a continuous linear map with $\|\mathcal{L}_R^{-1}\| \leq \tilde{\lambda}_s^{-1}$ where $\tilde{\lambda}_s$ is as defined in Definition III.4 (equation (III.13)). Now consider the map $(I+\mathcal{L}_R^{-1}N_{x_o})$ $: \mathcal{M} \to \mathcal{M}$. Clearly $\mathcal{L}_R^{-1}N_{x_o}$ is a compact map ($\mathcal{M}$ is finite dimensional and N_{x_o} is continuous with quasinorm 0. Hence, by Theorem (IV.1) $I + \mathcal{L}_R^{-1}N_{x_o}$ is onto $\mathcal{M}$ and further $\mathcal{L}_R + N_{x_o}$ is onto $\mathbb{R}^n$. Since $x_o \in \mathbb{R}^n$ and $t_o \in \mathbb{R}_+$ are arbitrary we have proved that the perturbed DDS of (V.1), (V.2) is uniformly completely controllable over T seconds.

Q.E.D.

Comments: (i) Strictly speaking strong uniform complete controllability of the FDLS is not required for the proof of the Theorem V.1 - uniform complete controllability suffices.

(ii) Since we have shown that the subspace $\mathcal{M}$ of controls is sufficient to steer the system from any x_o at t_o to any x_1 to t_o+T it should be possible to give the specific input required to make the required transfer. This question is taken up next.

V.2. Algorithm for the computation of an input to transfer the perturbed system from $x_o \in \mathbb{R}^n$ at t_o to $x_1 \in \mathbb{R}^n$ at $t_o + T$

Step 0. Set $k = 0, x^o(t) = x_o, u^o(t) = \theta_{n_i} \quad \forall t \in [t_o, t_o+T]$

Step 1. Define

$$x^{k+1}(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\tau)\, B(\tau)\, u^k(\tau)d\tau$$

$$+ \int_{t_o}^{t} \Phi(t,\tau)\, h(x^k(\tau),u^k(\tau),\tau)d\tau \qquad (V.10)$$

$$u^{k+1}(\cdot) = \mathcal{L}_R^*(\mathcal{L}_R\mathcal{L}_R^*)^{-1}[x_1 - \Phi(t_o+T,t_o)x_o$$

$$- \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\, h(x^k(\tau),u^k(\tau),\tau)d\tau] \qquad (V.11)$$

Step 2. Set $k = k+1$; go to Step 1

Proposition V.1. (Convergence of Algorithm)

(i) There exists at least one accumulation point (in the L_∞ sense) of the sequence of $(x^k(\cdot))_{k=1}^{\infty}$ say $x_\infty(\cdot)$ with corresponding input $u_\infty(\cdot)$ defined on $[t_o,t_o,T]$ satisfying $x_\infty(t_o) = x_o$ and $x_\infty(t_o+T) = x_1$.

(ii) For any accumulation point (in the L_∞ sense) of $(x^k(\cdot))_{k=1}^{\infty}$ say $\tilde{x}_\infty(\cdot)$, $\exists$ a control $\tilde{u}_\infty(\cdot)$ such that $\tilde{x}_\infty(t_o) = x_o$ and $\tilde{x}_\infty(t_o+T) = x_1$.

Proof. From equation (V.11) and the fact that

(i) $|h(x,u,t)| \leqq K_o \quad \forall x \in \mathbb{R}^n,\ u \in \mathbb{R}^{n_i},\ t \in \mathbb{R}_+$

(ii) $\mathcal{L}_R^* = B^*(\cdot)\ \phi^*(t_o+T,\cdot)$ with $B^*(\cdot)$ and $\phi^*(t_o+T,\cdot)$ bounded on $[t_o,t_o+T]$ (since the FDLS has a bounded realization) we may conclude that the $(u^k)_{k=1}^{\infty}$ are uniformly bounded on $[t_o,t_o+T]$, i. e. for some K_1 independent of k

$$|u^k(t)| \leqq K_1 \quad \forall t \in [t_o,t_o+T]$$

We use this bound in equation (V.10) to conclude that the sequence of continuous functions $(x^k(\cdot))_{k=1}^{\infty}$ is uniformly bounded on $[t_o,t_o+1]$:

i.e. $|x^k(t)| \leqq K_2 \quad \forall t \in [t_o,t_o+T],$

for some K_2 independent of k and that the sequence $(x^k(\cdot))_{i=1}^{\infty}$ is equicontinuous by the following series of inequalities with $t,s \in [t_o,t_o+T]$:

(i) $$|x^{k+1}(t) - x^{k+1}(s)| \leqq |\Phi(t,t_o)-\Phi(s,t_o)|_i\, |x_o|$$
$$+ \left|\int_{t_o}^{t} \Phi(t,\tau)\ B(\tau)\ u^k(\tau)d\tau - \int_{t_o}^{s} \Phi(s,\tau)\ B(\tau)\ u^k(\tau)d\tau\right|$$
$$+ \left|\int_{t_o}^{t} \Phi(t,\tau)\ h(x^k(\tau),u^k(\tau),\tau)d\tau\right.$$
$$\left. - \int_{t_o}^{s} \Phi(s,\tau)\ h(x^k(\tau),\ u^k(\tau),\tau)d\tau\right|$$

(ii) $$\left|\int_{t_o}^{t} \Phi(t,\tau)\ B(\tau)\, u^k(\tau)d\tau - \int_{t_o}^{s} \Phi(s,\tau)\ B(\tau)\, u^k(\tau)d\tau\right|$$
$$\leqq \left|\int_{s}^{t} \Phi(t,\tau)\ B(\tau)\, u^k(\tau)d\tau\right| + \left|\int_{t_o}^{s} [I-\Phi(t,s)]\Phi(s,\tau)B(\tau)u^k(\tau)d\tau\right|$$
$$\leqq K_4|t-s| + K_5|I-\Phi(t,s)|_i \quad \text{for some } K_4,K_5 \text{ independent of k.}$$

(iii) $|\Phi(t,t_o) - \Phi(s,t_o)|_i \; |x_o| \leq K_6 |I-\Phi(t,s)|_i$

(iv) By steps similar to (ii) above

$$\left|\int_{t_o}^{t} \Phi(t,\tau)\; h(x^k(\tau),\, u^k(\tau),\tau)d\tau - \int_{t_o}^{s} \Phi(s,\tau)\; h(x^k(\tau),\, u^k(\tau),\tau)d\tau\right|$$

$$\leq K_7|t-s| + K_8|I-\Phi(t,s)|_i \quad \text{for some } K_7,\, K_8 \text{ independent of } k.$$

Combining (i)-(iv) we have

$$|x^{k+1}(t)-x^{k+1}(s)| \leq (K_4+K_7)|t-s| + (K_5+K_6+K_8)(|I-\Phi(t,s)|_i)$$

showing uniform equicontinuity By the Arzela Ascoli theorem, there exists a subsequence of $(x^k(\cdot))$, say $(x^{k_i}(\cdot))_{i=1}^{\infty}$ converging uniformly on $[t_o, t_o+T]$. By an argument similar to the previous one the $(u^k(\cdot))_{k=1}^{\infty}$ are uniformly equicontinuous and bounded; so that there exists a further subsequence of $(u^{k_i}(\cdot))_{i=1}^{\infty}$ converging uniformly on $[t_o, t_o+T]$. Let the limits of these sequences be $x_\infty(\cdot)$ and $u_\infty(\cdot)$ respectively from the continuity of $h(x,u,t)$ we obtain

$$x_\infty(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\tau)\; B(\tau)\, u_\infty(\tau)d\tau$$

$$+ \int_{t_o}^{t} \Phi(t,\tau)\; h(x_\infty(\tau),\, u_\infty(\tau),\tau)d\tau \qquad \text{(V.12)}$$

and

$$u_\infty(\cdot) = \mathcal{L}_R^*(\mathcal{L}_R\mathcal{L}_R^*)^{-1}\left[x_1 - \int_{t_o}^{t_o+T} \phi(t_o+T,\tau)\; h(x_\infty(\tau),\, u_\infty(\tau),\tau)d\tau\right.$$

$$\left. - \phi(t_o+T,t_o)x_o\right] \qquad \text{(V.13)}$$

Using (V.13) in (V.12) we obtain $x_\infty(t_o+T) = x_1$ and it is clear that $u_\infty(\cdot)$ is the required control to transfer the system from x_o to x_1 on $[t_o, t_o+T]$. This proves part (i) of the Proposition. Part (ii) is procedural and is left to the reader. ¤

Comment.

(i) Notice that the sequence of $(u^k(\cdot))_{k=1}^{\infty} \in \mathcal{M}$, a closed n — dimensional subspace of $L_2^{n_i}[t_o,t_o+T]$, so that $u_\infty(\cdot) \in \mathcal{M}$, as claimed in the Proof of Theorem (V.1).

Section VI. Robustness of strong uniform complete controllability under quasibounded perturbations in the control channel and state dynamics

In this section we state conditions under which a strongly uniformly completely controllable (over T seconds) FDLS remains uniformly completely controllable under unbounded but quasibounded perturbations separately in (i) the control channel and (ii) the state dynamics. More explicitly, the FDLS (II.3.5) is said to have a quasibounded perturbation in its control channel if

$$\dot{x}(t) = A(t)x + B(t)u + f(u,t) \tag{VI.1}$$

where f is a C^o function from $\mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^n$ and for some constants $\gamma(f)$, $\beta(f) \in \mathbb{R}$,

$$|f(u,t)| \leq \gamma(f)\ |u| + \beta(f) \quad \forall u \in \mathbb{R}^{n_i},\ \ \forall t \in \mathbb{R}_+ \tag{VI.2}$$

By comment (iii) after Definition (IV.1) it follows that (VI.2) is equivalent to the quasiboundedness (uniformly in t) of f. Further, it is easy to verify that $\inf\{\gamma(f) : \exists\, \beta(f)$ such that (VI.2) holds$\}$ is the quasinorm (uniformly in t) of f. Thus $\gamma(f)$ can be chosen arbitrarily close to the quasinorm (uniformly in t) of f. The FDLS (II.3-5) is said to have a quasibounded perturbation in its state dynamics if

$$\dot{x}(t) = A(t)x + \psi(x,t) + B(t)u \tag{VI.3}$$

where ψ is a C^o function $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ which is globally Lipschitz continuous in its first argument (to guarantee uniqueness of solution of VI.3), and $\exists \gamma(\psi) < \infty$ and $\beta(\psi) < \infty$ such that

$$|\psi(x,t)| \leq \gamma(\psi)\ |x| + \beta(\psi) \qquad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_+ \tag{VI.4}$$

As before $\gamma(\psi)$ may be chosen arbitrarily close to the quasinorm (uniformly in t) of ψ.

VI.1. Robustness of uniform complete controllability under quasibounded perturbations in the control channel

We start with an FDLS with a bounded realization which is strongly uniformly completely controllable over T seconds. Then, define

$$\gamma(B) := \sup_{t \in \mathbb{R}_+} |B(t)|_i < \infty \tag{VI.5}$$

Recalling the definition of the map $\mathcal{L}$,

$$\mathcal{L}v = \int_{t_o}^{t_o+T} \phi(t_o+T,\tau)\ v(\tau)d\tau \qquad \forall v \in L_2^n([t_o,t_o+T]) \tag{VI.6}$$

we define the intrinsic grammian of the system.

Definition VI.1. (Intrinsic grammian)

The intrinsic grammian of the FDLS of (II.3), (II.4) with bounded realization is the matrix representation of the continuous linear map $\mathcal{L}\mathcal{L}^* : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$W[t_o,t_o+T] = \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\ \Phi^*(t_o+T,\tau)d\tau \tag{VI.7}$$

From this definition follows the idea of an intrinsic drift factor of an FDLS over T seconds.

Definition V.2. (Intrinsic drift factor)

The intrinsic drift factor μ of an FDLS with bounded realization over T secs is defined to be

$$\mu := \sup_{t_o \in \mathbb{R}_+} \sup_{\tau \in [0,T]} (\lambda_{max}(W[t_o, t_o+\tau))^{1/2} \qquad \text{(VI.8)}$$

¤

Comment. (i) From inequality (III.6) it follows that $\mu \leqq \left\{\frac{e^{2KT}-1}{2K}\right\}^{1/2}$ and thus is finite for a bounded realization. Using these notions, we have

Theorem VI.1. (Controllability of the system perturbed in the input channel)

If the FDLS with bounded realization represented by (II.3) is strongly uniformly controllable (over T seconds) then the perturbed DDS represented by equations (VI.1), (VI.2) is uniformly completely controllable (over T seconds) if

$$\gamma(f) < \frac{\tilde{\lambda}_s}{\mu} \qquad \text{(VI.9)}$$

where $\tilde{\lambda}_s$ is as defined in Definition (III.4) and μ is as defined above.

¤

Comments. (i) As may be easily checked by the reader, μ and $\tilde{\lambda}_L$ of Definition (III.4) are related by the inequality

$$\gamma(B)\mu \geqq \tilde{\lambda}_s \qquad \text{(VI.10)}$$

so that (VI.9) implies that

$$\frac{\gamma(f)}{\gamma(B)} < \frac{1}{\chi_R} \leqq 1 \qquad \text{(VI.11)}$$

where χ_R is the reachability condition number.

(ii) Comment (i) shows that if uniform complete controllability is preserved then the ratio of the gain of f to the gain of B is small compared to the inverse of the reachability condition number.

Proof of theorem: Fix $x_o \in \mathbb{R}^n$, the initial state and $t_o \in \mathbb{R}_+$. Let $x_1(t)$ and $x_2(t)$ be the state of the FDLS and the perturbed system at $t \in [t_o, t_o+T]$ in response to an input $u(\cdot) \in L_2^{n_i}([t_o, t_o+T])$. Then, we have

$$\dot{x}_1 = A(t)x_1 + B(t)u, \quad x_1(t_o) = x_o \tag{VI.12}$$

and

$$\dot{x}_2 = A(t)x_2 + B(t)u + f(u,t), \quad x_2(t_o) = x_o \tag{VI.13}$$

Defining $\Delta x := x_2 - x_1$ we have

$$\dot{\Delta x} = A(t)\Delta x + f(u,t), \quad \Delta x(t_o) = \theta_n \tag{VI.14}$$

By the same estimates as in Section V, and with the same notation

$$|\Delta x(t_o+T)| \leq |\mathcal{L}|_i \, \|v(\cdot)\|$$

with $\qquad v(t) := f(u(t),t).$

Now, from (VI.2)

$$\|v(\cdot)\| \leq \gamma(f) \, \|u(\cdot)\| + \beta(f).$$

and

$$|\mathcal{L}|_i \leq \mu$$

so that

$$|\Delta x(t_o+T)| \leq \mu \, \gamma(f) \, \|u(\cdot)\| + \mu\beta(f) \quad \forall t_o \in \mathbb{R}_+ \tag{VI.15}$$

we write

$$x_1(t_o+T) = \mathcal{L}_R(u) + \phi(t_o+T,t_o)x_o$$

and

$$x_2(t_o+T) = (\mathcal{L}_R+N)(u) + \phi(t_o+T,t_o)x_o$$

where $N(u) := \Delta x(t_o+T)$ is a quasibounded, continuous map (independent of x_o) with quasinorm $\leqq \mu\gamma(f)$. As before, to use Theorem IV.1 we restrict the domain of $\mathcal{L}_R$ to $\mathcal{M} := \mathcal{L}_R^*(\mathbb{R}^n)$ and define $\mathcal{L}_R^{-1}$ as before. Then $\mathcal{L}_R^{-1}N : \mathcal{M} \to \mathcal{M}$ is a continuous quasibounded map (between finite dimensional Banach spaces and hence compact) with quasinorm $\leqq \mu\gamma(f)|\mathcal{L}_R^{-1}|_i$ Further, $|\mathcal{L}_R^{-1}|_i \leqq \frac{1}{\tilde{\lambda}_s}$. By the same arguments as in Theorem (V.1) uniform complete controllability over T seconds is guaranteed if

$$\frac{\mu\gamma(f)}{\tilde{\lambda}_s} < 1$$

Q.E.D.

VI.2. Robustness of uniform complete controllability under quasibounded perturbations in state dynamics

Theorem VI.2. (Controllability of the system perturbed in state dynamics)

If the FDLS with bounded realization represented by (II.3) is strongly uniformly completely controllable over T seconds then the perturbed system represented by equations (VI.3), (VI.4) is uniformly completely controllable over T seconds if

$$\gamma(\psi) < \frac{1}{2\chi_R \mu T^{1/2}} \tag{VI.16}$$

Furthermore if the zero solution of the FDLS with no input is uniformly exponentially stable, with

$$|\Phi(t+\tau,t)|_i \leqq k_A e^{-C_A\tau} \qquad \forall t,\tau \in \mathbb{R}_+ \tag{VI.17}$$

then the perturbed system is uniformly completely controllable over T seconds if

$$\gamma(\psi) < \frac{\sqrt{C_A}}{\sqrt{2}\chi_R T^{1/2} k_A} \tag{VI.18}$$

If in addition the FDLS is time invariant and exponentially stable, then the perturbed system is uniformly completely controllable, if

$$\frac{\gamma(\psi)}{|\lambda_{min}(A+A^*)|^{1/2}} < \frac{1}{2\chi_R T^{1/2}} \tag{VI.19}$$

where $\lambda_{min}(A+A^*)$ is the smallest eigenvalue of $A+A^* \in \mathbb{R}^{nxn}$ ◻

Comment. Equation (V.19) supports the inituitive notion that in some sense uniform complete controllability should be preserved if the gain of ψ is small compared to the "gain of A." The theorem shows that "gain of A" should be replaced by the "gain of the symmetric part of A."

Proof of theorem. Fix $t_o \in \mathbb{R}_+$ and $x_o \in \mathbb{R}^n$, the initial state. Let $x_1(t)$, $x_2(t)$, $\Delta x(t)$ be defined as in the proof of Theorem (VI.1) with

$$\dot{x}_1 = A(t)x_1 + B(t)u, \; x_1(t_o) = x_o \tag{VI.20}$$

$$\dot{x}_2 = A(t)x_2 + \psi(x_2,t) + B(t)u, \; x_2(t_o) = x_o \tag{VI.21}$$

and

$$\Delta\dot{x} = A(t)\Delta x + \psi(x_2,t), \; \Delta x(t_o) = \theta_n \tag{VI.22}$$

By the same arguments as in Theorem (VI.1)

$$\sup_{\tau\in[t_o,t_o+T]} |\Delta x(\tau)| \leqq \mu\gamma(\psi)\, \|x_2(\cdot)\| + \mu\beta(\psi) \tag{VI.23}$$

and

$$\|x_2(\cdot)\| \leqq T^{1/2}[\sup_{t\in[t_o,t_o+T]} (|x_1(t)|+|\Delta x(t)|)] \tag{VI.24}$$

It is also relatively simple to realize that

$$\sup_{t\in[t_o,t_o+T]} |x_1(t)| \leq \tilde{\lambda}_L \|u(\cdot)\| + \sup_{t\in[t_o,t_o+T]} |\phi(t,t_o)x_o| \qquad \text{(VI.25)}$$

Using the estimates from (VI.25), (VI.24) in (VI.23) and noting that (VI.16) $\Rightarrow \mu\gamma(\psi)T^{1/2} < 1/2$ we obtain

$$\sup_{t\in[t_o,t_o+T]} |\Delta x(t)| \leq 2\mu\gamma(\psi)T^{1/2}\,\tilde{\lambda}_L\, \|u(\cdot)\| + c_1(x_o) \qquad \text{(VI.26)}$$

where $c_1(x_o) \in \mathbb{R}_+$ does not depend on $\|u(\cdot)\|$ but does depend on x_o. As before, we may write

$$x_2(t_o+T) = \mathcal{L}_R u + N_{x_o}(u) + \Phi(t_o+T,t_o)x_o.$$

where $N_{x_o}(u) := \Delta x(t_o+T)$ is a quasibounded continuous map with quasinorm $\leq 2\mu\gamma(\psi)T^{1/2}\,\tilde{\lambda}_L$ from (VI.26) and uniform complete controllability over T seconds is guaranteed for the perturbed system by Theorem IV.1 if

$$2\mu\gamma(\psi)T^{1/2}\,\tilde{\lambda}_L|\mathcal{L}_R^{-1}|_i < 1$$

or if

$$\gamma(\psi) < \frac{1}{2\chi_R\mu T^{1/2}} \qquad \text{(VI.16)}$$

(VI.17) and (VI.18) are procedural and are left to the reader. Q.E.D.

Clearly the results of Theorems (VI.1) and (VI.2) can be combined to state a condition for the uniform, complete controllability of a system perturbed both in state dynamics and control channel as

$$\dot{x} = A(t)x + \psi(x,t) + B(t)u + f(u,t) \qquad \text{(VI.27)}$$

where ψ and f satisfy the conditions listed previously. Then, we have

Theorem VI.3. (Controllability of the system perturbed in the input channel and state dynamics)

If the FDLS with bounded realization represented by (II.3) is strongly uniformly controllable over T seconds then the perturbed DDS represented by (VI.27) is uniformly completely controllable over T seconds if

$$\frac{\gamma(f)\mu}{\tilde{\lambda}_s} + 2\gamma(\psi)\mu T^{1/2}\chi_R[1 + \frac{\gamma(f)\mu}{\tilde{\lambda}_s}] < 1 \tag{VI.28}$$

Comment. (VI.28) in particular implies that

$$\frac{\gamma(f)}{\gamma(B)} + 2\gamma(\Psi)\mu T^{1/2}(1 + \frac{\gamma(f)}{\gamma(B)}) < 1/\chi_R \tag{VI.29}$$

Proof: Is routine and omitted for brevity. ¤

VI.3. Algorithm for the computation of an input to transfer the perturbed system from $x_o \in \mathbb{R}^n$ at t_o to $x_1 \in \mathbb{R}^n$ at $t_o + T$.

Under the conditions of Theorems (VI.1), (VI.2), and (VI.3) algorithms yielding at their accumulation points inputs belonging to the subspace $\mathcal{M}$ of $L_2^{n_i}([t_o,t_o+T])$ for making the requisite transfer may be obtained. To keep the section simple we prove the existence of limit points of the algorithm model for the case of the input perturbed system satisfying the conditions of Theorem (VI.1). The same algorithm model may however be used for the other two cases as well.

Algorithm Model

Step 0. Set $k = 0$; $x^o(t) = x_o, u^o(t) = \theta_{x_i}, \forall t \in [t_o,t_o+T]$

Step 1. Define

$$x^{k+1}(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\tau)\ B(\tau)\ u^k(\tau)d\tau.$$

$$+ \int_{t_o}^{t} \Phi(t,\tau)\ f(u^k(\tau),\tau)d\tau \qquad \text{(VI.30)}$$

$$u^{k+1}(\cdot) = \mathcal{L}_R^*(\mathcal{L}_R\mathcal{L}_R^*)^{-1}\ [x_1-\Phi(t_o+T,t_o)x_o - \int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\ f(u^k(\tau),\tau)d\tau]$$

Step 2. Set $k = k+1$; go to Step 1.

Proposition VI.1. (Convergence of Algorithm)

(i) There exists at least one accumulation point (in the L_∞ sense) of the sequence of $(x^k(\cdot))_{k=1}^{\infty}$ say $x_\infty(\cdot)$ with corresponding input $u_\infty(\cdot)$ defined on $[t_o,t_o+T]$ satisfying $x_\infty(t_o) = x_o$ and $x_\infty(t_o+T) = x_1$ for the control $u_\infty(\cdot)$.

(ii) For any accumulation point (in an L_∞ sense) of $(x^k(\cdot))_{k=1}^{\infty}$ say $\tilde{x}_\infty(\cdot)$ $\exists$ a control $\tilde{u}_\infty(\cdot)$ such that $\tilde{x}_\infty(t_o) = x_o$ and $\tilde{x}_\infty(t_o+T) = x_1$ for the control $\tilde{u}_\infty(\cdot)$.

Proof. The proof proceeds through a sequence of claims.

Claim 1. The sequence $(u^k(\cdot))_{k=1}^{\infty}$ is bounded in L_2 norm by $(|x_1|+|\phi(t_o+T,t_o)x_o|)(\tilde{\lambda}_s-\mu\gamma(f))^{-1} < \infty$ (by (VI.10)).

Proof. The (easy) proof is by induction. ⌑

Claim 2. The sequence $(u^k(\cdot))_{k=1}^{\infty}$ is bounded in L_∞ norm.

Proof. $$\sup_{t\in[t_o,t_o+T]} |u^k(t)| \leq \tilde{\lambda}_s^{-1}\ \gamma(B) \sup_{\tau\in[t_o,t_o+T]} |\phi(t_o+T,\tau)|_i\ \|u^k(\cdot)\|$$

because $u^k(\cdot) \in \mathcal{L}_R^*(\mathbb{R}^n)$

Since the sequence of norms $(\|u^k(.)\|)_{k=1}^{\infty}$ is bounded by Claim 1, the sequence $(u^k(\cdot))_{k=1}^{\infty}$ is bounded in L_∞ norm. ⌑

Claim 3. The sequence $(x^k(\cdot))_{k=1}^{\infty}$ is bounded in L_∞ norm and is uniformly equicontinuous.

Proof. Follows from the arguments of Proposition (V.1) and Claims 1 and 2. ◻

Now, the proof of the proposition follows exactly the same lines as that of Proposition (V.1). ◻

Section VII. Robustness of Strong Uniform Complete Controllability under Unbounded Lipschitz Continuous Perturbations

In this section we examine the uniform complete controllability over T seconds of an FDLS perturbed as in equation (VII.1)

$$\dot{x} = A(t)x + B(t)u + \varepsilon h(x,u,t) \qquad \text{(VII.1)}$$

with $\varepsilon \in \mathbb{R}$ and $h : \mathbb{R}^n \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^n$ is a Lipschitz continuous function satisfying

$$h(\theta_n, \theta_{n_i}, t) = \theta_n \quad \forall t \in \mathbb{R}_+ \qquad \text{(VII.2)}$$

and for some $c_o \in \mathbb{R}_+$, $\forall x \in \mathbb{R}^n$, $\forall u,v \in \mathbb{R}^{n_i}$ and $\forall t \in \mathbb{R}_+$,

$$|h(x,u,t) - h(y,v,t)| \leq c_o|u-v| + c_o|x-y| \qquad \text{(VII.3)}$$

Given that the FDLS is strongly uniformly controllable (in the DDS of (VII.1) with $\varepsilon = 0$) we will prove the existence of an interval I centered at 0 so that the system of (VII.1) is controllable for all $\varepsilon \in I$. It is of course clear that FDLS can actually lose the property of complete controllability from ε large as is evidenced by the scalar system

$$\dot{x} = u + \varepsilon\sqrt{u^2+x^2} \qquad x,u \in \mathbb{R} \qquad \text{(VII.4)}$$

losing complete controllability for $|\varepsilon| \geqq 1$ (indeed, in that case $\forall (x,u), \dot{x} \geqq 0$).

In Section (VII.1) we estimate the interval I on which the perturbed system remains uniformly completely controllable and in Section (VII.2) we give an algorithm which converges to a (unique) element u of $\mathcal{M} \subset L_2^{n_i}([t_o,t_o+T])$ required to transfer the system from x_o to x_1.

VII.1. Estimate of the interval I on which the perturbed system is uniformly completely controllable (over T seconds)

Theorem VII.1. (Controllability of the perturbed system)

If the FDLS with bounded realization represented by (II.3) is strongly uniformly completely controllable over T seconds then the perturbed system represented by (VII.1), (VII.2), (VII.3) is uniformly completely controllable on $[t_o,t_o+T]$ $\forall \varepsilon \in]-\varepsilon_o, \varepsilon_o[$ where

$$\frac{1}{\varepsilon_o} = \chi_R \cdot 2c_o \mu T^{1/2}(1+\gamma(B)\tilde{\lambda}_s \tilde{\lambda}_L^{-1} \sup_{t_o \in \mathbb{R}_+} \sup_{\tau \in [0,T]} |\Phi(t_o+T,t_o+\tau)|_i \qquad \text{(VII.5)}$$

Proof. Fix $t_o \in \mathbb{R}_+$; $x_o \in \mathbb{R}^n$. Define $x_1(t)$, $x_2(t)$, $\Delta x(t)$ as before with

$$\dot{x}_1 = A(t)x_1 + B(t)u, \quad x_1(t_o) = x_o \qquad \text{(VII.6)}$$

$$\dot{x}_2 = A(t)x_2 + B(t)u + \varepsilon h(x_2,u,t), \quad x_2(t_o) = x_o \qquad \text{(VII.7)}$$

and

$$\Delta\dot{x} = A(t)\Delta x + \varepsilon h(x_2,u,t), \quad \Delta x(t_o) = \theta_n \qquad \text{(VII.8)}$$

Using (VII.3) and the techniques of the previous section we obtain

$$\sup_{t \in [t_o,t_o+T]} |\Delta x(t)| \leqq \mu T^{1/2} \varepsilon c_o \sup_{t \in [t_o,t_o+T]} \{|\Delta x(t)| + |x_1(t)| + |u(t)|\} \qquad \text{(VII.9)}$$

Note that $|h(x_2,u,t)| \leq |h(x_2,u,t) - h(x_1,u,t)| + |h(x_1,u,t)|$ so that from (VII.2) and (VII.3) $|h(x_2,u,t)| \leq c_o|x(t)| + c_o|x_1(t)| + c_o|u(t)|$. Using the estimate

$$\sup_{t\in[t_o,t_o+T]} |x_1(t)| \leq \tilde{\lambda}_L \|u(\cdot)\| \tag{VII.10}$$

and the fact that $\varepsilon < \varepsilon_o \Rightarrow \varepsilon c_o\, T^{1/2}\mu < 1/2$ we obtain from (VII.9) that

$$\sup_{t\in[t_o,t_o+T]} |\Delta x(t)| \leq 2\varepsilon c_o \mu T^{1/2}\{c_1(x_o) + \tilde{\lambda}_L \|u(\cdot)\| + \sup_{t\in[t_o,t_o+T]} |u(t)|\} \tag{VII.11}$$

where $c_1(x_o)$ is some constant depending on x_o but not on u.

Also, as before, we define

$$N_{x_o} u := \Delta x(t_o+T)$$

we prove that N_{x_o} is quasibounded on $\mathcal{M}$.

On $\mathcal{M}$, $$\sup_{t\in[t_o,t_o+T]} |u(t)| \leq \gamma(B)\ \tilde{\lambda}_s^{-1} \sup_{t_o\in \mathbb{R}_+} \ \sup_{\tau\in[0,T]} |\Phi(t_o+T,t_o+\tau)|_i\ \|u(\cdot)\| \tag{VII.12}$$

Hence N_{x_o} is a quasibounded, continuous map on $\mathcal{M}$ with quasinorm less than or equal to

$$2\varepsilon c_o \mu T^{1/2} T^{1/2}\{\ \tilde{\lambda}_L^{1/2} + \gamma(B)\tilde{\lambda}_S^{-1} \sup_{t_o\varepsilon\mathbb{R}_+} \ \sup_{\tau\varepsilon[o,T]} |\Phi(t_o+T,k_o+\tau|_i\}$$

Clearly, theorem (V.1) guarantees uniform complete controllability of the perturbed system if equation (VII.5) is satisfied.

Q.E.D.

Comment.

In the instance that

$$|\phi(t,\tau)| \leq C_A \exp^{-K_A(t-\tau)}$$

for some C_A, K_A positive constants (i.e. the zero solution of the zero input FDLS is uniformly exponentially stable), equation (VII.5) may be restated as

$$\varepsilon_o = \frac{1}{\chi_R} \frac{\sqrt{K_A}}{\sqrt{2}\, c_o\, C_A} \{1 + \frac{\gamma(B) C_A}{\tilde{\lambda}_L \tilde{\lambda}_s}\} \tag{VII.13}$$

VII.2. Algorithm for the computation of an input to transfer the perturbed system from $x_o \in \mathbb{R}^n$ at t_o to $x_1 \in \mathbb{R}^n$ at $t_o + T$

Step 0. Set $k = 0$, $x^o(t) = x_o, u^o(t) = \theta_{n_i}$ $\forall t \in [t_o, t_o+T]$.

Step 1. Define $\dot{x}^{k+1}(\cdot)$ to be the state trajectory on $[t_o, t_o+T]$ satisfying the differential equation

$$\dot{x}^{k+1} = A(t)x^{k+1} + B(t) + \varepsilon h(x^{k+1}, u^k, t), x^{k+1}(t_o) = x_o$$

i.e.,

$$x^{k+1}(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,t_o)\; B(\tau)\; u^k(\tau)d\tau + \varepsilon\int_{t_o}^{t} \Phi(t,\tau)\; h(x^{k+1}(\tau),\; u^k(\tau),\tau)d\,\tau \tag{VII.14}$$

Also, define

$$u^{k+1}(\cdot) = \mathcal{L}_R^*(\mathcal{L}_R\mathcal{L}_R^*)^{-1}[x_1 - \Phi(t_o+T,t_o)x_o + \varepsilon\int_{t_o}^{t_o+T} \Phi(t_o+T,\tau)\; h(x^{k+1}(\tau),u^k(\tau),\tau)d\tau] \tag{VII.15}$$

Step 2. Set $k = k+1$; go to Step 1.

Comments. (i) The algorithm proposed above has interesting heuristics -- that of iterated "missed distance correction."

(ii) The reader will note that this algorithm is different from those proposed in Sections V and VI, as a comparison of equation (VII.14) with equation (V.10) and equation (VII.15) with equation (V.11) will show ($x^{k+1}(\cdot)$ is used in equation (VII.14) and (VII.15) and $x^k(\cdot)$ in equations (V.10) and (V.11)).

Proposition VII.1. (Convergence of Algorithm)

If the conditions of Theorem VII.1 are satisfied then the algorithm given above converges to a unique $u_\infty(\cdot) \in L_2^{n_i}([t_o,t_o+T])$ which transfers the system from x_o at t_o to x_1 at t_o+T.

Proof. Define $\Delta x_k := x^{k+1} - x^k$ and $\Delta u_k := u^{k+1} - u^k$. Then, from (VII.14) we have

$$\sup_{t\in[t_o,t_o+T]} |\Delta x_k(t)| \leq \tilde{\lambda}_L \|\Delta u_{k-1}(\cdot)\| + \varepsilon c_o T^{1/2} [\sup_{t\in[t_o,t_o+T]} |\Delta x_k(t)| + \sup_{t\in[t_o,t_o+T]} |\Delta u_{k-1}(t)|] \qquad \text{(VII.16)}$$

and from (VII.15) we have

$$\|\Delta u_k(\cdot)\| = \frac{1}{\tilde{\lambda}_s} \varepsilon c_o T^{1/2} \mu [\sup_{t\in[t_o,t_o+T]} (|\Delta x_k(t)| + |\Delta u_{k-1}(t)|)] \qquad \text{(VII.17)}$$

Using the fact that (VII.15) $\Rightarrow \varepsilon c_o T^{1/2} \mu < \frac{1}{2}$ in (VII.16) we have

$$\sup_{t\in[t_o,t_o+T]} |\Delta x_k(t)| \leq 2\tilde{\lambda}_L \|\Delta u_{k-1}(\cdot)\| + \sup_{t\in[t_o,t_o+T]} |\Delta u_{k-1}(t)| \qquad \text{(VII.18)}$$

Using (VII.18) in (VII.17) and inequality (VII.12) for $\sup_{t\in[t_o,t_o+T]} |\Delta u_{k-1}(t)|$

$$\|\Delta u_k(\cdot)\| \leq \frac{2\varepsilon C_o T^{1/2}\mu}{\tilde{\lambda}_s} [\tilde{\lambda}_L + \frac{\gamma(B) \sup_{\tau\in[t_o,t_o+T]} |\Phi(t_o+T,\tau)|_i}{\tilde{\lambda}_s}] \|\Delta u_{k-1}(\cdot)\| .$$

i.e. $\quad \|\Delta u_k(\cdot)\| \leq \rho\|\Delta u_{k-1}(\cdot)\|$

where by equation (VII.17) $\rho < 1$. Hence, by the contraction mapping theorem applied to $L_2^{n_i}([t_o,t_o+T], \lim_{k\to\infty} \|\Delta u_k(\cdot)\| = 0$ and the algorithm converges to a unique limit point say $u_\infty(\cdot) \in \mathcal{M}$ in the L_2 sense. From equation (VIII.16) and (VIII.15) and the continuity of h it is clear that the control $u_\infty(\cdot)$ is the control that drives the system from x_o at t_o to x_1 at t_o+T. It is not difficult to show that the convergence is also in the L_∞ sense since for all elements $u \in \mathcal{M}$

$$\sup_{t\in[t_o,t_o+T]} |u(t)| \leq \frac{\gamma(B) \sup_{\tau\in[t_o,t_o+T]} |\Phi(t_o+T,\tau)|_i \|u(\cdot)\|}{\tilde{\lambda}_s} \qquad \text{(VII.12)}$$

¤

Section VIII. Characterisation of zero-input observability for FDLS

Definition VII.1 (Zero-input uniform complete observability)

A differential dynamical system represented by equations (II·1), (II·2) is zero-input uniformly completely observable if $\exists\, T > 0$ such that

a) $\forall t_o \in \mathbb{R}_+$, the zero-input response of the DDS with initial state $x(t_o) = x_1$ say y_1 belong to $L_2^{n_o}([t_o, t_o+T])$, and b) this response y_1 is not identical to y_2, the zero-input response with any other initial state $x(t_o) = x_2 \neq x_1$.

To obtain a characterization of zero-input uniform complete observability for the FDLS with bounded realization represented by (II.3), (II.4) we define for given $t_o \in \mathbb{R}_+$, $\mathcal{L}_o$ to be the linear map (called the observability map) from $\mathbb{R}^n$ to $L_2^{n_o}([t_o,t_o+T])$ defined by

$$\mathcal{L}_o x = C(\cdot)\Phi(\cdot,t_o)x \qquad \text{(VIII.1)}$$

The adjoint of $\mathcal{L}_o$ denoted by $\mathcal{L}_o^*$ is the linear map from $L_2^{n_o}([t_o,t_o+T])$ to $\mathbb{R}^n$ defined by

$$\mathcal{L}_o^* u = \int_{t_o}^{t_o+T} \Phi^*(\tau,t_o)C^*(\tau)y(\tau)d\tau) \qquad \text{(VIII.2)}$$

As before, the boundedness of the realization guarantees that $\mathcal{L}_o$ and $\mathcal{L}_o^*$ are both continuous linear maps.

Theorem VIII.1 (Characterization of Zero Input Uniform Complete Observability for FDLS).

The FDLS with bounded realization described by equations (II.3) and (II.4) is zero-input uniformly completely observable $\Leftrightarrow \forall t_o \in \mathbb{R}_+$ the observability map $\mathcal{L}_o : \mathbb{R}^n \to L_2^{n_o}([t_o,t_o+T])$ is injective $\Leftrightarrow \forall t_o \in \mathbb{R}_+$ the composition of the adjoint of the observability map and the observability map $\mathcal{L}_o^* \mathcal{L}_o : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. ◻

Definition VIII.2 [Observability grammian].

Given $t_o \in \mathbb{R}_+$ the matrix representation of the linear map $\mathcal{L}_o^* \mathcal{L}_o : \mathbb{R}^n \to \mathbb{R}^n$ is the observability grammian, denoted $W_o[t_o, t_o+T]$ given by

$$W_o[t_o, t_o+T] = \int_{t_o}^{t_o+T} \Phi^*(\tau, t_o)C^*(\tau)C(\tau)\Phi(\tau, t_o)d\tau \qquad \text{(VIII.3)}$$

The FDLS with bounded realization is zero-input uniformly completely observable over T seconds iff $W_o[t_o, t_o+T]$ is nonsingular $\forall t_o \in \mathbb{R}$. However, its smallest eigenvalue may tend to 0 as $t_o \to \infty$. As before, we define a slightly stronger form of observability.

Definition VIII.3 (Zero-input strong uniform complete observability).

A FDLS with bounded realization represented by (II.3) and (II.4) is zero-input strongly uniformly completely observable if $\exists$ T, $\nu_s > 0$ such that $\forall t_o \in \mathbb{R}_I$

$$W_o\,[t_o, t_o+T] \geq \nu_s^2 I. \qquad \text{(VIII.4)}$$

Let $\tilde{\nu}_s$ be the largest ν_s satisfying (VIII.4). ⌑

Comment: As before, the boundedness of the realization guarantees that $\exists\, \nu_L \in \mathbb{R}_+$ such that $\forall t_o \in \mathbb{R}_+$

$$\nu_L^2 I \geq \sup_{\tau \in [0,T]} W_o\,[t_o, t_o+\tau] \qquad \text{(VIII.5)}$$

The next proposition shows how to identify the initial state of an FDLS at time t_o given the undriven output of the uniformly completely observable FDLS on $[t_o, t_o+T]$.

Proposition VIII.1 (Formula for initial state)

Given the zero-input response y on $[t_o, t_o+T]$ of a zero-input uniformly completely observable FDLS with bounded realization, the initial state $x(t_o) = x_o$ is given (uniquely) by

$$x_o = (\mathcal{L}_o^* \mathcal{L}_o)^{-1} \mathcal{L}_o^* y \qquad \text{(VIII.6)}$$

⌑

Section IX Robustness of zero-input strong uniform complete observability under perturbations in the state dynamics and output channel

In this section, we state conditions under which a zero-input strongly uniformly completely observable (over T seconds) FDLS remains uniformly completely observable under Lipschitz continuous perturbations in the state dynamics and continuous perturbations in the output channel. Specifically, we restrict attention to zero-input observability of the linear system perturbed in state dynamics represented by

$$\dot{x} = A(t)x + \psi(x,t) \tag{IX.1}$$

$$y = C(t)x \tag{IX.2}$$

where ψ is a C^o function: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ which is Lipschitz continuous in its first argument to guarantee uniqueness of solution of (IX.1) with

$$\psi(\theta_n, t) = \theta_n \quad \forall t \in \mathbb{R}_1 \tag{IX.3}$$

and for some $\gamma(\psi) < \infty$

$$|\psi(x_2,t) - \psi(x,t)| \leqq \gamma(\psi)|x_2 - x_1| \quad \forall t \in \mathbb{R}_+, \quad \forall x_1, x_2 \in \mathbb{R}^n \tag{IX.4}$$

The system perturbed in the output channel is represented by

$$\dot{x} = A(t)x \tag{IX.5}$$

$$y = C(t)x + f(x,t) \tag{IX.6}$$

where f is a C^o function: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ with

$$f(\theta_n, t) = \theta_{n_o} \quad \forall t \in \mathbb{R}_+ \tag{IX.7}$$

and

$$\sup_{t\in\mathbb{R}_+} \sup_{x\in\mathbb{R}^n} \frac{|f(x,t)|}{|x|} =: \gamma(f) < \infty \tag{IX.8}$$

Further, let

$$\gamma(C) := \sup_{t\in\mathbb{R}_+} |C(t)|_i < \infty \tag{IX.9}$$

IX.1 Robustness of observability

Theorem IX.1 (Observability of the system perturbed in state dynamics)

Given that the FDLS with bounded realization represented by (II.3), (II.4) is zero input strongly uniformly completely observable then the perturbed system represented by (IX.1), (IX.2) (for the instance that the input is zero) is uniformly completely observable over T seconds if

$$\gamma(\psi) < \frac{\tilde{\nu}_s}{2\mu T\gamma(C)\gamma(\Phi)} \tag{IX.10}$$

where

$$\gamma(\Phi) := \sup_{t_o\in\mathbb{R}_+} \sup_{\tau\in[0,T]} |\Phi(t_o,t_o+\tau)|_i \tag{IX.11}$$

Proof. Let $x_1(t)$, $x_2(t)$ be the state of the FDLS and the perturbed system respectively, i.e.

$$\dot{x}_1 = A(t)x_1, \quad x_1(t_o) = x_o \tag{IX.12}$$

and

$$\dot{x}_2 = A(t)x_2 + \psi(x_2,t), \quad x_2(t_o) = x_o \tag{IX.13}$$

With $\Delta x := x_2 - x_1$, we have

$$\Delta\dot{x} = A(t)\Delta x + \psi(x_2,t), \quad \Delta x(t_o) = \theta_n \tag{IX.14}$$

and

$$\sup_{t\in[t_o,t_o+T]} |\Delta x(t)| \leq \mu\, \gamma(\psi) T^{1/2} [\sup_{t\in[t_o,t_o+T]} |\Delta x(t)| + \gamma(\Phi)|x_o|$$

$$\forall t_o \in \mathbb{R}_+ \tag{IX.15}$$

by the same arguments as before.

$$x_o^{k+1} = (\mathcal{L}_o^* \mathcal{L}_o)^{-1} \mathcal{L}_o^* [y - C(\cdot) \int_{t_o}^{\cdot} \phi(\cdot,\tau)\phi(x^{k+1},\tau)d\tau] \qquad \text{(IX.24)}$$

Step 2 Set k = k + 1 and go to Step 1.

Proposition IX.1 (Convergence of Algorithm)

Under the conditions of Theorem (IX.1) the algorithm given above converges to a (unique) limit which is the required initial state.

Proof: The proof follows the same lines as the proof of the contraction mapping theorem and it is obvious that the limit of the sequence of $\{y^k(\cdot)\}_{k=1}^{\infty}$ namely $y_\infty(\cdot)=y(\cdot)$. The details are omitted for brevity. ¤

Comment (i) Similar algorithms can be obviously stated for systems satisfying Theorems (IX.2) and (IX.3).

Section X An application-emergency control of an interconnected power system

In this section we state a model for an interconnected power system and show how the results of Section V may be used to formulate control laws for steering the power system to an equilibrium point in the event of unanticipated line breakages. As we point out in Section X.1, the possibility of steering of the power system in the absence of constraints is easily established. In the event of constraints on the capacity of generation, frequency deviation of the generators, and the thermal (heating) limits of the lines the problem of steering the system is non-trivial and this is discussed in Section X.2.

X.1 Model of interconnected power systems and its controllability

The model we use for an interconnected power system is standard and may be found for instance in Elgerd [8]. We restate the model and assumptions explicitly to establish the notation. The power system is assumed to consist of a network of transmission lines interconnecting buses representing nodes of generation and supply.

(IX.10) implies in particular that $\gamma(\psi)\mu T^{1/2} < 1/2$ [since $\gamma(C)\gamma(\Phi)T^{1/2} \geq \tilde{\nu}_L \geq \tilde{\nu}_s$] so that (IX.15) yields

$$\sup_{t\in[t_o,t_o+T]} |\Delta x(t)| \leq 2\mu\ \gamma(\Phi)T^{1/2}\gamma(\psi)|x_o|.$$

If $y_1(\cdot)$ and $y_2(\cdot) \in L_2^{n_o}([t_o,t_o+T])$ are the outputs of the FDLS and the perturbed system respectively, then

$$\|y_1(\cdot)-y_2(\cdot)\| \leq 2\mu\ \gamma(\psi)\gamma(C)T\ \gamma(\Phi)|x_o| \tag{IX.16}$$

Now, $y_1(\cdot) = \mathcal{L}_o(x_o)$

and we represent

$$y_2(\cdot) = \mathcal{L}_o(x_o) + N(x_o)$$

where $N : \mathbb{R}^n \to L_2^{n_o}([t_o,t_o+T])$ is a continuous map. From the fact that $\psi(\theta_n,t) = \theta_n\ \forall t$ we have $N(\theta_n) \equiv 0$ and from the fact that ψ is Lipschitz we have from an argument similar to that leading to (IX.16) that

$$\|N(x_2)-N(x_1)\| \leq 2\mu\ \gamma(\psi)\gamma(C)T\ \gamma(\Phi)|x_2-x_1|$$

Given that the map $\mathcal{L}_o$ is injective from $\mathbb{R}^n$ to $L_2^{n_o}([t_o,t_o+T])$ (and in fact, a bijection from $\mathbb{R}$ to $\mathcal{L}_o^*(\mathbb{R}^n)$) we demand that $\mathcal{L}_o$ + N be one to one. By the contraction mapping theorem (see for example Marsden [17]), $\mathcal{L}_o$ + N is one to one if

$$2\mu\ \gamma(\psi)\gamma(C)T\ \gamma(\Phi)\quad \tilde{\gamma}_s^{-1} < 1 \tag{IX.17}$$

This completes the proof of the theorem. Q.E.D.

Theorem IX.2 (Observability of the system perturbed in the output channel)

If the FDLS with bounded realization represented by (II.3), (II.4) is zero input strongly uniformly completely observable over T seconds, then the perturbed system represented by (IX.5), (IX.6) (for the zero input case) is uniformly completely observable over T seconds if

$$\gamma(f) < \frac{\tilde{\nu}_s}{T^{1/2}\gamma(\Phi)} \qquad \text{(IX.18)}$$

Proof: is routine the left to the reader.

The results of Theorems (IX.1), (IX.2) can be combined for systems having nonlinear perturbations both in the state dynamics and output channel, that is

$$\dot{x} = A(t)x + \psi(x,t) \qquad \text{(IX.19)}$$

$$y = C(t)x + f(x,t) \qquad \text{(IX.20)}$$

with ψ and f satisfying (IX.3), (IX.4) and (IX.7), (IX.8) respectively.

Theorem IX.3 (Observability for the system perturbed both in state dynamics and output channel).

If the FDLS with bounded realization represented by equations (II.3), (II.4) is zero-input uniformly completely observable over T seconds then the perturbed system represented by (IX.19), (IX.20) (for zero input) is zero-input uniformly completely observable over T seconds if

$$\gamma(f)T^{1/2}\gamma(\Phi)\{1+2\mu\gamma(\psi)T^{1/2}\} + 2\gamma(C)T\gamma(\psi)\mu\ \gamma(\Phi) < \tilde{\gamma}_s \qquad \text{(IX.21)}$$

Proof: The proof is routine and omitted. ¤

IX.2 Algorithm for the identification of the initial state x_o of the perturbed system given the zero input response on $[t_o, t_o+T]$.

Algorithm

Given output $y \in L_2^{n_o}([t_o, t_o+T])$

Step 0 Set $k = 1$, $x_0^1 = (\mathcal{L}_o^* \mathcal{L}_o)^{-1} \mathcal{L}_o^* y$

Step 1 Define

$$\dot{x}^{k+1} = A(t)x^{k+1} + \psi(x^{k+1}, t), \quad x^{k+1}(t_o) = x_o^k \qquad \text{(IX.22)}$$

$$y^{k+1} = C(t)x^{k+1} \qquad \text{(IX.23)}$$

A1. (Generators at each node). We assume that there is a generator coupled to each bus.

As a consequence, the state space of the interconnected power system is a vector space. If there were load buses (i.e. without generators) in the network then nonlinear algebraic constraints on the angles of the generator buses would be present, and hence the state space of the interconnected power system with load buses may be a manifold, under appropriate transversality conditions.

A2. (Power delivered by the transmission line). If two buses i and j are connected by a line of susceptance B_{ij} (at the synchronous frequency of the system) and conductance G_{ij}, and if θ_i and θ_j are the phase angles of the buses with respect to a synchronously rotating reference frame, then the average power leaving bus i is given (approximately) by $B_{ij}\sin(\theta_i-\theta_j) + G_{ij}(1-\cos(\theta_i-\theta_j))$ (in per unit terms; assuming bus voltage magnitude to be 1 per unit and the average power leaving bus j is given by $B_{ij}\sin(\theta_j-\theta_i) + G_{ij}(1-\cos(\theta_i-\theta_j))$. Notice that the sum of these two powers is always ≥ 0 and represents the power lost (to heat) in the line.

A3. (Swing equations of the generators).

The classical swing equation model represents the dynamics of a synchronous generator. For our purposes, the transient reactance of the generator is neglected. Thus, for the ith generator we have

$$M_i\dot{\omega}_i + D_i\omega_i = -\sum_{j\neq i} [B_{ij}\sin(\theta_i-\theta_j) + G_{ij}(1-\cos(\theta_i-\theta_j)] + P_i \tag{X.1}$$

$$\dot{\theta}_i = \omega_i, \quad i = 1,\ldots,n \tag{X.2}$$

where

M_i, D_i = moment of inertia, damping constant with appropriate units

ω_i = angular speed of generator shaft

P_i = net input power at bus i. (Mechanical power input minus electrical demand at bus i.)

Note that $\sum_{j \neq i} [B_{ij} \sin(\theta_i - \theta_j) + G_{ij}(1-\cos(\theta_i - \theta_j))]$ is the output electrical power from bus i.

Equations (X.1), (X.2) constitute a state space model (DDS) for the interconnected power system. We wish to study the controllability of the DDS described by the 2n differential equations (X.1) and (X.2). Note that the model is time-invariant, hence complete controllability, if any, will be uniform. It is relatively simple to realize that the DDS described by the 2n differential equations (X.1) and (X.2) is completely controllable. In fact, given any trajectory in the state space $(\underline{\theta}, \underline{\omega}) \in \mathbb{R}^{2n}$ on [0,T] with the added requirement that $\underline{\dot{\theta}} = \underline{\omega}$, there exists a vector of controls $\underline{P}$ on [0,T] so as to steer the system along that trajectory and this vector of controls is given explicitly by equation (X.1) upon substitution of the desired trajectory (θ, ω).

However, in a physical power system there are constraints on the power generation capacity of each of the generators, so that controllability as we have established it so far for the interconnected power system is not very useful from a practical standpoint. Also, there are constraints on the initial and final values of the power generation $\underline{P} \in \mathbb{R}^n$ arising from the need to steer the power system to a specific equilibrium point. These constraints are described precisely in the next section and constrained steering (controllability) described therein.

X.2 Applications to the emergency control of power systems

Without going into the details of the exact definitions of emergency control of power systems (for these the reader is referred to Fink and Carlsen [26], Blankenship and Fink [27]) we will try to state as clearly as possible the control policy for an interconnected power system in the event of line breakages between buses.

In the event of line breakage the problem of emergency control involves steering the power system from the pre-disturbance equilibrium state vector $(\underset{\sim}{\theta}^1,0) \in \mathbb{R}^{2n}$ at time 0 to a post-disturbance equilibrium state vector $(\underset{\sim}{\theta}^2,0) \in \mathbb{R}^{2n}$ at time T with the constraint that the vector of power injections at time T be the same as that at time 0.

More precisely: at the pre-disturbance equilibrium for a vector of constant power injections $\underset{\sim}{P}^1 \in \mathbb{R}^n$, the state of the power system is the vector $(\underset{\sim}{\theta}^1,0) \in \mathbb{R}^{2n}$ satisfying the equilibrium load flow equations

$$P_i^1 = \sum_{j\neq i} [B_{ij} \text{Sin } (\theta_i^1 - \theta_j^2) + G_{ij} \; (1-\text{Cos}(\theta_i^1-\theta_j^1)] \quad i = \;,\ldots,n \tag{X.3}$$

and

$$\omega_i = 0 \qquad\qquad i = 1,\ldots,n. \tag{X.4}$$

Of course, one of the θ_i^1's in equation (X.3) may (arbitrarily) be chosen to be zero. With this in mind, it is assumed that for fixed $\underset{\sim}{P}^1$, the constant solution $(\underset{\sim}{\theta}^1,0)$ of equations (X.1) and (X.2) is asymptotically stable in the sense of Lyapunov. If, after line breakage, there exists a new asymptotically stable solution θ^2 of the load flow equations (with the B_{ij}, G_{ij} replaced by B'_{ij}, G'_{ij}), it is desirable (for reasons of optimal load dispatching power supply commitments) to steer the system from $(\theta^1,0)$ to $(\theta^2,0)$ through the P_i' s. The constraints are that $P_i(0) = P_i^1$ and $P_i(t) = P_i^1 \;\forall\; t \geqq T$ for i=1,..., n (so as to keep the power system in the equilibrium stable state $(\theta^2,0)$ $\forall t \geqq T$) and $P_i(\cdot)$ is smooth function of time (C^r for some r). To account for the new constraint we introduce a new set of variables $\underline{v} \in \mathbb{R}^n$ with

$$\dot{P}_i = v_i \qquad i = 1,\ldots,n \tag{X.5}$$

so as to make each P_i a state. We augment the DDS of (X.1), (X.2) (with B_{ij}, G_{ij} replaced by B'_{ij}, G'_{ij}) with the equations (X.5). The sate space is now of dimension 3n. To check the completely controllability of the DDS described by equaitons (X.1), (X.2) and (X.5) through the $\underline{v}$,we notice that the "nonlinear part" of equations (X.2), namely

$\sum_{j\neq i} B_{ij} \sin(\theta_i-\theta_j) + G_{ij}(1-\cos(\theta_i-\theta_j))$ is bounded by $\sum_{j\neq i} (|B_{ij}|+2G_{ij})$ so that the DDS of equations (X.1), (X.2) and (X.5) is completely controllable if the linear system described by the equations

$$M_i\dot{\omega}_i + D_i\omega_i = P_i \tag{X.6}$$

$$\dot{\theta}_i = \omega_i \tag{X.7}$$

$$\dot{P}_i = v_i \ , \quad i = 1,\ldots,n \tag{X.8}$$

is completely controllable. It is easy to check that this corresponds to n decoupled FDLS each in controllable canonical form and hence controllable $\forall T > 0$. Hence the DDS of equations (X.1), (X.2) and (X.5) is uniformly completely controllable $\forall T > 0$ and there exists a $v \in C^\infty$ (for time-invariant systems the functions in $\mathcal{L}_R^*(\mathbb{R}^n)$ are also C^∞) so as to drive the DDS from $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ to $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$ in T seconds. From the power system operators viewpoint the resulting trajectory of P(t) is the required control.

Often, however, the problem of steering the system from $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ to $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$ has added constraints: the thermal capacities of the lines, the maximum possible frequency deviation of the generators (ω_i) before frequency protective devices trip, the maximum generating capacity of the generators, etc. All of these introduce affine constraints on the region of $\mathbb{R}^{3n}$ in which the (θ,ω,P) trajectory can lie. For instance if C_{ij} is the thermal (heating) limit of the line between buses i and j; $|\theta_i-\theta_j| \leq \sin^{-1}((C_{ij}-2G_{ij})|B_{ij})$ could be the constraint on the state space of θ_i's. It is required that the control law $v(\cdot)$ keep the trajectory in the desirable region of the state space while making the transfer. It is clear that the desirable region of the state space is a collection (possibly more than one) of polytopes. If $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ and $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$ both belong to the same polytope we contend that it is possible to steer the system from $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ to $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$ with the trajectory staying inside the compact polytope.

Proposition X.1 (Emergency control of the power system)

Given two points $(\underset{\sim}{\theta}^1,0,P^1) \in \mathbb{R}^{3n}$ and $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1) \in \mathbb{R}^{3n}$ belonging to the interior of a compact polytope $\mathcal{P}$ of $\mathbb{R}^{3n}$ as initial and final states of an interconnected power system satisfying (X.1), (X.2) and (X.5) (with B_{ij},G_{ij} replaced by B'_{ij},G'_{ij}) there exists a control law v (·) defined on [0,T] which steers the power system from $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ to $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^2) \in$ interior $\mathcal{P}$ keeping the trajectory inside $\mathcal{P}, \forall t \in \mathbb{R}_+$.

Proof: Let $\mathcal{S}$ be the straight line connecting $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$ and $(\underset{\sim}{\theta}^2,0,\underset{\sim}{P}^1)$. Since $\mathcal{P}$ is a convex polytope, belongs to the interoir of $\mathcal{P}$ Now with center $(\underset{\sim}{\theta}^1,0,P^1)$ let $\mathcal{B}_1$ be a ball (Euclidean norm) lying wholly in $\mathcal{P}$(such A ball exists since $(\theta^1,0, \underset{\sim}{P}^1)$is interior $\mathcal{P}$. Let $\mathcal{S} \cap \partial B_1 = (\theta^3,\omega^3,\underset{\sim}{P}^3)$ where ∂B, denotes the boundary of $\mathcal{B}_1$. Let $\mathcal{B}_2 \subset \mathcal{P}$be a ball centered at $(\theta^3,\omega^3,\underset{\sim}{P}^3)$intersecting $\mathcal{S}$ in $(\theta^4,\omega^4,\underset{\sim}{P}^4)$and so on. Clearly from the compactness of $\mathcal{P}$ it follows that there exists a finite sequence of balls in $\mathcal{P}$ say $\mathcal{B}_1,\ldots,\mathcal{B}_N$ constructed as above with $(\underset{\sim}{\theta}^2,0, \underset{\sim}{P}^1) \in \mathcal{B}_N$. We will choose a (finite) sequence of controls $v^1,\ldots,v^N$ defined on $[0,T_1]$, $[T_1,T_1+T_2],\ldots$ so as to keep the trajectory in the balls $\mathcal{B}_1,\mathcal{B}_2$, etc. and to steer the power system from $(\underset{\sim}{\theta}^1,0,\underset{\sim}{P}^1)$ to $(\underset{\sim}{\theta}^3,\underset{\sim}{\omega}^3,\underset{\sim}{P}^3)$, $(\underset{\sim}{\theta}^3,\underset{\sim}{\omega}^3,\underset{\sim}{P}^3)$ to $(\underset{\sim}{\theta}^4,\underset{\sim}{\omega}^4,\underset{\sim}{P}^4)$, etc. But, it is easily verified from the estimates in the proof of Theorem (V.1) (inequality (V.7) and equations (V.7)) that $T_1,T_2,\ldots$ can be chosen so as to make this possible. ¤

Section XI Conclusions

In this paper we have shown that the uniform complete controllability of linear time varying finite dimensional systems is robust against a wide range of addiitve nonlinear perturbations, both bounded and unbounded which are not "too large". A measure of the degree of robustness that we are able to establish by our methods seems to be the reachability condition number defined in Section III. This is in keeping with standard numerical analysis intuition. Uniform zero-input observability of linear time varying finite dimensional systems is also shown to be robust against nonlinear perturbations in the dynamics and output channel. The nonlinear perturbations are restricted in some ways: the perturbations in the dynamics keeps the origin to be a fixed point of the flow (of the undriven "state" equation) and the additive perturbation in the output channel is unbiased (i.e. the origin of the state space is mapped to the origin of the output space)- but the necessity of this restriction is obvious. Finally, the techniques of our paper have been used to yield some preliminary results on the emergency control of power systems - which had been so far an analytically intractable problem. Further results in this direction are expected.

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Appendix. Fixed Point Theorems

A.1. The Rothe Fixed Point Theorems

Let $F: K \subset \mathcal{X} \to \mathcal{X}$ be a compact map defined on a closed ball K in a Banach space $\mathcal{X}$. If $F(\partial K)$, the image of the boundary ∂K of K, lies in K there exists at least one fixed point of the mapping F.

The Rothe fixed point theorem can be proved using the Schauder fixed point theorem, namely.

A.2. The Schauder Fixed Point Theorem

Let $F : K \subset \mathcal{X} \to \mathcal{X}$ be a compact map defined on a closed, convex subset K of a Banach space $\mathcal{X}$. If the image $F(K) \subset K$ then F has at least one fixed point.