HIERARCHICAL STABILITY AND ALERT STATE STEERING CONTROL OF INTERCONNECTED POWER SYSTEMS

by

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CONTROL OF INTERCONNECTED POWER SYSTEMS

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ABSTRACT

A state space model of an interconnected power system having both generator and load nodes is proposed. The resulting system of equations is interpreted as the degenerate limit of a singularly perturbed system. The model is used to devise a condition for the (local) asymptotic stability of an equilibrium. This condition decomposes in an intuitive way for subsystems interconnected via a backbone network. The model is used to formulate the problem of steering the power system from a post-disturbance alert state to a secure state, and a solution to the steering problem is also proposed.

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1. Introduction

We consider an interconnected power system in which there are both load and generator buses. A standard mathematical formulation of this system leads to a set of differential equations governing the generator frequencies and angles, and a set of nonlinear algebraic equations corresponding to the load buses. The presence of these algebraic equations makes analysis awkward. We propose to convert these into differential equations by supposing that the load depends upon frequency. The original system is then interpreted as the "degenerate" limit of the system with frequency-dependent loads as this dependency goes to zero. To permit such an interpretation we insist upon consistency in the sense of Hoppensteadt's work [1] on singularly perturbed systems. Section 2 is devoted to the development of the model.

We use such a model in two ways. First we derive a condition characterizing the (local) asymptotic stability of an equilibrium state. Perhaps surprisingly, this condition turns out to be the same as has been obtained [9] when there are no load buses. In some cases, several power subsystems are interconnected through a "backbone" transmission network. In such a case the stability condition for the overall system decomposes into a stability of the subsystems and of the backbone network. This decomposition has a very natural interpretation. Stability is discussed in Section 3.

Next we formalize three kinds of disturbances: line-switching, generator dropping and load change. We assume that the disturbance moves the state of the system from the secure to the alert region. We formulate the resulting control problem as one of finding a control which steers the system from the alert state to a (new) secure state while obeying the rated power flow capacities of the transmission lines. We assume that the control variables which can be manipulated, within certain limits, are the mechanical power inputs at various generating stations. We propose a solution to this steering control problem. While the solution is to a certain extent constructive it is as yet far from being implementable. This discussion occupies Section 4. Some concluding remarks are collected in Section 5.

2. Stability of interconnected power systems

2.1 Model of an interconnected power system

We consider a power system consisting of g generators and ℓ load nodes. Each generator is connected to a generator bus and each load node to a load bus. The buses are connected to each other by transmission lines. (See Figure 1).

Synchronous generators

The departure from synchronism of the ith generator is governed by the classical swing equation,

$$M_i\dot{\omega}_i + D_i\omega_i = P_i^m - P_i^g \qquad , i = 1,..,g \qquad (2.1)$$

$$\dot{\delta}_i = 2\pi\omega_i \qquad , i = 1,..,g \qquad (2.2)$$

where

$M_i(D_i)$ = moment of inertia (damping constant),

ω_i = departure of generator frequency from synchronous frequency ω_o,

δ_i = generator rotor angle (also assumed to equal generator bus angle) measured relative to a synchronously rotating reference,

P_i^m = exogenously specified mechanical input power minus power loss due to damping ($\omega_o D_i$) minus electrical power demanded at the generator bus,

P_i^g = electrical power output (determined by (2.3) below).

Decoupled load flow

The power flow in the network is modeled by non-linear load flow equations (described below), in which the determination of real power and phase angles is decoupled from the determination of reactive power and voltage magnitudes. In standard terminology this is summarized by saying we assume all buses are PV buses. Such an assumption is valid when transmission lines have a high ratio of reactance to resistance which we assume next.

Lossless transmission lines

All transmission lines are purely reactive.

Let θ_i, $i = 1,..,\ell$, be the phase angle of the ith load bus (measured relative to the same synchronously rotating reference as the δ_j). Let P_i^ℓ be the exogenously specified electrical power demanded there. By convention $P_i^\ell < 0$ while $P_i^g > 0$. Let Y_{ij}^{gg} be the admittance (susceptance) of the transmission line connecting the ith generator and jth generator buses. $Y_{ij}^{g\ell}$ and $Y_{ij}^{\ell\ell}$ are similarly defined. Of course, if two buses are not directly connected by a transmission line the corresponding Y is zero. With this notation the load flow equations are

$$P_i^g = f_i^g(\delta,\theta) = \sum_{j\neq i} Y_{ij}^{gg} \operatorname{Sin}(\delta_i - \delta_j) + \sum_k Y_{ik}^{g\ell} \operatorname{Sin}(\delta_i - \theta_k),\ i=1,..,g \tag{2.3}$$

$$P_i^\ell = f_i^\ell(\delta,\theta) = \sum_j Y_{ji}^{g\ell} \operatorname{Sin}(\theta_i - \delta_j) + \sum_{k\neq i} Y_{ik}^{\ell\ell} \operatorname{Sin}(\theta_i - \theta_k),\ i=1,..,\ell. \tag{2.4}$$

Here and throughout δ, θ, P^g, etc. denote vectors with components

δ_i, θ_i, P_i^g etc. Notice that because $Y_{ij}^{gg} = Y_{ji}^{gg}$, $Y_{ik}^{g\ell} = Y_{ki}^{\ell g}$ etc.
$\sum_i P_i^g + \sum_j P_j^\ell = 0$

Equations (2.1) - (2.4) constitute the model. We now give a preliminary interpretation of the model. Observe that since the vector P^m of mechanical power inputs and the vector P^ℓ of electrical power loads are exogenously specified, and since P^g is given by (2.3), there remain $2g + \ell$ "unknowns", namely ω, δ, θ, and $2g + \ell$ equations, namely (2.1), (2.2) and (2.4). Next, ω and δ, being governed by differential equations, cannot change instantaneously, so the correct interpretation should be that (2.4) must be "solved" to obtain θ in terms of δ and use this solution to get a system of 2g differential equations involving only ω and δ. Unfortunately for a given δ (and P^ℓ) equation (2.4) may yield no solution for θ or it may yield several solutions. <u>Therefore it is clear that if we wish to interpret (2.1) - (2.4) as a model of a dynamical system, so that we can analyze its stability and controllability properties etc., then we must be careful in specifying the underlying state space.</u> We do this next.

<u>State space of the power system</u>

The preceding discussion suggests the following definition for the state space X of the power system,

$$X = R^g \times M = \{\omega \in R^g\} \times \{(\delta,\theta) \in R^{g+\ell} | f^\ell(\delta,\theta) = P^\ell \text{ and } D_\theta f^\ell(\delta,\theta) \text{ nonsingular}\} \tag{2.5}$$

where $D_\theta f^\ell(\delta,\theta) = \frac{\partial f^\ell}{\partial \theta}(\delta,\theta)$. Thus we permit ω to be arbitrary but restrict δ,θ such that (i) they satisfy (2.4) and (ii) we can solve (2.4) for θ "smoothly" in terms of δ. Note that M depends upon the prespecified power demands P^ℓ and the admittances Y. If it is non-empty then M is a smooth, in fact analytic, manifold of dimension g. It is not connected and contains several "sheets" corresponding to the several solutions of (2.4) (see Figure 2). X is then a manifold of dimension 2g. It is possible to reduce this dimension to 2g - 1: observe in (2.3), (2.4) that the functions f^g, f^ℓ depend only upon the differences of the phase angles so that we could take one of the generator angles, say δ_g, as reference, define the remaining angles as deviations from δ_g, then eliminate the gth differential equation from (2.2) and adjust the others accordingly. Since nothing significant is gained by this reduction we prefer not to do so.

A point $(\omega^o, \delta^o, \theta^o) \in X$ is an equilibrium if $\omega^o = 0$, and $P^m - P^g = P^m - f^g(\delta^o, \theta^o) = 0$. Evidently, in the absence of any disturbances, the system stays forever at an equilibrium.

2.2 Model as degenerate limit of a singularly perturbed system

We wish to study the Lyapunov or local stability of an equilibrium. To carry out this study it is inconvenient to use directly the state space X introduced above. It is more suitable to augment the state space to all of $R^{2g+\ell} = \{(\omega, \delta, \theta) \mid \omega \in R^g, \delta \in R^g, \theta \in R^\ell\}$ and to augment the system dynamics appropriately so that (2.1) - (2.4) can be regarded as the degenerate form of a singularly perturbed system. As we will see this is not only mathematically convenient but it is also physically meaningful.

The key idea is to recognize that the load at the ith load bus is not the constant load P_i^ℓ as hypothesized previously but it is dependent on frequency, that is to say the 'true' load at the ith bus is

$$P_i^\ell - \varepsilon \dot{\theta}_i \tag{2.6}$$

where $\varepsilon > 0$ is a small parameter. Recall that $\dot{\theta}_i$ is the deviation from the synchronous frequency of the frequency at the ith load bus. Hence if $\dot{\theta}_i < 0$, i.e. there is a drop in frequency, then the true load $|P_i^\ell + \varepsilon\dot{\theta}_i|$ drops below the nominal load $|P_i^\ell|$, whereas if $\dot{\theta}_i > 0$, then the true load rises above the nominal load. Such a load-frequency characteristic is in conformity with empirical observation, although a more accurate representation may require a nonlinear characteristic. However for small magnitudes of $\dot{\theta}_i$ the linear characterization (2.6) should be adequate. With this assumption the system dynamics are governed by (2.1), (2.2), (2.3) and (2.7) replacing (2.4):

$$P_i^\ell - \varepsilon\,\dot{\theta}_i = f_i^\ell\,(\delta,\theta) \quad , \qquad i = 1,..,\ell\,. \tag{2.7}$$

We call the system described by (2.1) - (2.3), (2.7) the <u>perturbed system</u>. Its state space is $R^{2g+\ell}$. The system described by (2.1) - (2.4) is called the <u>degenerate system</u>. These terms are borrowed from the theory of singularly perturbed systems (see [1]). To justify their use we must show that for $\varepsilon > 0$ the degenerate system does approximate the perturbed system. This is <u>not</u> generally the case unless the system is in the neighborhood of a stable equilibrium point, as we now demonstrate.

Observe that the set of equilibrium points of the perturbed and degenerate systems are the <u>same</u>. Consider one such equilibrium $(\omega^o = 0,\ \delta^o,\ \theta^o)$. The equations of the perturbed system, linearized around this equilibrium, are

$$M\Delta\dot{\omega} + D\Delta\omega = -D_\delta f^g(\delta^o,\theta^o)\Delta\delta - D_\theta f^g(\delta^o,\theta^o)\Delta\theta, \tag{2.8}$$

$$\Delta\dot{\delta} = 2\pi\Delta\omega, \tag{2.9}$$

$$\varepsilon\Delta\dot{\theta} = -D_\delta f^\ell(\delta^o,\theta^o)\Delta\delta - D_\theta f^\ell(\delta^o,\theta^o)\Delta\theta, \tag{2.10}$$

where M, D are diagonal matrices with entries M_i, D_i respectively, $\Delta\omega = \omega - \omega^o$, $\Delta\delta = \delta - \delta^o$, $\Delta\theta = \theta - \theta^o$.

Similarly the equations (2.1) - (2.4) of the degenerate system, linearized around the same equilibrium, are

$$M\Delta\dot{\tilde{\omega}} + D\Delta\tilde{\omega} = - D_\delta f^g(\delta^o,\theta^o)\Delta\tilde{\delta} - D_\theta f^g(\delta^o,\theta^o)\Delta\tilde{\theta}, \tag{2.11}$$

$$\Delta\dot{\tilde{\delta}} = 2\pi\Delta\tilde{\omega}, \tag{2.12}$$

$$0 = - D_\delta f^\ell(\delta^o,\theta^o)\Delta\tilde{\omega} - D_\theta f^\ell(\delta^o,\theta^o)\Delta\tilde{\theta}. \tag{2.13}$$

Theorem 2.1 Suppose that

$$D_\theta f^\ell(\delta^o,\theta^o) > 0 \tag{2.14}$$

$$A = D_\delta f^g(\delta^o,\theta^o) - D_\delta f^g(\delta^o,\theta^o)\,[D_\theta f^\ell(\delta^o,\theta^o)]^{-1}\, D_\delta f^\ell(\delta^o,\theta^o) \geq 0, \tag{2.15}$$

and furthermore that the matrix A has exactly one zero eigenvalue with corresponding eigenvector 1^g, the vector in R^g all of whose components are unity. Let $(\omega^\varepsilon(t),\delta^\varepsilon(t),\theta^\varepsilon(t))$ be a solution of the perturbed system with initial condition $\omega(0)$, $\delta(0)$, $\theta(0)$. If $|\omega(0)-\omega^o| + |\delta(0)-\delta^o| + |\theta(0)-\theta^o| = R$ is sufficiently small then, as $\varepsilon \to 0+$, $(\omega^\varepsilon(t),\delta^\varepsilon(t),\theta^\varepsilon(t))$ converges to a solution of the degenerate system, uniformly on $[T,\infty)$ for $T > 0$.

Proof. See Appendix I ⌑

Comment: Recall that f^g, f^ℓ depend only upon differences between the various bus angles, that is for all real α

$$f^g(\delta^o+\alpha 1^g,\theta^o+\alpha 1^\ell) = f^g(\delta^o,\theta^o),$$
$$f^\ell(\delta^o+\alpha 1^g,\theta^o+\alpha 1^\ell) = f^\ell(\delta^o,\theta^o).$$

Differentiating at δ^o, θ^o gives

$$D_\delta f^g 1^g + D_\theta f^g 1^\ell = 0,\quad D_\delta f^\ell 1^g + D_\theta f^\ell 1^\ell = 0, \tag{2.16}$$

from which

$$\{D_\delta f^g - D_\theta f^g [D_\theta f^\ell]^{-1}\, D_\delta f^\ell\}\, 1^g = 0\ .$$

Thus the matrix in (2.15) always has a zero eigenvalue with eigenvector 1^g.

Theorem 2.1 says that the description of the interconnected power system as given by the degenerate equations is consistent with its description given by the perturbed equations in the neighborhood of certain equilibria. These equilibria are stable as shown later. It is possible to show consistency in the neighborhood of unstable equilibria provided however that the initial state of the perturbed system lies on the stable manifold of these equilibria (see [2]). However such a generalization does not seem to us to make sense in the power system context.

Conditions (2.14), (2.15) can be combined. Let $f(\delta,\theta) = (f^g(\delta,\theta), f^\ell(\delta,\theta))$ and let Df be its Jacobian. It is a matrix of dimension $g + \ell$.

Proportion 2.1 The hypothesis of Theorem 2.1 holds if and only if

$$J = Df(\delta^o,\theta^o) \geq 0$$

and further J has exactly one zero eigenvalue (corresponding to $1^{g+\ell}$).

Proof. See Appendix I ◻

2.3 Asymptotic stability

Definition 2.1 An equilibrium $(\omega^o = 0,\delta^o,\theta^o)$ of the degenerate system is asymptotically stable if there is $\varepsilon^o>0$ such that (i) for all $0<\varepsilon<\varepsilon^o$, and for all $\Delta\omega$, $\Delta\delta$, $\Delta\theta$ sufficiently small, the solution of the perturbed system corresponding to the initial condition $(\omega^o+\Delta\omega,\delta^o+\Delta\delta,\theta^o+\Delta\theta)$ converges asymptotically to the one dimensional subspace $\{(0,\delta^o+\alpha 1^g,\theta^o+\alpha 1^\ell) \mid \alpha \in R\}$, and (ii) as $\varepsilon\to 0+$, these solutions converge to a solution of the degenerate system uniformly on $[T,\infty)$ for all $T>0$.

Theorem 2.2 An equilibrium ($\omega^o = 0, \delta^o, \theta^o$) of the degenerate system is asymptotically stable if $J = Df(\delta^o, \theta^o) \geq 0$ and furthermore this matrix has exactly one zero eigenvalue.

Proof. See Appendix I $\square$

Theorem 2.2 justifies the use of the degenerate system as an approximation to the perturbed system in a neighborhood of a stable equilibrium. For a fixed $\varepsilon > 0$, the stability of the augmented system has been analyzed in [3]. Note that the stability condition is the same as that which has been obtained for a network where there are only generator nodes.

3. Hierarchial stability

Interconnected power systems usually consist of several networks each operated by different utilities coupled via tie lines. Thus we may visualize a power network N as consisting of subnetworks $N^1,..,N^k$ interconnected by a "backbone" network N^o . N^o is obtained from N by coalescing the nodes within each of the subnetworks (see Figure 3). We suppose that each of the N^i has its own control center (area controller) while a central controller is provided for the backbone network N^o. Thus control is shared by two levels. It is desirable that the central controller have only limited information about the subnetworks and each area controller have only limited information about other subnetworks.

With these considerations in mind we derive the stability condition obtained earlier, in terms of conditions on the subnetworks $N^1,..,N^k$ and the backbone network N^o. In section 3.1 we consider the simple case where each N^i is connected to other subnetworks through a single bus. In section 3.2 we consider arbitrary interconnections.

3.1 A special class of interconnected systems

The ith subnetwork consists of g^i generator buses and ℓ^i load buses with state variables ω^i, δ^i, θ^i. There is a privileged bus, called the boundary bus at node b^i say, such that N^i is connected to the other N^j only through b^i. In this case N^o consists precisely of the nodes $b^1,\ldots,b^k$ and the transmission lines interconnecting them (see Figure 4). Let $g = \Sigma g^i$, $\ell = \Sigma \ell^i$, let $\omega = (\omega^1,\ldots,\omega^k) \in R^g$ $\delta = (\delta^1,\ldots,\delta^k) \in R^g$ and $\theta = (\theta^1,\ldots,\theta^k) \in R^\ell$ be the state variables for the interconnected network N with $(\omega^i, \delta^i, \theta^i)$ the variables for N^i. Finally let $(\omega^o, \delta^o, \theta^o)$ be the variables corresponding to the backbone network N^o.

We denote by f, respectively f^i, the power flow functions for N, respectively N^i, $i = 0,1,\ldots,k$. Let $(\omega^o = 0, \delta^o, \theta^o) \in R^{2g+\ell}$ be an equilibrium of N, and let $(\omega^{io} = 0, \delta^{io}, \theta^{io})$ be the corresponding values for the network N^i, $i = 0,1,\ldots,k$.

Theorem 3.1 The following statements are equivalent.

(i) $J = Df(\delta^o, \theta^o) \geq 0$ and the matrix has exactly one zero eigenvalue.

(ii) For each $i = 0,\ldots,k$, $J^i = Df^i(\delta^{io}, \theta^{io}) \geq 0$ and the matrix has exactly one zero eigenvalue.

Proof. See Appendix II

$\square$

It is easy to understand why the stability of the network N is equivalent to the stability of the subnetworks $N^1,\ldots,N^k$ and the backbone network N^o. Consider a stable equilibrium of N. Since each subnetwork N^i, $i \geq 1$, is connected to the rest only through a single bus, therefore if we take the angle at that bus to be a reference, then N^i will be essentially "decoupled" from the rest of the network so that its equilibrium must be stable. As far as the backbone network is concerned, each N^i may be replaced by its single boundary bus with a load equal to the amount of electrical power flowing into N^i (through its boundary bus) at equilibrium, so that N^o must be stable as well.

3.2 Arbitrary interconnected systems

Generally the subnetwork N^i is connected to the rest through several boundary buses. Let b^i_j, $j = 1,..,n^i$ be those boundary buses. As before let f be the flow function for N and f^i the flow function for N^i, $i = 1,..,k$. (Note that a flow function for N^o is not defined if for some $i, n^i > 1$). Let $(\omega^o = 0, \delta^o, \theta^o)$ be an equilibrium of N and let $(\omega^{io} = 0, \delta^{io}, \theta^{io})$ be the corresponding equilibria for N^i, $i = 1,..,k$.

Theorem 3.2. Suppose conditions (i), (ii) and (iii) hold.

(i) For each $i = 1,..,k$, $Df^i(\delta^{io}, \theta^{io}) \geq 0$ and the matrix has exactly one zero eigenvalue.

(ii) For each $i = 1,..,k$ there exists a distinguished boundary bus $\tilde{b}^i \in \{b^i_1, \cdot\cdot, b^i_{n^i}\}$ such that if $\tilde{N}^o$ is the reduced backbone network obtained from N^o by deleting all transmission lines of N^o except those which connect the distinguished buses $\tilde{b}^i$, $1 \leq i \leq k$, and if $\tilde{f}^o$ in the (now well-defined) flow function for $\tilde{N}^o$ then $D\tilde{f}^o(\tilde{\delta}^{oo}, \tilde{\theta}^{oo}) > 0$ and the matrix has exactly one zero eigenvalue. (The vector $\tilde{\delta}^{oo}$, $\tilde{\theta}^{oo}$ consist of the components of δ^o, θ^o corresponding to $\tilde{N}^o$).

(iii) The angle difference $|\theta_{b^i_j} - \theta_{b^k_\ell}|$ across the transmission lines in N^o that are deleted is strictly less than $\frac{\pi}{2}$.

Proof. See Appendix II ◻

The arguments used in Section 3.1 may be used to obtain stability conditions for other than the special class considered there. For example consider the two interconnected subnetworks one of which has a single boundary bus and the other has several (see Figure 5a). It is easy to show that the interconnected network is stable if and only if the networks N_1' and $\hat{N}_2$ of Figure 5b are stable.

4. Steering of power system in alert state

Disturbances in the power system may drive it from a normal or secure operating state to an insecure or alert state. A more extensive description of the alert region is given in [4,5]. Here we merely note that when the system is in the alert region, active control measures must be undertaken to steer the system to a secure state, because otherwise there is a danger of system breakdown. In this section we use the model introduced above to formulate and analyze a particular steering control strategy.

We suppose that initially the system is in a secure state and that some disturbance has caused it to move to an alert state. Three kinds of disturbances are considered: line switching, generator dropping, and load change. The steering problem for line switching is discussed in detail in sections 4.1 and 4.2, and the two remaining cases are briefly treated in section 4.3.

4.1. Line switching

The switching (i.e. opening or re-closing) at time t=0 of a line connecting buses i and j is represented, in our model, by a change in the admittance from Y_{ij} to Y'_{ij}. Now if either i or j is a load bus then the algebraic relations (2.4) change and so the manifold defined by (2.5) changes from M to say M'. We assume that prior to the disturbance the system is at a stable equilibrium $(\omega^o,\delta^o,\theta^o)$ with $\omega^o = 0$ and $(\delta^o,\theta^o) \in M$. Immediately after switching, at $t = 0+$, the generator frequencies and angles cannot change from their pre-fault values, that is we must have $(\omega(0+),\delta(0+)) = (\omega^o,\delta^o)$. Hence the only viable interpretation of the degenerate system model is that the load bus angles change instantaneously to a new value $\theta(0+) \neq \theta^o$ such that $(\delta^o,\theta(0+)) \in M'$ i.e. satisfies (2.4).

Our first problem is to determine this value $\theta(0+)$. Once this has been determined the post-fault behavior of the degenerate model will be governed by the same equations (2.1)-(2.4) (with Y_{ij} replaced by Y'_{ij}), but with new initial conditions $(\omega(0+),\delta(0+),\theta(0+)) = (\omega^o,\delta^o,\theta(0+))$.

We face two difficulties in finding the correct value of $\theta(0+)$ such that $(\delta^o,\theta(0+)) \in M'$. In the first place there may exist no such value in which case no further discussion can be conducted within the terms of the degenerate model. So, to proceed further, we need to assume the existence of θ' such that $(\delta^o,\theta') \in M'$. The second difficulty stems from the fact that there may be several such θ'; which of these should we choose as the "correct" value of $\theta(0+)$? We propose the following answer to this question. Observe that for any $\varepsilon > 0$, the post-fault trajectory, of the perturbed system $(\delta_\varepsilon(t),\theta_\varepsilon(t))$, $t > 0$, is uniquely determined, since $(\delta_\varepsilon(0+),\theta_\varepsilon(0+)) = (\delta^o,\theta^o)$. So we propose to define the initial value $(\delta(0+),\theta(0+))$ for the post-fault degenerate system as the following limit, provided it exists:

$$(\delta(0+),\theta(0+)) = \lim_{t\to 0+} \lim_{\varepsilon\to 0+} (\delta_\varepsilon(t),\theta_\varepsilon(t)).$$

It is easy to see that

$$\delta(0+) = \lim_{t\to 0+} \lim_{\varepsilon\to 0+} \delta_\varepsilon(t) = \delta^o$$

as expected; whereas, if the limit below exists then

$$\theta(0+) = \lim_{t\to 0+} \lim_{\varepsilon\to 0+} \theta_\varepsilon(t) = \lim_{\varepsilon\to 0+} \tilde{\theta}_\varepsilon(t),\ t > 0, \tag{4.1}$$

where $\tilde{\theta}_\varepsilon(t)$, $t \geq 0$, is the trajectory of the "boundary layer" system given by (2.7) with δ fixed at δ^o, i.e.,

$$\varepsilon\dot{\tilde{\theta}}_\varepsilon(t) = P^\ell - f'^\ell(\delta^o,\tilde{\theta}_\varepsilon),\ \tilde{\theta}_\varepsilon(0) = \theta^o \tag{4.2}$$

(f'^{ℓ} is the post-fault load flow function.) From (4.2) we see that the trajectories corresponding to different ε are related by a change of time scale,

$$\tilde{\theta}_{\varepsilon_1}(t) = \tilde{\theta}_{\varepsilon_2}(\varepsilon_2 t/\varepsilon_1), \quad t \geq 0 \tag{4.3}$$

and so it follows from (4.1) that

$$\theta(0+) = \lim_{t\to\infty} \tilde{\theta}_{\varepsilon}(t), \tag{4.4}$$

provided this limit exists in which case, by (4.3) it is independent of $\varepsilon > 0$. We now investigate when such a limit exists. Rewrite (4.2) as

$$\varepsilon\dot{\tilde{\theta}}_{\varepsilon}(t) = -\nabla V(\delta^o,\tilde{\theta}_{\varepsilon}(t)),\ \tilde{\theta}_{\varepsilon}(0) = \theta^o, \tag{4.5}$$

where $\nabla V(\delta^o,\theta)$ denotes the gradient with respect to θ of the "potential" function V given by the path integral

$$\begin{aligned} V(\delta^o,\theta) &= -\int_0^{\theta} [P^{\ell}-f'^{\ell}(\delta^o,\xi)]^t d\xi \\ &= -\sum_i P_i^{\ell}\theta_i - \sum_i\{\sum_j Y_{ji}'^{g\ell}\cos(\theta_i-\delta_j^o) + \sum_{k\neq i} Y_{ik}'^{\ell\ell}\cos(\theta_i-\theta_k)\} + K, \end{aligned} \tag{4.6}$$

using (2.4). Here K is some constant and the primes denote post-fault values. Thus the trajectory $\tilde{\theta}_{\varepsilon}(t)$ follows a path of steepest descent of the potential V starting at θ^o. Therefore, if this trajectory converges at all it must converge to a point θ' such that $\nabla V(\delta^o,\theta') = P^{\ell}-f'^{\ell}(\delta^o,\theta') = 0$ i.e. $(\delta^o,\theta') \in M'$. (Recall that we have already assumed the existence of θ' with $(\delta^o,\theta') \in M'$.) Suppose that $\theta_{\varepsilon}(t)$ does converge to θ' so that $\nabla V(\delta^o,\theta') = 0$. Then θ' is either a local minimum of V, and then $D_{\theta}f'^{\ell}(\delta^o,\theta') \geq 0$, or θ' is a point of inflexion of V. In the latter case, even though $\tilde{\theta}_{\varepsilon}(t)$, starting at θ^o, converges to

θ', the slightest change from the initial condition θ^o can lead to trajectories which will not converge to θ'. (In other words, the equilibrium point θ' of (4.5) is a saddle.) Such a delicate dependence on initial conditions is exhibited by (4.5) but it seems to us unreasonable to suppose that this fragile behavior accurately reflects the "real" system. We shall assume instead that there exists a θ' with $(\delta^o,\theta') \in M'$ and $D_\theta f'^{\ell}(\delta^o,\theta') > 0$ so that θ' is an asymptotically stable equilibrium of (4.5) and θ^o belongs to the attractor of θ'. To guarantee this stability we restrict our attention to a subset of the state space called the principal polytope.

The principal polytope is the convex open set C of $R^{g+\ell}$ given by

$$C = \{(\delta,\theta) \big| |\delta_i-\theta_j| \le \frac{\pi}{2} \text{ if } Y_{ij}^{g\ell} > 0,\ |\delta_i-\delta_j| \le \frac{\pi}{2} \text{ if } Y_{ij}^{gg} > 0,\ |\theta_i-\theta_j| < \frac{\pi}{2} \text{ if } Y_{ij}^{\ell\ell} > 0\}.$$

Evidently C depends upon Y. Before returning to the main discussion we present two properties of the principal polytope.

Lemma 4.1. If $(\delta,\theta) \in M \cap C$, then $D_\theta f^{\ell}(\delta,\theta) > 0$. In particular, if $M \cap C$ is nonempty, then it is a smooth g dimensional submanifold of $R^{g+\ell}$.

Proof. See Appendix III. ⌑

Let C, C' denote the pre- and post-fault principal polytopes. (Observe from (4.7) that if the fault consists of the opening of one of the lines then $C \subset C'$.)

Lemma 4.2. Suppose there exists θ' such that $(\delta^o,\theta') \in M' \cap C'$. Then such θ' is unique. Thus $M' \cap C'$ is a connected manifold.

Proof. See Appendix III. ⌑

We return to the main discussion. We make

Assumption A. The pre-fault equilibrium values $(\delta^o,\theta^o) \in M \cap C$ is such that there exists θ', necessarily unique, such that $(\delta^o,\theta') \in M' \cap C'$.

By Lemma 4.1 $D_\theta f'^{\ell}(\delta^o,\theta') > 0$ and so θ' is an asymptotically stable equilibrium of the differential equation (4.5), hence the solution of (4.5) satisfies

$$\lim_{t\to\infty} \tilde{\theta}_\varepsilon(t) = \theta'$$

provided $\tilde{\theta}_\varepsilon(0) = \theta^0$ belongs to the attractor of θ'. This leads to

Assumption B. θ^o belongs to the attractor of θ'.

Under these assumptions the limit in (4.1) is $\theta(0+) = \theta'$. This justifies the following choice for the post-fault initial condition for the degenerate system model.

Definition 4.1. Suppose the pre-fault equilibrium $(\omega^o,\delta^o,\theta^o)$ satisfies Assumptions A, B. Then the post-fault state is defined as

$$(\omega(0+),\delta(0+),\theta(0+)) = (\omega^o,\delta^o,\theta').$$

Thus at the instant of the disturbance the various bus angles of the degenerate system model jump from $(\delta^o,\theta^o) \in M \cap C$ to $(\delta^o,\theta') \in M' \cap C'$ (see Figure 6). The new initial state $(\omega^o,\delta^o,\theta')$ will not generally be an equilibrium for the post-fault system. Let $(\omega(t),\delta(t),\theta(t))$ be the trajectory issuing from the new initial state. Three qualitatively different behaviors are possible.

The worst outcome is that the post-fault system possesses no equilibrium at all (at the specified loads and rated generating capacities). In the words of [4] the situation now calls for heroic control action. Since our model is quite inadequate for discussing such control action we assume away this outcome by presupposing the existence of an equilibrium. We make

Assumption C. There exists a set of net mechanical power inputs $P^{m1} = (P_1^{m1},\cdot\cdot,P_g^{m1})$ and a post-fault equilibrium state $(\omega^1,\delta^1,\theta^1)$ with $\omega^1 = 0$ and $(\delta^1,\theta^1) \in M' \cap C'$.

Observe that this assumption implies aggregate power balance,

$$\sum_{i=1}^{g} P_i^{m1} = \sum_{j=1}^{\ell} P_j^{\ell}. \tag{4.7}$$

Using assumption C an argument almost identical to that used in the proofs of Lemmas 4.1, 4.2 shows that $(\delta^1,\theta^1) \in M' \cap C'$ is unique and $Df'(\delta^1,\theta^1) \geq 0$ with this matrix having exactly one zero eigenvalue. By Theorem 2.2 the equilibrium $(\omega^1,\delta^1,\theta^1)$ is asymptotically stable in the sense of Definition 2.1.

Two possibilities remain. The favorable outcome is that the initial state $(\omega^o,\delta^o,\theta')$ is in the normal or secure region: this means that (i) $(\omega^o,\delta^o,\theta')$ belongs to the attractor of $(\omega^1,\delta^1,\theta^1)$ and (ii) along the convergent trajectory $(\omega(t),\delta(t),\theta(t))$, the deviation of the instantaneous frequency from synchronous frequency and the magnitudes of instantaneous power flows over transmission lines, are both within rated tolerances. In terms of [4] we would say that the "inequality constraints" are respected. Clearly in this situation no active control action is necessary.

The last possibility is that the initial state $(\omega^o,\delta^o,\theta')$ is in the alert region which means that one or both of the aforementioned conditions characterizing the normal region are absent. It is then necessary to design a control over some interval [0,T] to be selected which steers the state from $(\omega^o,\delta^o,\theta')$ at time 0 to $(\omega^1,\delta^1,\theta^1)$ at time T, without violating the inequality constraints. In the next section we first present a formal description of the steering problem and then propose a solution.

4.2. The steering problem

It is assumed that control is to be exercised by varying the (net) mechanical power inputs $P^m(t)$ at the generating stations. There is a complex set of physical limitations imposed by the dynamics and capacities of power generating plants (succinctly discussed in [6]) which limits the extent to which P^m can be varied. We abstract from these considerations and impose two restrictions: there is a maximum value which $P_i^m(t)$ can take and the rate of change $|\dot{P}_i^m(t)|$ must be bounded. Formally we have

Constraint A. (Admissible control constraint) $P_i^m(t)$, $0 \le t \le T$, is admissible if for each $i = 1,..,g$

$$0 < P_i^m(t) < \pi_i, \quad 0 \le t \le T \tag{4.8}$$

and if $v_i(t) = \dot{P}_i^m(t)$ satisfies

$$\sup_{0 \le t \le T} |v_i(t)| < \infty \tag{4.9}$$

Here π_i are prespecified generating capacity limits.

To take this constraint into account it is convenient to augment the system state by P^m making v the control variable. Then the perturbed model becomes

$$\begin{aligned} M\dot{\omega} + D\omega &= P^m - f'^g(\delta,\theta) \\ \dot{\delta} &= 2\pi\omega \\ \dot{P}^m &= v \\ \varepsilon\dot{\theta} &= P^\ell - f'^\ell(\delta,\theta) \end{aligned} \tag{4.10}$$

As before, the degenerate model is obtained from (4.10) by setting $\varepsilon = 0$.

For convenience let $x = (\omega,\delta,P^m,\theta) \in R^{3g+\ell}$ denote the state. In terms of the discussion of Section 2, we interpret a degenerate system trajectory as the limit as $\varepsilon \to 0+$ of the perturbed system trajectory. This is formalized as

Constraint B. (Consistency constraint) Let $v(t)$, $0 \le t \le T$ be a control function. Then $x(t)$, $0 \le t \le T$ is a corresponding degenerate system trajectory if it satisfies (4.10) with $\varepsilon = 0$, and if, for each $\varepsilon > 0$, there is a perturbed system trajectory $x_\varepsilon(t)$ such that

$$\lim_{\varepsilon\to 0+} x_\varepsilon(t) = x(t), \quad 0 \le t \le T. \tag{4.11}$$

Finally, we have the inequality constraints.

Constraint C. (Inequality constraint) We require that

$$|\omega_i| < \Omega, \; i = 1,\cdots,g \tag{4.12}$$

$$|Y'^{gg}_{ij} \sin(\delta_i-\delta_j)| < C^{gg}_{ij}, \; |Y'^{g\ell}_{ik} \sin(\delta_i-\theta_k)| < C^{g\ell}_{ik}, \; |Y'^{\ell\ell}_{km} \sin(\theta_k-\theta_m)| < C^{\ell\ell}_{km},$$

$$i,j = 1,\cdots g; \; k,m = 1,\cdots,\ell. \tag{4.13}$$

Here Ω is the maximum permissible deviation of the instantaneous frequency and the various C_{ij} are the maximum permissible power flows. Let

$$\Sigma = \{x = (\omega,\delta,P^m,\theta) | x \text{ satisfies } (4.8), (4.12), (4.13)\}.$$

We say that a degenerate system trajectory $x(t)$, $0 \le t \le T$, is admissible if

$$x(t) \in \Sigma, \quad 0 \le t \le T \tag{4.14}$$

Definition 4.2. Given two states $x^I \in \Sigma$, $x^F \in \Sigma$ the steering problem $SP(x^I,x^F)$ is to find an admissible control $v(t)$, $0 \le t \le T$ and a corresponding admissible degenerate system trajectory x(t) such that

$$x(0) = x^I, \; x(T) = x^F.$$

In terms of this definition the steering problem following line switching is given by

$$x^I = (\omega^o, \delta^o, P^{mo}, \theta'), \; x^F = (\omega^1, \delta^1, P^{m1}, \theta^1) \tag{4.15}$$

where P^{mo} is the pre-fault power input. Let

$$\Sigma' = \{x = (\omega, \delta, P^m, \theta) \in \Sigma \,|\, (\delta, \theta) \in M' \cap C'\} \tag{4.16}$$

where C' is the post-fault principal polytope. Our main result is

<u>Theorem 4.1</u>. The steering problem $SP(x^I, x^F)$ can be solved if x^I, x^F are both in Σ'.

<u>Proof</u>. See Appendix IV. ◻

We remark here that the spirit of the proof is constructive i.e., it does suggest a way of finding a steering control. However much work needs to be done before such a control can be computed and implemented in real time.

<u>Corollary 4.1</u>. Under Assumptions A, B, C the steering problem following line switching can be solved.

<u>Proof</u>. By hypothesis x^I and x^F given by (4.15) are both in Σ'. ◻

4.3. <u>Generator dropping and load change</u>

Suppose that generator 1 say is suddenly disconnected from the network at time 0. Within our framework this situation is modelled as follows. The state space (see (2.5)) changes instantaneously from

$X = R^g \times M$ to $X' = R^{g-1} \times M'$ where M' is now a (g-1)-dimensional manifold. The bus to which generator 1 was connected now becomes a load bus so that the dimension of θ is augmented by one. The angles and frequencies of the remaining generators ω_i, δ_i, $i \geq 2$, cannot change instantaneously. The disturbance is therefore reflected in an instantaneous change in the load bus angles $\delta_1, \theta_1, \ldots, \theta_\ell$, whose values at $t = 0+$ must be determined according to the discussion of Section 4.1. If the new initial state at $t = 0+$ is in the alert region, then we are again faced with a steering problem of the kind discussed in Section 4.2.

Suppose that at time 0 there is a sudden change in the load from $P^{\ell 0}$ to $P^{\ell 1}$ say. This again shifts the manifold M to M' (see (2.5)) and will cause an instantaneous shift in the load bus angles. The situation is now exactly the same as described in Sections 4.1 and 4.2.

5. Concluding remarks

This paper is written with two objectives in mind. First we wished to present a model of an interconnected power system in which the structure of the load bus subnetwork is preserved. This was done by arguing that the algebraic constraints introduced by the static loads could be consistently regarded as a degenerate limit (boundary layer) of dynamic loads which depend on frequency. We suggest the utility of this approach by producing a stability criterion which is identical to the one obtained for an all-generator network.

Second, we wished to formulate the problem of controlling the network following a disturbance as one of steering the system from an alert state to a secure region. Our attempt can be viewed as a mathematization of informal discussions of this problem. We believe such a formalization is essential to a thorough understanding of the problem of emergency control.

Appendix I. Proofs of assertions in Section 2

I.1. Proof of Theorem 2.1. We apply the result of Hoppensteadt [1]. To do this it is convenient to choose as reference one of the generator bus angles, say δ_g, and define the others in terms of it. Accordingly let $\gamma = (\gamma_1,..,\gamma_{g-1})$ with $\gamma_i = \delta_i - \delta_g$, and $\phi = (\phi_1,..,\phi_\ell)$ with $\phi_i = \theta_i - \delta_g$. Also define the power flow functions h^g, h^ℓ by

$$h^g(\gamma,\phi) = f^\delta(\delta,\theta),\ h^\ell(\gamma,\phi) = f^\ell(\delta,\theta) \tag{I.1}$$

In the state (ω,γ,ϕ) the perturbed system is described by

$$M_i\dot{\omega}_i + D_i\omega_i = P_i^m - h_i^g(\gamma,\phi), \qquad i = i,..,g, \tag{I.2}$$

$$\dot{\gamma}_i = 2\pi(\omega_i - \omega_g), \qquad i = 1,..,g-1, \tag{I.3}$$

$$\varepsilon(\dot{\phi}_i - 2\pi\omega_g) = P_i^\ell - h_i^\ell(\gamma,\phi), \qquad i = 1,..,\ell.$$

The degenerate system is obtained by simply setting $\varepsilon = 0$ in these equations.

In addition it is necessary to consider the so-called "boundary layer" system. This is obtained from the perturbed system by letting $s = t/\varepsilon$ and then setting $\varepsilon = 0$, giving

$$M_i\frac{d\omega_i}{ds} = 0 \qquad , \ i = 1,...,g,$$

$$\frac{d\gamma_i}{ds} = 0 \qquad , \ i = 1,...,g-1,$$

$$\frac{d}{ds}\phi_i - 2\pi\omega_g = P_i^\ell - h_i^\ell(\gamma,\phi), \ i = 1,...,\ell,$$

which simplifies to

$$\frac{d}{ds}\phi_i = P_i^\ell - h_i(\bar{\gamma},\phi) + 2\pi\bar{\omega}_g, \ i = 1,...,\ell, \tag{I.4}$$

and $\omega(s) = \bar{\omega}$, $\gamma(s) = \bar{\gamma}$ where $\bar{\omega}$, $\bar{\gamma}$ are constants.

Now let $(\omega^o=0,\delta^o,\theta^o)$ be an equilibrium of the original equations (2.1)-(2.4). Let $\gamma_i^o = \delta_i^o - \delta_g^o$, $\phi_i^o = \theta_i^o - \delta_g^o$. First we show that $(\omega^o,\gamma^o,\phi^o)$ is an asymptotically stable equilibrium of the degenerate system. Its equations, linearized around this equilibrium, are

$$M\Delta\dot{\tilde{\omega}} + D\Delta\tilde{\omega} = -D_\gamma h^g(\gamma^o,\phi^o)\Delta\tilde{\gamma} - D_\phi h^g(\gamma^o,\phi^o)\Delta\tilde{\phi} \ , \tag{I.5}$$

$$\Delta\dot{\tilde{\gamma}}_i = 2\pi(\Delta\tilde{\omega}_i - \Delta\tilde{\omega}_g), \ i = 1,..,g-1, \tag{I.6}$$

$$0 = -D_\gamma h^\ell(\gamma^o,\phi^o)\Delta\tilde{\gamma} - D_\phi h^\ell(\gamma^o,\phi^o)\Delta\tilde{\phi}, \tag{I.7}$$

which may be compared with (2.11)-(2.13). Solving for $\Delta\tilde{\phi}$ from (I.7) and using (I.1) these equations can be rewritten more compactly as

$$\begin{bmatrix} \Delta\dot{\tilde{\gamma}} \\ \Delta\dot{\tilde{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & 2\pi T \\ -M^{-1}A_R & -M^{-1}D \end{bmatrix} \begin{bmatrix} \Delta\tilde{\gamma} \\ \Delta\tilde{\omega} \end{bmatrix} = H \begin{bmatrix} \Delta\tilde{\gamma} \\ \Delta\tilde{\omega} \end{bmatrix} \quad \text{say} \tag{I.8}$$

where the $(g-1) \times g$ matrix $T = [I \ \vdots \ -1^{g-1}]$ with $1^{g-1} \in R^{g-1}$ the vector all of those components are unity, and A_R is given by

$$A_R T = A = D_\delta f^g(\delta^o,\theta^o)[D_\delta f^\ell(\delta^o,\theta^o)]^{-1} D_\delta f^\ell(\delta^o,\theta^o).$$

Suppose λ is an eigenvalue of H with eigenvector (x,y). Then $2\pi Ty = x$, $-M^{-1}A_R x - M^{-1}Dy = \lambda y$, from which it follows that

$$y^*(\lambda^2 M+\lambda D+2\pi A)y = a\lambda^2 + b\lambda^2 + c = 0,$$

with $a > 0$, $b > 0$ and $c \geq 0$. This implies Re $\lambda < 0$ or $\lambda = 0$. Moreover $\lambda = 0$ is an eigenvalue only if $y^*Ay = 0$ which, by (2.15), requires $y = 1^g$. Then $2\pi Ty = 2\pi T1^g = 0$, and $1^g = -D^{-1}A_R x$ so that $TD^{-1}A_R x = 0$ and so $x = 0$ since $TD^{-1}A_R$ is nonsingular. But then $y = 0$ also and so $\lambda = 0$ cannot be an eigenvalue of H. Thus (I.8) is asymptotically stable and it follows that $(\omega^o,\gamma^o,\phi^o)$ is an asymptotically stable solution of the degenerate system.

Next we study the boundary layer for (ω,γ,ϕ) inside a small sphere S_R of radius R centered at $(\omega^o,\gamma^o,\phi^o)$. From (I.1) we see that

$$D_\theta f^\ell(\delta,\theta) = D_\phi h^\ell(\gamma,\phi) \tag{I.9}$$

and so, by (2.14) and the Implicit Function Theorem, there is for $R > 0$ small, a smooth function $(\bar{\omega},\bar{\gamma}) \mapsto \phi = \Phi(\bar{\omega},\bar{\gamma})$ such that $P_i^\ell - h_i^\ell(\bar{\gamma},\phi) + 2\pi\bar{\omega}_g = 0$, $i = 1,\ldots,\ell$ and $(\bar{\omega},\bar{\gamma},\phi) \in S_R$ if and only if $\phi = \Phi(\bar{\omega},\bar{\gamma})$. We show now that the equilibrium $\phi = \Phi(\bar{\omega},\bar{\gamma})$ of (I.4) is asymptotically stable uniformly for $|\omega^o-\bar{\omega}| + |\gamma^o-\bar{\gamma}| < r$ for small r. The solution of (I.4) linearized around this equilibrium is governed by

$$\frac{d}{ds}\Delta\phi = -Dh_\phi^\ell(\bar{\gamma},\Phi(\bar{\omega},\bar{\gamma}))\Delta\phi.$$

From (I.9) and (2.14) it follows that if r is small enough, then

$$|\Delta\phi(s)| \leq Ke^{-\rho s}|\Delta\phi(0)|$$

for some $K < \infty$ and $\rho > 0$. This implies that the solution $\phi(s)$ of (I.4) converges asymptotically to $\Phi(\bar{\omega},\bar{\gamma})$ uniformly for $(\bar{\omega},\bar{\gamma},\phi(0)) \in S_R$ for R small.

By Hoppensteadt's result the asymptotic stability of the degenerate system and the uniform asymptotic stability of the boundary layer system implies that if $|\omega^\varepsilon(0)-\omega^o| + |\gamma^\varepsilon(0)-\gamma^o| + |\phi^\varepsilon(0)-\phi^o|$ is sufficiently small, then $(\omega^\varepsilon(t),\gamma^\varepsilon(t),\phi^\varepsilon(t))$ of the perturbed system (I.1)-(I.3) converges, as $\varepsilon\to 0+$, uniformly on $[T,\infty)$ for $T > 0$, to the solution $(\omega(t),\gamma(t),\phi(t))$ of the degenerate system. The theorem follows readily.

□

I.2. <u>Proof of Proposition 2.1</u>. We have

$$J = \begin{bmatrix} D_\delta f^g & D_\theta f^g \\ D_\delta f^\ell & D_\theta f^\ell \end{bmatrix}$$

evaluated at (δ^o,θ^o). Notice that if $1^{g+\ell}$ is the only eigenvector of J corresponding to the eigenvalue 0 then $D_\theta f^\ell$ must be nonsingular. Hence (2.14) and (2.18) both imply $D_\theta f^\ell$ is nonsingular. For any $(x,y) \in R^{g+\ell}$, we find

$$(x,y)'J(x,y) = x'Ax + |[D_\theta f^\ell]^{-1/2} D_\delta f^\ell x + [D_\theta f^\ell]^{1/2} y|^2 ,$$

from which the equivalence of (2.14), (2.15) and (2.18) follows readily. ¤

I.3. Proof of Theorem 2.2. Part (ii) of Definition 2.1 follows from Theorem 2.1, and so it only remains to verify part (i). In terms of the notation introduced in Subsection I.1 above, this is equivalent to finding $\varepsilon^o > 0$ such that $(\omega^o,\gamma^o,\phi^o)$ is an asymptotically stable equilibrium of the perturbed system for $0 < \varepsilon < \varepsilon^o$.

To prove this it suffices to show that for sufficiently small $\varepsilon > 0$, the perturbed system linearized around $(\omega^o,\gamma^o,\phi^o)$ is asymptotically stable at the origin. Now the eigenvalues of this linear system are the same as those of the system (2.8) - (2.10) except that the latter has one additional zero eignevalue.

Let λ be an eigenvalue of (2.8) - (2.10) with eigenvector (x,y,z) in $R^{2g+\ell}$. An algebraic manipulation leads to the relation

$$\{\lambda^2 2\pi M + \lambda D + D_\delta f^g - D_\theta f^g[D_\theta f^\ell - \varepsilon\lambda I]^{-1} D_\theta f^\ell\} x = 0 \tag{I.10}$$

The matrix inverse in (I.10) exists for ε small since $D_\theta f^\ell > 0$.

The solution $\lambda(\varepsilon)$ of (I.10) is of the form

$$\lambda(\varepsilon) = \lambda_o + O(\varepsilon),$$

where λ_o satisfies

$$\{\lambda_o^2\, 2\pi M + \lambda_o D + D_\delta f^g - D_\theta f^g[D_\theta f^\ell]^{-1} D_\theta f^\ell\} x = 0$$

We have already seen that this implies $\lambda_o \geq 0$. If $\lambda_o > 0$ then, from (I.11), $\lambda(\varepsilon) > 0$ for small ε. On the other hand if $\lambda_o = 0$ then we know from (2.16) that $x = 1^g$ and

$$D_\delta f^g - D_\theta f^g - D_\theta f^g [D_\theta f^\ell]^{-1} D_\theta f^\ell \} \, 1^g = 0$$

Expanding (I.10) in power of ε and using this relation then shows $\lambda = 0$. The result is proved. ◻

Appendix II. Proof of assertions in Section 3.

II.1. Proof of Theroem 3.1. For $i = 1,..,k$ define the reduced Jacobian matrix J_R^i by deleting the row and column of J^i corresponding to the boundary bus b^i, and define J_R^o, J_R by deleting from J^o and J respectively the row and column corresponding to b_k. Condition (i) is then equivalent to $J_R > 0$ and (ii) is equivalent to $J_R^i > 0$ for $i = 0,\ldots,k$.

J_R and $J_R^1,\ldots,J_R^k$ are related as in (II.1).

$$J_R = \begin{array}{c|cccccc|} & & b_1 & & b_2 & & \\ \hline & \boxed{J_R^1} & x & & & & \\ & & \vdots & & & & \\ b_1 & x \cdots & x & & x & & \\ & & & \boxed{J_R^2} & \vdots & & \\ b_2 & & x & \cdots & x & & \\ & & & & & \ddots & \\ & & & & & & \boxed{J_R^k} \\ \hline \end{array} \qquad \text{(II.1)}$$

Suppose $J_R > 0$. It is immediate from (II.1) that $J_R^i > 0$ for $i = 1,\ldots,k$. The $(k-1)x(k-1)$ matrix J_R^o with rows and columns corresponding to the boundary buses $b_1,..,b_{k-1}$, is obtained from J_R in the following way. Its off-diagonal entries corresponding to (b_i,b_j), $i \neq j$ is the same as the (b_i,b_j)th entry of J_R. Its (b_i,b_i)th diagonal entry is obtained by adding to the (b_i,b_i)th entry of J_R all those entry of the b_ith row which correspond to buses in the ith subnetwork. With this in mind, it follows that for $0 \neq x = (x_1,\ldots,x_{k-1})'$

$$x'J_R^o x = z'J_R z > 0$$

where $z = (x_1,\ldots,x_1,x_2,\ldots,x_2,\ldots,x_{k-1},\ldots,x_{k-1},1,\ldots,1)'$. Hence $J_R^o > 0$.

Conversely suppose $J_R^i > 0$, $i = 0,\ldots,k$. Let $0 \neq z = (z_1,x_1,\ldots,x_{k-1},z_k)' \in R^{g+\ell-1}$, with $z_i \in R^{g_i+\ell_i-1}$, $x_i \in R$. From (II.1) we can see that

$$z'J_R z = \sum_{i=1}^{k-1} y_i'J^i y_i + x'J_R^o x + z_k'J_R^k z_k \tag{II.2}$$

where $y_i' = (z_i,x_i) \in R^{g_i+\ell_i}$, $x' = (x_1,\ldots,x_{k-1})$. It is easily seen from (II.2) that $J_R > 0$. ¤

II.2. Proof of Theorem 3.2. We consider the simplest case where exactly one line connecting $\hat{b}_1$ and $\hat{b}_2$ is deleted to obtain $\hat{N}^o$ from N^o. The general case is similar but notationally awkward. J_R has the form shown in (II.3). It is the same as in (II.1) except for the additional entries corresponding to $\hat{b}_1$, $\hat{b}_2$.

$$J_R = \begin{array}{c} \\ \hat{b}_1 \\ \\ \hat{b}_2 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{b}_1 & & \hat{b}_2 & & & \end{array} \\ \left[\begin{array}{cccccc} \boxed{J_R^1} & \begin{matrix} x \\ \vdots \end{matrix} & x & & & \\ x \;\cdots & x & & x & & \\ x & & \boxed{J_R^2} & \vdots & & \\ & x & \cdots & x & & \\ & & & & \ddots & \\ & & & & & \boxed{J_R^k} \end{array}\right] \end{array} \tag{II.3}$$

Let $0 \neq z' = (z_1, x_1, \ldots, x_{k-1}, z_k)' \in R^{g+\ell-1}$ as above. Then

$$z' J_R z = \sum_{i=1}^{k-1} y_i' \tilde{J}^i y_i' + x' \tilde{J}_R^o x + z J_R^k z$$

$$+ [z_{1,\hat{b}_1}, z_{2,\hat{b}_2}] \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \begin{bmatrix} z_{1,\hat{b}_1} \\ z_{2,\hat{b}_2} \end{bmatrix} \tag{II.4}$$

In (II.4), $y_i' = (z_i, x_i) \in R^{g_i + \ell_i}$, $\tilde{J}^i$ is the Jacobian for the network N^i with the line connecting $\hat{b}_1$, $\hat{b}_2$ deleted and $\hat{J}_R^o$ is the reduced Jacobian for the network $\hat{N}^o$; finally $z_{i,\hat{b}_i}$ is the $\hat{b}_i$th component of z_i and $a = Y_{\hat{b}_1 \hat{b}_2} \cos(\theta_{\hat{b}_1} - \theta_{\hat{b}_2}) > 0$ by condition (iii). Conditions (i), (ii) imply that $J_R > 0$. ◻

Appendix III. Proof of Lemmas 4.1, 4.2.

III.1. Proof of Lemma 4.1.

The entries of the matrix $D_\theta f^\ell(\delta,\theta)$ are given by

$$(D_\theta f^\ell)_{ii} = \sum_{i \neq j} Y_{ij}^{\ell\ell} \cos(\theta_i - \theta_j) + \sum_k Y_{ik}^{\ell g} \cos(\theta_i - \delta_k), \quad i = 1, \ldots, \ell \tag{III.1}$$

$$(D_\theta f^\ell)_{ij} = -Y_{ij}^{\ell\ell} \cos(\theta_i - \theta_j), \quad i \neq j. \tag{III.2}$$

Since $(\delta,\theta) \in C$, (4.2) implies that $\cos(\theta_i - \theta_j) > 0$, $\cos(\theta_i - \theta_k) > 0$. Therefore $D_\theta f^\ell(\delta,\theta)$ is diagonally dominant, hence positive definite. ◻

III.2. Proof of Lemma 4.2. Suppose $(\delta^o, \theta^i) \in M' \cap C'$, $i = 1,2,$. By (2.5)

$$f'^\ell(\delta^o, \theta^1) = f'^\ell(\delta^o, \theta^2) = P^\ell. \tag{III.3}$$

Since C' is convex, $(\delta^o,\theta(u)) = (\delta^o,u\theta^1+(1-u)\theta^2) \in C'$ for $0 \leq u \leq 1$. By (III.2),

$$\int_0^1 D_\theta f'^{\ell}(\delta^o,\theta(u))\,\frac{\partial\theta(u)}{\partial u}du = f'^{\ell}(\delta^o,\theta^1) - f'^{\ell}(\delta^o,\theta^2) = 0 \tag{III.4}$$

But $\frac{\partial\theta(u)}{\partial u} = \theta^1 - \theta^2$ and $D_\theta f'^{\ell}(\delta^o,\theta(u)) > 0$ by Lemma 4.1. Hence

$$(\theta^1-\theta^2)' \int_0^1 D_\theta f'^{\ell}(\delta^o,\theta(u))(\theta^1-\theta^2)du > 0$$

if $\theta^1 \neq \theta^2$ which would contradict (III.4). So we must have $\theta^1 = \theta^2$, proving the uniqueness of θ'. ⌑

Appendix IV. Proof of Theorem 4.1

The proof is carried out in several steps. We first prove a result, Lemma IV. 2, which has some independent interest. Consider the system

$$\dot{x} = Ax + Bu + H(x,u) \tag{IV.1}$$

with $x \in R^n$, $u \in R^m$ and where H is a bounded continuous function which is Lipschitz in x. Let $R_T(x^I)$ be the set of states reachable in time T, starting at x^I, and using controls $u \in L^2[0,T]$.

Lemma IV.1 Suppose (A,B) is controllable. Then $R_T(x^I) = R^n$.

Proof. $\quad x(T) = (\exp TA)x^I + L_T(u) + N_T(u)$

where L_T, N_T are maps from $L_2[0,T]$ to R^n satisfying

$$L_T(u) = \int_o^T[\exp(T-t)A]\;Bu(t)dt,$$

$$N_T(u) = \int_o^T[\exp(T-t)A]\;H(x(t),\;u(t))]$$

Since (A,B) is controllable the range of L_T is all of R^n. Since H is bounded, so is N_T, and it then follows from a result on quasi-bounded maps due to Granas [8] that the range of $L_T + N_T$ is also R^n. The result is then immediate. ¤

We now find a control which steers (IV.1) from prespecified x^I to x^F in time T. Consider the following sequence of controls $u^k \in L^2[0,T]$.

For $k = 0$, set $u^k \equiv 0$.

Suppose u^k has been selected. Let

$$x^k(t) = (\exp tA)x^I + L_t(u^k) + N_t(u^k), \quad 0 \le t \le T \tag{IV.2}$$

be the corresponding trajectory. Then let

$$u^{k+1} = L_T^* [L_T L_T^*]^{-1} [x^F - (\exp TA)x^I - N_T(u^k)]. \tag{IV.3}$$

Here $L_T^* : R^n \to L^2[0,T]$ is the adjoint of the map L_T. That is

$$L_T^*(x)(t) = B'[\exp(T-t)A']x,$$
$$L_T L_T^* = \int_0^T [\exp(T-t)A]\, BB'[\exp(T-t)A']dt.$$

Now if $u = L_T^* [L_T L_T^*]^{-1}x$ then $|u(t)| \le K_1 T^{-1/2}|x|$, $0 \le t \le T$, for some constant K_1 depending on A and B. Also, since H is bounded $|N_T(u)| \le K_2 T^{1/2}$. Hence

$$|u^{k+1}(t)| \le K_1\, T^{-1/2}\, \{|x^F - (\exp TA)x^I| + K_2 T^{1/2}\}. \tag{IV.4}$$

It is now possible to show following the argument of [7, Proposition V.1] that there exists a subsequence of the u^k and x^k which converges to u and x. From (IV.2) it follows that

$$x(t) = (\exp tA)x^I + L_t(u) + N_t(u) \quad , \; 0 \le t \le T$$

so that x is the trajectory corresponding to u and from (IV.3) it follows that

$$x(T) = x^F$$

so that u steers (IV.1) from x^I to x^F. Also u (t) satisfies the bound (IV.4). Finally

$$\begin{aligned} |x(t) - (\exp tA)x^I| &\le |\, L_t(u)| + |\, N_t(u)| \\ &\le K_3 T^{1/2} \, |\, u \,| + K_2 T^{1/2} \\ &\le K_1 K_3 \, \{ \, |x^F - (\exp TA)x^I| + K_2 T^{1/2} \} + K_2 T^{1/2}. \end{aligned} \tag{IV.5}$$

<u>Lemma IV.2</u> Let X be a bounded set and $\alpha > 0$. There exists $T < \infty$, $\beta > 0$ such that for all $x^I \in X$ and all x^F with $|x^F - x^I| < \beta$, there is a bounded control $u(t)$, $0 \le t \le T$, which steers (IV.1) from x^I to x^F such that $|x(t) - x^I| < \alpha$, $0 \le t \le T$.

<u>Proof</u>. Immediate using (IV.5) ¤

Now consider the degenerate system model (see (4.10))

$$\begin{aligned} M\dot{\omega} + D\omega &= P^m - f'^g(\delta,\theta) \\ \dot{\delta} &= 2\pi\omega \\ \dot{P}^m &= \nu \\ 0 &= P^\ell - f'^\ell(\delta,\theta) \end{aligned}$$

By Lemmas 4.1, 4.2 there is a smooth function $\phi : R^g \to R^\ell$ such that $(\delta,\theta) \in M' \cap C'$ if and only if $\theta = \phi(\delta)$, so that in this region the model above may be rewritten as

$$
\begin{aligned}
M\dot{\omega} + D\omega &= P^m - f'^g(\delta,\phi(\delta)) \\
\dot{\delta} &= 2\pi\omega \\
\dot{P}^m &= \nu
\end{aligned}
\tag{IV.6}
$$

We can now prove a local version of Therorem 4.1. For $\alpha > 0$ let

$$
\Sigma'_\alpha = \{x = (\omega,\delta,P^m,\theta) \in \Sigma' \mid |(\tilde{\omega},\tilde{\delta},\tilde{P}^m) - (\omega,\delta,P^m)| < \alpha \text{ implies } (\tilde{\omega},\tilde{\delta},\tilde{P}^m, \phi(\tilde{\delta})) \in \Sigma'\}
$$

<u>Lemma IV.3</u> There exists $\beta > 0$ and $T < \infty$ such that for every pair x^I, x^F in Σ'_α with $|x^I - x^F| < \beta$ there exists a solution to the steering problem $SP(x^I,x^F)$ on $[0,T]$.

<u>Proof</u> The system (IV.6) is in the form (IV.1) with its linear part give by $M\dot{\omega} + D\omega = P^m$, $\dot{\delta} = 2\pi\omega$, $\dot{P}^m = \nu$. This linear system is easily seen to be controllable. Also, from (2.3) we can see that $f'^g(\delta,\phi(\delta))$ is a bounded function of δ. Hence by Lemma IV.2, there is a control function $v(t)$, $0 \le t \le T$ which steers (IV.6) from x^I to x^F along a trajectory $x(t)$ satisfying $|x(t) - x^I(t)| < \alpha$. By the definition of Σ'_α this implies $x(t) \in \Sigma'$, $0 \le t \le T$, so that (4.14) is satisfied. It only remains to show that the trajectory $x(t)$ is admissible i.e. it satisfies constraint B. Now, $x(t) = (\omega(t),\delta(t),P^m(t),\theta(t)) \in \Sigma'$ implies $(\delta(t),\theta(t)) \in M' \cap C'$. An argument along the same lines as in the proof of Lemma 4.1 shows that $Df'(\delta(\theta),\theta(t)) \ge 0$ and this matrix has exactly one zero eigenvalue. Hoppensteadt's result can now be applied in exactly the same way as in the proof of Theorem 2.1 to prove (4.11). ⌑

Proof of Theorem 4.1. Let x^I, x^F be in Σ'. Then there is $\alpha > 0$ such that x^I, x^F are in Σ'_α. Let $\beta > 0$ and $T < \infty$ be as in Lemma IV.3. We can find a sequence $x^I = x^0, x^1, \cdots, x^K = x^F$ in Σ'_α with $|x^{k-1} - x^k| < \beta$, and control functions v^k which solve respectively $SP(x^{k-1},x^k)$ over $[0,T]$.

Clearly the control function

$$v((k-1)T+t) = v^k(t), \ 0 \leq t < T, \ k = 1, \cdots, K$$

defined on the interval $[0,KT]$ solves $SP(x^I,x^F)$. ¤

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Figure Captions

Fig. 1. Interconnected power system.

Fig. 2. State space of degenerate system.

Fig. 3. Hierarchical decomposition of interconnected power system.

Fig. 4. Representative of the class of power networks under investigation in section 3.1.

Fig. 5. Power system with two utilities; one of them having several boundary buses.

Fig. 6. Two stage alert state control of power system.

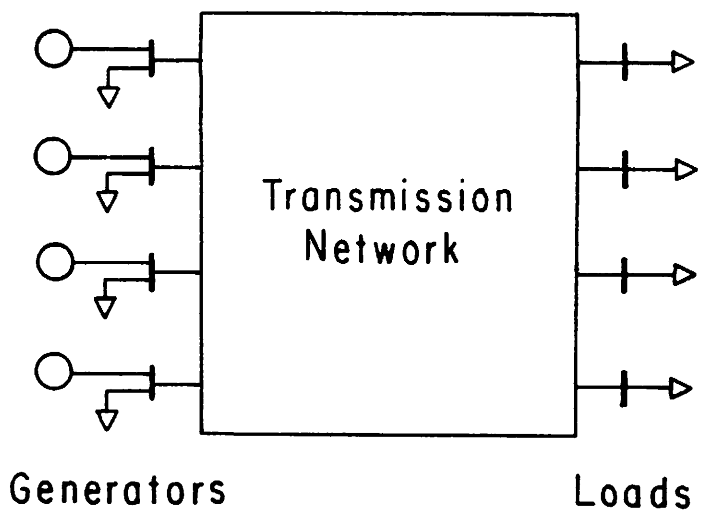

Figure 1. INTERCONNECTED POWER SYSTEM

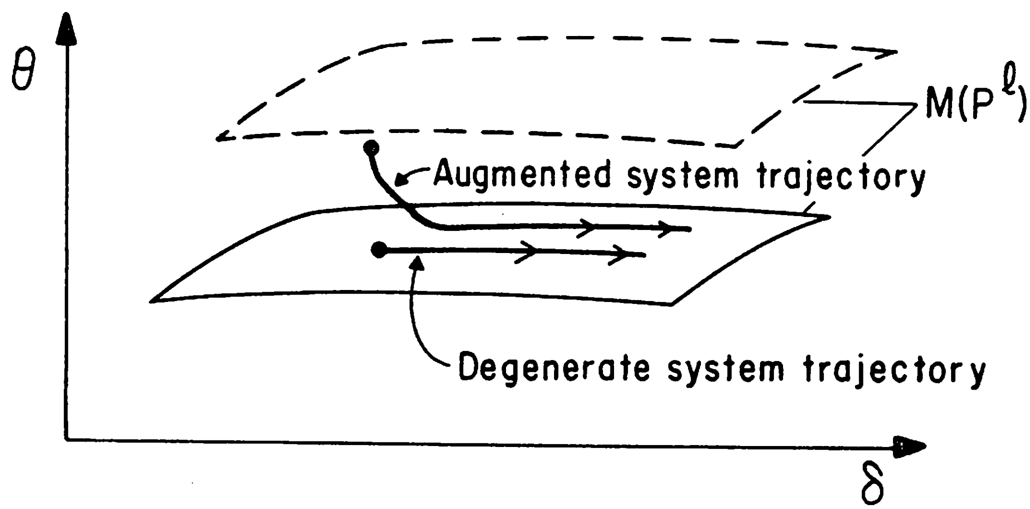

Figure 2. STATE SPACE OF DEGENERATE SYSTEM

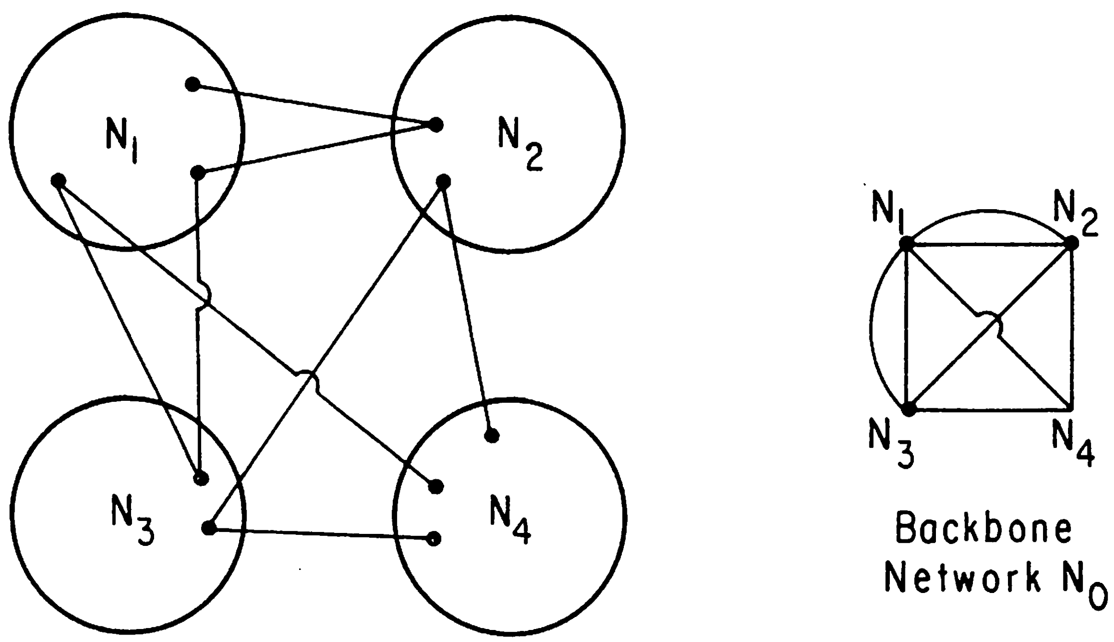

Figure 3. HIERARCHICAL DECOMPOSITION OF INTERCONNECTED POWER SYSTEM

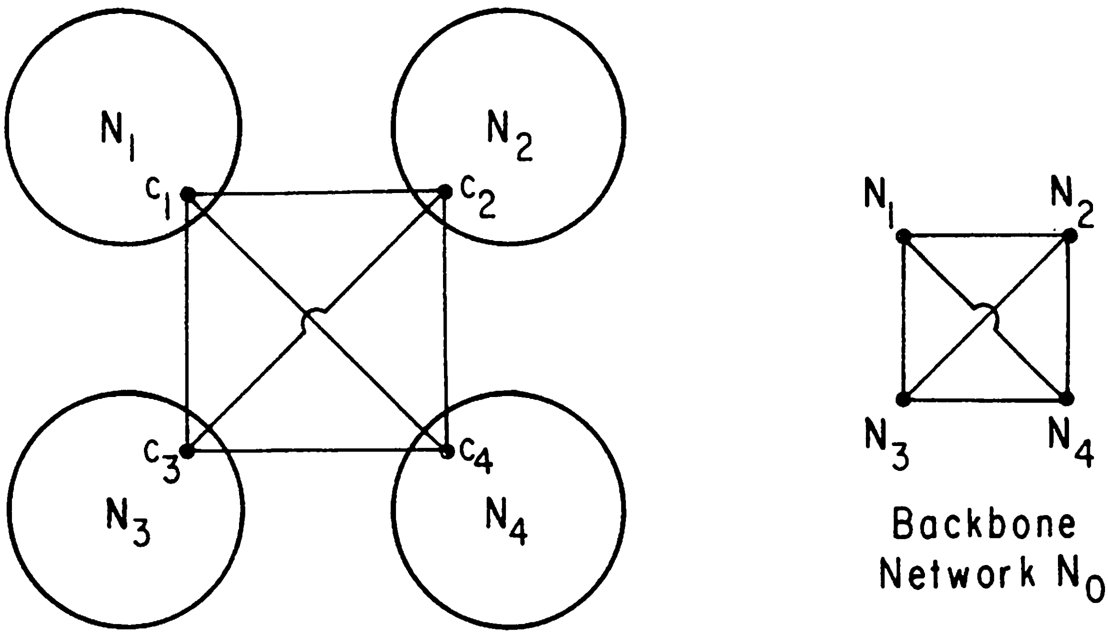

Figure 4. REPRESENTATIVE OF THE CLASS OF POWER NETWORKS UNDER INVESTIGATION IN SECTION 3.1

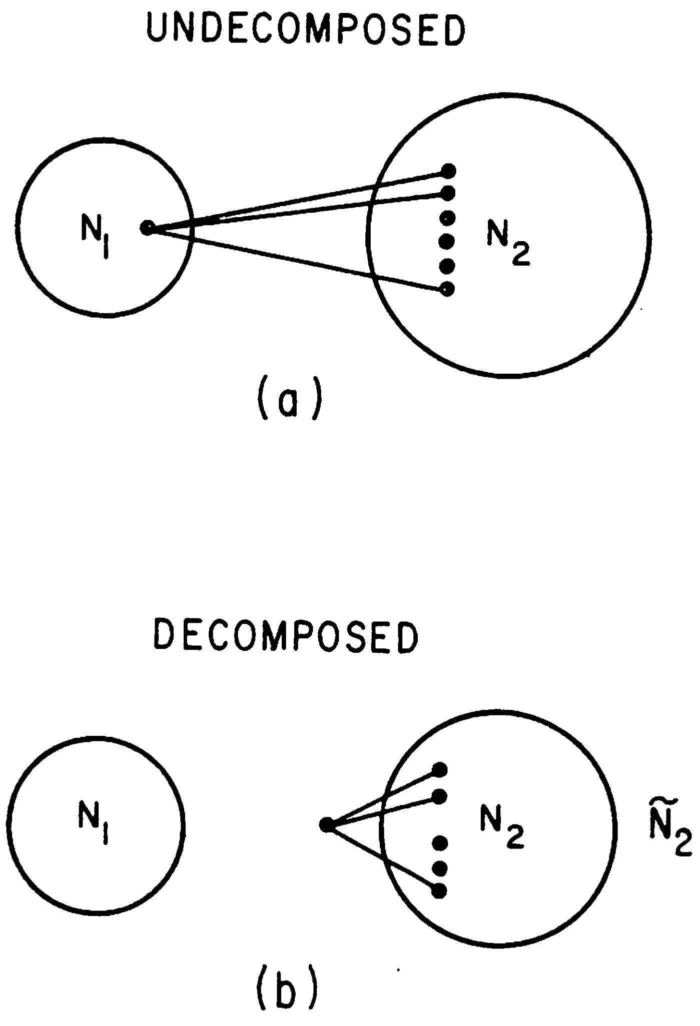

Figure 5. POWER SYSTEM WITH TWO UTILITIES; ONE OF THEM HAVING SEVERAL BOUNDARY BUSES

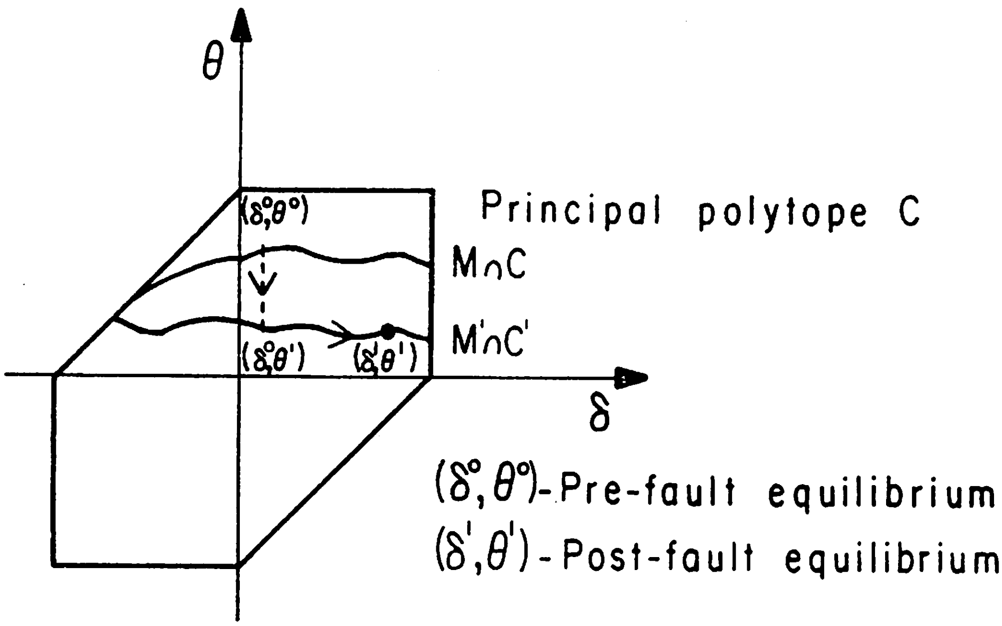

Figure 6. TWO STAGE ALERT STATE CONTROL OF POWER SYSTEM