DESIGN OF NONLINEAR FEEDBACK CONTROLLERS

by

E. Polak and D. Q. Mayne

Memorandum No. UCB/ERL M80/12

25 February 1980

ELECTRONIC RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

# DESIGN OF NONLINEAR FEEDBACK CONTROLLERS[1]

E. Polak[2] and D. Q. Mayne[3]


## Abstract

This paper shows how the design of feedback controllers for nonlinear systems may be formulated as an optimization problem with infinite dimensional constraints for which known algorithms may be employed. An important aspect is a method for reducing the time interval, required to insure stability, to a finite value.


---

[1] Research supported by United Kingdom Science Research Council and the National Science Foundation ECS-79-13148.

[2] Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

[3] Department of Computing and Control, Imperial College, London, SW7 England.

## 1. Introduction

It has been shown (for example in [1-3]) that many (computer aided) design problems may be formulated as determining a point which satisfies infinite dimensional, as well as conventional, inequalities or as optimizing some criterion function subject to these inequalities.

The specific forms which these inequalities may have are as follows:

$$\max_{y \in Y_j} \phi^j(z,y) \leq 0, \; j = 1,2,\ldots,m,$$

where $\phi^j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and $Y_j$ is a compact subset of $\mathbb{R}^m$. Thus, we can express in this form constraints such as

$$\max_{\substack{x_0 \in X \\ t \in T \\ \alpha \in A}} \phi^j(z,x_0,\alpha,t), \; j = 1,2,..,m$$

where z is the design parameter, $\phi^j$ is the time (or frequence) response to an input characterized by the parameter $\alpha$, and $x_0$ is an initial state. A variety of algorithms [4,5,6,7,8,9] have been developed to solve such problems. In this paper we examine some of the less obvious implications of these algorithms to control system design. In particular, we show how the problem of designing feedback controllers for non-linear systems may be formulated as a design problem with infinite dimensional constraints. We shall consider constraints which ensure closed loop system stability and satisfactory responses to a class of polynomial inputs. These constraints are, basically, inequalities which must be satisfied for all initial states lying in a compact subset X of $\mathbb{R}^n$ and, possibly for all times in the <u>infinite</u> interval $[0,\infty)$; stability constraints, for example, have this structure. Normally, satisfaction of such a constraint is assessed by examining the trajectory $x(t;x_0,r)$ of the closed loop system for all initial states $x_0$ in X, for all inputs r in the class R and all t in $[0,\infty)$. Since simulation of non-linear systems is computationally expensive, every effort must be made to reduce it. The number of initial states and inputs r for which the responses must be examined depends on the global optimization algorithm [10,11] that one must use, on the structure of the nonlinearity and on whether certain Lipschitz constant estimates are available or not. Hence we have only a qualitative control over this aspect of the computational work. However (and this is one of the main contributions of this paper), it is possible to reduce the duration of simulation from the infinite interval from $[0,\infty)$ to a finite one $[0,T]$, where T may be quite small. This results in a considerable reduction in computation.

## 2. The Stability Constraint

The single most important constraint in control system design is that which ensures stability. Since the Lyapunov approach is generally impractical, stability of a non-linear system is usually assessed in practice by repeated simulations. We shall formalize this approach and show how the associated computation can be substantially reduced.

Suppose that the system to be controlled is described by:

$$\dot{x}(t) = \bar{f}(x(t),u(t)) \tag{1}$$

where $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. Suppose that the set of initial states of interest is a compact subset X of $\mathbb{R}^n$. We will assume that the origin is the desired (equilibrium) state, i.e., that $\bar{f}(0,0) = 0$. We assume further that a feedback control structure has been chosen, so that u(t) in (1) is replaced by $h(x(t),z)$, where the controller parameter z has to be chosen. The function $h : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ is assumed to be continuously differentiable. With this control structure (1) may be replaced by:

$$\dot{x}(t) = f(x(t), z) \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is defined by:

$$f(x,z) \triangleq \bar{f}(x, h(x,z)) \tag{3}$$

and is continuously differentiable.

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function with the following properties:

(i) $V(x) \geq 0$ for all $x \in \mathbb{R}^n$;

(ii) $V(\alpha x) = \alpha V(x)$ for all $\alpha \in [0,\infty)$, all $x \in \mathbb{R}^n$;

(iii) $V(x) = 0 \Leftrightarrow x = 0$.

An example of such a function is $x \longmapsto (x^T Q x)^{1/2}$ where $Q$ is positive definite. For all $x$ in $\mathbb{R}^n$ let the set $B(x)$ be defined by:

$$B(x) \triangleq \{x' \in \mathbb{R}^n \mid v(x') \leq V(x)\} \tag{4}$$

We obtain immediately that:

$$V(x') \leq V(x) \Rightarrow B(x') \subset B(x), \tag{5}$$

and:

$$B(\beta x) \subset B(x) \tag{6}$$

for all $\beta \in (0,1)$. Also

$$B(0) = \{0\}.$$

It can be shown that $B(x)$ is compact for all $x \in \mathbb{R}^n$. This property is possessed by the above example.

Choose $\bar{x}$, and $\bar{V} = V(\bar{x})$, so that $X \subset B(\bar{x})$. For any initial state $x_0$ at $t = 0$ let $x(t,x_0,z)$ denote the solution of (2) at time t. If z can be chosen so that $V(x)$ (or $V(x)^2$) strictly decreases along all solutions of (2) (i.e. $[V(x(t,x_0,z)^2]_x f(x(t,x_0,z),z) < 0$ for all $t \in [0,\infty)$ all $x_0$ in $B(\bar{x})$ excluding the origin) then the (closed-loop) system (2) is asymptotically stable and $B(\bar{x})$ is a domain of attraction. However such a z does not usually exist (except, perhaps, if $B(\bar{x})$ is small). Hence we adopt another approach in which $V(x)$ is permitted to <u>increase</u> for a finite period T along solutions of (2).

We need the following property of $B(x)$:

<u>Proposition 1</u>.

For all $\alpha \in (0,\infty)$, all x:

$$B(\alpha x) = \alpha B(x) \tag{7}$$

<u>Proof</u>:

$$\begin{aligned} B(\alpha x) &= \{x' \mid v(x') \leq v(\alpha x)\} \\ &= \{x' \mid v(x') \leq \alpha v(x)\} \\ &= \{x' \mid v(\tfrac{x'}{\alpha}) \leq v(x)\} \\ &= \{\alpha x'' \mid v(x'') \leq v(x)\} \\ &= \alpha B(x) \end{aligned}$$

¤

We can now state a stability theorem which permits $V(x)$ to increase, along solutions of (2), for a limited time T:

<u>Theorem 1</u>.

Let $\beta \in (0,1)$, $\gamma \in (1,\infty)$ and $B(\bar{x}) \supset X$ be given. If there exists a $z \in \mathbb{R}^p$ and a $T \in (0,\infty)$ such that:

(i) $x(t,x,z) \in \gamma\, B(x_0)$ for all $x_0 \in B(\bar{x})$ and all $t \in [0,T]$,

(ii) $x(T,x_0,z) \in \beta\, B(x_0)$ for all $x_0 \in B(\bar{x})$,

then

(a) $x(t,x_0,z) \in \gamma\, B(\bar{x})$ for all $x_0 \in B(\bar{x})$, all $t \in [0,\infty)$,

(b) $x(t,x_0,z) \to 0$ as $t \to \infty$, for all $x_0 \in B(\bar{x})$.

Proof:

If $x_0 \in \beta^k\, B(\bar{x}) = B(\beta^k\, \bar{x})$, then $B(x_0) \subset B(\beta^k\, \bar{x})$ so that, from (ii), $x(T,x_0,z) \in \beta\, B(\beta^k\, \bar{x}) = \beta^{k+1}\, B(\bar{x})$. Since f is time invariant it follows that $x(kT,x_0,z) \in \beta^k\, B(\bar{x})$ for all $x_0 \in B(\bar{x})$. It follows from (i) that $x(t,x_0,z) \in \gamma\beta^k\, B(\bar{x})$ for all $t \in [kT,(k+1)T]$ and, hence, that

(i) $x(t,x_0,z) \in \gamma B(\bar{x})$ for all $x_0 \in B(\bar{x})$, all $t \in [0,\infty)$,

(ii) $x(t,x_0,z) \to 0$ as $t \to \infty$, for all $x_0 \in B(\bar{x})$. ¤

The consequence of this theorem is that asymptotic stability, with a domain of attraction $B(\bar{x})$, is assured if the infinite dimensional inequalities in hypothesis (i) and (ii) of Theorem 1 are satisfied. The inequalities require all initial states in $B(\bar{x})$ to be investigated via a global optimization algorithm - this appears inevitable in non-linear system design - but requires the solution of the differential equation (2) for the period $[0,T]$ only, where T may be quite small. The theorem is a Lyapunov type result, in that $V(x(k+1)T,x_0,z)) < V(x(kT,x_0,z))$ for all $k = 0,1,2,\ldots$ even though $V(x(t,x_0,z))$ is not necessarily less than $V(x(kT,x_0,z))$ in the interval $[kT, (k+1)T)$.

## 3. Choice of Performance Constraints for Regulators

In the case of a regulator, the most obvious performance criteria is rapidity of response - this corresponds to the speed at which the initial state $x_0$ is steered to the origin. Since in a regulator satisfying (i) and (ii) of Theorem 1, $x(kT,x_0,z) \in \beta^k B(\bar{x})$, this speed is clearly controlled by the choice of $\beta \in (0,1)$ and $T \in (0,\infty)$. For computational reasons it is desirable to choose T as small as possible (but large enough to ensure that the feasible set - i.e. the set of z satisfying inequalities (i) and (ii) in Theorem 1 - is not empty). Hence speed of response can be increased by reducing $\beta$ - the control design problem could, for example, be expressed as minimizing $\beta$ subject to the inequalities specified in hypotheses (i) and (ii) of Theorem 1.

The exponential type of response ($x(kT) \in \beta^k B(\bar{x})$), while appropriate for linear systems, may not be suitable for certain non-linear systems: a value of $\beta$ feasible for states far from the origin may be unnecessarily large (corresponding to a slow response) for states close to the origin. In such cases it may be desirable to replace the constant $\beta$ by a continuous function $\tilde{\beta} : \mathbb{R}^n \to [\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in (0,1)$, $\delta_2 > \delta_1$ and $\tilde{\beta}$ has the property: $V(x') > V(x) \Rightarrow \tilde{\beta}(x') > \tilde{\beta}(x)$.

The following result shows that replacing $\beta$ by $\tilde{\beta}$ does not destroy stability.

Theorem 2.

Let $\gamma \in (1,\infty)$ and $B(\bar{x}) \supset X$ be given. Let $\tilde{\beta}$ map $B(\bar{x})$ into $[\delta_1, \delta_2]$ where $\delta_1, \delta_2 \in (0,1)$. If there exists a $z \in \mathbb{R}^p$ and a $T \in (0,\infty)$ such that:

(i) $x(t,x_0,z) \in \gamma\, B(x_0)$ for all $x_0 \in B(\bar{x})$ and all $t \in [0,T]$

(ii) $x(T,x_0,z) \in \tilde{\beta}(x_0)\, B(x_0)$ for all $x_0 \in B(\bar{x})$ then, if $x(0) \in B(\bar{x})$; $x(kT) \in \tilde{\beta}(x(0))\, \tilde{\beta}(x(T)) \ldots \tilde{\beta}(x((k-1)T)\, B(\bar{x}) \subset \delta_2^k\, B(\bar{x})$ where $x(iT)$ denotes $x(iT,x_0,z)$, $i = 0,\ldots,k$, for all $k = 0,1,2,\ldots$ .

Proof:

Let $\beta_i$ denote $\tilde{\beta}(x(iT))$, $i = 0,1,2,\ldots$ . Suppose $x(iT) \in \beta_0, \beta_1 \cdots \beta_{i-1}$ $B(\bar{x}) = B(\beta_0,\beta_1 \cdots \beta_{i-1}\ \bar{x})$. From (ii), $x((i+1)T) \in \beta_i\ B(\beta_0,\beta_1 \cdots \beta_{i-1}\ \bar{x})$ $= \beta_0,\beta_1 \cdots \beta_i B(\bar{x})$. Since $x_0 \in B(\bar{x})$, by induction $x(kT) \in \beta_0,\beta_1 \cdots \beta_k\ B(\bar{x})$ for all $k = 0,1,2,\ldots$ thus proving the theorem. ⌑

The conditions on $\beta$ and $\tilde{\beta}$ in Theorems 1 and 2 are not the only ones which lead to asymptotic stability. For example, let $\tilde{\beta} : B(\bar{x}) \to [0,1]$ be such that $\tilde{\beta}(x) \leq \max\ \{\alpha,\ 1-V(x)/V(\bar{x})\}$ with $\alpha \in (0,1)$. Then we see that $\tilde{\beta}(x) \to 1$ as $V(x) \to 0$. It is easy to see that for this function, too, the conclusions of Theorem 2 hold. For suppose $V(x(iT)) \not\to 0$ as $i \to \infty$. Let $\hat{\beta}(x) \triangleq \max\ \{\alpha,\ 1-V(x)/V(\bar{x})\}$. Then there exists an infinite subsequence, with $i \in K$, and a constant $b \in (0,1)$, such that $\frac{V(x(iT))}{V(\bar{x})} \geq b$ for all $i \in K$, and hence $\tilde{\beta}(x(iT)) \leq \hat{\beta}(x(iT)) \leq \max\ \{\alpha,1-b\} \triangleq \beta^* < 1$ for all $i \in k$. Since $\tilde{\beta}(x(iT)) \in [0,1]$ for all $i$, $V(x((i+1)T) \leq V(x(iT))$ for all $i$ and since $V(x(i+1)T) \leq \beta^*\ V(x(iT)$ for all $i \in K$, we conclude that $V(x(iT)) \to 0$. We see that we have just constructed a contradiction, and hence our claim, that any function $\tilde{\beta} : B(\bar{x}) \to [0,1]$, such that $\tilde{\beta}(x) \leq \hat{\beta}(x)$, can be used in Theorem 2, must be correct.

Accuracy in following a reference, $y_r$ say, is sometimes a requirement. In such problems the feedback term is $x \longrightarrow h(x,y_r,z)$ and (2) is replaced by:

$$x(t) = f(x(t),y_r,z) \tag{8}$$

The equilibrium state $x_e(y_r)$ now depends on $y_r$ and is the solution of

$$0 = f(x,y_r,z) \tag{9}$$

If the output of the system is $g(x)$ then a performance criteria of the form $\|g(x_e(y_r)) - y_r\| \leq \varepsilon\|y_r\|$ can be imposed, for selected values of $y_r$ or a whole set.

Another performance requirement is robustness, for example maintenance of stability as a plant parameter p ranges over a set P. The trajectory is now a function of $(t,x_0,z;p)$ and the stability constraints become:

(i) $x(t,x,z,p) \in \gamma\, B(x_0)$ for all $x_0 \in B(\bar{x})$ all $t \in [0,T]$ and all $p \in P$.

(ii) $x(T,x_0,z,P) \in \beta\, B(x_0)$ for all $x_0 \in B(\bar{x})$, all $p \in P$.

Other performance criteria can be similarly formulated.

## 4. The Response Function V

The procedures outlined above depend on the choice of the function V and associated set valued function B; a wise choice will permit a small value for T. The following choice should be suitable for many applications.

Linearize the system (1) about the equilibrium point yielding:

$$\dot{z}(t) = A\, z(t) + B\, u(t) \tag{10}$$

where $A \triangleq \bar{f}_x(0,0)$ and $B \triangleq \bar{f}_u(0,0)$ if the equilibrium point is the origin. In addition the linearized output equation is:

$$y(t) = Cz(t) \tag{11}$$

where $C \triangleq g_x(0)$ if the equilibrium point is the origin. Design a linear controller:

$$u(t) = -Kz(t) + v(t) \tag{12}$$

so that the closed loop system:

$$\dot{z}(t) = (A-BK)\, z(t) + Bv(t) \tag{13}$$

satisfies the various stability and performance constraints. (If the controller is dynamic its states should be included in (1) and (10)). For some $Q > 0$ compute the positive definite solution $P$ of the Lyapunov equation

$$(A-BK)^T P + P(A-BK) = -Q \tag{14}$$

The response function is then defined by:

$$V(x) = (x^T Px)^{1/2}. \tag{15}$$

$V$ is a Lyapunov function for (13). It will also be one for the non-linear system (2)(with the same linear controller) in some neighborhood of the origin. In this neighborhood $T = 0$ will suffice. It is therefore plausible that this choice of $V$ will permit a relatively small value of $T$ to be chosen for the non-linear design problem.

## 5. Polynomial Inputs

We shall now show that constraints on the response to a polynomial input can also be cast into a form compatible with the new algorithms. Suppose, without loss of generality, that the input $u(\cdot)$ of (1) is scalar valued and that an input $r(\cdot)$ is to be applied to the closed loop system, characterized by the closed loop structural law $u(t) = h(x(t),z,r(t))$ where, as before, $z$ is a finite dimensional parameter to be chosen. This leads to the following obvious extension of (2):

$$\dot{x}(t) = f(x(t),z,r(t)) \tag{16}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^1 \to \mathbb{R}^n$ is defined by

$$f(x,z,r) \triangleq \bar{f}(x,h(x,z,r)) \tag{17}$$

Now, suppose that we are concerned with the class of inputs $r(\cdot)$ defined by:

$$r(\cdot) \in \{p(\cdot)|p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k,\ \alpha \in A\} \tag{18}$$

where $\alpha \triangleq (\alpha_0,\dots,\alpha_k)^T$ and $A$ is a compact subset of $\mathbb{R}^k$. In this case $r(\cdot)$ has a finite parametric representation:

$$r(t) = \langle \alpha, \hat{p}(t)\rangle \tag{19}$$

where $\hat{p}(t) \triangleq (1,t,t^2,..,t^k)^T$. Thus, the differential equation (16) can be rewritten as

$$\dot{x}(t) = \tilde{f}(x(t),z,\alpha,t) . \tag{20}$$

Now suppose again that the system output is given by

$$y(t) \triangleq g(x(t)). \tag{21}$$

The instantaneous tracking error can be defined as

$$e(t,x_0,\alpha,z) = [g(x(t,x_0,\alpha) - g(x_0) - \langle \alpha,\hat{p}(t)\rangle]^2 \tag{22}$$

Hence, given a compact set $B(\bar{x})$, as in the earlier sections, and assuming that the functions f,g and h satisfy the usual differentiability assumptions we can employ the new algorithms either to solve minimization problems with a cost function $c(z)$, of the form

$$c(z) \triangleq \max_{\substack{x_0 \in X \\ \alpha \in A \\ t \in T}} e(t,x_0,\alpha) \tag{23}$$

where $T \triangleq [0,t']$ is an interval of interest, or they can be used to solve minimization problems or systems of inequalities which include an inequality, in z, of the form

$$\max_{\substack{x_0 \in X \\ \alpha \in A \\ t \in T}} e(t,x_0,\alpha,z) - e_0 \leq 0 \qquad (24)$$

## 6. Conclusion

We have explored some of the less obvious applications of a new family of optimization and inequality solving algorithms to nonlinear control system design. The measure of confidence that one can derive from such a design procedure depends greatly on the success one has in solving global optimization problems of the form $\max_{\nu \in N} \phi(z,\nu)$, with N a compact set, which must be solved reasonably accurately, at least towards the end of the computation, as a subprocess of the algorithms in [5,7]. Thus, we see that we are largely dependent on the state of the art of global optimization techniques. Currently, some of the better global optimization techniques use a mixture of random initialization of deterministic maximization algorithms. We expect that in many design situations these techniques will be economically feasible design tools.

## References


1. V. Zakian and U. Al-Naibk, "Design of Dynamical and Control Systems by the Method of Inequalities," Proc. IEE, Vol. 120, No. 11, pp. 1421-1427, 1973.
2. J. W. Bandler and H. L. Abdul-Malek, "Advances in the mathematical Programming Approach to Design Centering, Tolerancing and Tuning," Proc. 1978 JACC, Philadelphia, pp. 329-344, Oct. 14-21, 1978.
3. R. G. Becker, A. J. Hurnis, D. Q. Mayne, "Computer Dated Design of Control Systems via Optimization," Proc. IEE, Vol 126, No. 6, pp. 573-578, June 1979.
4. E. Polak and D. Q. Mayne, "An Algorithm for Optimization Problems with Functional Inequaltiy Constraints," Trans. IEEE, AC-21, pp. 186-193, 1976.
5. D. Q. Mayne, E. Polak and R. Trahan, "An Outer Approximation Algorithm for Computer Aided Design Problems," Journal of Optimization Theory and Application, Vol. 28, No. 5, pp. 331-352, 1979.
6. C. Gonzaga, E. Polak and R. Trahan, "An Improved Algorithm for Optimization Problems with Functional Inequaltiy Constraints," Electronics Research Laboratory, University of California, Berkeley, Memorandum UCB/ERL M 78/56. IEEE Trans. Automat. Contr., in press.
7. C. Gonzaga and E. Polak, "On Constraint Dropping Schemes and Optimality Functions for a Class of Outer Approximations Algorithms," SIAM J. Control and Optimization, Vol. 17, No. 4, 1979.