ENERGY-RELATED CONCEPTS FOR NONLINEAR TIME-VARYING N-PORTS: PASSIVITY AND LOSSLESSNESS

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ABSTRACT

Definitions of passivity and losslessness are presented which apply to n-port networks which are not necessarily linear, time-invariant, or lumped; in fact, these definitions apply to any n-port which has a dynamical system representation. For lumped, nonlinear n-port networks which can be mathematically represented by a finite-order dynamical system, conditions for passivity and losslessness are formulated in terms of properties of the state equation function, the output function, etc. These conditions can be verified without solving the state equation, and can be viewed as nonlinear generalizations of the well-known time-domain and frequency-domain passivity and losslessness conditions for linear time-invariant lumped n-port networks.

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I. Introduction

The energy-related concepts of passivity and losslessness play important roles in the highly developed classical theory of linear time-invariant lumped (LTIL) n-ports [1]-[3]. In the more recent and less developed theory of nonlinear n-ports, there is no consensus as to how these energy-related concepts should be defined; moreover, there are just a few results (most of which apply only to narrow classes of nonlinear n-ports) that can be utilized to determine whether a given nonlinear n-port is passive or active, lossless or lossy (we define activity to be the negation of passivity and lossyness to be the negation of losslessness).

The first goal of this research is to provide consistent definitions of the concepts of passivity and losslessness which apply to any n-port which has a dynamical system representation. By "consistent," we mean that these definitions should have the following properties:

(i) For those classes of n-ports where definitions of these concepts have already been established (e.g., LTIL n-ports), our definitions should agree with the established definitions. In cases where a definition does not agree, there must be a good reason why our definition should supplant the established definition.

(ii) An n-port should be unambiguously classified as passive or active, lossless or lossy.

Consistent definitions for the nonlinear time-invariant case have already been presented in references [4] and [5]. The definition of passivity presented in [4] has a straightforward generalization to the nonlinear time-varying case, and we present that generalization in Subsection 4.1. The definition of losslessness presented in

[5] has no obvious generalization to the nonlinear time-varying case; however, in Subsection 5.1 we succeed in devising a consistent theory of losslessness which applies to both time-invariant and time-varying nonlinear n-ports. We consider the material in Subsection 5.1 to be one of the significant contributions of this research.

A large class of lumped, nonlinear n-port networks can be mathematically represented by a special type of dynamical system which we call a "finite-order dynamical system": in essence, such a dynamical system is one in which the state lies in an m-dimensional Euclidean space and its evolution over time is governed by a so-called "state equation" $\dot{x} = f(x,u,t)$, where x denotes the state, u denotes the input, and t denotes time. The second goal of this research is as follows: for n-ports which can be mathematically represented by a finite-order dynamical system, find conditions for passivity and losslessness (in terms of properties of the state equation function $f(\cdot,\cdot,\cdot)$, the output function $g(\cdot,\cdot,\cdot)$, etc.) which can be verified without solving the state equation. Such results can be viewed as nonlinear generalizations of the well-known time-domain and frequency-domain passivity and losslessness conditions for LTIL n-ports [1]. For some classes of finite-order dynamical systems we shall find sufficient conditions for passivity and/or losslessness, for other classes we shall find necessary conditions, and for still others we shall find conditions which are both necessary and sufficient.

Our results involving finite-order dynamical systems can be viewed as contributions to the theory of optimal control. This is especially true of our results dealing with passivity. As will be discussed in Subsection 4.1, the question of passivity for a time-invariant dynamical system is essentially the following nonstandard optimal control problem: find conditions for a certain optimal-value function defined on the state space (the negative of the "available energy") to be finite-valued at each point of its domain. In the usual optimal-control problem, it is assumed a priori that the optimal-value function is finite-valued. This assumption frequently takes the form of a restriction on the class of allowable cost-functional integrands, e.g., one might deal only with cost-functional integrands which are nonnegative. In the theory of passivity for dynamical systems, we cannot base our entire theory on the assumption that the available energy function is finite-valued; indeed, the question of whether the available energy function is finite-valued is precisely the question we are trying to answer in the theory of passivity.

Summarizing the paper, Section II gives the basic definitions and assumptions which will be used throughout this paper. Most of our notation is also defined in Section II. Subsection 2.1 deals mostly with dynamical systems, while Subsection 2.2 gives a precise definition of the term "n-port"--one that is both useful and meaningful within the framework of our theory.

Section III contains various technical lemmas. The main result, Lemma 3.1.7 in Subsection 3.1, is a decidedly non-trivial analytical result for finite-order time-invariant dynamical systems which has applications in the studies of both passivity and losslessness. For first-order dynamical systems, Lemma 3.1.7 can be strengthened considerably: this stronger result is contained in Subsection 3.2.

Section IV is devoted to the concept of passivity. Subsection 4.1 is essentially the basic theory of passivity found in reference [4], with straightforward generalizations to the time-varying case. One noteworthy item in Subsection 4.1 is the condition (Suff. 4.1.4) contained in Lemma 4.1.4: this is an obvious sufficient algebraic passivity condition for finite-order dynamical systems which is equivalent to the existence of a C^1 internal energy function. Another noteworthy item in Subsection 4.1 is Lemma 4.1.8, which shows that (unlike the time-invariant case) conditions more restrictive than mere reachability must be imposed in order for the required energy function to yield an internal energy function for a passive time-varying dynamical system.

A conjecture is introduced in Subsection 4.2 which we call the "Smoothness Conjecture." Roughly speaking, this conjecture says that a passive, controllable C^∞ finite-order dynamical system has at least one C^1 internal energy function. On several occasions the first author has heard the Smoothness Conjecture, or some minor variation of it, in his discussions with optimal control theorists. Also, the truth of the Smoothness Conjecture seems to have been assumed in references [6] and [7], although it was not explicitly stated in either of those two references. If the Smoothness Conjecture were true, then (Suff. 4.1.4) would be a necessary (as well as sufficient) algebraic passivity condition for the class of controllable C^∞ finite-order dynamical systems. Unfortunately, the Smoothness Conjecture is <u>false</u>: this is proved in Subsection 4.2 by producing a counterexample.

Hence, even if we restrict ourselves to controllable C^∞ finite-order dynamical systems, (Suff. 4.1.4) is not a necessary condition for passivity. Therefore it is of interest to obtain a sufficient algebraic passivity condition for finite-order dynamical systems which is not as restrictive as (Suff. 4.1.4). Such a condition can be obtained by applying Stalford's [8] results from optimal control theory: this is done in Subsection 4.3.

In the following subsection, 4.4, the technical results from Subsection 3.1 are applied to obtain some original sufficient activity conditions for finite-order dynamical systems. Note that the negations of these conditions are necessary conditions for passivity.

The technical results from Subsection 3.2 are applied in Subsection 4.5 to obtain an easily-verifiable necessary and sufficient passivity condition for controllable first-order time-invariant dynamical systems. This result was first published by the authors in reference [4]. We also present a new result which shows that a passive, controllable first-order time-invariant dynamical system has an internal energy function which possesses certain smoothness properties. Although first-order dynamical systems are of little practical interest, they have been explicitly analyzed in this paper because for this class of dynamical systems we are able to obtain necessary and sufficient conditions for both passivity and losslessness which can be verified without solving the state equation; hence, they can be used to test the validity of various conjectures regarding passivity and losslessness for finite-order dynamical systems. Indeed, the insight gained by studying passive first-order dynamical systems enabled the first author to devise the counterexample to the Smoothness Conjecture presented in Subsection 4.2.

Section V is devoted to the concept of losslessness. A thorough treatment of the general theory of losslessness for time-invariant nonlinear n-ports is given in [5], but there is no obvious extension of that theory to time-varying nonlinear n-ports. In Subsection 5.1, we present a consistent theory of losslessness which applies to time-varying as well as time-invariant nonlinear n-ports. In our theory, the question of whether a time-varying nonlinear n-port $\mathcal{N}$ is lossless reduces to the question of whether a certain observable time-invariant dynamical system associated with $\mathcal{N}$ is lossless. One notable consequence of our theory is that a linear time-varying 1-port capacitor is not lossless: this is the same classification that Penfield [9] has argued for.

The algebraic condition (Suff. 5.1.20) in Lemma 5.1.20 of Subsection 5.1 is an obvious sufficient losslessness condition for finite-order time-invariant dynamical systems, but we do not know whether (Suff. 5.1.20) is a necessary condition for losslessness. Therefore it is of interest to obtain a sufficient algebraic losslessness condition which is not as restrictive as (Suff. 5.1.20). In Subsection 5.2 we obtain such a condition by applying Stalford's [8] results from optimal control theory.

The technical results from Subsection 3.1 are applied in Subsection 5.3 to obtain an original necessary losslessness condition for finite-order time-invariant dynamical systems.

In the final subsection of this paper, 5.4, a necessary and sufficient algebraic losslessness condition for first-order time-invariant dynamical systems is presented. The authors have also published this material in reference [5].

It should be noted that all the general nonlinear passivity and losslessness conditions in this paper are valid for LTIL n-ports (we use the adjective "nonlinear" to mean "not necessarily linear"); however, the special properties of LTIL n-ports allow us to derive passivity and losslessness conditions of a much more explicit nature. These conditions are fairly standard, but (contrary to wide-spread belief) there does not seem to be any treatment of this topic in the literature which is totally satisfactory in terms of completeness and rigor. For this reason the authors are writing a companion paper [10] which gives complete, rigorous proofs of the passivity and losslessness conditions for LTIL n-ports.

II. Definitions, Assumptions, and the Mathematical Representation of N-Ports

Our basic notation is standard, and is completely defined in reference [11]. Some of our nonstandard notation is as follows. The symbol $\mathbb{R}^+$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}^+ \triangleq [0,\infty)$, while $\mathbb{R}^e$ denotes the set of extended real numbers [12, p.34]; symbolically, this is approximately denoted as follows: $\mathbb{R}^e \triangleq \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. Finally, $\mathbb{R}^2_+$ denotes the subset of all $(t_1,t_0) \in \mathbb{R} \times \mathbb{R}$ such that $t_1 \geq t_0$; symbolically, $\mathbb{R}^2_+ \triangleq \{(t_1,t_0) \in \mathbb{R} \times \mathbb{R} : t_1 \geq t_0\}$.

2.1 Dynamical Systems

We are interested in the class of n-port[1] networks which can be mathematically represented by a "dynamical system" -- a mathematical abstraction which is defined as follows.

2.1.1 Definition. A dynamical system, denoted S, is a septuplet $\{U,\mathcal{U},\Sigma,\phi(\cdot,\cdot,\cdot,\cdot),Y,g(\cdot,\cdot,\cdot),\omega(\cdot,\cdot)\}$, where

[1] A precise definition of the term "n-port" is given in Subsection 2.2.

(i) U is a nonempty set called the set of admissible input values.

(ii) $\mathcal{U}$ is a set of functions mapping $\mathbb{R}$ into U called the set of admissible inputs. The set $\mathcal{U}$ is assumed to be translation invariant, i.e., if $u(\cdot) \in \mathcal{U}$, then the function $u_\tau: \mathbb{R} \to U$ defined by $u_\tau(t) \triangleq u(t-\tau)$ also belongs to $\mathcal{U}$ for every $\tau \in \mathbb{R}$. Moreover, $\mathcal{U}$ is assumed to be closed under concatenation. This means that if $u_1(\cdot), u_2(\cdot) \in \mathcal{U}$, then the functions $u_{12\tau}: \mathbb{R} \to U$ and $\hat{u}_{12\tau}: \mathbb{R} \to U$ defined by

$$u_{12\tau}(t) \triangleq \begin{cases} u_1(t), & \text{if } t \leq \tau, \\ u_2(t), & \text{if } t > \tau, \end{cases}$$

$$\hat{u}_{12\tau}(t) \triangleq \begin{cases} u_1(t), \text{ if } t < \tau, \\ u_2(t), \text{ if } t \geq \tau, \end{cases}$$

also belong to $\mathcal{U}$. Finally, we assume that $\mathcal{U}$ contains all the constant functions mapping $\mathbb{R}$ to U, i.e., for every $u_0 \in U$, the function $u: \mathbb{R} \to U$ defined by $u(t) \triangleq u_0$ belongs to $\mathcal{U}$.

(iii) Σ is a nonempty set called the state space.

(iv) $\phi: \mathbb{R}_+^2 \times \Sigma \times \mathcal{U} \to \Sigma$ is called the state transition function.

It obeys the following axioms.

(a) Consistency: $\phi(t_0, t_0, x_0, u(\cdot)) = x_0$ for all $t_0 \in \mathbb{R}$, $x_0 \in \Sigma$, and $u(\cdot) \in \mathcal{U}$.

(b) Determinism: $\phi(t_1, t_0, x_0, u_1(\cdot)) = \phi(t_1, t_0, x_0, u_2(\cdot))$ for all $(t_1, t_0, x_0) \in \mathbb{R}_+^2 \times \Sigma$ and all $u_1(\cdot), u_2(\cdot) \in \mathcal{U}$ satisfying $u_1(t) = u_2(t)$ for $t \in [t_0, t_1]$.

(c) Semi-group: for all $(t_2, t_0, x_0, u(\cdot)) \in \mathbb{R}_+^2 \times \Sigma \times \mathcal{U}$ and all $t_1 \in [t_0, t_2]$,
$\phi(t_2, t_0, x_0, u(\cdot)) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u(\cdot)), u(\cdot))$.

(v) Y is a nonempty set called the set of output values.

(vi) $g: \Sigma \times U \times \mathbb{R} \to Y$ is called the output function.

(vii) $\omega: U \times Y \to \mathbb{C}^n \times \mathbb{C}^n$ is called the port variables read-out function. It defines the port voltage read-out function $V: U \times Y \to \mathbb{C}^n$ and the port current read-out function $I: U \times Y \to \mathbb{C}^n$ by the following equation:

$$(V(u,y),\ I(u,y)) \triangleq \omega(u,y) \quad . \qquad (2.1.1.1)$$

The class of n-port networks which can be mathematically represented by a dynamical system is quite broad: it includes networks with nonlinear time-varying distributed elements as well as networks with the familiar linear time-invariant lumped elements of classical network theory (resistors, capacitors, inductors, etc.).

The only significant difference between our definition of a dynamical system and most others which have appeared in the literature (e.g., reference [13]), is the inclusion of the port variables read-out function $\omega(\cdot,\cdot)$. The equation $(v,i) = \omega(u,y)$ gives the values of the port voltage and current vectors, v and i, respectively, as a function of the instantaneous values of the input (independent) variable u and the output (dependent) variable y. It may seem strange that we allow complex-valued port variables, i.e., that we allow $\omega(\cdot,\cdot)$ to take values in $\mathbb{C}^n \times \mathbb{C}^n$. Of course, if a dynamical system is a mathematical representation of a real, physical n-port, then $\omega(\cdot,\cdot)$ must take values in $\mathbb{R}^n \times \mathbb{R}^n \subset \mathbb{C}^n \times \mathbb{C}^n$; we will see in a companion paper [10], however, that dynamical systems with complex-valued port variables can be useful theoretical tools.

2.1.2 Example. Regarding the port variables read-out function $\omega(\cdot,\cdot)$, the most common situation is where $U=\mathbb{R}^n$, $Y=\mathbb{R}^n$, and $\omega: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a linear bijective function characterized by a nonsingular coordinate transformation matrix $\Omega \in \mathbb{R}^{2n \times 2n}$ as follows:

$$\begin{bmatrix} v \\ i \end{bmatrix} = \Omega \begin{bmatrix} y \\ u \end{bmatrix} \triangleq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \qquad (2.1.2.1)$$

where v stands for V(u,y), i stands for I(u,y), and Ω is partitioned into four $n \times n$ submatrices a, b, c, and d as shown on the right-hand side of (2.1.2.1). The interesting special cases are as follows.

(i) The impedance representation. Here u=i and y=v; hence, a = I, b = 0, c = 0, and d = I (note: I denotes the $n \times n$ identity matrix).

(ii) The admittance representation. Here u=v and y=i; hence, a = 0, b = I, c = I, and d = 0.

(iii) The hybrid representation. In this case, one of the following two conditions is satisfied for each $k \in \{1,2,\ldots,n\}$: either $v_k = y_k$ and $i_k = u_k$, or else $v_k = u_k$ and $i_k = y_k$, where v_k is the k-th component of v, etc. Thus

$$a = d = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}, \quad b = c = \begin{bmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{bmatrix},$$

where, for each $k \in \{1,2,\ldots,n\}$, either $a_{kk}=1$ and $b_{kk}=0$, or else $a_{kk}=0$ and $b_{kk}=1$.

(iv) For the scattering representation, the matrices a,b,c, and d have the following form:

$$a = b = \begin{bmatrix} \sqrt{r_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{r_n} \end{bmatrix}, \quad d = -c = \begin{bmatrix} \frac{1}{\sqrt{r_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{r_n}} \end{bmatrix},$$

where the real positive constants r_k are called the port normalizing numbers.

(v) For the transmission representation, n must be even and a, b, c, and d have the following form:

$$a = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}, \qquad d = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

where I denotes the $\frac{n}{2} \times \frac{n}{2}$ identity matrix.

Before proceeding to the next definition, we must define some additional notation. If $w,z \in \mathbb{C}^n$, then $\langle w,z \rangle \triangleq \sum_{j=1}^{n} \overline{w}_j z_j$, where w_j denotes the j-th component of w, etc., and $\overline{w}_j$ denotes the complex conjugate of w_j; also, we define $\|w\| \triangleq \sqrt{\langle w,w \rangle}$. Finally, if $s \in \mathbb{C}$, then Re s and Im s denote the real and imaginary parts of s, respectively.

2.1.3 Definition. Let S denote a dynamical system. Associated with S is a function $p: \Sigma \times U \times R \to \mathbb{R}$, called the power input function, which is defined by

$$p(x,u,t) \triangleq \mathrm{Re}\langle V(u,g(x,u,t)), I(u,g(x,u,t)) \rangle \quad . \qquad (2.1.3.1)$$

We will always use the associated reference directions for assigning the polarity of the port variables of an n-port [2, pp. 4-5]; hence, if a dynamical system S is a mathematical representation for a real, physical n-port N, then $V(u,g(x,u,t))$ and $I(u,g(x,u,t))$ have real-valued components and $p(x,u,t)$ gives the net power flowing into the ports of

N when the state of S is x, the input value is u, and the time is t (note that p(x,u,t) can be positive, negative, or zero).

2.1.4 Definitions. Let S denote a dynamical system, and let $(t_0,x_0,u(\cdot)) \in \mathbb{R} \times \Sigma \times U$. Define $x: [t_0,\infty) \to \Sigma$ by

$$x(t) \triangleq \phi(t,t_0,x_0,u(\cdot)) \ . \tag{2.1.4.1}$$

Then[2] $x(\cdot)|[t_0,\infty)$ is called the state trajectory of S with $x(t_0) = x_0$ which is generated by $u(\cdot)$. Define $y: [t_0,\infty) \to Y$, $v: [t_0,\infty) \to \mathbb{C}^n$, and $i: [t_0,\infty) \to \mathbb{C}^n$ by

$$y(t) \triangleq g(x(t),\ u(t),\ t) \tag{2.1.4.2}$$

$$v(t) \triangleq V(u(t),\ y(t)) \tag{2.1.4.3}$$

$$i(t) \triangleq I(u(t),\ y(t)) \ . \tag{2.1.4.4}$$

Then $y(\cdot)|[t_0,\infty)$ (resp., $v(\cdot)|[t_0,\infty)$; resp., $i(\cdot)|[t_0,\infty)$) is called the output (resp., port voltage; resp., port current) of S with initial state x_0 which is generated by $u(\cdot)$. Moreover, $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ (resp., $\{u(\cdot),y(\cdot)\}|[t_0,\infty)$) is called an input-trajectory (resp., input-output) pair of S with initial state x_0, while $\{v(\cdot),i(\cdot)\}|[t_0,\infty)$ is called the voltage-current pair of S with initial state x_0 which is generated by $u(\cdot)$. Let $t_1 \in [t_0,\infty)$, and define $x_1 \triangleq x(t_1)$. Then $x(\cdot)|[t_0,t_1]$ is called the state trajectory of S from x_0 to x_1 which is generated by $u(\cdot)$, and we say that $u(\cdot)$ "drives" or "steers" S from x_0 to x_1 over the time interval $[t_0,t_1]$. Moreover, $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ (resp., $\{u(\cdot),y(\cdot)\}|[t_0,t_1]$) is called an input-trajectory (resp., input-output) pair of S from x_0 to x_1, while $\{v(\cdot),i(\cdot)\}|[t_0,t_1]$ is

[2] The notation $f(\cdot)|A$ denotes the restriction of a function $f(\cdot)$ to a subset A of its domain.

called the voltage-current pair of S from x_0 to x_1 which is generated by $u(\cdot)$.

Before proceeding, we must define some additional notation and terminology. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set, let W be a subset of a normed vector space X (with the topology of W the relative topology that it inherits from X), and let $0 < p < \infty$. Then $L^p(A \to W)$ denotes the set of all Lebesgue measurable functions f: $A \to W$ such that $\int_A \|f(t)\|^p dt < \infty$, where $\int_A \|f(t)\|^p dt$ denotes the Lebesgue integral of the function $t \to \|f(t)\|^p$ over the set A. (If A is an interval [a,b], then we use the standard notation $\int_a^b$ to denote $\int_A$.) The notation $L^p_{loc}(A \to W)$ denotes the set of all Lebesgue measurable functions f: $A \to W$ such that $\int_B \|f(t)\|^p dt < \infty$ for every compact Lebesgue measurable set $B \subset A$. Also, the notation $L^\infty(A \to W)$ denotes the set of all Lebesgue measurable functions f: $A \to W$ for which there exists a finite constant $M(f(\cdot)) > 0$ (which depends on $f(\cdot)$) such that $\|f(t)\| \le M(f(\cdot))$ for a.a.t $\in$ A (the notation "a.a.t" stands for "almost all t," it means for all t with the possible exception of some t which form a set of Lebesgue measure zero). We use $L^\infty_{loc}(A \to W)$ to denote the set of all Lebesgue measurable functions f: $A \to W$ which satisfy the following condition: for every compact Lebesgue measurable set $B \subset A$, there exists a finite constant $M(f(\cdot),B) > 0$ (which depends on $f(\cdot)$ and B) such that $\|f(t)\| \le M(f(\cdot),B)$ for a.a.t $\in$ B. For $0 < p \le \infty$, we call $L^p(A \to W)$ the set of L^p functions mapping A to W, and we call $L^p_{loc}(A \to W)$ the set of locally L^p functions mapping A to W. Finally, when we use terminology such as "measurable set," "measurable function," and "integral," it will always be understood that we mean "Lebesgue measurable set," "Lebesgue measurable function," and "Lebesgue integral," respectively.

2.1.5 Assumption. Let S denote a dynamical system, and let $p(\cdot,\cdot,\cdot)$ denote the power input function associated with S (Def. 2.1.3). It is assumed that for every $(t_0,x_0,u(\cdot)) \in \mathbb{R} \times \Sigma \times \mathcal{U}$, the mapping $t \to p(x(t),u(t),t)$ belongs to $L^1_{loc}([t_0,\infty) \to \mathbb{R})$, where $x(\cdot)|[t_0,\infty)$ is the state trajectory of S with $x(t_0) = x_0$ which is generated by $u(\cdot)$.

2.1.6 Definition. Let S denote a dynamical system, and let $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ denote an input-trajectory pair of S. For every $t_1 \in [t_0,\infty)$, the energy consumed by $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ is defined to be the quantity $\int_{t_0}^{t_1} p(x(t),u(t),t)dt$. (By Assumption 2.1.5, $\int_{t_0}^{t_1} p(x(t),u(t),t)dt$ exists and is finite for every finite $t_1 \geq t_0$; moreover, note that this integral can be positive, negative, or zero.)

2.1.7 Definition. (Reachability) Let S denote a dynamical system. We say that a state x_1 of S is reachable from a state x^* of S if for each $t_1 \in \mathbb{R}$, there exists (for some $t_0 \in (-\infty,t_1]$) an input-trajectory pair $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ of S from x^* to x_1. We say that S is reachable from x^* if every state of S is reachable from x^*.

2.1.8 Definition. (Controllability) Let S denote a dynamical system. We say that S is controllable if for each $(x_0,x_1,t_1) \in \Sigma \times \Sigma \times \mathbb{R}$, there exists (for some $t_0 \in (-\infty,t_1]$) an input-trajectory pair $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ of S from x_0 to x_1. S is defined to be uncontrollable if it is not controllable.

2.1.9 Definition. Let S and S' denote two (not necessarily distinct) dynamical systems. State x of S and state x' of S' are defined to be equivalent at $t_0 \in \mathbb{R}$ if the set of voltage-current pairs $\{v(\cdot), i(\cdot)\}|[t_0,\infty)$ of S with initial state x is the same as the set of voltage-current pairs $\{v'(\cdot), i'(\cdot)\}|[t_0,\infty)$ of S' with initial state x'.

2.1.10 Definition. (Equivalence) Two dynamical systems, S and S', are defined to be equivalent if the following condition is satisfied for each $t_0 \in \mathbb{R}$: for every state x of S, there exists a state x' of S' which is equivalent at t_0 to the state x of S, and conversely, for every state x' of S', there exists a state x of S which is equivalent at t_0 to the state x' of S'.

Note that two dynamical systems which are equivalent by Def. 2.1.10 have the same external behavior, i.e., they have the same behavior with respect to the port voltage v and the port current i. For this reason, we consider two dynamical systems to be (equally valid) mathematical representations for the same n-port if and only if they are equivalent according to Def. 2.1.10. More discussion on this matter, including a precise definition of the term "n-port," is given in Subsection 2.2.

2.1.11 Definition. Let S denote a dynamical system. We say that S is input-observable if the following condition holds for every $(t_1,t_0,x_0) \in \mathbb{R}_+^2 \times \Sigma$: if $u_a(\cdot)$ and $u_b(\cdot)$ are any two inputs such that $\{v_a(t),\ i_a(t)\} = \{v_b(t),\ i_b(t)\}$ for all $t \in [t_0,t_1]$; where $\{v_a(\cdot),\ i_a(\cdot)\}|[t_0,\infty)$ and $\{v_b(\cdot),\ i_b(\cdot)\}|[t_0,\infty)$ are the voltage-current pairs of S with common initial state x_0 which are generated by $u_a(\cdot)$ and $u_b(\cdot)$, respectively; then $u_a(t) = u_b(t)$ for all $t \in [t_0,t_1]$. We say that S is input-distinguishable if $\omega(\cdot,\cdot)$ is injective (recall from Def. 2.1.1 that $\omega(\cdot,\cdot)$ is the port variables read-out function for S).

Note that if a dynamical system S has a port variables read-out function of the class described in Example 2.1.2, then S is input-distinguishable.

The following lemma shows that the set of input-distinguishable dynamical systems is a subset of the set of input-observable dynamical systems.

2.1.12 Lemma. Let S denote a dynamical system. If S is input-distinguishable, then S is input-observable.

The proof is given in the Appendix.

2.1.13 Definition. (Observability) A dynamical system S is defined to be observable[3] if both of the following conditions are satisfied:

(i) S is input-observable (Def. 2.1.11).

(ii) For each $t_0 \in \mathbb{R}$, the equivalence at t_0 of any two states x_1 and x_2 of S (Def. 2.1.9) implies that $x_1 = x_2$.

Next, we are going to briefly discuss linearity. A dynamical system (Def. 2.1.1) is defined to be linear[4] if all of the following four conditions are satisfied:

(i) $U, \mathcal{U}, \Sigma$, and Y are linear vector spaces [14, p. 5] over the same field F, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

(ii) For every $(t,t_0) \in \mathbb{R}_+^2$, $\phi(t,t_0,\cdot,\cdot)$ is a linear map of $\Sigma \times \mathcal{U}$ into Σ.

(iii) For every $t \in \mathbb{R}$, $g(\cdot,\cdot,t)$ is a linear map of $\Sigma \times U$ into Y.

(iv) $\omega(\cdot,\cdot)$ is a linear map of $U \times Y$ into $\mathbb{C}^n \times \mathbb{C}^n$.

[3]This property is called "total observability" in [5].

[4]This definition of linearity is more restrictive than some others in the literature (e.g., reference [13]), which would require only that S be externally linear.

In our terminology, the adjective "nonlinear" will mean "not necessarily linear." Hence, phrases such as "dynamical system" and "nonlinear dynamical system" mean exactly the same thing: we will choose the latter over the former only if we wish to emphasize that the dynamical system under consideration need not be linear.

2.1.14 Definition. (Time-Invariant and Time-Varying Dynamical Systems) Let S denote a dynamical system. For each $(u(\cdot),\tau) \in \mathcal{U} \times \mathbb{R}$, define $u_\tau: \mathbb{R} \to U$ by $u_\tau(t) \triangleq u(t-\tau)$. S is defined to be time-invariant[5] if both of the following conditions are satisfied:

(i) For all $\tau \in \mathbb{R}$ and all $(t_1,t_0,x_0,u(\cdot)) \in \mathbb{R}_+^2 \times \Sigma \times \mathcal{U}$, $\phi(t_1+\tau,\ t_0+\tau,\ x_0,\ u_\tau(\cdot)) = \phi(t_1,t_0,x_0,u(\cdot))$.

(ii) $g(x,u,t)$ does not depend on t (this being the case, we usually write the output function value as $g(x,u)$ rather than $g(x,u,t)$).

Moreover, S is defined to be time-varying if it is not time-invariant.

Note that for a time-invariant dynamical system, $p(x,u,t)$ is independent of t; this being the case, we usually write the function value as $p(x,u)$ instead.

Observe that the set $\mathcal{I}_T$ of input-trajectory pairs of a time-invariant dynamical system has the following property: $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ belongs to $\mathcal{I}_T$ if and only if $\{u_\tau(\cdot),x_\tau(\cdot)\}|[t_0+\tau,\infty)$ belongs to $\mathcal{I}_T$ for all $\tau \in \mathbb{R}$. Similar comments apply to the sets of input-output and voltage-current pairs of a time-invariant dynamical system.

2.1.15 Definition. (Canonical Time-Invariant Dynamical System) Let $S = \{U,\mathcal{U},\Sigma,\phi(\cdot,\cdot,\cdot,\cdot),Y,\ g(\cdot,\cdot,\cdot),\omega(\cdot,\cdot)\}$ denote a dynamical system (Def. 2.1.1). The canonical time-invariant dynamical system associated

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[5] This definition of time-invariance is more restrictive than some others in the literature (e.g., reference [13]), which would require only that S be externally time-invariant.

with S, denoted $S^* = \{U^*, \mathcal{U}^*, \Sigma^*, \phi^*(\cdot,\cdot,\cdot,\cdot), Y^*, g^*(\cdot,\cdot), \omega^*(\cdot,\cdot)\}$, is defined as follows.

(i) $U^* \triangleq U, \quad \mathcal{U}^* \triangleq \mathcal{U}, \quad Y^* \triangleq Y, \quad \omega^*(\cdot,\cdot) \triangleq \omega(\cdot,\cdot)$.

(ii) $\Sigma^* \triangleq \Sigma \times \mathbb{R}$.

(iii) $\phi^*: \mathbb{R}_+^2 \times \Sigma^* \times \mathcal{U} \to \Sigma^*$ is defined by

$$\phi^*(t_1,t_0,(x_0,\sigma_0),u(\cdot)) \triangleq (\phi(t_1-t_0+\sigma_0,\sigma_0,x_0,u_{\sigma_0-t_0}(\cdot)),\ t_1-t_0+\sigma_0) \tag{2.1.15.1}$$

(recall that for $\tau \in \mathbb{R}$, $u_\tau(t) \triangleq u(t-\tau)$; thus, $u_{\sigma_0-t_0}(t) \triangleq u(t+t_0-\sigma_0)$). It is straightforward to verify that $\phi^*(\cdot,\cdot,\cdot,\cdot)$ satisfies the consistency, determinism, and semi-group axioms of Def. 2.1.1. Moreover, it is easy to see that $\phi^*(\cdot,\cdot,\cdot,\cdot)$ satisfies property (i) of Def. 2.1.14.

(iv) $g^*: \Sigma^* \times U \to Y$ is defined by

$$g^*((x,\sigma),u) \triangleq g(x,u,\sigma). \tag{2.1.15.2}$$

Also, the power input function for S^*, $p^*: \Sigma^* \times U \to \mathbb{R}$, is given by

$$p^*((x,\sigma),u) \triangleq p(x,u,\sigma) \tag{2.1.15.3}$$

where $p(\cdot,\cdot,\cdot)$ is the power input function for S (Def. 2.1.3).

2.1.16 Remarks. Roughly speaking, we obtain S^* from S by letting time be one of the state variables. Note the following:

(a) Regardless of whether or not S is controllable, S^* is never controllable. This is because the "time" state variable $\sigma(\cdot)$ always increases monotonically with t.

(b) If S is time-invariant, then S and S^* are equivalent.

(c) If S is time-varying, then S and S^* are not equivalent (i.e., they do not have the same external behavior). However, the external behavior of S^* subsumes that of S in the following sense. Let $t_0 \in \mathbb{R}$ be any initial time, let x_0 be the state of S at time t_0, and let (x_0,σ_0) be the state of S^* at time t_0. By the definition of a dynamical system, we are free to let σ_0 take any value whatsoever. If we happen to choose $\sigma_0 = t_0$, then the set of voltage-current pairs $\{v^*(\cdot),i^*(\cdot)\}|[t_0,\infty)$ of S* with initial state $(x_0,\sigma_0) = (x_0,t_0)$ is identical to the set of voltage-current pairs $\{v(\cdot),i(\cdot)\}|[t_0,\infty)$ of S with initial state x_0 (similar comments can be made regarding the sets of input-output and input-trajectory pairs of S and S*; of course, in the latter case one must make obvious modifications necessitated by the technical fact that S and S* have different state spaces).

The preceding remarks show how one can often reduce a problem involving a time-varying dynamical system to a similar problem involving a time-invariant dynamical system. This procedure is frequently used in optimal control theory to allow one to state theoretical results exclusively in terms of time-invariant dynamical systems, without any loss of generality.

<u>2.1.17 Definition</u>. (Finite-Order Dynamical System) By definition, a <u>finite-order dynamical system</u> is a dynamical system S (Def. 2.1.1) which satisfies the following additional conditions:

(i) There exists a positive integer m, called the <u>order</u> of S, such that $\Sigma \subset \mathbb{R}^m$.

(ii) $U \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$.

(iii) $g(\cdot,\cdot,\cdot)$ and $\omega(\cdot,\cdot)$ are continuous, and $\omega(\cdot,\cdot)$ takes values in $\mathbb{R}^n \times \mathbb{R}^n$ (as opposed to $\mathbb{C}^n \times \mathbb{C}^n$).

(iv) The elements of $\mathcal{U}$ are measurable functions mapping $\mathbb{R}$ to U.

(v) There exists a continuous function $f: \Sigma \times U \times \mathbb{R} \to \mathbb{R}^m$ with the following property: for each $(t_0,x_0,u(\cdot)) \in \mathbb{R} \times \Sigma \times \mathcal{U}$, there is a <u>unique</u> function $x: [t_0,\infty) \to \Sigma$ with

$$x(t_0) = x_0 \qquad (2.1.17.1)$$

which is absolutely continuous [12,p. 104] over $[t_0,t_1]$ for every $t_1 \geq t_0$, satisfies

$$\dot{x}(t) = f(x(t),u(t),t) \text{ for a.a. } t \in [t_0,\infty), \qquad (2.1.17.2)$$

and satisfies

$$x(t) = \phi(t,t_0,x_0,u(\cdot)) \text{ for all } t \in [t_0,\infty). \qquad (2.1.17.3)$$

2.1.18 Definition. Let S denote a finite-order dynamical system. Then the equation

$$\dot{x}(t) = f(x(t),u(t),t) \qquad (2.1.18.1)$$

is called the state equation of S, and the equation

$$y(t) = g(x(t),u(t),t) \qquad (2.1.18.2)$$

is called the output equation of S.

2.1.19 Remark. It is clear that a finite-order dynamical system is time-invariant if and only if both $f(x,u,t)$ and $g(x,u,t)$ are independent of t; this being the case, we usually write these function values as $f(x,u)$ and $g(x,u)$, respectively.

2.2 A Precise Definition for the Term "N-Port"

The term "n-port" is commonly used in electrical network theory to denote any one of three distinct concepts. One concept is that of a physical n-port, i.e., the actual "real-world" electrical network under consideration. This electrical network has n ports[6] (hence, the term "n-port"), and from our point of view it interacts with the outside world exclusively through its ports. Another concept is that of a model of a physical n-port, which we shall call an n-port model.

[6] The n is used in the generic sense -- we do not assume that every such network has the same number of ports.

This concept is an idealization of a physical n-port; it represents an attempt to extract all the relevant physical effects. One usually identifies an n-port model with a network graph drawn on a piece of paper. The network graph consists of certain idealized elements interconnected by lines which represent ideal interconnection wires. Finally, the term "n-port" is sometimes used to denote a mathematical representation of an n-port model.

In formulating a definition of the term "n-port" which is both useful and meaningful within the framework of our theory, we have rejected all three of the concepts mentioned in the preceding paragraph. The last alternative mentioned above -- that of defining an n-port to be a mathematical representation of an n-port model -- is the most troublesome. It seems to ignore the fact that an n-port model usually has an infinite number of equally-valid mathematical representations (if it has any at all), and it often leads to inconsistencies in defining the concepts of passivity and losslessness (see references [4] and [5] for examples).

To motivate our definition of the term "n-port," first note that our theory is limited to the mathematical representations of Def. 2.1.1 -- the so-called "dynamical systems." Suppose that a given n-port model, denoted $\mathcal{N}_m$, has a dynamical system representation S. If S' is any dynamical system which is equivalent to S (Def. 2.1.10), then S and S' have the same external behavior; hence, S' is an equally-valid mathematical representation for $\mathcal{N}_m$. Note that the equivalence relation of Def. 2.1.10 is a true equivalence relation in the set-theoretical sense [12, p. 22]; hence, it partitions the universe of all dynamical systems into equivalence classes. Recall that these equivalence classes are disjoint subsets of dynamical systems which have the following property: two dynamical systems are equivalent if and only if they belong to the same equivalence class. Thus N_m can be identified with the unique equivalence class of dynamical systems which contains S. Note that there may exist an n-port model N'_m, distinct from N_m, which can also be identified with the equivalence class containing S; but since N_m and N'_m have the same external (port) behavior, they are indistinguishable for our purposes. These observations justify the following definition of the term "n-port."

2.2.1 Definition. An n-port is an equivalence class [12, p. 22] of dynamical systems, where the equivalence relation is given in Def. 2.1.10.

2.2.2 A Note on Terminology. Let N denote an n-port, and let S denote a dynamical system. We say that "S is a dynamical system representation for N," or that "N has the dynamical system representation S," if S is an element of the equivalence class N.

A linear n-port is one with a linear dynamical system representation. The phrase "nonlinear n-port" means an n-port which is not necessarily linear.

2.2.3 Definitions. Let N denote an n-port. N is defined to be time-invariant if it has a time-invariant dynamical system representation. N is defined to be time-varying if it is not time-invariant.

We conclude this chapter with a mild technical assumption which will play a role in our theory of losslessness.

2.2.4 Assumption. Let A denote the class of all n-ports which have at least one input-distinguishable dynamical system representation (Def. 2.1.11), and let A_{ti} denote the class of all time-invariant n-ports which have at least one input-distinguishable time-invariant dynamical system representation. Our theory of n-ports will be restricted to those n-ports which belong to $\mathcal{A}$; moreover, our theory of time-invariant n-ports will be restricted to those n-ports which belong to $\mathcal{A}_{ti}$.

III. Technical Lemmas

3.1 The Main Result

The purpose of this subsection is to present a new analytical result for finite-order time-invariant dynamical systems. As will be demonstrated in the following two sections, this result has applications in the studies of both passivity and losslessness.

3.1.1 Notation. $S^m \triangleq \{x \in \mathbb{R}^m : \|x\| = 1\}$. $\mathcal{P}(U)$ will denote the collection of all subsets of $U \subset \mathbb{R}^n$.

3.1.2 Definitions. For a finite-order time-invariant dynamical system (Def. 2.1.17, Remark 2.1.19), define $\hat{U}: \Sigma \times S^m \to \mathcal{P}(U)$ by

$$\hat{U}(x,\alpha) \triangleq \{u \in U: f(x,u) - \alpha\|f(x,u)\| = 0\} \cap \{u \in U: \|f(x,u)\| > 0\} \qquad (3.1.2.1)$$

and define $h: \Sigma \times S^m \to \mathbb{R}^e$ by

$$h(x,\alpha) \triangleq \inf\left\{\frac{p(x,u)}{\|f(x,u)\|}: u \in \hat{U}(x,\alpha)\right\} . \qquad (3.1.2.2)$$

We follow the convention that the infimum over the empty set is ∞; thus, $h(x_0,\alpha_0) = \infty$ if and only if $\hat{U}(x_0,\alpha_0) = \emptyset$ ($\emptyset$ denotes the empty set).

3.1.3 Definitions. Let $\gamma_0: \mathbb{R}^+ \to \Sigma$ be a function which is absolutely continuous on $[0,T_0]$ for some $T_0 \in \mathbb{R}^+$. Then

$\gamma_1\colon [0,T_1] \to \Sigma$ is defined to be a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$ if $\gamma_1(t) = \gamma_0(\sigma(t))$ for all $t \in [0,T_1]$, where $\sigma\colon [0,T_1] \to [0,T_0]$ is an absolutely continuous function such that $\sigma(0) = 0$, $\sigma(T_1) = T_0$, and $\dot{\sigma}(t) > 0$ for a.a. $t \in [0,T_1]$. The set of all re-parametrizations of $\gamma_0(\cdot)|[0,T_0]$ is denoted $\mathcal{R}[\gamma_0(\cdot)|[0,T_0]]$. Moreover, if $\gamma_0(\cdot)$ is a state trajectory of a finite-order time-invariant dynamical system S, then any re-parametrization of $\gamma_0(\cdot)|[0,T_0]$ is called an admissible curve of S.

3.1.4 Remarks. It follows that $\gamma_1(\cdot) = (\gamma_0 \circ \sigma)(\cdot)$ is absolutely continuous, since $\gamma_0(\cdot)$ and $\sigma(\cdot)$ are absolutely continuous and $\sigma(\cdot)$ is increasing [15, p. 95, Theorem I.4.42]. Also, note that every state trajectory of S is an admissible curve of S, but the converse is not true in general.

3.1.5 Integration Conventions. We are using Lebesgue integrals. Let $g\colon \mathbb{R} \to \mathbb{R}^e$ be a function. Define $g^+\colon \mathbb{R} \to \mathbb{R}^e$ and $g^-\colon \mathbb{R} \to \mathbb{R}^e$ by $g^+(t) \triangleq \max\{0,g(t)\}$, $g^-(t) \triangleq \max\{0,-g(t)\}$. Let $E \subset \mathbb{R}$ be a (Lebesgue) measurable set. Then the function $g(\cdot)$ is defined to be integrable in the extended sense over E if both of the following conditions are satisfied: (a) $g(\cdot)$ is (Lebesgue) measurable, and (b) either $\int_E g^+(t)dt < \infty$ or $\int_E g^-(t)dt < \infty$. This being the case, $\int_E g(t)dt$ is assigned the value

$$\int_E g(t)dt \triangleq \int_E g^+(t)dt - \int_E g^-(t)dt \,. \qquad (3.1.5.1)$$

Now consider an integral of the form $\int_0^{T_0} L(\gamma_0(t),\dot{\gamma}_0(t))dt$, where $\gamma_0\colon [0,T_0] \to \Sigma$ is absolutely continuous. This integral is defined to be parametrization independent provided that $\int_0^{T_0} L(\gamma_0(t),\dot{\gamma}_0(t))dt = \int_0^{T_1} L(\gamma_1(t),\dot{\gamma}_1(t))dt$ for every re-parametrization $\gamma_1(\cdot)|[0,T_1]$ of $\gamma_0(\cdot)|[0,T_0]$.

3.1.6 Lemma (Change of Variables). Let $k: [\tau_0,\tau_1] \to \mathbb{R}$ be absolutely continuous, $\dot{k}(\tau) > 0$ for a.a. $\tau \in [\tau_0,\tau_1]$, and let $a(\cdot) \in L^1([k(\tau_0),k(\tau_1)] \to \mathbb{R})$; then the following statements are true:

(i) $k(\cdot)$ is a bijection of $[\tau_0,\tau_1]$ onto $[k(\tau_0),k(\tau_1)]$,

(ii) $k^{-1}(\cdot)$ is absolutely continuous,

(iii) the function $\tau \to a(k(\tau))\dot{k}(\tau)$ belongs to $L^1([\tau_0,\tau_1] \to \mathbb{R})$,

and (iv) $\int_{k(\tau_0)}^{k(\tau_1)} a(t)dt = \int_{\tau_0}^{\tau_1} a(k(\tau))\dot{k}(\tau)d\tau$.

The proof is given by Warga [15, p. 98, Theorem I.4.43].

Remark #1. Instead of assuming that $a(\cdot) \in L^1([k(\tau_0),k(\tau_1)] \to \mathbb{R})$, we can assume that the function $\tau \to a(k(\tau))\dot{k}(\tau)$ belongs to $L^1([\tau_0,\tau_1] \to \mathbb{R})$; it then follows that $a(\cdot) \in L^1([k(\tau_0),k(\tau_1)] \to \mathbb{R})$. To see this, let $\sigma(\cdot) \triangleq k^{-1}(\cdot)$ (hence, $\dot{\sigma}(t) = 1/\dot{k}(\sigma(t)) > 0$ for a.a. $t \in [k(\tau_0),k(\tau_1)]$). We know from statement (ii) that $\sigma(\cdot)$ is absolutely continuous; thus, we can apply statement (iii) with the function $\tau \to a(k(\tau))\dot{k}(\tau)$ taking the role of $a(\cdot)$ and $\sigma(\cdot)$ taking the role of $k(\cdot)$, and we conclude that $t \to a(k(\sigma(t)))\dot{k}(\sigma(t))\dot{\sigma}(t)$ $= a(t)$ is an element of $L^1([k(\tau_0),k(\tau_1)] \to \mathbb{R})$.

Remark #2. Consider Def. 3.1.3 again. Lemma 3.1.6 shows that $\gamma_1(\cdot)|[0,T_1]$ is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$ if and only if $\gamma_0(\cdot)|[0,T_0]$ is a re-parametrization of $\gamma_1(\cdot)|[0,T_1]$.

3.1.7 Lemma. Let S denote a finite-order time-invariant dynamical system with U a closed subset of $\mathbb{R}^n$ and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to U)$.

(a) Let an admissible curve $\gamma(\cdot)$ of S and a real number $T \geq 0$ be such that $\dot{\gamma}(t) \neq 0$ for a.a. $t \in [0,T]$. Then the integral $\int_0^T h\left(\gamma(t),\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\right)\|\dot{\gamma}(t)\|dt$ exists in the extended sense, its value is either finite or $-\infty$, and it is parametrization independent.

(b) Let $\gamma_0(\cdot)|[0,T_0]$ be an admissible curve of S with $\dot{\gamma}_0(t) \neq 0$ for a.a. $t \in [0,T_0]$. Then

$$\inf_{T\geq 0}\left\{\int_0^T p(x(t),u(t))dt: x(\cdot)|[0,T] \in \mathcal{R}[\gamma_0(\cdot)|[0,T_0]]\right\}$$
$$= \int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt \qquad (3.1.7.1)$$

where the expression on the left-hand side of (3.1.7.1) denotes the infimum of $\int_0^T p(x(t),u(t))dt$ over all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[0,T]$ of S, where $T \geq 0$ is <u>not</u> fixed, subject to the restriction that $x(\cdot)|[0,T]$ is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$.

The proof is given in the Appendix.

3.2 Special Case: First-Order Dynamical Systems

In this subsection we consider first-order time-invariant dynamical systems, i.e., those for which $\Sigma \subset \mathbf{R}$.

3.2.1 <u>Definitions</u>. Let $\Sigma \subset \mathbf{R}$, with the topology of Σ the relative topology that it inherits from $\mathbf{R}$. A function $\phi: \Sigma \rightarrow \mathbf{R}^e$ is defined to be <u>upper semicontinuous</u> if the set $\{x\in\Sigma: \phi(x) < \alpha\}$ is open (in the topology of Σ) for all $\alpha \in \mathbf{R}$ [16,pp. 38-39]. Likewise, $\phi(\cdot)$ is defined to be <u>lower semicontinuous</u> if the set $\{x\in\Sigma: \phi(x) > \alpha\}$ is open for all $\alpha \in \mathbf{R}$. Note that $\phi(\cdot)$ is upper semicontinuous if and only if $-\phi(\cdot)$ is lower semicontinuous; also, $\phi(\cdot)$ is continuous if and only if it is both upper and lower semicontinuous.

3.2.2 <u>Lemma</u>. The infimum of any collection of upper semicontinuous functions is upper semicontinuous. The supremum of any collection of lower semicontinuous functions is lower semicontinuous.

The proof is given in the Appendix.

3.2.3 Lemma. Let $\phi: \Sigma \to \mathbf{R}^e$ and let K be a compact subset of Σ. If $\phi(\cdot)$ is upper semicontinuous and $\phi(x) \neq \infty$ for all $x \in K$, then $\phi(\cdot)$ is bounded above on K. If $\phi(\cdot)$ is lower semicontinuous and $\phi(x) \neq -\infty$ for all $x \in K$, then $\phi(\cdot)$ is bounded below on K.

The proof is given in the Appendix.

3.2.4 Definitions. Recall the mappings $\hat{U}: \Sigma \times S^m \longrightarrow P(U)$ and $h: \Sigma \times S^m \longrightarrow \mathbf{R}^e$ in Def. 3.1.2. Consider a first-order time-invariant dynamical system (i.e., $m = 1$). Define, for each $x \in \Sigma$,

$$U_x^+ \triangleq \hat{U}(x,1) = \{u \in U: f(x,u) > 0\} \tag{3.2.4.1}$$

$$U_x^- \triangleq \hat{U}(x,-1) = \{u \in U: f(x,u) < 0\} \tag{3.2.4.2}$$

$$\bar{h}(x) \triangleq h(x,1) = \inf_{u \in U_x^+} \left\{\frac{p(x,u)}{f(x,u)}\right\} \tag{3.2.4.3}$$

$$\underline{h}(x) \triangleq -h(x,-1) = \sup_{u \in U_x^-} \left\{\frac{p(x,u)}{f(x,u)}\right\} \quad . \tag{3.2.4.4}$$

3.2.5 Lemma. For a first-order time-invariant dynamical system, $\bar{h}(\cdot)$ is upper semicontinuous and $\underline{h}(\cdot)$ is lower semicontinuous.

The proof is given in the Appendix.

The functions $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are continuous in the special case when $\bar{h}(x) = \underline{h}(x)$ for all $x \in \Sigma$ (this follows from Lemma 3.2.5 and the comments in Def. 3.2.1). In general, however, neither $\bar{h}(\cdot)$ nor $\underline{h}(\cdot)$ will be continuous. The following example shows that these functions can be quite bizarre.

3.2.6 Example. Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of the rational numbers. Consider the finite-order dynamical system with the state equation

$$\dot{x} = u^3$$

and the power input function

$$p(x,u) = u^3 \mathrm{sgn}(u) \sum_{n=1}^{\infty} \frac{1}{2^n} \exp(-u^2(x-r_n)^2) ,$$

where $\mathrm{sgn}(u) \triangleq u/|u|$ for $u \neq 0$, $\mathrm{sgn}(0) \triangleq 0$. Here $\Sigma = U = \mathbb{R}$ and $\mathcal{U} = L^{\infty}_{loc}(\mathbb{R} \to \mathbb{R})$. It is straightforward to verify that

$$\bar{h}(x) = \begin{cases} 0 , & \text{if } x \text{ is irrational} \\ \frac{1}{2^n} , & \text{if } x = r_n \end{cases}$$

$$\underline{h}(x) = \begin{cases} 0 , & \text{if } x \text{ is irrational} \\ -\frac{1}{2^n} , & \text{if } x = r_n \end{cases}$$

Note that $\bar{h}(\cdot)$ is upper semicontinuous and $\underline{h}(\cdot)$ is lower semicontinuous (in agreement with Lemma 3.2.5); however, both $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are discontinuous at each rational number.

3.2.7 Notation. For any dynamical system, let $R(x_0)$ denote the set of states reachable from $x_0 \in \Sigma$ (Def. 2.1.7). For a first-order dynamical system, let $R^+(x_0) \triangleq \{x \in R(x_0): x > x_0\}$ and $R^-(x_0) \triangleq \{x \in R(x_0): x < x_0\}$. For any subset A of a Euclidean space, let $\mathrm{int}\, A$ denote the set of interior points of A.

3.2.8 Lemma. Let S denote a first-order time-invariant dynamical system with U a closed subset of $\mathbb{R}^n$ and $\mathcal{U} = L^{\infty}_{loc}(\mathbb{R} \to U)$. Let x_0 be any element of Σ.

(a) If $x_1 \in \mathrm{int}\, R^+(x_0)$, then there exists at least one state trajectory $x(\cdot)|[0,T]$ from x_0 to x_1 with $\dot{x}(t) > 0$ for a.a. $t \in [0,T]$; moreover, $\int_{x_0}^{x_1} \bar{h}(x)dx$ exists in the extended sense, its value is either finite or $-\infty$, and

$$\inf_{\substack{x_0 \to x_1 \\ \dot{x}>0 \\ T>0}} \left\{\int_0^T p(x(t),u(t))dt\right\} = \int_{x_0}^{x_1} \bar{h}(x)dx \ . \qquad (3.2.8.1)$$

The expression on the left-hand side of (3.2.8.1) denotes that the infimum is taken over all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[0,T]$ of S from x_0 to x_1, where $T > 0$ is not fixed, subject to the restriction that $\dot{x}(t) > 0$ for a.a. $t \in [0,T]$.

(b) If $x_2 \in \text{int } R^-(x_0)$, then there exists at least one state trajectory $x(\cdot)|[0,T]$ from x_0 to x_2 with $\dot{x}(t) < 0$ for a.a. $t \in [0,T]$; moreover, $\int_{x_0}^{x_2} \underline{h}(x)dx$ exists in the extended sense, its value is either finite or $-\infty$, and

$$\inf_{\substack{x_0 \to x_2 \\ \dot{x}<0 \\ T>0}} \left\{\int_0^T p(x(t),u(t))dt\right\} = \int_{x_0}^{x_2} \underline{h}(x)dx \ . \qquad (3.2.8.2)$$

The expression on the left-hand side of (3.2.8.2) denotes that the infimum is taken over all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[0,T]$ of S from x_0 to x_2, where $T > 0$ is not fixed, subject to the restriction that $\dot{x}(t) < 0$ for a.a. $t \in [0,T]$.

The proof is given in the Appendix.

Remark. The integral on the right-hand side of (3.2.8.2) may be slightly confusing. Since $x_2 < x_0$, $\int_{x_0}^{x_2} \underline{h}(x)dx$ will be negative if $\underline{h}(\cdot)$ is positive on the interval $[x_2,x_0]$.

IV. Passivity

4.1 General Theory

The basic theory of passivity for nonlinear time-invariant n-ports has been exhaustively discussed in [4]. In this subsection we briefly review the theory, and make straightforward generalizations to time-varying n-ports.

4.1.1 Definition. Let S denote a dynamical system (Def. 2.1.1). The available energy for S, $E_A : \Sigma \times \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$, is defined by

$$E_A(x_0,t_0) \triangleq -\inf_{\substack{x_0\to \\ t_1\geq t_0}} \left\{\int_{t_0}^{t_1} p(x(t),u(t),t)dt\right\}$$

$$= \sup_{\substack{x_0\to \\ t_1\geq t_0}} \left\{-\int_{t_0}^{t_1} p(x(t),u(t),t)dt\right\} \qquad (4.1.1.1)$$

where the notation $\inf_{\substack{x_0\to \\ t_1\geq t_0}}$ (resp., $\sup_{\substack{x_0\to \\ t_1\geq t_0}}$) denotes that the infimum (resp., supremum) is taken over all $t_1 \geq t_0$ and all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ of S with $x(t_0) = x_0$.

For a time-invariant dynamical system, the available energy $E_A(x_0,t_0)$ is independent of the time variable t_0 and will usually be written $E_A(x_0)$ instead.

4.1.2 Definition. A dynamical system is passive if $E_A(x,t) < \infty$ for all $(x,t) \in \Sigma \times \mathbb{R}$, where $E_A(\cdot\,,\cdot)$ is the available energy for S. An n-port is passive if it has a passive dynamical system representation. Finally, a dynamical system or an n-port is active if it is not passive.

A complete justification of this definition, and a comparison of it with other passivity definitions which have appeared in the literature, is given in [4].

Note that the infimum which arises in the definition of the available energy (Def. 4.1.1) can be viewed as the "optimal value" function of an optimal control problem; however, the type of optimal control problem which arises in the theory of passivity for dynamical systems is not conventional. In virtually all of optimal control theory, it is assumed a priori that the optimal value function is finite-valued. This is usually guaranteed by requiring the "cost functional integrand," which in our case is the power input function $p(\cdot,\cdot,\cdot)$, to have some special property; e.g., one might deal only with cost functional integrands which are nonnegative. In the theory of passivity for dynamical systems, we cannot base our entire theory on the assumption that $E_A(\cdot\,,\cdot)$ is finite-valued; indeed, the question of whether $E_A(\cdot\,,\cdot)$ is finite-valued is precisely the question of whether the given dynamical system is passive.

Note that the passivity of a dynamical system S is equivalent to the existence of a (finite-valued) function $E : \Sigma \times \mathbb{R} \to \mathbb{R}^+$ such that

$$\int_{t_0}^{t_1} p(x(t),u(t),t)dt + E(x(t_0),t_0) \geq 0 \qquad (4.1.2.1)$$

for all $(t_1,t_0) \in \mathbb{R}_+^2$ and all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ of S. To see this, suppose first that $E_A(x,t) < \infty$ for all $(x,t) \in \Sigma \times \mathbb{R}$. Then (4.1.2.1) can be satisfied by choosing $E(\cdot\,,\cdot) = E_A(\cdot\,,\cdot)$. Now suppose that (4.1.2.1) is satisfied by a finite-valued function $E(\cdot\,,\cdot)$. Then $E_A(x,t) \leq E(x,t) < \infty$ for all $(x,t) \in \Sigma \times \mathbb{R}$.

4.1.3 Lemma. An n-port N is passive if and only if all dynamical system representations for N are passive.

The proof is given in the Appendix.

Lemma 4.1.3 shows that Def. 4.1.2 is truly consistent in the sense that it is based solely on the behavior of N as viewed at its ports — it does not depend upon which dynamical system one chooses to represent N. Not all definitions of passivity have this desirable property (see [4] for examples).

Before proceeding to the next lemma, we shall briefly discuss some of our notation and terminology. Let A be a subset of $\mathbb{R}^p$, with the topology of A the relative topology it inherits from $\mathbb{R}^p$. A function $w : A \to \mathbb{R}^q$ is defined to be C^0 if it is continuous. Now suppose that A is an open subset of $\mathbb{R}^p$, and let k be a positive integer. Then $w(\cdot)$ is defined to be C^k if it has continuous partial derivatives of all orders up to and including k; moreover, $w(\cdot)$ is defined to be C^∞ if it is C^k for every positive integer k. Now suppose $A \subset \mathbb{R}^p \times \mathbb{R}^r$. If we make the natural identification of $\mathbb{R}^p \times \mathbb{R}^r$ with $\mathbb{R}^{p+r}$, then there should be no confusion as to what is meant by statements such as "A is an open subset of $\mathbb{R}^p \times \mathbb{R}^r$," or "$w : A \to \mathbb{R}^q$ is C^k," etc.

The following is an obvious sufficient passivity condition for finite-order dynamical systems.

4.1.4 Lemma (Sufficient Condition for Passivity). Let S denote a finite-order dynamical system (Def. 2.1.17). Let (Suff. 4.1.4) denote the following condition:

(Suff. 4.1.4) There exists an open subset G of $\mathbb{R}^m \times \mathbb{R}$ with $\Sigma \times \mathbb{R} \subset G$ and a (nonnegative) C^1 function $\psi : G \to \mathbb{R}^+$ such that

$$p(x,u,t) \geq \langle \nabla_x \psi(x,t),\ f(x,u,t) \rangle + \frac{\partial \psi(x,t)}{\partial t} \qquad (4.1.4.1)$$

for all $(x,u,t) \in \Sigma \times U \times \mathbb{R}$.

If S satisfies (Suff. 4.1.4), then S is passive.

The proof is given in the Appendix. The first inequality in (4.1.4.3) of that proof motivates the following definition.

4.1.5 Definition. Let S denote a dynamical system. A function $E_I : \Sigma \times \mathbb{R} \to \mathbb{R}^+$ is an internal energy function for S if for all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ of S,

$$\int_{t_0}^{t_1} p(x(t),u(t),t)dt \geq E_I(x(t_1),t_1) - E_I(x(t_0),t_0) . \quad (4.1.5.1)$$

If S is time-invariant and has an internal energy function $E_I(\cdot,\cdot)$, we will always assume that $E_I(x,t)$ is independent of the time variable t, and we will usually write the function value as $E_I(x)$ instead.

4.1.6 Lemma. Let S denote a dynamical system. Then S is passive if and only if it has an internal energy function. Moreover, if S has an internal energy function $E_I(\cdot , \cdot)$, then $0 \leq E_A(\cdot , \cdot) \leq E_I(\cdot , \cdot)$ and the available energy $E_A(\cdot , \cdot)$ is itself an internal energy function for S.

The proof is given in the Appendix.

4.1.7 Definition. Let S denote a dynamical system. For each $x^* \in \Sigma$, define the required energy (from x^*), $E_{Rx^*} : \Sigma \times \mathbb{R} \to \mathbb{R}^e$, by

$$E_{Rx^*}(x_1,t_1) \triangleq \inf_{\substack{x^* \to x_1 \\ t_0 \leq t_1}} \left\{ \int_{t_0}^{t_1} p(x(t),u(t),t)dt \right\} \quad (4.1.7.1)$$

where the notation $\inf_{\substack{x^* \to x_1 \\ t_0 \leq t_1}}$ denotes that the infimum is taken over all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ of S from x^* to x_1, where $t_0 \leq t_1$ is not fixed.

Note that $E_{Rx^*}(x,t) < \infty$ for all $(x,t) \in \Sigma \times \mathbb{R}$ if and only if S is reachable from x^* (Def. 2.1.7).

In the time-invariant case, $E_{Rx^*}(x,t)$ is independent of the time variable t and is usually written $E_{Rx^*}(x)$ instead.

4.1.8 Lemma. Let S denote a passive dynamical system, and suppose that there exists a state $x^* \in \Sigma$ from which S is reachable (Def. 2.1.7).

(a) Let $E_I(\cdot\,,\cdot)$ be any internal energy function for S, and define $\bar{E}_I : \Sigma \times \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\bar{E}_I(x,t) \triangleq \sup_{\tau \leqslant t} E_I(x,\tau) \quad . \tag{4.1.8.1}$$

Then,

$$E_I(x,t) - \bar{E}_I(x^*,t) \leqslant E_{Rx^*}(x,t) \tag{4.1.8.2}$$

for all $(x,t) \in \Sigma \times \mathbb{R}$.

(b) Define $E_A^* : \Sigma \to \mathbb{R}^+ \cup \{\infty\}$ by

$$E_A^*(x) \triangleq \sup_{t \in \mathbb{R}} E_A(x,t) \tag{4.1.8.3}$$

and define $\Lambda_{x^*} : \Sigma \times \mathbb{R} \to \mathbb{R}^e$ by

$$\Lambda_{x^*}(x,t) \triangleq E_{Rx^*}(x,t) + E_A^*(x^*) \quad . \tag{4.1.8.4}$$

If $E_A^*(x^*) < \infty$, then $\Lambda_{x^*}(\cdot\,,\cdot)$ is an internal energy function for S.

The proof is given in the Appendix.

The following corollary, which applies in the special case when S is time-invariant, is an immediate consequence of Lemma 4.1.8.

4.1.9 Corollary. Let S denote a passive, time-invariant dynamical system. Suppose that there exists a state $x^* \in \Sigma$ from which S is reachable.

(a) Let $E_I(\cdot)$ be any internal energy function for S. Then,

$$E_I(x) - E_I(x^*) \leq E_{Rx^*}(x) \quad . \qquad (4.1.9.1)$$

(b) Define $\Lambda_{x^*} : \Sigma \to \mathbb{R}^+$ by

$$\Lambda_{x^*}(x) \triangleq E_{Rx^*}(x) + E_A(x^*) \quad . \qquad (4.1.9.2)$$

Then $\Lambda_{x^*}(\cdot)$ is an internal energy function for S.

4.1.10 <u>Definitions</u>. Let A denote a (not necessarily open) subset of $\mathbb{R}^p$, and let $w : A \to \mathbb{R}^q$. If $A \subset B \subset \mathbb{R}^p$ and if $\hat{w} : B \to \mathbb{R}^q$ is a function which satisfies $\hat{w}(x) = w(x)$ for all $x \in A$, then $\hat{w}(\cdot)$ (along with its domain B) is defined to be an <u>extension of $w(\cdot)$</u> (to the domain B). Let $1 \leq k \leq \infty$. The function $w(\cdot)$ is defined to be <u>differentiable</u> (resp., <u>C^k</u>) if there exists an open subset G of $\mathbb{R}^p$ with $A \subset G$ and a differentiable (resp., C^k) extension of $w(\cdot)$ to the domain G; this concept is sometimes expressed by saying that, "$w(\cdot)$ can be extended to a differentiable (resp., C^k) function with domain G." Note that such an extension, if it exists, if not necessarily unique; in fact, if $w_1(\cdot)$ and $w_2(\cdot)$ are differentiable extensions of $w : A \to \mathbb{R}^q$, then it is <u>not</u> necessarily true that $Dw_1(x) = Dw_2(x)$ for $x \in A$.

4.1.11 <u>Lemma</u>. Let S denote a (necessarily passive) finite-order dynamical system with a differentiable internal energy function $E_I(\cdot\,,\cdot)$. Let $\psi : G \to \mathbb{R}$ denote a differentiable extension of $E_I(\cdot\,,\cdot)$ (thus G is an open subset of $\mathbb{R}^m \times \mathbb{R}$, $\Sigma \times \mathbb{R} \subset G$, $\psi(\cdot\,,\cdot)$ is differentiable, and $\psi(x,t) = E_I(x,t)$ for all $(x,t) \in \Sigma \times \mathbb{R}$). Then

$$p(x,u,t) \geq \langle \nabla_x \psi(x,t),\ f(x,u,t,) \rangle + \frac{\partial \psi(x,t)}{\partial t} \qquad (4.1.11.1)$$

for all $(x,u,t) \in \Sigma \times U \times \mathbb{R}$.

The proof is given in the Appendix.

Remark. If $\hat{\psi} : \hat{G} \to \mathbb{R}$ is any other differentiable extension of $E_I(\cdot\ ,\cdot)$, then the proof of Lemma 4.1.11 in the Appendix shows that the quantity $\langle \nabla_x \hat{\psi}(x,t), f(x,u,t) \rangle + \frac{\partial \hat{\psi}(x,t)}{\partial t}$ equals the quantity $\langle \nabla_x \psi(x,t), f(x,u,t) \rangle + \frac{\partial \psi(x,t)}{\partial t}$. This follows because for any $(x_0, u_0, t_0) \in \Sigma \times U \times \mathbb{R}$, (4.1.11.2) shows that both quantities are equal to $\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \left(E_I(x(t_0+\Delta t), t_0+\Delta t) - E_I(x(t_0), t_0) \right)$ at $(x,u,t) = (x_0, u_0, t_0)$.

Note that Lemma 4.1.11 is not the converse of Lemma 4.1.4. Lemma 4.1.11 merely gives a necessary condition that a differentiable extension of an internal energy function must satisfy. We do not know how to identify the class of passive finite-order dynamical systems which possess a differentiable internal energy function (cf. Subsection 4.2).

4.2 The Smoothness Conjecture and a Counterexample

In this subsection we will show that a common conjecture concerning finite-order dynamical systems is false. For simplicity, we will assume that the dynamical systems under consideration are time-invariant.

Recall the definition of controllability, Def. 2.1.8. The following additional controllability concepts will also be of interest (cf. [6], [17]).

4.2.1 Definition (Local Controllability) Let S denote a finite-order time-invariant dynamical system. S is locally controllable if the following condition is satisfied: for each $x_0 \in \Sigma$, there exists $\delta_0(x_0) > 0$ such that if $0 \leq \delta \leq \delta_0(x_0)$ and if $\|x_1 - x_0\| \leq \delta$, then there exists an input-trajectory pair $\{u(\cdot), x(\cdot)\}|[0,t_1]$ of S from x_0 to x_1 with $\|x(t) - x_0\| \leq \delta$ for all $t \in [0,t_1]$.

4.2.2 Definition (Local Continuous Controllability) Let S denote a finite-order time-invariant dynamical system. S has the property of local continuous controllability if the following condition is satisfied: for each $x_0 \in \Sigma$ and each $\varepsilon > 0$, there exists $\delta_0(x_0,\varepsilon) > 0$ such that if $0 \leq \delta \leq \delta_0(x_0,\varepsilon)$ and if $\|x_1-x_0\| \leq \delta$, then there exists an input-trajectory pair $\{u(\cdot),x(\cdot)\}|[0,t_1]$ of S from x_0 to x_1 with $\|x(t)-x_0\| \leq \delta$ for all $t \in [0,t_1]$ and $|\int_0^{t_1} p(x(t),u(t))dt| < \varepsilon$.

Thus local continuous controllability is a special case of local controllability.

4.2.3 Definition Let S denote a finite-order time-invariant dynamical system. Suppose that $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are C^k functions (Def. 4.1.10) for some $1 \leq k \leq \infty$. Then S is called a C^k finite-order time-invariant dynamical system.

On more than one occasion the first author has come across the following conjecture (or some minor variation of it) in discussions with optimal control theorists. Also, the truth of the following conjecture seems to have been assumed in references [6] and [7], although it was not explicitly stated in either of those two references.

4.2.4 The Smoothness Conjecture Let S denote a C^∞ finite-order time-invariant dynamical system with the properties of controllability and local continuous controllability. Suppose further that $\Sigma = \mathbb{R}^m$, $U = \mathbb{R}^n$, and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to \mathbb{R}^n)$ (for some unspecified integers $m \geq 1$ and $n \geq 1$). Under these conditions, if S is passive, then S has at least one C^1 internal energy function.

If the Smoothness Conjecture were true, then Lemma 4.1.11 would show that (Suff. 4.1.4) is a necessary (as well as sufficient) passivity condition for the class of C^∞ dynamical systems described in 4.2.4. This would be a highly desirable result, since the dynamical systems described in 4.2.4 form a broad class of interesting dynamical systems, and the question of passivity for such dynamical systems would then reduce to the question of whether $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy the algebraic condition in (Suff. 4.1.4).

Let S denote a passive, controllable, time-invariant dynamical system. Lemma 4.1.6 shows that $E_A(\cdot)$ is always an internal energy function for S, and Corollary 4.1.9 shows that for any $x^* \in \Sigma$, the function $x \to \Lambda_{x^*}(x) \overset{\Delta}{=} E_{Rx^*}(x) + E_A(x^*)$ is also internal energy function for S. Moreover, $E_A(\cdot)$ and $E_{Rx^*}(\cdot)$ are bounds on the set of all possible internal energy functions in the following sense: If $E_I(\cdot)$ is any internal energy function for S, then

$$E_A(x) \leqslant E_I(x) \leqslant E_{Rx^*}(x) + E_I(x^*) \tag{4.2.4.1}$$

for all $x \in \Sigma$. Note that $-E_A(\cdot)$ and $E_{Rx^*}(\cdot)$ can be viewed as "optimum value" functions for an optimal control problem (cf. Defs. 4.1.1 and 4.1.7); hence, it is not surprising that optimal control theorists would have something to say about the properties of these functions. The following variation of the Smoothness Conjecture is the version that the first author has heard most often in his discussions with optimal control theorists.

4.2.5 The Smoothness Conjecture--Variation A Let S denote a C^k finite-order time-invariant dynamical system with the properties of controllability and local continuous controllability. Suppose further that $\Sigma = \mathbb{R}^m$, $U = \mathbb{R}^n$, and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to \mathbb{R}^n)$ (for some unspecified integers $m \geqslant 1$ and $n \geqslant 1$). Under these conditions, if S is passive, then the available energy and the required energy (from any state $x^* \in \Sigma$) are C^k functions.

We shall introduce one more version of the Smoothness Conjecture, as follows.

4.2.6 The Smoothness Conjecture--Variation B Let S denote a C^∞ finite-order time-invariant dynamical system with the properties of controllability and local continuous controllability. Suppose further that $\Sigma = \mathbb{R}^m$, $U = \mathbb{R}^n$, and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to \mathbb{R}^n)$ (for some unspecified integers $m \geqslant 1$ and $n \geqslant 1$). Under these conditions, if S is passive, then S has at least one differentiable internal energy function.

The only difference between the Smoothness Conjecture 4.2.4 and Variation B in 4.2.6 is the following: Variation B asserts merely that S has a differentiable internal energy function, as opposed to a continuously differentiable (C^1) internal energy function. Thus Variation B is weaker than (i.e., is implied by) the Smoothness Conjecture 4.2.4; moreover, it is clear that Variation B is weaker than Variation A as well.

We will show that the Smoothness Conjecture and its two variations are false. This will be done by producing a counterexample to Variation B.

4.2.7 Proposition The Smoothness Conjecture 4.2.4, its Variation A in 4.2.5, and its Variation B in 4.2.6, are all false.

Proof The proposition is proved by producing a counterexample to Variation B. Let $f:\mathbb{R} \to \mathbb{R}$ be defined by

$$f(u) \triangleq \begin{cases} \exp\left(\dfrac{-1}{(u-1)^2}\right), & \text{if } u > 1, \\ 0, & \text{if } 0 \leqslant u \leqslant 1, \\ -f(-u), & \text{if } u < 0. \end{cases} \tag{4.2.7.1}$$

It is well-known [18, p. 7, problem 18] (and it can be shown in a straightforward manner) that $f(\cdot)$ is C^∞. Define $\alpha:\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\alpha(x,u) \triangleq \begin{cases} -\left(\frac{1}{1+x^2}\right)\left(\frac{\exp(ux)-1}{\exp(ux)+1}\right)\left(\frac{\exp(ux)}{1+\exp(ux)}\right), & \text{if } u > 0, \\ \frac{-2\exp(-ux)}{1+\exp(-ux)}, & \text{if } u \leq 0. \end{cases} \quad (4.2.7.2)$$

Clearly, $\alpha(\cdot,\cdot)$ is C^∞ in $\{(x,u) \in \mathbb{R} \times \mathbb{R} : u \neq 0\}$; but it is apparently not differentiable along the line $\{(x,u) \in \mathbb{R} \times \mathbb{R} : u = 0\}$. Define $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$p(x,u) \triangleq \alpha(x,u)f(u). \quad (4.2.7.3)$$

Then $p(\cdot,\cdot)$ is C^∞, because $f(u)$ is zero for $u \in [-1,1]$.

Let S denote a first-order time-invariant dynamical system with the state equation

$$\dot{x} = f(u) \quad (4.2.7.4)$$

where $f(\cdot)$ is given in (4.2.7.1). Here $\Sigma = U = \mathbb{R}$ and $\mathcal{U}=L^\infty_{loc}(\mathbb{R} \to \mathbb{R})$. The power input function $p(\cdot,\cdot)$ for S is given in (4.2.7.3). Thus S is a C^∞ finite-order time-invariant dynamical system.

If $x_1 > x_0 \in \Sigma$, then any constant input $u_0 > 1$ will drive the state of S in a strictly monotone manner from x_0 to x_1 over some finite time interval $[t_0,t_1]$. The energy consumed by the input-trajectory pair $\{u_0,x(\cdot)\}|[t_0,t_1]$ of S from x_0 to x_1 is

$$\int_{t_0}^{t_1} p(x(t),u_0)dt = \int_{t_0}^{t_1} \alpha(x(t),u_0)f(u_0)dt$$

$$= \int_{t_0}^{t_1} \alpha(x(t),u_0)\dot{x}(t)dt = \int_{x_0}^{x_1} \alpha(x,u_0)dx . \quad (4.2.7.5)$$

Note that the integral on the right-hand side of (4.2.7.5) approaches zero as $x_1 \to x_0$. Similar comments hold if $x_1 < x_0$, the only difference being that we must choose a constant input value $u_0 < -1$ to drive the state of S from x_0 to x_1. This shows that S has the properties of controllability and local continuous controllability.

Before proceeding with the proof, we shall review the following facts from analysis. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function which is discontinuous at a point $x_0 \in \mathbb{R}$. Then $\beta(\cdot)$ is said to have a discontinuity of the <u>first kind</u> at x_0 if $\beta(x_0^+) \triangleq \lim_{\Delta x \to 0^+} \beta(x_0 + \Delta x)$ and $\beta(x_0^-) \triangleq \lim_{\Delta x \to 0^+} \beta(x_0 - \Delta x)$ both exist. Otherwise, the discontinuity is said to be of the <u>second kind</u> [19, p. 81]. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at <u>every</u> point $x \in \mathbb{R}$, and let $\psi' : \mathbb{R} \to \mathbb{R}$ denote the derivative of $\psi(\cdot)$. Then $\psi'(\cdot)$ is not necessarily continuous, but $\psi'(\cdot)$ <u>cannot</u> have any discontinuities of the first kind [19, p. 93, Corollary to Theorem 5.12].

It will be shown shortly that S is passive; first, however, it will be shown that S does not have any differentiable internal energy functions. In particular, it will be shown that S cannot have an internal energy function which is differentiable at $x = 0$.

From Lemma 4.1.11, we know that a differentiable internal energy function $\psi(\cdot)$, if it exists, must satisfy $\psi'(x)f(u) \leqslant \alpha(x,u)f(u)$ for all $(x,u) \in \mathbb{R} \times \mathbb{R}$. Hence, let us investigate the question of whether there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\beta(x)f(u) \leqslant \alpha(x,u)f(u) \qquad (4.2.7.6)$$

for all $(x,u) \in \mathbb{R} \times \mathbb{R}$. Inequality (4.2.7.6) is equivalent to the following two inequalities taken together:

$$\beta(x) \leqslant \alpha(x,u) \text{ for all } (x,u) \in \mathbb{R} \times (1,\infty)\ , \qquad (4.2.7.7a)$$

$$\beta(x) \geqslant \alpha(x,u) \text{ for all } (x,u) \in \mathbb{R} \times (-\infty,-1)\ . \qquad (4.2.7.7b)$$

Recall the functions $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ in Def. 3.2.4. In this case, these functions are given by

$$\overline{h}(x) = \inf_{u>1} \alpha(x,u) \qquad (4.2.7.8a)$$

$$\underline{h}(x) = \sup_{u<-1} \alpha(x,u). \qquad (4.2.7.8b)$$

It is easy to verify from (4.2.7.8) that

$$\overline{h}(x) = \begin{cases} \dfrac{-1}{1+x^2}, & \text{if } x > 0 , \\ 0, & \text{if } x \leqslant 0 . \end{cases} \qquad (4.2.7.9a)$$

$$\underline{h}(x) = \begin{cases} \dfrac{-2\exp(x)}{1+\exp(x)}, & \text{if } x \geqslant 0 , \\ 0, & \text{if } x < 0 . \end{cases} \qquad (4.2.7.9b)$$

Note that

$$\underline{h}(x) \leqslant \overline{h}(x) \qquad (4.2.7.10)$$

for all $x \in \mathbb{R}$. From (4.2.7.7), (4.2.7.8), and (4.2.7.10), it follows that there exists a function $\beta(\cdot)$ which satisfies (4.2.7.6); moreover, $\beta(\cdot)$ satisfies (4.2.7.6) if and only if

$$\underline{h}(x) \leqslant \beta(x) \leqslant \overline{h}(x) \qquad (4.2.7.11)$$

for all $x \in \mathbb{R}$.

From (4.2.7.9) and (4.2.7.11), it follows that $\beta(0^+) = -1$ and $\beta(0^-) = 0$; thus, any function $\beta(\cdot)$ which satisfies (4.2.7.6) must have a discontinuity of the first kind at $x = 0$. It follows that $\beta(\cdot)$ cannot be the derivative of a differentiable function $\psi : \mathbb{R} \to \mathbb{R}$.

To show that S is passive, define $\overline{H} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{H}(x) \triangleq \int_0^x \overline{h}(z)dz = \begin{cases} -\text{Arctan } x, & \text{if } x > 0 , \\ 0, & \text{if } x \leqslant 0 . \end{cases} \qquad (4.2.7.12)$$

Let $\{u(\cdot), x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S. Note that

$$\overline{h}(x(t))\dot{x}(t) = \frac{d}{dt}\overline{H}(x(t)) \text{ for a.a. } t \in [t_0,t_1]. \tag{4.2.7.13}$$

Since $\overline{h}(\cdot)$ is bounded on $\mathbf{R}$ and $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$, it follows that $t \to \overline{H}(x(t))$ is absolutely continuous on $[t_0,t_1]$[15, pp. 95-96, Theorem I.4.42]. Hence

$$\int_{t_0}^{t_1} \overline{h}(x(t))\dot{x}(t)dt = \int_{t_0}^{t_1} \frac{d}{dt}[\overline{H}(x(t))]dt$$

$$= \overline{H}(x(t_1)) - \overline{H}(x(t_0)) . \tag{4.2.7.14}$$

Since $\overline{h}(\cdot)$ satisfies (4.2.7.6), we have

$$\int_{t_0}^{t_1} p(x(t),u(t))dt \geq \int_{t_0}^{t_1} \overline{h}(x(t))\dot{x}(t)dt$$

$$= \overline{H}(x(t_1)) - \overline{H}(x(t_0))$$

$$\geq -\frac{\pi}{2} - \overline{H}(x(t_0)) . \tag{4.2.7.15}$$

It follows from (4.2.7.15) that $E_A(x) \leq \frac{\pi}{2} + \overline{H}(x) < \infty$ for all $x \in \mathbf{R}$, i.e., S is passive. Q.E.D.

4.3 A Less Restrictive Sufficient Condition

In this subsection we present a sufficient passivity condition for finite-order dynamical systems which is significantly less restrictive than the condition (Suff. 4.1.4) in Lemma 4.1.4. The results presented here follow along the lines of Stalford's work in optimal control theory [8].

For simplicity, the results will be stated for the time-invariant case. They can easily be extended to the time-varying case by using the procedure of letting one state variable be the time (see Def. 2.1.15 and Remarks 2.1.16).

4.3.1 Definition Let W be open in $\mathbb{R}^m$. A function $F : W \to \mathbb{R}^p$ is defined to be locally Lipschitzian if for each $x_0 \in W$, there exists a neighborhood of x_0, $N(x_0) \subset W$, and a constant $K(x_0) \geq 0$ such that

$$\| F(x') - F(x'') \| \leq K(x_0) \| x'-x'' \| \qquad (4.3.1.1)$$

for all $x', x'' \in N(x_0)$.

Note: a neighborhood of $x_0 \in \mathbb{R}^m$ is a set $N \subset \mathbb{R}^m$ with $x_0 \in \text{int } N$.

4.3.2 Definition Let $A \subset \mathbb{R}^m$. A function $F : A \to \mathbb{R}^p$ is defined to be locally Lipschitzian (resp., locally Lipschitzian and differentiable) if there exists an open set W of $\mathbb{R}^m$ with $A \subset W$ such that $F(\cdot)$ can be extended to a locally Lipschitzian (resp., locally Lipschitzian and differentiable) function with domain W.

4.3.3 Definition (Decomposition of a State Space) A decomposition D of a set $\Sigma \subset \mathbb{R}^m$ (which could be the state space of a finite-order dynamical system) is defined to be a countable collection of subsets of Σ whose union is Σ. This is written $D = \{\Sigma_j : j \in J\}$, where J is a countable index set and each Σ_j is called a member of the decomposition D.

Note: it is not required that the members of D be pairwise disjoint.

4.3.4 Definition. Let $\Sigma \subset \mathbb{R}^m$, and let $D = \{\Sigma_j : j \in J\}$ be a decomposition of Σ. A function $F : \Sigma \to \mathbb{R}^p$ is defined to be locally Lipschitzian (resp., locally Lipschitzian and differentiable) with respect to D if, for each $j \in J$, $F(\cdot)|\Sigma_j$ (i.e., the restriction of $F(\cdot)$ to Σ_j) is locally Lipschitzian (resp., locally Lipschitzian and differentiable); that is, there exists a collection $\{(W_j, F_j(\cdot)) : j \in J\}$ such that W_j is an open set in $\mathbb{R}^m$ containing Σ_j, $F_j : W_j \to \mathbb{R}^p$ is locally Lipschitzian (resp., locally Lipschitzian and differentiable), and $F_j(x) = F(x)$ for all

$x \in \Sigma_j$. The collection $\{(W_j, F_j(\cdot)) : j \in J\}$ (which is not necessarily unique) is said to be associated with F(·) and D.

4.3.5 Remarks on Integration Theory. Consider a function $\alpha : [a,b] \to \mathbb{R}$. If $\alpha(\cdot)$ is C^1, then the Fundamental Theorem of Calculus shows that

$$\alpha(t) - \alpha(a) = \int_a^t \frac{d\alpha(\tau)}{d\tau} d\tau \quad \text{for all} \quad t \in [a,b]. \quad (4.3.5.1)$$

More generally, in the Lebesgue theory of integration (4.3.5.1) holds if and only if $\alpha(\cdot)$ is absolutely continuous [16,p.178].

Suppose that $\alpha(\cdot)$ is continuous and differentiable almost everywhere with $t \to \frac{d\alpha(t)}{dt}$ integrable on [a,b]. Under these rather restrictive assumptions, the reader may be surprised to learn that (4.3.5.1) does not hold in general. There exists a function $\alpha(\cdot)$ defined on [0,1] (the "Cantor ternary function") which is continuous, monotone increasing with $\alpha(0) = 0$, $\alpha(1) = 1$, and differentiable almost everywhere with $\frac{d\alpha(t)}{dt} = 0$ wherever it exists [16,p.179]. Such a function does not satisfy (4.3.5.1) since $\int_0^1 \frac{d\alpha(t)}{dt} dt = 0$, yet $\alpha(1) - \alpha(0) = 1$.

It is hoped that the preceding remarks on integration theory will help the reader appreciate the significance of the following lemma.

4.3.6 The Monotonicity Lemma. Let $\Sigma \subset \mathbb{R}^m$, and let $D = \{\Sigma_j : j \in J\}$ be a (countable) decomposition of Σ. Let $\gamma : [t_0, t_1] \to \Sigma$ be absolutely continuous and let $h : [t_0, t_1] \to \mathbb{R}$ be integrable. Let $\psi : \Sigma \to \mathbb{R}$ be continuous on Σ and locally Lipschitzian with respect to D. Let $\{(W_j, \psi_j(\cdot)) : j \in J\}$ be a collection which is associated with $\psi(\cdot)$ and D. For $j \in J$, define $T_j \triangleq \{t \in [t_0, t_1] : \gamma(t) \in \Sigma_j\}$. Suppose that for each $j \in J$,

$$h(t) - \frac{d}{dt}(\psi_j \circ \gamma)(t) \geq 0 \quad \text{for a.a.} t \in T_j . \tag{4.3.6.1}$$

Define $\beta : [t_0, t_1] \to \mathbb{R}$ by

$$\beta(t) \triangleq \int_{t_0}^{t} h(\tau) d\tau - \psi(\gamma(t)) \quad . \tag{4.3.6.2}$$

Then $\beta(\cdot)$ is monotone increasing (i.e., $a < b$ implies $\beta(a) \leq \beta(b)$) and absolutely continuous.

The proof, which is quite involved, is given by Stalford [8, pp.56-59]. The fact that $\frac{d}{dt}(\psi_j \circ \gamma)(t)$ exists for a.a.$t \in T_j$ is also shown by Stalford in [8, p.55]. Note that the lemma is stated in terms of a particular collection $\{(W_j, \psi_j(\cdot)) : j \in J\}$ associated with $\psi(\cdot)$ and D. The hypothesis (4.3.6.1) will also be satisfied by every other collection $\{(\hat{W}_j, \hat{\psi}_j(\cdot)) : j \in J\}$ associated with $\psi(\cdot)$ and D, because it is shown in [8, p.55] that $\frac{d}{dt}(\psi_j \circ \gamma)(t) = \frac{d}{dt}(\hat{\psi}_j \circ \gamma)(t)$ for a.a.$t \in T_j$. Finally, if the continuous function $\psi(\cdot)$ is locally Lipschitzian and differentiable with respect to D, then (4.3.6.1) becomes

$$h(t) - \langle \nabla\psi_j(\gamma(t)), \dot{\gamma}(t) \rangle \geq 0 \quad \text{for a.a.} t \in T_j . \tag{4.3.6.1'}$$

We are now ready to present the main result of this subsection.

4.3.7 Theorem (Sufficient Condition for Passivity). Let S denote a finite-order time-invariant dynamical system. Let (Suff. 4.3.7) denote the following condition:

(Suff. 4.3.7) There exists a continuous (nonnegative) function $\psi : \Sigma \to \mathbb{R}^+$ along with a (countable) decomposition $D = \{\Sigma_j : j \in J\}$ of Σ such that $\psi(\cdot)$ is locally Lipschitzian and differentiable with respect to D, and a collection $\{(W_j, \psi_j(\cdot)) : j \in J\}$ associated with $\psi(\cdot)$ and D such that for each $j \in J$,

$$p(x,u) - \langle \nabla\psi_j(x),\ f(x,u)\rangle \geq 0 \qquad (4.3.7.1)$$

for all $(x,u) \in \Sigma_j \times U$.

If S satisfies (Suff. 4.3.7), then S is passive.

Proof. Suppose that S satisfies (Suff. 4.3.7). Let $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S. By Lemma 4.3.6,

$$\int_{t_0}^{t_1} p(x(t),u(t))dt - \psi(\gamma(t_1)) \geq -\psi(\gamma(t_0)) . \qquad (4.3.7.2)$$

Since $\psi(\cdot)$ is nonnegative, (4.3.7.2) shows that $\psi(\cdot)$ is an internal energy function for S. By Lemma 4.1.6, S is passive. Q.E.D.

4.3.8 Remark. Note that the counterexample presented in the proof of Proposition 4.2.7 satisfies (Suff. 4.3.7). We can choose $\psi(x) = \frac{\pi}{2} + \int_0^x \bar{h}(x)dx$ and $D = \{(-\infty,0],(0,\infty)\}$. This provides another proof of passivity for that counterexample.

4.4 Sufficient Conditions for Activity

In this subsection we apply the technical results from Section III to obtain several sufficient conditions for activity (recall from Def. 4.1.2 that activity is the negation of passivity). All but the first of these conditions are, to the authors' knowledge, entirely new to the literature. It should be noted that one can obtain necessary conditions for passivity by negating these sufficient conditions for activity.

The results of this subsection are stated for finite-order time-invariant dynamical systems; however, they can be extended to finite-order time-varying dynamical systems by using the procedure of letting one state variable be the time (see Def. 2.1.15 and Remarks 2.1.16).

4.4.1 Theorem (Sufficient Conditions for Activity). Let S denote a finite-order time-invariant dynamical system with U a closed subset of $\mathbb{R}^n$ and $\mathcal{U} = L^{\infty}_{loc}(\mathbb{R} \to U)$. Recall the function $h : \Sigma \times S^m \to \mathbb{R}^e$ in Def. 3.1.2.

(a) If $p(x_0,u_0) < 0$ at some point $(x_0,u_0) \in \Sigma \times U$ where $f(x_0,u_0) = 0$, then S is active.

(b) Let $\gamma:[0,T] \to \Sigma$ be a state trajectory of S with $\dot{\gamma}(t) \neq 0$ for a.a. $t \in [0,T]$. If

$$\int_0^T h\left(\gamma(t), \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\right) \|\dot{\gamma}(t)\| \, dt = -\infty , \quad (4.4.1.1)$$

then S is active.

(c) Let $\gamma:[0,T] \to \Sigma$ be a state trajectory of S with $\dot{\gamma}(t) \neq 0$ for a.a. $t \in [0,T]$. If the mapping $t \to h\left(\gamma(t), \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\right)$ is equal to $-\infty$ over some subset of $[0,T]$ with positive measure, then S is active.

(d) Let (x_0,u_0) be an element of $\Sigma \times U$ such that $f(x_0,u_0) \neq 0$. If the mapping $x \to h\left(x, \frac{f(x,u_0)}{\|f(x,u_0)\|}\right)$ is equal to $-\infty$ in some neighborhood $N(x_0) \subset \Sigma$ of x_0,[7] then S is active.

Proof. (a) Suppose that $p(x_0,u_0) < 0$ at some point $(x_0,u_0) \in \Sigma \times U$ where $f(x_0,u_0) = 0$. Then $\{u_0,x_0\}|[0,T]$ is a valid input-trajectory pair of S for all $T \geq 0$. The energy consumed by this input-trajectory pair is

$$\int_0^T p(x_0,u_0)dt = p(x_0,u_0)T \to -\infty \quad \text{as} \quad T \to \infty . \quad (4.4.1.2)$$

Therefore $E_A(x_0) = \infty$, i.e., S is active.

[7] The phrase "some neighborhood $N(x_0) \subset \Sigma$ of x_0" is intended to mean that $N(x_0)$ is some set whose interior relative to the topology of Σ contains x_0, where the topology of Σ is the relative topology that it inherits from $\mathbb{R}^m$.

(b) If the hypotheses in (b) are satisfied, then it follows immediately from assertion (b) of Lemma 3.1.7 that $E_A(\gamma(0)) = \infty$, i.e., S is active.

(c) Let $\gamma:[0,T] \to \Sigma$ be a state trajectory of S with $\dot{\gamma}(t) \neq 0$ for a.a. $t \in [0,T]$. Recall from assertion (a) of Lemma 3.1.7 that the integral

$$\int_0^T h\left(\gamma(t), \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\right) \|\dot{\gamma}(t)\| \, dt \tag{4.4.1.3}$$

exists in the extended sense, its value being either finite or $-\infty$. If the mapping $t \to h\left(\gamma(t), \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\right)$ is equal to $-\infty$ over some subset of $[0,T]$ with positive measure, then the integral (4.4.1.3) is equal to $-\infty$. By assertion (b) of the present theorem, S is active.

(d) Suppose that the hypotheses in (d) are satisfied. Let $\gamma_0(\cdot)|[0,\infty)$ denote the state trajectory of S with initial state $\gamma_0(0) = x_0$ generated by the constant input $u(t) \equiv u_0$. Since $f(x_0,u_0) \neq 0$, there exists $T > 0$ such that $\dot{\gamma}_0(t) \neq 0$ for all $t \in [0,T]$. Since $N(x_0)$ is a neighborhood of x_0, there exists a time $t_1 \in (0,T]$ such that $\gamma_0(t) \in N(x_0)$ for all $t \in [0,t_1]$. Thus the mapping $t \to h\left(\gamma_0(t), \frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)$ is equal to $-\infty$ for all $t \in [0,t_1]$. By assertion (c) of the present theorem, S is active. Q.E.D.

The following example illustrates the use of Theorem 4.4.1.

4.4.2 <u>Example</u>. Let S denote a second-order time-invariant dynamical system with $\Sigma = \mathbb{R}^2$, $U = \mathbb{R}$, and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to \mathbb{R})$. The state equation for S is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x,u) \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u\cos(x_1+x_2+u) \\ u\sin(x_1+x_2+u) \end{bmatrix} \qquad (4.4.2.1)$$

and the power input function for S is

$$p(x,u) \triangleq -(\sqrt{x_1^2+x_2^2}+u)^2 \exp[-u^2(x_1\sin(x_1+x_2+u) - x_2\cos(x_1+x_2+u))^2] \ . \qquad (4.4.2.2)$$

The problem of determining whether S is passive or active is nontrivial; indeed, the authors are not aware of any results in the published literature which could handle this problem. We are going to prove that S is active: this will be done by applying Theorem 4.4.1.

It is well-known that the mapping $z \to [\cos z, \sin z]'$ from $\mathbb{R}$ to $\mathbb{R}^2$ is periodic with period 2π and maps $\mathbb{R}$ onto the unit circle in $\mathbb{R}^2$. Thus for each fixed $[x_1,x_2]' \in \mathbb{R}^2$, the equation

$$\left(\sqrt{x_1^2+x_2^2}\cos(x_1+x_2+u) - x_1\right)^2 + \left(\sqrt{x_1^2+x_2^2}\sin(x_1+x_2+u) - x_2\right)^2 = 0 \qquad (4.4.2.3)$$

has a solution $u = u^* \in \mathbb{R}$ (which depends on x_1 and x_2); moreover, $u = u^* + 2k\pi$ is also a solution of (4.4.2.3) for every integer k.

Now let $x_0 \triangleq [x_1,x_2]' \neq [0,0]'$, and let u_1 be any positive solution of (4.4.2.3). For each integer $k \geq 2$, define

$$u_k \triangleq u_1 + 2(k-1)\pi \ . \qquad (4.4.2.4)$$

Since each u_k is positive and a solution of (4.4.2.3), it follows that

$$\frac{f(x_0,u_k)}{\|f(x_0,u_k)\|} = \frac{f(x_0,0)}{\|f(x_0,0)\|} \qquad (4.4.2.5)$$

for all $k \geq 1$; therefore

$$u_k \in \hat{U}\left(x_0, \frac{f(x_0,0)}{\|f(x_0,0)\|}\right) \tag{4.4.2.6}$$

for all $k \geq 1$ (cf. Def. 3.1.2). From the definition of $h(\cdot,\cdot)$, we have

$$h\left(x_0, \frac{f(x_0,0)}{\|f(x_0,0)\|}\right) \leq \inf_{k\geq 1} \frac{p(x_0,u_k)}{\|f(x_0,u_k)\|}$$

$$= \inf_{k\geq 1} \left\{ \frac{-(\sqrt{x_1^2+x_2^2}+u_k)^2}{\sqrt{x_1^2+x_2^2}+u_k} \right\} \quad \text{(since } u_k \text{ satisfies (4.4.2.3))}$$

$$= \inf_{k\geq 1} \left\{ -\left(\sqrt{x_1^2+x_2^2}+u_k\right) \right\} = -\infty \ . \tag{4.4.2.7}$$

Since x_0 was an arbitrary nonzero element of $\mathbb{R}^2$, we have shown that $h\left(x, \frac{f(x,0)}{\|f(x,0)\|}\right) = -\infty$ for all $x \in \mathbb{R}^2\setminus\{0\}$. By assertion (d) of Theorem 4.4.1, S is active.

4.5 First-Order Time-Invariant Dynamical Systems

In this subsection we will apply the technical results of Subsection 3.2 to obtain an easily verifiable necessary and sufficient passivity condition for first-order time-invariant dynamical systems. This condition has been previously published by the authors in [4]. We also present a new result for first-order time-invariant dynamical systems which deals with the question of the existence of an internal energy function with certain smoothness properties.

For simplicity, the results in this subsection will deal only with a first-order time-invariant dynamical system S which satisfies the following assumption.

4.5.1 Assumption. S is controllable, with Σ an open interval in $\mathbb{R}$. (We allow the possibility of Σ being an unbounded open interval; indeed, Σ could be $\mathbb{R}$ itself.)

The more general case is treated in [4].

Assumption 4.5.1 greatly simplifies the proofs of the following results, and most interesting examples will satisfy it. The following observations deal with the question of verifying Assumption 4.5.1.

4.5.2 Observations. Let S denote a first-order time-invariant dynamical system with state space Σ.

(a) If S is controllable, then Σ is an interval in $\mathbb{R}$ and for each $x \in \operatorname{int}\Sigma$, $U_x^+ \neq \phi$ and $U_x^- \neq \phi$.

(b) If $U_x^+ \neq \phi$ and $U_x^- \neq \phi$ for all $x \in \Sigma$, then Σ is open in $\mathbb{R}$.

(c) If Σ is an interval in $\mathbb{R}$ with $U_x^+ \neq \phi$ and $U_x^- \neq \phi$ for all $x \in \Sigma$, then Σ is an open interval in $\mathbb{R}$ and S is controllable.

Observations (a) and (b) are trivial. The assertion that S is controllable in observation (c) can be proved by considering two arbitrary states, $x_0, x_1 \in \Sigma$, and constructing a piecewise constant control, as in the proof of Lemma 3.2.8, which drives the dynamical system from x_0 to x_1.

4.5.3 Theorem. Let S denote a first-order time-invariant dynamical system with U a closed subset of $\mathbb{R}^n$ and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to U)$; moreover, suppose that S satisfies Assumption 4.5.1. Under these conditions, S is passive if and only if all three of the following conditions are satisfied:

(i) $p(x,u) \geq 0$ for every $(x,u) \in \Sigma \times U$ such that $f(x,u) = 0$.

(ii) $\underline{h}(x) \leq \bar{h}(x)$ for all $x \in \Sigma$.

(iii) There exists a (finite-valued) function $W : \Sigma \to \mathbb{R}^+$ such that, for each $x_0 \in \Sigma$,

$$\int_{x_0}^{x_1} \bar{h}(x)dx + W(x_0) \geqslant 0 \qquad \text{for every } x_1 \in (x_0,\infty) \cap \Sigma \,, \tag{4.5.3.1}$$

$$\int_{x_0}^{x_2} \underline{h}(x)dx + W(x_0) \geqslant 0 \qquad \text{for every } x_2 \in (-\infty,x_0) \cap \Sigma \,. \tag{4.5.3.2}$$

The proof is given in the Appendix.

4.5.4 Corollary. Under the hypotheses of Theorem 4.5.3, S is passive if and only if there exists a measurable function $\alpha : \Sigma \to \mathbb{R}$ which is bounded on every compact subset of Σ and a (finite-valued) function $E : \Sigma \to \mathbb{R}^+$ such that both of the following conditions are satisfied:

(i) $p(x,u) \geqslant \alpha(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$.

(ii) $\int_{x_0}^{x_1} \alpha(x)dx + E(x_0) \geqslant 0$ for all $x_0, x_1 \in \Sigma$.

The proof is given in the Appendix.

4.5.5 Corollary. Under the hypotheses of Theorem 4.5.3, S is passive if and only if it has an internal energy function $E_I(\cdot)$ which is differentiable almost everywhere in Σ and for which the mapping $x \to dE_I(x)/dx$ belongs to $L^\infty_{loc}(\Sigma \to \mathbb{R})$.

The proof is given in the Appendix.

Remark. The counterexample presented in the proof of Proposition 4.2.7 shows that we cannot strengthen the conclusion of Corollary 4.5.5 to say that $E_I(\cdot)$ is differentiable everywhere. That counterexample is passive and satisfies the hypotheses of Theorem 4.5.3, yet every internal energy function for it fails to be differentiable at $x = 0$.

V. Losslessness

5.1 General Theory

Reference [5] gives an exhaustive treatment of the general theory of losslessness for time-invariant n-ports, but there was no obvious extension of that theory to time-varying n-ports; indeed, it has been suggested that the problem of devising a consistent theory of losslessness which applies to both time-invariant and time-varying n-ports is quite formidable.[8] These authors do not share that view. It will be shown in this subsection that the general theory of losslessness for time-invariant n-ports presented in [5] can be extended in a straightforward manner to time-varying n-ports. The basic concepts required for this extension are the canonical time-invariant dynamical system (Def. 2.1.15) and the "canonical observable dynamical system," to be defined shortly.

We begin by discussing losslessness in the time-invariant case.

5.1.1 Definition. Let S denote a time-invariant dynamical system. Then S is lossless if the following condition is satisfied for any two input-trajectory pairs of S, $\{u_a(\cdot), x_a(\cdot)\}|\,[0, T_a]$ and $\{u_b(\cdot), x_b(\cdot)\}|[0, T_b]$, for which $x_a(0) = x_b(0)$ and $x_a(T_a) = x_b(T_b)$:

$$\int_0^{T_a} p(x_a(t), u_a(t))dt = \int_0^{T_b} p(x_b(t), u_b(t))dt. \qquad (5.1.1.1)$$

S is lossy if it is not lossless.

[8] Suggested by Professor John Wyatt, senior author of reference [5], in a private conversation.

Equivalently, the time-invariant dynamical system S is lossless if and only if the following condition is satisfied: if x_0 and x_1 are any two states of S, then all input-trajectory pairs of S from x_0 to x_1 (if there are any) consume the same energy (Def. 2.1.6).

5.1.2 Observation. Every time-invariant dynamical system $S = \{U, \mathcal{U}, \Sigma, \phi(\cdot,\cdot,\cdot,\cdot), Y, g(\cdot,\cdot), \omega(\cdot,\cdot)\}$ (Def. 2.1.1, Def. 2.1.14) is equivalent (Def. 2.1.10) to a lossless time-invariant dynamical system $S' = \{U, \mathcal{U}, \Sigma', \phi'(\cdot,\cdot,\cdot,\cdot), Y, g'(\cdot,\cdot), \omega(\cdot,\cdot)\}$. To see this, let $\Sigma' \triangleq \Sigma \times \mathbb{R}$ and define ϕ': $\mathbb{R}_+^2 \times \Sigma' \times \mathcal{U} \to \Sigma'$ by

$$\phi'(t,t_0,(x_0,e_0),u(\cdot)) \triangleq \left(\phi(t,t_0,x_0,u(\cdot)),\ e_0 + \int_{t_0}^{t} p(x(\tau),u(\tau))d\tau\right), \qquad (5.1.2.1)$$

where $x(t) \triangleq \phi(t,t_0,x_0,u(\cdot))$. Also, define g': $\Sigma' \times U \to Y$ by

$$g'((x,e),u) = g(x,u) . \qquad (5.1.2.2)$$

Clearly, S' is equivalent to S: S' is obtained from S simply by attaching an artificial state variable $e(\cdot)$ which is "unobservable" in the sense that the value of $e(\cdot)$ does not affect the output. Note that $e(\cdot)$ measures the change in the input energy; hence, it is obvious that S' is lossless by Def. 5.1.1.

The point of Observation 5.1.2 is the following: Def. 5.1.1 is merely a formal definition which says <u>absolutely nothing</u> about the external behavior of S. It is only when we restrict ourselves to <u>observable</u> dynamical systems that the concept of losslessness becomes meaningful in terms of external behavior. This is the content of the following two lemmas.

5.1.3 Lemma. Let S_1 and S_2 be equivalent time-invariant dynamical systems, with S_1 observable (Def. 2.1.13) and S_2 input-observable (Def. 2.1.11). Under these conditions, if S_1 is lossless, then S_2 is lossless.

The proof, which is straightforward but rather lengthy, is given in [5, Theorem 3.1].[9]

5.1.4 Lemma. Let S_1 and S_2 be equivalent, observable, time-invariant dynamical systems. Under these conditions, S_1 is lossless if and only if S_2 is lossless.

The proof is immediate from Lemma 5.1.3. Lemma 5.1.4 motivates the following definition.

5.1.5 Definition. A time-invariant n-port N is lossless if it has a lossless, observable, time-invariant dynamical system representation. N is lossy if it is not lossless.

A complete justification of this definition, and a comparison of it with other losslessness definitions which have appeared in the literature, is given in [5].

Let N denote a time-invariant n-port, and let S denote a given input-distinguishable time-invariant dynamical system representation for N (such an S exists by Assumption 2.2.4). Suppose that we know whether S is lossless or lossy, observable or not observable. Then what can we conclude about N? If S is lossless and observable, then N is lossless: this is simply Def. 5.1.5. If S is lossy, then N is lossy (regardless of whether S is observable): this is a consequence of Def. 5.1.5 and Lemma 5.1.3. The one remaining possibility is that S is lossless but not observable; in this case, no immediate conclusion

[9] Note that our terminology is slightly different from that in reference [5].

can be drawn. What our theory needs is a canonical method for reducing the state space of a time-invariant dynamical system which eliminates the "unobservable modes." Such a procedure could, in principle, be applied to our problem in order to obtain a canonical observable time-invariant dynamical system S_0 which is equivalent to S. The question of whether N is lossless would then reduce to the question of whether S_0 is lossless. The following lemma states that such a canonical observable time-invariant dynamical system exists.

5.1.6 Lemma. Let S denote an input-distinguishable time-invariant dynamical system. Then there exists an observable time-invariant dynamical system S_0 which is equivalent to S.

The proof is given in the Appendix.

5.1.7 Definition. Let S denote an input-distinguishable time-invariant dynamical system. Let S_0 denote the observable time-invariant dynamical system equivalent to S which is constructed in the proof of Lemma 5.1.6 (see Appendix). Then S_0 is called the canonical observable dynamical system equivalent to S.

The theoretical results we have established so far allow us to prove the following lemma, which is analogous to Lemma 4.1.3 in the general theory of passivity. Note that such a lemma was not possible in the framework of reference [5], because the theory in that reference was restricted to finite-order dynamical systems (as opposed to the abstract dynamical systems of Def. 2.1.1).

5.1.8 Lemma. A time-invariant n-port N is lossless if and only if all input-distinguishable time-invariant dynamical system representations for N are lossless.

The proof is given in the Appendix.

We are now ready to begin extending our theory of losslessness to the time-varying case.

5.1.9 Definition. Let S denote a (possibly time-varying) dynamical system, and let S* denote the canonical time-invariant dynamical system associated with S (Def. 2.1.15). Then S is wide-sense lossless if S* is lossless (Def. 5.1.1).

Equivalently, S is wide-sense lossless if and only if the following condition is satisfied for any two input-trajectory pairs of S, $\{u_a(\cdot),x_a(\cdot)\}|[T_0, T_1]$ and $\{u_b(\cdot),x_b(\cdot)\}|[T_0, T_1]$, for which $x_a(T_0) = x_b(T_0)$ and $x_a(T_1) = x_b(T_1)$:

$$\int_{T_0}^{T_1} p(x_a(t),u_a(t),t)dt = \int_{T_0}^{T_1} p(x_b(t),u_b(t),t)dt \ . \qquad (5.1.9.1)$$

The concept of wide-sense losslessness is essentially the concept of "losslessness" proposed in references [20] and [21] for certain classes of time-varying dynamical systems. When applied to the class of time-invariant dynamical systems, the only difference between Def. 5.1.1 and Def. 5.1.9 is that the time intervals associated with the two input-trajectory pairs are the same in the latter case. Thus losslessness implies wide-sense losslessness for time-invariant dynamical systems. The following example shows, however, that the converse does not hold.

5.1.10 Example. Consider the first-order time-invariant dynamical system S with the following state and output equations

$$\dot{x} = ux \qquad (5.1.10.1)$$

$$y = x \qquad (5.1.10.2)$$

where $\Sigma = (0,\infty)$, $U = \mathbb{R}$, and $\mathcal{U} = L^1_{loc}(\mathbb{R} \to \mathbb{R})$ (note that the state trajectories of S are given by $x(t) = x(t_0)\exp(\int_{t_0}^{t} u(\tau)d\tau))$. The port voltage and port current are given by

$$v = y \tag{5.1.10.3}$$

$$i = \frac{1}{y} \ . \tag{5.1.10.4}$$

It is clear that S is wide-sense lossless, because the energy consumed by an input-trajectory pair $\{u(\cdot),x(\cdot)\}|[T_0, T_1]$ is

$$\int_{T_0}^{T_1} v(t)i(t)dt = \int_{T_0}^{T_1} x(t)\left(\frac{1}{x(t)}\right)dt = \int_{T_0}^{T_1} dt = T_1 - T_0 \ ; \tag{5.1.10.5}$$

however, S is <u>not</u> lossless. To see this, let $x_0 \in \Sigma$. Then $\{0,x_0\}|[0,T]$ is a valid input-trajectory pair of S from x_0 to x_0 for every $T \geq 0$. In particular, if $T_a \neq T_b$, then the energy consumed by $\{0,x_0\}|[0,T_a]$ is <u>not</u> equal to the energy consumed by $\{0,x_0\}|[0,T_b]$.

Example 5.1.10 shows that a theory of time-varying losslessness based on wide-sense losslessness alone would be inadequate, since the time-varying theory must be consistent with the time-invariant theory. For this reason we introduce the following more restrictive version of time-varying losslessness.

<u>5.1.11 Definitions</u>. Let S denote an input-distinguishable (possibly time-varying) dynamical system. Let S* denote the canonical time-invariant dynamical system associated with S (Def. 2.1.15). Finally, let S_0^* denote the canonical observable dynamical system equivalent to S* (Def. 5.1.7). We shall call S_0^* the <u>canonical observable time-invariant dynamical system associated with S</u>, and we shall say that S is <u>narrow-sense lossless</u> if S_0^* is lossless (Def. 5.1.1).

<u>5.1.12 Lemma</u>. Let S denote an input-distinguishable (possibly time-varying) dynamical system. Let S* denote the canonical time-invariant dynamical system associated with S (Def. 2.1.15).

(a) Let S^*_{oo} denote any given dynamical system which is observable, time-invariant, and equivalent to S^*. Then S is narrow-sense lossless if and only if S^*_{oo} is lossless.

(b) Suppose that S^* is observable. Under these conditions, S is narrow-sense lossless if and only if S is wide-sense lossless.

The proof is given in the Appendix.

The point of assertion (a) is the following: suppose that by one method or another we can obtain an observable time-invariant dynamical system S^*_{oo} which is equivalent to S^*; then we need only check whether S^*_{oo} is lossless in order to find out whether S is narrow-sense lossless. In other words, it is not necessary to check the canonical observable time-invariant dynamical system S^*_o; indeed, it is enough to check S^* if S^* happens to be observable.

Assertion (b) of Lemma 5.1.12 merely says that narrow-sense losslessness and wide-sense losslessness are equivalent concepts for that class of dynamical systems for which S^* is observable.

Now let S denote a lossy, time-invariant input-distinguishable dynamical system. Let S' denote the lossless time-invariant dynamical system equivalent to S which is constructed in Observation 5.1.2. By construction, S' is lossless; but Lemma 5.1.3 and Def. 5.1.11 show that S' is <u>not</u> narrow-sense lossless. This proves that losslessness and narrow-sense losslessness are not equivalent concepts for time-invariant dynamical systems. However, every time-invariant input-distinguishable dynamical system which is narrow-sense lossless is lossless as well. This assertion, and several others, is stated in the following lemma.

5.1.13 Lemma.

(a) For the class of (possibly time-varying) input-distinguishable dynamical systems, narrow-sense losslessness ⇒ wide-sense losslessness.

(b) For the class of time-invariant input-distinguishable dynamical systems, narrow-sense losslessness ⇒ lossnessness ⇒ wide-sense losslessness.

The proof is given in the Appendix.

5.1.14 Lemma. A time-invariant n-port *N* is lossless (Def. 5.1.5) if and only if it has an input-distinguishable time-invariant dynamical system representation which is narrow-sense lossless (Def. 5.1.11).

The proof is given in the Appendix.

Lemma 5.1.14 gives an alternative definition of losslessness for time-invariant n-ports, and it has an immediate generalization to time-varying n-ports. For this reason we have adopted the following definition.

5.1.15 Definition. A (possibly time-varying) n-port *N* is lossless if it has an input-distinguishable dynamical system representation which is narrow-sense lossless (Def. 5.1.11). *N* is lossy if it is not lossless.

Remarks. As shown by Lemma 5.1.14, Def. 5.1.15 is consistent with Def. 5.1.5 when applied to time-invariant n-ports. Note that in the time-varying theory, losslessness has been defined only for n-ports: we have not assigned (and will not assign) any meaning to the term "losslessness" as applied to time-varying dynamical systems.

In analogy with Lemmas 4.1.3 and 5.1.8, we have the following result in the theory of time-varying losslessness.

5.1.16 Lemma. A (possibly time-varying) n-port N is lossless if and only if all input-distinguishable dynamical system representations for N are narrow-sense lossless.

The proof is given in the Appendix.

5.1.17 Observation. The following lossyness criterion is an immediate consequence of Lemma 5.1.16 and assertion (a) of Lemma 5.1.13: A (possibly time-varying) n-port is lossy if it has at least one input-distinguishable dynamical system representation which is not wide-sense lossless.

In practice, it may be very hard to verify whether a given time-varying n-port is lossless, but the lossyness criterion of Observation 5.1.17 gives us a relatively easy method of proving that certain time-varying n-ports are not lossless. The use of this criterion is illustrated in the following example.

5.1.18 Example. A linear time-varying 1-port capacitor is a 1-port with a linear time-varying first-order dynamical system representation S characterized by state and output equations of the form

$$\dot{q}(t) = i(t) \qquad (5.1.18.1a)$$

$$v(t) = K(t)q(t) \qquad (5.1.18.1b)$$

where $\Sigma = U = Y = \mathbb{R}$, $\mathcal{U} = L^1_{loc}(\mathbb{R} \to \mathbb{R})$, and $K: \mathbb{R} \to \mathbb{R}$ is a nonconstant C^1 function. We will prove that S is not wide-sense lossless.

Since $K(\cdot)$ is C^1 and not constant, there exists $t_0 \in \mathbb{R}$ such that $\left.\frac{dK(t)}{dt}\right|_{t=t_0} \neq 0$. By the continuity of $t \to \frac{dK(t)}{dt}$, there exist times $T_0, T_1 \in \mathbb{R}$ with $T_0 < t_0 < T_1$ such that the mapping $t \to \frac{dK(t)}{dt}$ is sign-definite on $[T_0, T_1]$.

Define $\{i(t),q(t)\} \triangleq \left\{\frac{2\pi}{T_1-T_0}\cos\left[\frac{2\pi}{T_1-T_0}(t-T_0)\right], \sin\left[\frac{2\pi}{T_1-T_0}(t-T_0)\right]\right\}$ for $t \in \mathbb{R}$. Then $\{i(\cdot),q(\cdot)\}|[0,\infty)$ is a valid input-trajectory pair of S -- note that $q(T_0) = q(T_1) = 0$. The energy consumed by $\{i(\cdot),q(\cdot)\}|[T_0,T_1]$ is

$$\int_{T_0}^{T_1} v(t)i(t)dt = \int_{T_0}^{T_1} K(t)q(t)\dot{q}(t)dt$$

$$= \int_{T_0}^{T_1} K(t)\frac{d}{dt}\left[\frac{1}{2}q^2(t)\right]dt$$

$$= K(T_1)\left[\frac{1}{2}q^2(T_1)\right] - K(T_0)\left[\frac{1}{2}q^2(T_0)\right] - \int_{T_0}^{T_1}\frac{1}{2}q^2(t)\frac{dK(t)}{dt}dt$$

$$= -\int_{T_0}^{T_1}\frac{1}{2}q^2(t)\frac{dK(t)}{dt}dt \neq 0 . \tag{5.1.18.2}$$

The fact that the last integral in (5.1.18.2) is nonzero follows since $q(\cdot)$ is nonzero on (T_0,T_1) and $t \to \frac{dK(t)}{dt}$ is sign-definite on $[T_0,T_1]$. Thus $\{i(\cdot),q(\cdot)\}|[T_0,T_1]$ consumes a nonzero quantity of energy.

Now define $\{\hat{i}(t),\hat{q}(t)\} \triangleq \{0,0\}$ for $t \in \mathbb{R}$. Note that $\{\hat{i}(\cdot), \hat{q}(\cdot)\}|[T_0,T_1]$ is a valid input-trajectory pair of S which consumes zero energy. Since $\hat{q}(T_0) = q(T_0)$ and $\hat{q}(T_1) = q(T_1)$, it follows that S is not wide-sense lossless.

What Example 5.1.18 shows is the following: a linear time-varying 1-port capacitor is <u>not</u> lossless by our definition. It should be noted that this classification of a linear time-varying 1-port capacitor is the same classification that Penfield [9, p. 43] has argued for.

The next example is included simply to illustrate how one might apply our theory in order to verify that a given time-varying n-port is lossless.

5.1.19 Example. Let N denote a time-varying n-port with a first-order dynamical system representation S characterized by state and output equations of the form

$$\dot{x}(t) = x(t)(v(t) + t)v(t) \tag{5.1.19.1}$$

$$i(t) = x^2(t)(v(t) + t) \tag{5.1.19.2}$$

where $\Sigma = (0,\infty)$, $U=Y=\mathbb{R}$, and $\mathcal{U} = L^2_{loc}(\mathbb{R} \to \mathbb{R})$ (note that the state trajectories of (5.1.19.1) are given by $x(t) = x(t_0)\exp\left(\int_{t_0}^{t} (v(\tau) + \tau)\cdot v(\tau)d\tau\right)$). Let S* denote the canonical time-invariant dynamical system associated with S (Def. 2.1.15). Then S* is characterized by state and output equations of the form

$$(\dot{x},\dot{\sigma}) = (x(v + \sigma)v,\ 1) \tag{5.1.19.3}$$

$$i = x^2(v + \sigma) \tag{5.1.19.4}$$

where $\Sigma^* = (0,\infty) \times \mathbb{R}$, $U^*=Y^*=\mathbb{R}$, and $\mathcal{U}^* = L^2_{loc}(\mathbb{R} \to \mathbb{R})$. Let $\{v(\cdot), (x(\cdot), \sigma(\cdot))\}|[0,T]$ denote an input-trajectory pair of S*, with $i(\cdot)|[0,T]$ the corresponding current (output). Then

$$\begin{aligned}\int_0^T v(t)i(t)dt &= \int_0^T x^2(t)(v(t) + \sigma(t))v(t)dt \\ &= \int_0^T x(t)\dot{x}(t)dt = \int_0^T \frac{d}{dt}\left(\frac{1}{2}x^2(t)\right)dt \\ &= \frac{1}{2}x^2(0) - \frac{1}{2}x^2(T).\end{aligned} \tag{5.1.19.5}$$

Thus S^* is lossless, which means that S is wide-sense lossless (Def. 5.1.9). We claim that S^* is observable as well. To prove this claim, consider two initial states $(x_1,\sigma_1),(x_2,\sigma_2) \in \Sigma^*$ with $(x_1,\sigma_1) \neq (x_2,\sigma_2)$ (i.e., either $x_1 \neq x_2$, $\sigma_1 \neq \sigma_2$, or both). If $x_1 = x_2$, then $\sigma_1 \neq \sigma_2$ and we define $v(t) \triangleq v_a$ for $t \geq 0$, where v_a is any constant. If $x_1 \neq x_2$, define $v(t) = v_b$ for $t \geq 0$, where $v_b \neq (x_2^2\sigma_2 - x_1^2\sigma_1)/(x_1^2 - x_2^2)$. Let $(x'(\cdot),\sigma'(\cdot))|[0,\infty)$ (resp., $(x''(\cdot),\sigma''(\cdot))|[0,\infty)$) denote the state trajectory of S^* with $(x'(0),\sigma'(0)) = (x_1,\sigma_1)$ (resp., $(x''(0),\sigma''(0)) = (x_2,\sigma_2)$) which is generated by $v(\cdot)$. From the definition of $v(\cdot)$, it is easy to verify that

$$x'(0)^2(v(0) + \sigma'(0)) \neq x''(0)^2(v(0) + \sigma''(0)) \; ; \qquad (5.1.19.6)$$

in other words, the corresponding currents (outputs) are unequal at $t = 0$. This shows that S^* is observable. By Lemma 5.1.12, S is narrow-sense lossless; hence, N is lossless (Def. 5.1.15).

To sum up the theory of time-varying losslessness, we have shown that the question of whether a time-varying n-port N is lossless is the question of whether any given input-distinguishable dynamical system representation S for N is narrow-sense lossless. This in turn reduces to the question of whether any given observable time-invariant dynamical system equivalent to S^* is lossless. Thus the question of whether a time-varying n-port is lossless reduces to the question of whether an associated time-invariant dynamical system is lossless; so for most of the rest of this section, we shall deal with the question of losslessness only for <u>time-invariant</u> dynamical systems.

The following is an obvious sufficient losslessness condition for finite-order time-invariant dynamical systems.

5.1.20 Lemma. (Sufficient Condition for Losslessness) Let S denote a finite-order time-invariant dynamical system. Let (Suff. 5.1.20) denote the following condition:

(Suff. 5.1.20) There exists an open subset G of $\mathbb{R}^m$ with $\Sigma \subset G$ and a C^1 function ϕ: $G \to \mathbb{R}$ such that

$$p(x,u) = \langle \nabla\phi(x), f(x,u) \rangle \tag{5.1.20.1}$$

for all $(x,u) \in \Sigma \times U$.

If S satisfies (Suff. 5.1.20), then S is lossless.

The proof is given in the Appendix.

An immediate corollary to Lemma 5.1.20 is the following: a finite-order time-varying dynamical system is wide-sense lossless if there exists an open subset G of $\mathbb{R}^m \times \mathbb{R}$ with $\Sigma \times \mathbb{R} \subset G$ and a C^1 function ϕ: $G \to \mathbb{R}$ such that

$$p(x,u,t) = \langle \nabla_x\phi(x,t), f(x,u,t) \rangle + \frac{\partial\phi(x,t)}{\partial t} \tag{5.1.20.3}$$

for all $(x,u,t) \in \Sigma \times U \times \mathbb{R}$.

Eq. (5.1.20.2) in the Appendix motivates the following definition.

5.1.21 Definition. Let S denote a time-invariant dynamical system. A function ϕ: $\Sigma \to \mathbb{R}$ is called a conservative potential energy function for S if

$$\int_0^T p(x(t),u(t))dt = \phi(x(T)) - \phi(x(0)) \tag{5.1.21.1}$$

for all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[0,\infty)$ of S and for all $T \geq 0$.

The following observation is trivial. A formal proof of the second half is given in [5, Lemma 2.2].

5.1.22 Observation. Let S denote a time-invariant dynamical system. If S has a conservative potential energy function, then S is lossless. If S is lossless and reachable from some state $x^* \in \Sigma$ (Def. 2.1.7), then S has a conservative potential energy function.

Thus losslessness and the existence of a conservative potential energy function are equivalent concepts for the class of reachable time-invariant dynamical systems.

5.1.23 Lemma. Let S denote a (necessarily lossless) finite-order time-invariant dynamical system with a differentiable conservative potential energy function $\phi(\cdot)$. Let $\psi: G \to \mathbb{R}$ denote a differentiable extension of $\phi(\cdot)$ (thus G is an open subset of $\mathbb{R}^m$, $\Sigma \subset G$, $\psi(\cdot)$ is differentiable, and $\psi(x) = \phi(x)$ for $x \in \Sigma$). Then

$$p(x,u) = \langle \nabla\psi(x), f(x,u) \rangle \qquad (5.1.23.1)$$

for all $(x,u) \in \Sigma \times U$.

The proof is given in the Appendix.

Note that Lemma 5.1.23 is not the converse of Lemma 5.1.20. Lemma 5.1.23 merely gives a necessary condition that a differentiable extension of a conservative potential energy function must satisfy. We do not claim that the existence of a differentiable conservative potential energy function is a necessary condition for losslessness. Even if we restrict ourselves to controllable C^∞ finite-order dynamical systems, the existence of a differentiable conservative potential energy function is still not known to be a necessary condition for losslessness.

5.2 A Less Restrictive Sufficient Condition

Let S denote a time-invariant finite-order dynamical system. From Lemma 5.1.20, we know that (Suff. 5.1.20) is a sufficient losslessness condition for S; as discussed in Subsection 5.1, however, we do not know whether (Suff. 5.1.20) is a necessary condition for S to be lossless. It is therefore of interest to obtain a sufficient losslessness condition for S which is not as restrictive as (Suff. 5.1.20). In this subsection we shall apply Stalford's [8] results from optimal control theory to obtain such a condition (cf. Subsection 4.3).

The following result is a special case of the Monotonicity Lemma 4.3.6.

5.2.1 Lemma. Let $\Sigma \subset \mathbb{R}^m$, and let $D = \{\Sigma_j: j \in J\}$ be a (countable) decomposition of Σ. Let $\gamma: [t_0,t_1] \to \Sigma$ be absolutely continuous and let $h: [t_0,t_1] \to \mathbb{R}$ be integrable. Let $\phi: \Sigma \to \mathbb{R}$ be continuous and locally Lipschitzian with respect to D. Let $\{(W_j,\phi_j(\cdot)): j \in J\}$ be a collection which is associated with $\phi(\cdot)$ and D. Define $T_j \triangleq \{t \in [t_0,t_1]: \gamma(t) \in \Sigma_j\}$ for $j \in J$. Suppose that for each $j \in J$,

$$h(t) - \frac{d}{dt}(\phi_j \circ \gamma)(t) = 0 \quad \text{for a.a.} \, t \in T_j. \tag{5.2.1.1}$$

Define $\beta: [t_0,t_1] \to \mathbb{R}$ by

$$\beta(t) \triangleq \int_{t_0}^{t} h(\tau)d\tau - \phi(\gamma(t)). \tag{5.2.1.2}$$

Then $\beta(\cdot)$ is constant, i.e., $\beta(t) = \beta(t_0)$ for all $t \in [t_0,t_1]$.

The proof is given in the Appendix.

5.2.2 Theorem (Sufficient Condition for Losslessness). Let S denote a finite-order time-invariant dynamical system. Let (Suff. 5.2.2) denote the following condition:

(Suff. 5.2.2) There exists a continuous function $\phi: \Sigma \to \mathbb{R}$ along with a (countable) decomposition $D = \{\Sigma_j: j \in J\}$ of Σ such that $\phi(\cdot)$ is locally Lipschitzian and differentiable with respect to D, and a collection $\{(W_j, \phi_j(\cdot)): j \in J\}$ associated with $\phi(\cdot)$ and D such that for every $j \in J$,

$$p(x,u) - \left\langle \nabla\phi_j(x),\ f(x,u) \right\rangle = 0 \qquad (5.2.2.1)$$

for all $(x,u) \in \Sigma_j \times U$.

If S satisfies (Suff. 5.2.2), then S is lossless.

Proof. Suppose that S satisfies (Suff. 5.2.2). Let $\{u(\cdot), x(\cdot)\}|[0, T]$ be any input-trajectory pair of S. By Lemma 5.2.1,

$$\int_0^T p(x(t),u(t))dt - \phi(x(T)) = - \phi(x(0)). \qquad (5.2.2.2)$$

This shows that $\phi(\cdot)$ is a conservative potential energy function for S (Def. 5.1.21); hence, S is lossless (Observation 5.1.22). Q.E.D.

5.3 A Necessary Condition

In this subsection we apply the technical results from Section III to obtain a necessary losslessness condition for finite-order time-invariant dynamical systems. This condition is, to the authors' knowledge, entirely new to the literature.

5.3.1 Theorem (Necessary Condition for Losslessness). Let S denote a finite-order time-invariant dynamical system with U a closed subset of $\mathbb{R}^n$ and $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to U)$. Let (Nec. 5.3.1) denote the following condition:

(Nec. 5.3.1) There exists a function $h: \Sigma \times S^m \to \mathbb{R}^e$ with the following property: for each $u \in U$, there exists a dense subset $\Sigma_d \subset \Sigma$ (depending on u) such that

$$p(x,u) = h\left(x, \frac{f(x,u)}{\|f(x,u)\|}\right)\|f(x,u)\| \qquad (5.3.1.1)$$

for all $x \in \Sigma_d$. (If $(x,u) \in \Sigma \times U$ is such that $f(x,u) = 0$, then the right-hand side of (5.3.1.1) is set equal to zero, and (5.3.1.1) holds at all such (x,u).)

If S is lossless, then S satisfies (Nec. 5.3.1).

The proof is given in the Appendix.

5.3.2 Remark. Since $(x,u) \to p(x,u)$ and $(x,u) \to \|f(x,u)\|$ are continuous functions, it might be conjectured that (Nec. 5.3.1) implies that (5.3.1.1) holds at every $(x,u) \in \Sigma \times U$. The authors have not been able to prove this conjecture, because the function $h(\cdot,\cdot)$ in Def. 3.1.2 has no *a priori* continuity properties.

The following example illustrates the application of Theorem 5.3.1.

5.3.3 Example. Consider the second-order time-invariant dynamical system S in Example 4.4.2. The authors are not aware of any results in the published literature which could be directly applied to the problem of determining whether S is lossless or lossy; however, we can prove that S is lossy by a simple application of Theorem 5.3.1. Since it has been shown in Example 4.4.2 that

$h\left(x, \frac{f(x,0)}{\|f(x,0)\|}\right) = -\infty$ for all $x \in \mathbb{R}^2\backslash\{0\}$, S cannot satisfy (Nec. 5.3.1); therefore, S is lossy.

5.4 First-Order Time-Invariant Dynamical Systems

In this subsection we will present an easily-verifiable necessary and sufficient losslessness condition for first-order time-invariant dynamical systems. This condition has also been published by the authors in reference [5].

5.4.1 Definition. Let S denote a finite-order time-invariant dynamical system. A state x_0 is called a singular state (of S) if $f(x_0,u) = 0$ for all $u \in U$. A state which is not singular is called a nonsingular state (of S).

Note that all states of S are nonsingular if S is controllable.

5.4.2 Theorem. Let S denote a first-order time-invariant dynamical system. Suppose that U is a closed subset of $\mathbb{R}^n$ and that $\mathcal{U} = L^\infty_{loc}(\mathbb{R} \to U)$. Under these conditions, S is lossless if and only if there exists a function $h: \Sigma \to \mathbb{R}$ (which is necessarily continuous at each nonsingular state) such that $p(x,u) = h(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$.

The proof is given in the Appendix.

Let S denote a first-order time-invariant dynamical system which is lossless and controllable. In this case, Σ will be an interval in $\mathbb{R}$ (Observation 4.5.2). By Theorem 5.4.2, there exists a continuous function $h: \Sigma \to \mathbb{R}$ such that $p(x,u) = h(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$. Define $\phi: \Sigma \to \mathbb{R}$ by $\phi(x) \triangleq \int_{x_0}^{x} h(x')dx'$, where x_0 is any fixed point in Σ. Then $\phi(\cdot)$ is a C^1 function which satisfies $p(x,u) = \frac{d\phi(x)}{dx} f(x,u)$ for all $(x,u) \in \Sigma \times U$. If the interval Σ is closed at either endpoint, then $\frac{d\phi(x)}{dx}$ at that endpoint is taken to be the appropriate one-sided derivative. Obviously, if Σ is not open, then $\phi(\cdot)$ can be extended to a C^1 function whose domain G is an open interval containing Σ. Hence, (Suff. 5.1.20) is both a necessary and sufficient losslessness condition for controllable first-order time-invariant dynamical systems (cf. the discussion at the end of Subsection 5.1).

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Appendix

Proof of Lemma 2.1.12

Let $(t_1, t_0, x_0) \in \mathbb{R}_+^2 \times \Sigma$, and let $u_a(\cdot)$ and $u_b(\cdot)$ denote any two inputs such that

$$\{v_a(t),\ i_a(t)\} = \{v_b(t),\ i_b(t)\} \quad \text{for all } t \in [t_0, t_1], \qquad (2.1.12.1)$$

where $\{v_a(\cdot),\ i_a(\cdot)\}|[t_0,\infty)$ and $\{v_b(\cdot),\ i_b(\cdot)\}|[t_0,\infty)$ are the voltage-current pairs of S with common initial state x_0 which are generated by $u_a(\cdot)$ and $u_b(\cdot)$, respectively. Let $y_a(\cdot)|[t_0,\infty)$ and $y_b(\cdot)|\ [t_0,\infty)$ denote the outputs of S with common initial state x_0 which are generated by $u_a(\cdot)$ and $u_b(\cdot)$, respectively. Then (2.1.12.1) gives

$$\omega(u_a(t),\ y_a(t)) = \omega(u_b(t),\ y_b(t)) \text{ for all } t \in [t_0, t_1]. \quad (2.1.12.2)$$

If $\omega(\cdot,\cdot)$ is injective, then (2.1.12.2) shows that $u_a(t) = u_b(t)$ for all $t \in [t_0, t_1]$. Q.E.D.

Proof of Lemma 3.1.7

Assertions (a) and (b) will be proved simultaneously.

Let $\{u_0(\cdot), \gamma_0(\cdot)\}|[0,T_0]$ be an input-trajectory pair of S with $\dot{\gamma}_0(t) \neq 0$ for a.a. $t \in [0,T_0]$. Let $J \subset [0,T_0]$ denote the subset of all $t \in [0,T_0]$ where $\dot{\gamma}_0(t)$ exists, is nonzero, and satisfies $\dot{\gamma}_0(t) = f(\gamma_0(t), u_0(t))$; hence,[10] $[0,T_0]\backslash J$ is a set of measure zero. Define $\alpha_0\colon J \to S^m$ by

$$\alpha_0(t) \triangleq \dot{\gamma}_0(t)/\|\dot{\gamma}_0(t)\| . \qquad (3.1.7.2)$$

(Note that $\alpha_0(\cdot)$ is measurable.)

Let $K_{T_0} \subset U$ be a compact set such that $\dot{\gamma}_0(t) \in f(\gamma_0(t), K_{T_0})$

[10] For two sets A and B, the notation A\B denotes those elements of A (if any) which are not in B.

for a.a. $t \in [0,T_0]$ (the existence of such a compact set follows since $u_0(\cdot) \in \mathcal{U} = L^{\infty}_{loc}(\mathbb{R} \to U)$ and U is a closed subset of $\mathbb{R}^n$). For each $q \geq 0$, define $B_q \triangleq \{u \in \mathbb{R}^n : \|u\| \leq q\}$. Then since K_{T_0} is compact, there exists an integer $N \geq 1$ such that $K_{T_0} \subset B_q$ for all $q \geq N$. For each integer $q \geq N$, define $C_q: J \longrightarrow P(U)$ by

$$C_q(t) \triangleq \{u \in B_q \cap U: f(\gamma_0(t),u) - \alpha_0(t)\|f(\gamma_0(t),u)\| = 0\}$$
$$\cap \{u \in B_q \cap U: \|f(\gamma_0(t),u)\| - \|\dot{\gamma}_0(t)\|/q \geq 0\} . \qquad (3.1.7.3)$$

Note that $C_q(t)$ is compact for all $t \in J$ because it is a closed subset of the compact set $B_q \cap U$, and it is nonempty for all $t \in J$ because $u_0(t) \in C_q(t)$ for all $t \in J$. Also, note that

$$C_q(t) \subset C_{q+1}(t) \subset \cdots \subset \hat{U}(\gamma_0(t),\alpha_0(t)) \qquad (3.1.7.4)$$

and

$$\hat{U}(\gamma_0(t),\alpha_0(t)) = \bigcup_{q=N}^{\infty} C_q(t) \qquad (3.1.7.5)$$

for all $t \in J$.

For each $q \geq N$, define $h_q: J \longrightarrow \mathbb{R}$ by

$$h_q(t) \triangleq \min\left\{\frac{p(\gamma_0(t),u)}{\|f(\gamma_0(t),u)\|}: u \in C_q(t)\right\} . \qquad (3.1.7.6)$$

It follows from (3.1.7.4) and (3.1.7.5) that

$$h_q(t) \geq h_{q+1}(t) \geq \cdots \geq h(\gamma_0(t),\alpha_0(t)) \qquad (3.1.7.7)$$

and

$$h(\gamma_0(t),\alpha_0(t)) = \lim_{q \to \infty} h_q(t) \qquad (3.1.7.8)$$

for all $t \in J$. The next step is to show that $h_q(\cdot)$ is measurable. It then follows from (3.1.7.8) that the function $t \longrightarrow h(\gamma_0(t),\alpha_0(t))$ is measurable as well, since the limit of a sequence of measurable functions is measurable [12, p. 67, Theorem 20].

For any pair of integers $q \geq N$ and $i \geq 1$, define $G_{qi}: J \to P(U)$ by

$$G_{qi}(t) \triangleq \{u \in B_q \cap U: \|(f(\gamma_0(t),u) - \alpha_0(t)\|f(\gamma_0(t),u)\|)\| < \tfrac{1}{i}\}$$
$$\cap \{u \in B_q \cap U: \|f(\gamma_0(t),u)\| - \|\dot{\gamma}_0(t)\|(\tfrac{1}{q} - \tfrac{1}{q+i}) > 0\}. \quad (3.1.7.9)$$

Note that $G_{qi}(t)$ is open in the relative topology that $B_q \cap U$ inherits from $\mathbb{R}^n$; also,

$$G_{q1}(t) \supset G_{q2}(t) \supset \cdots \supset C_q(t) \quad (3.1.7.10)$$

and

$$C_q(t) = \bigcap_{i=1}^{\infty} G_{qi}(t) \quad (3.1.7.11)$$

for all $t \in J$.

For any $u \in \mathbb{R}^n$ and any $A \subset \mathbb{R}^n$, define

$$d(u,A) \triangleq \inf_{w \in A} \|u-w\| . \quad (3.1.7.12)$$

We have the following obvious inequalities:

$$d(u_0,A) \leq d(u_1,A) + \|u_1-u_0\|$$
$$d(u_1,A) \leq d(u_0,A) + \|u_1-u_0\| ;$$

therefore,

$$-\|u_1-u_0\| \leq d(u_1,A) - d(u_0,A) \leq \|u_1-u_0\| . \quad (3.1.7.13)$$

Inequality (3.1.7.13) shows that the mapping $u \to d(u,A)$ is continuous.

We are going to prove that for every $q \geq N$ and every $t \in J$,

$$\sup\{d(u,C_q(t)): u \in G_{qi}(t)\} \to 0 \text{ as } i \to \infty . \quad (3.1.7.14)$$

To see this, suppose on the contrary that there exists $\delta_0 > 0$, $q_0 \geq N$, and $t_0 \in J$ with the following property: for every $i \geq 1$, there exists $u_i \in G_{q_0 i}(t)$ such that $d(u_i, C_{q_0}(t_0)) > \delta_0$. Since u_i belongs to the compact set $B_{q_0} \cap U$ for all i, there exists a subsequence, still denoted u_i, such that $u_i \rightarrow \bar{u} \in B_{q_0} \cap U$. From the continuity of $f(\cdot,\cdot)$, it follows that $\bar{u} \in C_{q_0}(t_0)$; thus, $d(\bar{u}, C_{q_0}(t_0)) = 0$. But this contradicts the continuity of the mapping $u \rightarrow d(u, C_{q_0}(t_0))$; hence, (3.1.7.14) must hold for every $q \geq N$ and every $t \in J$.

Next we are going to show that

$$\lim_{i\to\infty}\left[\inf_{u \in G_{qi}(t)}\left(\frac{p(\gamma_0(t),u)}{\|f(\gamma_0(t),u)\|}\right)\right] = h_q(t) \qquad (3.1.7.15)$$

for all $q \geq N$ and all $t \in J$. First note that the quantity in the square brackets on the left-hand side of (3.1.7.15) is monotone increasing as a function of i; hence, the limit as $i \to \infty$ exists in the extended sense. Furthermore, the left-hand side of (3.1.7.15) is a priori less than the right-hand side. Fix $q_0 \geq N$, $t_0 \in J$, and let $\varepsilon > 0$ be given. It remains to show that there exists an integer $k \geq 1$ (depending on q_0 and t_0) such that

$$\inf_{u \in G_{q_0 k}(t_0)}\left(\frac{p(\gamma_0(t),u)}{\|f(\gamma_0(t_0),u)\|}\right) \geq h_{q_0}(t_0) - \varepsilon . \qquad (3.1.7.16)$$

Let $\overline{G_{q_0 1}(t_0)}$ denote the closure of $G_{q_0 1}(t_0)$ in $\mathbb{R}^n$. Since $\overline{G_{q_0 1}(t_0)}$ is a closed subset of the compact set $B_{q_0} \cap U$, the former is compact as well. Moreover, $f(\gamma_0(t_0),u) \neq 0$ for $u \in \overline{G_{q_0 1}(t_0)}$; therefore, $u \rightarrow r(u) \triangleq p(\gamma_0(t_0),u)/\|f(\gamma_0(t),u)\|$ is a continuous mapping when u is restricted to $\overline{G_{q_0 1}(t_0)}$. By continuity and compactness, there exists a $\delta > 0$ such that if u_1 and u_2 are any elements of $\overline{G_{q_0 1}(t_0)}$ such that $\|u_1 - u_2\| < \delta$, then $|r(u_1) - r(u_2)| < \varepsilon$.

By (3.1.7.14), we can choose k so large that

$$\sup\{d(u,C_{q_0}(t_0)) : u \in G_{q_0 k}(t_0)\} < \delta . \qquad (3.1.7.17)$$

For such a choice of k, (3.1.7.16) will be satisfied.

Since $B_q \cap U \subset \mathbb{R}^n$ and $\mathbb{R}^n$ is a separable metric space, it follows that $B_q \cap U$ is a separable metric space in the topology that it inherits from $\mathbb{R}^n$ [12, p. 138, Proposition 13]. In other words, there exists a countable dense subset $\{\hat{u}_j\}_{j=1}^{\infty} \subset B_q \cap U$. Since $G_{qi}(t)$ is open in the relative topology of $B_q \cap U$, it follows that $\{\hat{u}_j\}$ is also dense in $G_{qi}(t)$.

For each $\hat{u}_k \in \{\hat{u}_j\}_{j=1}^{\infty}$, each $q \geq N$, and each $i \geq 1$, define $h_{qi}(\cdot,\hat{u}_k) : J \to \mathbb{R}^e$ by

$$h_{qi}(t,\hat{u}_k) \triangleq \begin{cases} \dfrac{p(\gamma_0(t),\hat{u}_k)}{\|f(\gamma_0(t),\hat{u}_k)\|} , & \text{if } \hat{u}_k \in G_{qi}(t) , \\ \infty , & \text{if } \hat{u}_k \notin G_{qi}(t) . \end{cases} \qquad (3.1.7.18)$$

Since $\{\hat{u}_j\}$ is dense in $G_{qi}(t)$, it follows that

$$\inf_{k \in \mathbb{N}} \{h_{qi}(t,\hat{u}_k)\} = \inf_{u \in G_{qi}(t)} \left\{ \frac{p(\gamma_0(t),u)}{\|f(\gamma_0(t),u)\|} \right\} \qquad (3.1.7.19)$$

where $\mathbb{N} \triangleq \{1,2,3,\ldots\}$, the set of natural numbers. From (3.1.7.15) and (3.1.7.19), we have

$$h_q(t) = \lim_{i \to \infty} [\inf_{k \in \mathbb{N}} h_{qi}(t,\hat{u}_k)] . \qquad (3.1.7.20)$$

If we can show that $h_{qi}(\cdot,\hat{u}_k) : J \to \mathbb{R}^e$ is measurable, it then follows from (3.1.7.20) that $h_q(\cdot)$ is measurable, since infimums and limits of (countable) sequences of measurable functions are measurable

[12, p. 67, Theorem 20]. Thus, let $\beta \in \mathbb{R}$. Then

$$\begin{aligned}
&\{t \in J:\ h_{qi}(t,\hat{u}_k) \leq \beta\} \\
&\quad = \{t \in J:\ \|(f(\gamma_0(t),\hat{u}_k) - \alpha_0(t)\|f(\gamma_0(t),\hat{u}_k)\|)\| < \tfrac{1}{i}\} \\
&\qquad \cap \{t \in J:\ \|f(\gamma_0(t),\hat{u}_k)\| - \|\dot{\gamma}_0(t)\|(\tfrac{1}{q} - \tfrac{1}{q+i}) > 0\} \\
&\qquad \cap \{t \in J:\ p(\gamma_0(t),\hat{u}_k) - \beta\|f(\gamma_0(t),\hat{u}_k)\| \leq 0\}\ .
\end{aligned} \tag{3.1.7.21}$$

Each of the sets on the right-hand side of (3.1.7.21) is a measurable subset of J; hence, their intersection is a measurable subset of J. Since β was an arbitrary element of $\mathbb{R}$, it follows that $h_{qi}(\cdot,\hat{u}_k)$ is measurable [12, p. 66]; thus, $h_q(\cdot)$ is measurable.

For each $q \geq N$, define $D_q\colon J \to P(U)$ by

$$D_q(t) \triangleq \{u \in C_q(t):\ \frac{p(\gamma_0(t),u)}{\|f(\gamma_0(t),u)\|} = h_q(t)\}\ . \tag{3.1.7.22}$$

It follows from the definition of $h_q(t)$, (3.1.7.6), that $D_q(t)$ is nonempty for all $t \in J$; moreover, $D_q(t)$ is compact for all $t \in J$ because it is a closed subset of the compact set $C_q(t)$. Let $t \in J$, and choose $u_q(t) \in D_q(t)$ as follows: let $u_q(t)$ be the element of $D_q(t)$ with the smallest first component; if more than one such element exists, choose the one among these with the smallest second component; if more than one such element exists, choose the one among these with the smallest third component, etc. The process eventually terminates, since $u_q(t)$ has only n components. In this way we define a unique $u_q(t) \in D_q(t)$ for each $t \in J$. Note that the function $u_q(\cdot)$ satisfies the relation

$$h_q(t) = \frac{p(\gamma_0(t),u_q(t))}{\|f(\gamma_0(t),u_q(t))\|} \tag{3.1.7.23}$$

for all $t \in J$. The next step is to show that $u_q(\cdot)$ is an admissible input over the time interval $[0,T_0]$; i.e., $u_q(\cdot) \in L^\infty([0,T_0] \to U)$.

The function $u_q(\cdot)$ obviously has the required boundedness properties, since $u_q(t)$ is an element of the compact set $B_q \cap U$ for all $t \in J$. It remains to show that $u_q: J \to U$ is measurable (i.e., each component of $u_q(\cdot)$ is measurable). This will be shown by induction. Let $u_q^r(\cdot)$, $r = 1,\ldots,n$, denote the r-th component of $u_q(\cdot)$. Assume that $u_q^r(\cdot)$ is measurable for $r = 1,\ldots,(s-1)$ (if $s = 1$ we are assuming nothing at all). For each integer $\ell \geq 1$, let E_ℓ be a closed subset of J such that $u_q^r(\cdot)$, $r = 1,\ldots,(s-1)$, $h_q(\cdot)$, and $\dot{\gamma}_0(\cdot)$ are continuous when restricted to E_ℓ and

$$m(J \backslash E_\ell) < \frac{1}{2^\ell}, \qquad (3.1.7.24)$$

where m is the Lebesgue measure restricted to $J \subset \mathbb{R}$. The existence of such a set is a general property of measurable functions [15, p. 70, Theorem I.4.19]. Let $\beta \in \mathbb{R}$. We claim that the set $\{t \in E_\ell: u_q^s(t) \leq \beta\}$ is a closed (and therefore measurable) subset of E_ℓ. To prove this claim, let $\{t_k\}_{k=1}^\infty$ be a sequence in E_ℓ such that $t_k \to \bar{t} \in E_\ell$ and $u_q^s(t_k) \leq \beta$. Now $u_q(t_k) \in B_q \cap U$ for all k. Since $B_q \cap U$ is compact, it follows that there exists a subsequence, still denoted t_k, such that

$$u_q(t_k) \to \bar{u} \in B_q \cap U . \qquad (3.1.7.25)$$

Since $f(\cdot,\cdot)$ and $\gamma_0(\cdot)$ are continuous and $\dot{\gamma}_0(\cdot)$ is continuous when restricted to E_ℓ, it follows that

$$\|f(\gamma_0(t_k),u_q(t_k))\| - \frac{\|\dot{\gamma}_0(t_k)\|}{q} \xrightarrow[k\to\infty]{} \|f(\gamma_0(\bar{t}),\bar{u})\| - \frac{\|\dot{\gamma}_0(\bar{t})\|}{q} \qquad (3.1.7.26)$$

and

$$f(\gamma_0(t_k),u_q(t_k)) - \alpha_0(t_k)\|f(\gamma_0(t_k),u_q(t_k))\| \xrightarrow[k\to\infty]{} f(\gamma_0(\bar{t}),\bar{u}) - \alpha_0(\bar{t})\|f(\gamma_0(\bar{t}),\bar{u})\| \ . \tag{3.1.7.27}$$

Since $u_q(t_k) \in C_q(t_k)$, it follows that the left-hand sides of (3.1.7.26) and (3.1.7.27) are nonnegative and zero, respectively; hence

$$\|f(\gamma_0(\bar{t}),\bar{u})\| - \frac{\|\dot{\gamma}_0(\bar{t})\|}{q} \geq 0 \tag{3.1.7.28}$$

$$f(\gamma_0(\bar{t}),\bar{u}) - \alpha_0(\bar{t})\|f(\gamma_0(\bar{t}),\bar{u})\| = 0 \ , \tag{3.1.7.29}$$

and therefore

$$\bar{u} \in C_q(\bar{t}) \ . \tag{3.1.7.30}$$

From (3.1.7.28), $f(\gamma_0(\bar{t}),\bar{u}) \neq 0$; thus from the continuity of $f(\cdot,\cdot)$, $p(\cdot,\cdot)$, and $\gamma_0(\cdot)$,

$$\frac{p(\gamma_0(t_k),u_q(t_k))}{\|f(\gamma_0(t_k),u_q(t_k))\|} \xrightarrow[k\to\infty]{} \frac{p(\gamma_0(\bar{t}),\bar{u})}{\|f(\gamma_0(\bar{t}),\bar{u})\|} \ . \tag{3.1.7.31}$$

By the given continuity on E_ℓ,

$$h_q(t_k) \longrightarrow h_q(\bar{t}) \tag{3.1.7.32}$$

and

$$u_q^r(t_k) \longrightarrow u_q^r(\bar{t}) = \bar{u}^r \ , \quad r = 1,\ldots,(s-1) \ . \tag{3.1.7.33}$$

Combining (3.1.7.23), (3.1.7.31), and (3.1.7.32), we obtain

$$h_q(\bar{t}) = \frac{p(\gamma_0(\bar{t}),\bar{u})}{\|f(\gamma_0(\bar{t}),\bar{u})\|} \ . \tag{3.1.7.34}$$

Recall that $u_q^s(t_k) \leq \beta$. It follows that $\bar{u}^s \leq \beta$. Now from (3.1.7.33), (3.1.7.34), and the definition of $u_q(\bar{t})$, it follows that $u_q^s(\bar{t}) \le \bar{u}^s \le \beta$. This shows that the set $\{t \in E_\ell : u_q^s(t) \le \beta\}$ is a closed subset of E_ℓ. Hence $u_q^s(\cdot)$ is measurable on E_ℓ. By induction, $u_q(\cdot)$ is measurable on E_ℓ. It follows that $u_q(\cdot)$ is measurable on $\bigcup_{\ell=1}^{\infty} E_\ell$. By (3.1.7.24), $\bigcup_{\ell=1}^{\infty} E_\ell$ differs from J (and hence $[0,T_0]$) by a set of measure zero. Therefore $u_q(\cdot)$ is measurable on $[0,T_0]$ (the fact that $u_q(\cdot)$ has not been defined on $[0,T_0]\backslash J$ is not significant because $[0,T_0]\backslash J$ is a set of measure zero). This completes the proof that $u_q(\cdot) \in L^\infty([0,T_0] \to U)$.

The next step in the proof is to show that the function $t \to h_q(t)\|\dot{\gamma}_0(t)\|$ is an element of $L^1([0,T_0] \to \mathbb{R})$. Note that

$$\begin{aligned} |h_q(t)\|\dot{\gamma}_0(t)\|| &= |p(\gamma_0(t),u_q(t))| \frac{\|\dot{\gamma}_0(t)\|}{\|f(\gamma_0(t),u_q(t))\|} \\ &\le q|p(\gamma_0(t),u_q(t))| \end{aligned} \qquad (3.1.7.35)$$

for all $t \in J$. The last inequality follows since $u_q(t) \in C_q(t)$, and hence, $\|f(\gamma_0(t),u_q(t))\| - \|\dot{\gamma}_0(t)\|/q \ge 0$ for all $t \in J$. Now $p(\gamma_0(t),u_q(t)) \in p(\gamma_0([0,T_0]),B_q \cap U)$ for all $t \in J$. Recall that $B_q \cap U$ is compact; and since $\gamma_0(\cdot)$ is continuous, $\gamma_0([0,T_0])$ is also compact. By Tychonoff's theorem [2, p. 166], or by more elementary means, $\gamma_0([0,T_0]) \times (B_q \cap U)$ is a compact subset of $\Sigma \times U$. Since $p(\cdot,\cdot)$ is continuous, $p(\gamma_0([0,T_0]),B_q \cap U)$ is compact. Hence the function $t \to h_q(t)\|\dot{\gamma}_0(t)\|$ is an element of $L^\infty([0,T_0] \to \mathbb{R}) \subset L^1([0,T_0] \to \mathbb{R})$.

From (3.1.7.7), (3.1.7.8), and the Monotone Convergence Theorem [12, p. 84], it follows that

$$\int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt = \lim_{q\to\infty}\int_0^{T_0} h_q(t)\|\dot{\gamma}_0(t)\|dt \;. \tag{3.1.7.36}$$

We have shown that the integral $\int_0^{T_0} h_q(t)\|\dot{\gamma}_0(t)\|dt$ exists and is finite. Since $h_q(t)\|\dot{\gamma}_0(t)\|$ decreases monotonically to $h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|$ as $q \to \infty$, it follows that the integral on the left-hand side of (3.1.7.36) exists in the extended sense, and its value is either finite or $-\infty$. Let $\sigma\colon [0,T_1] \longrightarrow [0,T_0]$ be an absolutely continuous function such that $\sigma(0) = 0$, $\sigma(T_1) = T_0$, and $\dot{\sigma}(t) > 0$ for a.a. $t \in [0,T_1]$. Let $\gamma_1(\cdot) \triangleq (\gamma_0\circ\sigma)(\cdot)$. Then $\gamma_1(\cdot)|[0,T_1]$ is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$. Note that

$$\begin{aligned}
&\int_0^{T_1} h\left(\gamma_1(\tau),\frac{\dot{\gamma}_1(\tau)}{\|\dot{\gamma}_1(\tau)\|}\right)\|\dot{\gamma}_1(\tau)\|d\tau \\
&= \int_0^{T_1} h\left(\gamma_0(\sigma(\tau)),\frac{\dot{\gamma}_0(\sigma(\tau))}{\|\dot{\gamma}_0(\sigma(\tau))\|}\right)\|\dot{\gamma}_0(\sigma(\tau))\|\dot{\sigma}(\tau)d\tau \\
&= \lim_{q\to\infty}\int_0^{T_1} h_q(\sigma(\tau))\|\dot{\gamma}_0(\sigma(\tau))\|\dot{\sigma}(\tau)d\tau \\
&= \lim_{q\to\infty}\int_0^{T_0} h_q(t)\|\dot{\gamma}_0(t)\|dt \\
&= \int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt \;.
\end{aligned} \tag{3.1.7.37}$$

The penultimate step in (3.1.7.37) is an application of the Change of Variables Lemma 3.1.6; this application is justified because it has been shown that the function $t \longrightarrow h_q(t)\|\dot{\gamma}_0(t)\|$ is an element of $L^1([0,T_0]\longrightarrow \mathbf{R})$. The final step in (3.1.7.37) is a restatement of (3.1.7.36). Thus the integral on the left of (3.1.7.36) is parametrization independent. This completes the proof of assertion (a).

Now we proceed to prove assertion (b). Let $\{u(\cdot),x(\cdot)\}|[0,T]$ be any input-trajectory pair of S such that $x(\cdot)|[0,T]$ is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$. Then, from Def. 3.1.2,

$$\int_0^T p(x(t),u(t))dt \geq \int_0^T h\left(x(t),\frac{\dot{x}(t)}{\|\dot{x}(t)\|}\right)\|\dot{x}(t)\|dt \ . \qquad (3.1.7.38)$$

From assertion (a), the integral on the right-hand side of (3.1.7.38) is parametrization independent; thus,

$$\int_0^T p(x(t),u(t))dt \geq \int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt. \qquad (3.1.7.39)$$

It follows from (3.1.7.39) that in order to prove assertion (b), we need only construct a sequence of input-trajectory pairs $\{\hat{u}_q(\cdot),\hat{x}_q(\cdot)\}|[0,T_q]$ of S such that $\hat{x}_q(\cdot)|[0,T_q]$ is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$ and

$$\lim_{q\to\infty}\int_0^{T_q} p(\hat{x}_q(t),\hat{u}_q(t))dt$$
$$= \int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt \ . \qquad (3.1.7.40)$$

In order to do this, let $q \geq N$ and define $\sigma_q\colon [0,T_0] \to \mathbf{R}$ by

$$\sigma_q(t) \triangleq \int_0^t \frac{\|\dot{\gamma}_0(t')\|}{\|f(\gamma_0(t'),u_q(t'))\|}dt' \leq qt < \infty \ . \qquad (3.1.7.41)$$

The inequality $\sigma_q(t) \leq qt$ follows since $u_q(t') \in C_q(t')$ for all $t' \in [0,t]\cap J$. Note that $t_1 < t_2$ implies that $\sigma_q(t_1) < \sigma_q(t_2)$, therefore $\sigma_q(\cdot)$ is a bijection of $[0,T_0]$ onto $[0,\sigma_q(T_0)]$. Let $k_q\colon [0,\sigma_q(T_0)] \to [0,T_0]$ be defined by $k_q(\tau) \triangleq \sigma_q^{-1}(\tau)$. Then

$$\dot{k}_q(\tau) = \frac{1}{\dot{\sigma}_q(k_q(\tau))} = \frac{\|f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))\|}{\|\dot{\gamma}_0(k_q(\tau))\|} \tag{3.1.7.42}$$

for a.a. $\tau \in [0,\sigma_q(T_0)]$ (the symbol $\dot{\sigma}_q(k_q(\tau))$ means $\frac{d\sigma_q(t)}{dt}$ evaluated at $t = k_q(\tau)$; likewise for $\dot{\gamma}_0(k_q(\tau))$). From (3.1.7.42), it follows that

$$\begin{aligned}\frac{d\gamma_0(k_q(\tau))}{d\tau} &= \dot{\gamma}_0(k_q(\tau))\dot{k}_q(\tau) \\ &= \frac{\dot{\gamma}_0(k_q(\tau))}{\|\dot{\gamma}_0(k_q(\tau))\|}\|f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))\| \\ &= f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))\end{aligned} \tag{3.1.7.43}$$

for a.a. $\tau \in [0,\sigma_q(T_0)]$. The last step in (3.1.7.43) follows since $u_q(k_q(\tau)) \in C_q(k_q(\tau))$, and hence

$$\frac{\dot{\gamma}_0(k_q(\tau))}{\|\dot{\gamma}_0(k_q(\tau))\|} = \frac{f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))}{\|f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))\|}$$

for a.a. $\tau \in [0,\sigma_q(T_0)]$ (we are using the fact that $\sigma_q\colon [0,T_0] \to [0,\sigma_q(T_0)]$ maps sets of measure zero to sets of measure zero, see [12, p. 108, problem 14]). Equation (3.1.7.43) shows that, for each $q \geq N$, $\{(u_q \circ k_q)(\cdot),(\gamma_0 \circ k_q)(\cdot)\}|[0,\sigma_q(T_0)]$ is an input-trajectory pair of S for which the state trajectory is a re-parametrization of $\gamma_0(\cdot)|[0,T_0]$. It follows from (3.1.7.23) and (3.1.7.42) that

$$\begin{aligned}&\int_0^{\sigma_q(T_0)} p(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))d\tau \\ &\quad = \int_0^{\sigma_q(T_0)} h_q(k_q(\tau))\|f(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))\|d\tau \\ &\quad = \int_0^{\sigma_q(T_0)} h_q(k_q(\tau))\|\dot{\gamma}_0(k_q(\tau))\|\dot{k}_q(\tau)d\tau \ .\end{aligned} \tag{3.1.7.44}$$

Since $t \to h_q(t)\|\dot{\gamma}_0(t)\|$ belongs to $L^1([0,T_0] \to \mathbb{R})$, we can apply the Change of Variables Lemma 3.1.6 to the right-hand side of (3.1.7.44) to obtain

$$\int_0^{\sigma_q(T_0)} p(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))d\tau = \int_0^{T_0} h_q(t)\|\dot{\gamma}_0(t)\|dt \ . \tag{3.1.7.45}$$

Combining (3.1.7.36) with (3.1.7.45) gives

$$\lim_{q\to\infty} \int_0^{\sigma_q(T_0)} p(\gamma_0(k_q(\tau)),u_q(k_q(\tau)))d\tau = \int_0^{T_0} h\left(\gamma_0(t),\frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt \ . \tag{3.1.7.46}$$

Q.E.D.

Proof of Lemma 3.2.2

Let $\{\phi_\beta(\cdot): \beta \in B\}$ be a collection of upper semicontinuous functions, where the index set B may be finite, countable, or uncountable. Let $\phi(\cdot) \triangleq \inf\{\phi_\beta(\cdot): \beta \in B\}$. Then $\{x: \phi(x) < \alpha\} = \bigcup_{\beta\in B} \{x: \phi_\beta(x) < \alpha\}$. Thus $\{x: \phi(x) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$, i.e., $\phi(\cdot)$ is upper semicontinuous. The proof of the other assertion is similar. Q.E.D.

Proof of Lemma 3.2.3

Let $\phi: \Sigma \to \mathbb{R}^e$ be upper semicontinuous, and let $K \subset \Sigma$ be compact. Suppose $\phi(x) \neq \infty$ for all $x \in K$. For $\alpha \in \mathbb{R}$, define $V_\alpha \triangleq \{x \in \Sigma: \phi(x) < \alpha\}$. Note that each set V_α is open, and the collection $\{V_\alpha: \alpha \in \mathbb{R}\}$ covers K. Since K is compact, there exists a finite subcover $\{V_{\alpha_1},\ldots,V_{\alpha_n}\}$. Let $M = \max\{\alpha_1,\ldots,\alpha_n\}$. Then

$\phi(x) < M < \infty$ for all $x \in K$. The proof of the other assertion is similar. Q.E.D.

Proof of Lemma 3.2.5.

We shall prove that $\bar{h}(\cdot)$ is upper semicontinuous: the proof of the other assertion is similar. For every $u \in U$, define $\bar{h}_u: \Sigma \to \mathbf{R}^e$ by

$$\bar{h}_u(x) \triangleq \begin{cases} \frac{p(x,u)}{f(x,u)}, & \text{if } u \in U_x^+ , \\ \infty , & \text{if } u \in U \backslash U_x^+ . \end{cases}$$

It is clear that $\bar{h}(x) = \inf\{\bar{h}_u(x): u \in U\}$. Let $u \in U$ and $\alpha \in \mathbf{R}$ be fixed. Then

$$\{x \in \Sigma: \bar{h}_u(x) < \alpha\} = \{x \in \Sigma: f(x,u) > 0\} \cap \{x \in \Sigma: p(x,u) - \alpha f(x,u) < 0\} . \quad (3.2.5.1)$$

From the continuity of the functions $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$, both sets on the right-hand side of (3.2.5.1) are open; thus, their intersection is open. Hence $\bar{h}(\cdot)$ is the infimum of a collection of upper semi-continuous functions. It follows from Lemma 3.2.2 that $\bar{h}(\cdot)$ is upper semicontinuous. Q.E.D.

Proof of Lemma 3.2.8

We shall prove assertion (a) only: the proof of (b) is similar. Let $x_0 \in \Sigma$, and suppose that $x_1 \in \text{int } R^+(x_0)$. To prove that there exists a state trajectory $x(\cdot)|[0,T]$ from x_0 to x_1 with $\dot{x}(t) > 0$ for a.a. $t \in [0,T]$, let $\hat{x} \in [x_0,x_1]$ and note that (since $f(\cdot,\cdot)$ is continuous and $x_1 \in \text{int } R^+(x_0)$) there exists an input value $u_{\hat{x}}$ and an interval $I_{\hat{x}} \subset [x_0,x_1]$ which contains $\hat{x}$ and is open in the relative topology of $[x_0,x_1]$ such that

$$f(x,u_{\hat{x}}) > 0 \text{ for all } x \in I_{\hat{x}} . \qquad (3.2.8.3)$$

Observe that $\{I_{\hat{x}}: \hat{x}\in[x_0,x_1]\}$ is an open covering of the compact interval $[x_0,x_1]$; hence, there exists a finite subcover $\{I_{\hat{x}_1},\ldots,I_{\hat{x}_k}\}$. Without loss of generality, we may assume that $\hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_k$ and that $I_{\hat{x}_i} \cap I_{\hat{x}_j} \neq \emptyset$ if and only if $|i-j| \leq 1$. For each $i \in \{1,\ldots,k-1\}$, choose $\bar{x}_i \in I_{\hat{x}_i} \cap I_{\hat{x}_{i+1}}$. Suppose that the state of S at $t = 0$ is $x(0) = x_0$. Construct a piecewise constant input $u(\cdot)$ as follows: set $u(t) = u_{\hat{x}_1}$ for $t \in [0,T_1]$, where T_1 is the unique time such that $x(T_1) = \bar{x}_1$. Likewise, set $u(t) = u_{\hat{x}_2}$ for $t \in (T_1,T_2]$, where T_2 is the unique time such that $x(T_2) = \bar{x}_2$, etc. Continuing in this manner, we construct a piecewise constant input which drives S from x_0 at $t = 0$ to x_1 at some finite time T; moreover, the corresponding state trajectory $x(\cdot)|[0,T]$ satisfies $\dot{x}(t) > 0$ for a.a. $t \in [0,T]$.

Let $T_0 > 0$, and define $\gamma_0: [0,T_0] \to [x_0,x_1]$ by

$$\gamma_0(\tau) \triangleq x_0 + \frac{\tau}{T_0}(x_1-x_0) . \qquad (3.2.8.4)$$

Then $\gamma_0(\cdot)|[0,T_0]$ is an admissible curve of S; in fact, it is a re-parametrization of <u>any</u> state trajectory $\gamma_1(\cdot)|[0,T_1]$ of S from x_0 to x_1 with $\dot{\gamma}_1(t) > 0$ for a.a. $t \in [0,T_1]$. To see this, let

$$\sigma(t) = \frac{T_0}{x_1-x_0}(\gamma_1(t)-x_0) . \qquad (3.2.8.5)$$

Then clearly,

$$\gamma_1(t) = \gamma_0(\sigma(t)) \qquad (3.2.8.6)$$

for all $t \in [0,T_1]$; moreover, $\sigma(\cdot)$ is absolutely continuous with $\sigma(0) = 0$, $\sigma(T_1) = T_0$, and $\dot{\sigma}(t) = \frac{T_0}{x_1-x_0}\dot{\gamma}_1(t) > 0$ for a.a. $t \in [0,T_1]$. Thus $\gamma_1(\cdot)|[0,T_1]$ is a re-parameterization of

$\gamma_0(\cdot)|[0,T_0]$ (Def. 3.1.3). Conversely, Lemma 3.1.6 and Remark #2 following it show that $\gamma_0(\cdot)|[0,T_0]$ is a re-parameterization of the state trajectory $\gamma_1(\cdot)|[0,T_1]$; hence, $\gamma_0(\cdot)|[0,T_0]$ is an admissible curve of S (Def. 3.1.3). Thus

$$\inf_{\substack{x_0\to x_1\\ \dot{x}>0\\ T>0}} \left\{\int_0^T p(x(t),u(t))dt\right\}$$

$$= \inf_{T>0}\left\{\int_0^T p(x(t),u(t))dt:\ x(\cdot)|[0,T]\in\mathcal{R}[\gamma_0(\cdot)|[0,T_0]]\right\}$$

$$= \int_0^{T_0} \hbar\left(x_0+\frac{\tau}{T_0}(x_1-x_0)\right)\frac{x_1-x_0}{T_0}\,d\tau \qquad (3.2.8.7)$$

where the last step follows from Lemma 3.1.7. By making the affine change of variables[11] $x = x_0+\frac{\tau}{T_0}(x_1-x_0)$, the integral in (3.2.8.7) can be rewritten

$$\int_0^{T_0} \hbar\left(x_0+\frac{\tau}{T_0}(x_1-x_0)\right)\frac{x_1-x_0}{T_0}\,d\tau = \int_{x_0}^{x_1}\hbar(x)dx\ . \qquad (3.2.8.8)$$

Combining (3.2.8.7) and (3.2.8.8), we obtain (3.2.8.1). Q.E.D.

Proof of Lemma 4.1.3

The "if" part is immediate from Def. 4.1.2. To prove the "only if" part, suppose that N is passive. Then, by Def. 4.1.2, N has a passive dynamical system representation S_1. Let S_2 be any other dynamical system representation for N (note that S_2 is necessarily equivalent (Def. 2.1.10) to S_1). Let $(x_2,t_0)\in\Sigma_2\times\mathbb{R}$. Then, by equivalence,

[11]Note that it is always permissible to make an affine change of variables in an integral. For other transformations, however, one must be careful (cf. Lemma 3.1.6).

there exists a state $x_1 \in \Sigma_1$ such that the set of voltage-current pairs $\{v(\cdot),i(\cdot)\}|[t_0,\infty)$ of S_1 with initial state x_1 is identical to the set of voltage-current pairs $\{v(\cdot),i(\cdot)\}|[t_0,\infty)$ of S_2 with initial state x_2. Hence, S_2 is passive because S_1 is. Q.E.D.

Proof of Lemma 4.1.4

Suppose that S satisfies (Suff. 4.1.4). If $\{u(\cdot),x(\cdot)\}|[t_0,\infty)$ is any input trajectory pair of S, then (4.1.4.1) gives

$$p(x(t),u(t),t) \geq \frac{d}{dt} \psi(x(t),t) \tag{4.1.4.2}$$

for a.a. $t \in [t_0,\infty)$. Since $\psi(\cdot,\cdot)$ is C^1 and $x(\cdot)$ is absolutely continuous, the mapping $t \to \psi(x(t),t)$ is absolutely continuous over $[t_0,t_1]$ for every $t_1 \geq t_0$. Thus (4.1.4.2) can be integrated with the following result:

$$\int_{t_0}^{t_1} p(x(t),u(t),t)dt \geq \psi(x(t_1),t_1) - \psi(x(t_0),t_0) \geq -\psi(x(t_0),t_0) . \tag{4.1.4.3}$$

The last step follows since $\psi(\cdot,\cdot)$ is nonnegative. It is clear from (4.1.4.3) that $E_A(x_0,t_0) \leq \psi(x_0,t_0) < \infty$ for all $(x_0,t_0) \in \Sigma \times \mathbb{R}$. Q.E.D.

Proof of Lemma 4.1.6

Suppose that $E_I(\cdot,\cdot)$ is an internal energy function for S. Then, since $E_I(\cdot,\cdot)$ is nonnegative, (4.1.5.1) gives

$$\int_{t_0}^{t_1} p(x(t),u(t),t)dt \geq -E_I(x(t_0),t_0) \tag{4.1.6.1}$$

for all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ of S. It is clear from (4.1.6.1) that $E_A(x,t) \leq E_I(x,t) < \infty$ for all $(x,t) \in \Sigma \times \mathbb{R}$. This shows that S is passive, and it also shows that $E_A(\cdot,\cdot) \leq E_I(\cdot,\cdot)$.

Now suppose that S is passive. To prove that $E_A(\cdot\,,\cdot)$ is an internal energy function for S, let $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S. Then we have the following obvious inequality:

$$\sup_{\substack{x(t_0)\to \\ t'\geq t_0}} \left\{-\int_{t_0}^{t'} p(x'(t),u'(t),t)dt\right\} \geq -\int_{t_0}^{t_1} p(x(t),u(t),t)dt$$

$$+ \sup_{\substack{x(t_1)\to \\ t''\geq t_1}} \left\{-\int_{t_1}^{t''} p(x''(t),u''(t),t)dt\right\} \quad . \qquad (4.1.6.2)$$

Substituting the definition of $E_A(\cdot\,,\cdot)$ into (4.1.6.2) gives

$$E_A(x(t_0),t_0) \geq -\int_{t_0}^{t_1} p(x(t),u(t),t)dt + E_A(x(t_1),t_1) \quad . \qquad (4.1.6.3)$$

Q.E.D.

Proof of Lemma 4.1.8

(a) Define $E_{rx^*} : \Sigma \times \mathbb{R}_+^2 \to \mathbb{R}$ by

$$E_{rx^*}(x_1,t_1,t_0) \triangleq \inf_{x^*\to x_1} \left\{\int_{t_0}^{t_1} p(x(t),u(t),t)dt\right\} \qquad (4.1.8.5)$$

where the expression on the right-hand side of (4.1.8.5) indicates that the infimum is taken over all input-trajectory pairs $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ from x^* to x_1, where t_0 and t_1 are fixed. Note that

$$E_{Rx^*}(x_1,t_1) = \inf_{t_0\leq t_1} E_{rx^*}(x_1,t_1,t_0) \quad . \qquad (4.1.8.6)$$

Let $(x_1,t_1) \in \Sigma \times \mathbb{R}$ be arbitrary, and let $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S from x^* to x_1. Then, since $E_I(\cdot\,,\cdot)$ is an internal energy function,

$$E_I(x_1,t_1) - E_I(x^*,t_0) \leq \int_{t_0}^{t_1} p(x(t),u(t),t)dt \quad . \tag{4.1.8.7}$$

This immediately gives

$$E_I(x_1,t_1) - E_I(x^*,t_0) \leq E_{rx^*}(x_1,t_1,t_0) \quad . \tag{4.1.8.8}$$

Taking the infimum over $t_0 \leq t_1$ on both sides of (4.1.8.8), we obtain

$$E_I(x_1,t_1) - \bar{E}_I(x^*,t_1) \leq E_{Rx^*}(x_1,t_1) \quad . \tag{4.1.8.9}$$

(b) By reachability, $E_{Rx^*}(\cdot\,,\cdot) < \infty$. Suppose that $E_A^*(x^*) < \infty$; then $\Lambda_{x^*}(\cdot\,,\cdot) < \infty$. Note that

$$\begin{aligned}
\Lambda_{x^*}(x,t) &\triangleq E_{Rx^*}(x,t) + E_A^*(x^*) \\
&\geq E_{Rx^*}(x,t) + \bar{E}_A(x^*,t) \\
&\geq E_A(x,t) \qquad \text{(by assertion (a))} \\
&\geq 0 \quad .
\end{aligned} \tag{4.1.8.10}$$

Thus $0 \leq \Lambda_{x^*}(\cdot\,,\cdot) < \infty$, which is a necessary condition for $\Lambda_{x^*}(\cdot\,,\cdot)$ to be an internal energy function.

Let $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S. Then we have the following obvious inequality:

$$\inf_{\substack{x^* \to x(t_1) \\ t' \leq t_1}} \left\{ \int_{t'}^{t_1} p(x'(t),u'(t),t)dt \right\} \leq \inf_{\substack{x^* \to x(t_0) \\ t'' \leq t_0}} \left\{ \int_{t''}^{t_0} p(x''(t),u''(t),t)dt \right\} + \int_{t_0}^{t_1} p(x(t),u(t),t)dt \quad . \tag{4.1.8.11}$$

Substituting the definition of $E_{Rx^*}(\cdot\,,\cdot)$ into (4.1.8.11) gives

$$E_{Rx^*}(x(t_1),t_1) \leq E_{Rx^*}(x(t_0),t_0) + \int_{t_0}^{t_1} p(x(t),u(t),t)dt \; . \quad (4.1.8.12)$$

Thus,

$$\Lambda_{x^*}(x(t_1),t_1) - \Lambda_{x^*}(x(t_0),t_0) = E_{Rx^*}(x(t_1),t_1) - E_{Rx^*}(x(t_0),t_0)$$

$$\leq \int_{t_0}^{t_1} p(x(t),u(t),t)dt \quad , \quad (4.1.8.13)$$

which completes the proof that $\Lambda_{x^*}(\cdot\,,\cdot)$ is an internal energy function.

Q.E.D.

Proof of Lemma 4.1.11

Let $(x_0,u_0,t_0) \in \Sigma \times U \times \mathbb{R}$, and let $x(\cdot)|[t_0,\infty)$ denote the state trajectory of S with $x(t_0) = x_0$ which is generated by the constant input $u(t) \equiv u_0$. We have

$$\left\{\langle \nabla_x \psi(x,t),\, f(x,u,t)\rangle + \frac{\partial\psi(x,t)}{\partial t}\right\}\Bigg|_{(x,u,t)=(x_0,u_0,t_0)}$$

$$= \frac{d\psi(x(t),t)}{dt}\Bigg|_{t=t_0}$$

$$= \lim_{\Delta t \to 0^+} \frac{\psi(x(t_0+\Delta t),t_0+\Delta t) - \psi(x(t_0),t_0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0^+} \frac{E_I(x(t_0+\Delta t),t_0+\Delta t) - E_I(x(t_0),t_0)}{\Delta t}$$

$$\leq \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} p(x(t),u_0,t)dt \qquad \text{(by (4.1.5.1))}$$

$$= p(x_0,u_0,t_0) \quad . \tag{4.1.11.2}$$

The last step in (4.1.11.2) is simply an application of the Fundamental Theorem of Calculus, which is justified because $p(\cdot,\cdot,\cdot)$ is continuous (Def. 2.1.17). Q.E.D.

Proof of Theorem 4.5.3

(Necessity). Suppose that S is passive. The necessity of (i) follows immediately from assertion (a) of Theorem 4.4.1.

By Assumption 4.5.1, $U_x^- \neq \phi$ and $U_x^+ \neq \phi$ for all $x \in \Sigma$. To prove the necessity of (ii), suppose on the contrary that there exists an $x_0 \in \Sigma$ such that

$$\underline{h}(x_0) > \bar{h}(x_0) \quad .$$

Then there exists $u_1 \in U_{x_0}^+$ and $u_2 \in U_{x_0}^-$ such that

$$\frac{p(x_0,u_2)}{f(x_0,u_2)} > \frac{p(x_0,u_1)}{f(x_0,u_1)} \quad .$$

By continuity, there exists $\delta > 0$ such that

$$f(x,u_1) > 0 \qquad \text{for every} \quad x \in [x_0, x_0+\delta] \quad ,$$

$$f(x,u_2) < 0 \qquad \text{for every} \quad x \in [x_0, x_0+\delta] \quad ,$$

$$\frac{p(x,u_2)}{f(x,u_2)} > \frac{p(x,u_1)}{f(x,u_1)} \qquad \text{for every} \quad x \in [x_0, x_0+\delta] \quad .$$

Hence the constant input value u_1 will drive S from state x_0 at time $t=0$ to state $x_0+\delta$ at some finite time $t_1 > 0$, and the constant input value u_2

will drive S from state $x_0+\delta$ at time t_1 to state x_0 at some finite time $t_2 > t_1$. Define an input $u(\cdot)|[0,t_2]$ as follows:

$$u(t) \triangleq \begin{cases} u_1, & \text{for } 0 \leq t \leq t_1 \quad, \\ u_2, & \text{for } t_1 < t \leq t_2 \quad. \end{cases}$$

Let $x(\cdot)|[0,t_2]$ denote the state trajectory with $x(0) = x_0$ which is generated by $u(\cdot)$. Note that $x(t_2) = x_0$; thus, $x(\cdot)|[0,t_2]$ is a "loop" from x_0 to $x_0+\delta$ and back again. We have

$$\int_0^{t_2} p(x(t),u(t))dt = \int_0^{t_1} \frac{p(x(t),u_1)}{f(x(t),u_1)} \dot{x}(t)dt + \int_{t_1}^{t_2} \frac{p(x(t),u_2)}{f(x(t),u_2)} \dot{x}(t)dt$$

$$= \int_{x_0}^{x_0+\delta} \frac{p(x,u_1)}{f(x,u_1)} dx + \int_{x_0+\delta}^{x_0} \frac{p(x,u_2)}{f(x,u_2)} dx$$

$$= \int_{x_0}^{x_0+\delta} \left[\frac{p(x,u_1)}{f(x,u_1)} - \frac{p(x,u_2)}{f(x,u_2)}\right] dx < 0 \quad .$$

The last inequality follows because the integrand is strictly negative on the interval $[x_0,x_0+\delta]$. This shows that $E_A(x_0) = \infty$, because we can drive the state repeatedly in the above mentioned loop between x_0 and $x_0+\delta$ and thereby extract an unbounded amount of energy. But $E_A(x_0) = \infty$ contradicts the assumption that S is passive; therefore, condition (ii) must be satisfied.

Finally, if we choose $W(\cdot) = E_A(\cdot)$, the available energy function, then condition (iii) follows immediately from Lemma 3.2.8.

(Sufficiency). Suppose that conditions (i), (ii), and (iii) are satisfied.

By Assumption 4.5.1, $\bar{h}(x) < \infty$ and $\underline{h}(x) > -\infty$ for all $x \in \Sigma$. By Lemma 3.2.3, Lemma 3.2.5, and condition (ii), the functions $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are bounded on every compact subset of Σ. From conditions (i), (ii), and Def. 3.2.4,

$$p(x,u) \geqslant \bar{h}(x)\ f(x,u) \qquad (4.5.3.3a)$$

$$p(x,u) \geqslant \underline{h}(x)\ f(x,u) \qquad (4.5.3.3b)$$

for all $(x,u) \in \Sigma \times U$.

Now choose $x_0 \in \Sigma$. We want to show that $E_A(x_0) < \infty$. For each $x \in \Sigma$, define

$$\bar{H}(x) \triangleq \int_{x_0}^{x} \bar{h}(z)dz \quad , \qquad (4.5.3.4a)$$

$$\underline{H}(x) \triangleq \int_{x_0}^{x} \underline{h}(z)dz \quad . \qquad (4.5.3.4b)$$

From the boundedness properties of $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$, it follows that $\bar{H}(x)$ and $\underline{H}(x)$ are well-defined and finite for all $x \in \Sigma$. Let $\{u(\cdot),x(\cdot)\}|[0,T]$ be any input-trajectory pair of S with initial state $x(0) = x_0$. Note that

$$\bar{h}(x(t))\ \dot{x}(t) = \frac{d}{dt}\ \bar{H}(x(t)) \quad , \qquad (4.5.3.5a)$$

$$\underline{h}(x(t))\ \dot{x}(t) = \frac{d}{dt}\ \underline{H}(x(t)) \quad , \qquad (4.5.3.5b)$$

for a.a.$t \in [0,T]$. Since $x(\cdot)$ is continuous, it follows that $x([0,T])$ is a compact subset of Σ. Since $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are bounded on compact subsets of Σ and $x(\cdot)$ is absolutely continuous, it follows that the mappings $t \to \bar{H}(x(t))$ and $t \to \underline{H}(x(t))$ are absolutely continuous [15, pp.95-96, Theorem I.4.42]. Hence, from (4.5.3.4) and (4.5.3.5) we obtain

$$\int_0^T \bar{h}(x(t))\,\dot{x}(t)dt = \bar{H}(x(T)) - \bar{H}(x_0) = \bar{H}(x(T)) \quad (4.5.3.6a)$$

$$\int_0^T \underline{h}(x(t))\,\dot{x}(t)dt = \underline{H}(x(T)) \quad . \quad (4.5.3.6b)$$

Suppose that $x(T) \geq x_0$. Then (4.5.3.3a), (4.5.3.6a), and condition (iii) give

$$\int_0^T p(x(t),u(t))dt \geq \int_0^T \bar{h}(x(t))\dot{x}(t)dt$$

$$= \bar{H}(x(T)) = \int_{x_0}^{x(T)} \bar{h}(z)dz$$

$$\geq -W(x_0) \quad . \quad (4.5.3.7)$$

On the other hand, if $x(T) < x_0$, we have from (4.5.3.3b), (4.5.3.6b), and condition (iii) that

$$\int_0^T p(x(t),u(t))dt \geq \int_0^T \underline{h}(x(t))\dot{x}(t)dt$$

$$= \underline{H}(x(T)) = \int_{x_0}^{x(T)} \underline{h}(z)dz$$

$$\geq -W(x_0) \quad . \quad (4.5.3.8)$$

Equations (4.5.3.7) and (4.5.3.8) show that $E_A(x_0) \leq W(x_0) < \infty$. Q.E.D.

<u>Proof of Corollary 4.5.4</u>

(Sufficiency). Condition (i) of Corollary 4.5.4 implies condition (i) of Theorem 4.5.3. Also, condition (i) of Corollary 4.5.4 and Def. 3.2.4 imply that

$$\underline{h}(x) \leq \alpha(x) \leq \bar{h}(x) \quad (4.5.4.1)$$

for all $x \in \Sigma$. This gives condition (ii) of Theorem 4.5.3. Finally, (4.5.4.1) along with condition (ii) of Corollary 4.5.4 gives condition (iii) of Theorem 4.5.3, with $W(x_0) = E(x_0)$. Therefore S is passive.

(Necessity). Suppose that S is passive. Then S satisfies the three conditions of Theorem 4.5.3. From the sufficiency portion of the proof of Theorem 4.5.3, we know that $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are bounded on compact subsets of Σ and satisfy (4.5.3.3). Choose $z_0 \in \Sigma$ and define $\alpha : \Sigma \to \mathbb{R}$ by

$$\alpha(x) \triangleq \begin{cases} \bar{h}(x) \quad , & \text{for } x \geqslant z_0 \, , \\ \underline{h}(x) \quad , & \text{for } x < z_0 \, . \end{cases} \tag{4.5.4.2}$$

Clearly, $\alpha(\cdot)$ is bounded on compact subsets of Σ and satisfies $p(x,u) \geqslant \alpha(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$. Since $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are semicontinuous (Lemma 3.2.5), they are measurable; hence, $\alpha(\cdot)$ is measurable as well.

Now define $A : \Sigma \to \mathbb{R}^+$ by

$$A(x_0) \triangleq \begin{cases} \sup \left\{ \int_z^{x_0} \bar{h}(x)dx \, : \, z \in [z_0,x_0] \right\} , & \text{if } x_0 \geqslant z_0 , \\ -\inf \left\{ \int_{x_0}^{z} \underline{h}(x)dx \, : \, z \in [x_0,z_0] \right\} , & \text{if } x_0 < z_0 , \end{cases}$$

and define $E : \Sigma \to \mathbb{R}^+$ by

$$E(x_0) \triangleq W(x_0) + W(z_0) + A(x_0) \quad , \tag{4.5.4.3}$$

where $W(\cdot)$ is the function appearing in Theorem 4.5.3. It is straightforward to verify that $\alpha(\cdot)$ and $E(\cdot)$, as defined in (4.5.4.2) and (4.5.4.3), satisfy condition (ii) of Corollary 4.5.4. Q.E.D.

Proof of Corollary 4.5.5

Sufficiency is just a special case of Lemma 4.1.6. To prove necessity, suppose that S is passive. Let $\alpha(\cdot)$ and $E(\cdot)$ be the functions in Corollary 4.5.4. Choose $z_0 \in \Sigma$, and define $E_I : \Sigma \to \mathbb{R}^+$ by

$$E_I(x) \triangleq \int_{z_0}^{x} \alpha(z)dz + E(z_0) \quad . \tag{4.5.5.1}$$

(Note that $E_I(\cdot)$ is nonnegative by condition (ii) of Corollary 4.5.4.) From (4.5.5.1), $E_I(\cdot)$ is differentiable at almost every $x \in \Sigma$, and

$$\frac{dE_I(x)}{dx} = \alpha(x) \qquad \text{for a.a.} x \in \Sigma \ . \tag{4.5.5.2}$$

Since $\alpha(\cdot)$ is bounded on compact subsets of Σ, it follows that the mapping $x \to dE_I(x)/dx$ belongs to $L^\infty_{loc}(\Sigma \to \mathbb{R})$.

Now let $\{u(\cdot),x(\cdot)\}|[t_0,t_1]$ be any input-trajectory pair of S. Note that

$$\alpha(x(t))\dot{x}(t) = \frac{dE_I(x(t))}{dt} \quad ; \tag{4.5.5.3}$$

moreover, since $\alpha(\cdot)$ is bounded on compact subsets of Σ and $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$, it follows that $t \to E_I(x(t))$ is absolutely continuous on $[t_0,t_1]$ [15, pp.95-96, Theorem I.4.42]: thus

$$\int_{t_0}^{t_1} \alpha(x(t))\dot{x}(t)dt = E_I(x(t_1)) - E_I(x(t_0)) \quad . \tag{4.5.5.4}$$

From condition (i) of Corollary 4.5.4 and (4.5.5.4), we obtain

$$\int_{t_0}^{t_1} p(x(t),u(t))dt \geq \int_{t_0}^{t_1} \alpha(x(t))\dot{x}(t)dt$$

$$= E_I(x(t_1)) - E_I(x(t_0)) \quad . \qquad (4.5.5.5)$$

Hence, $E_I(\cdot)$ is an internal energy function for S. Q.E.D.

<u>Proof of Lemma 5.1.6</u>

Let $S = \{U,\mathcal{U},\Sigma,\phi(\cdot,\cdot,\cdot,\cdot),Y,g(\cdot,\cdot),\omega(\cdot,\cdot)\}$ denote an input-distinguishable time-invariant dynamical system. Since $\omega(\cdot,\cdot)$ is injective (Def. 2.1.11), it follows that two states x' and x" of S are equivalent if and only if the set of <u>input-output</u> pairs $\{u'(\cdot),y'(\cdot)\}|[0,\infty)$ of S with initial state x' is the same as the set of <u>input-output</u> pairs $\{u''(\cdot),y''(\cdot)\}|[0,\infty)$ of S with initial state x" (cf. Def. 2.1.9). Let $E: \Sigma \to P(\Sigma)$ denote the map which takes each $x \in \Sigma$ to the equivalence class of x which is defined by the equivalence relation of Def. 2.1.9 (thus for each $x \in \Sigma$, $E(x)$ is the set of all states equivalent to x). The collection of all such equivalence classes is denoted Σ_0. Define $\phi_0: \mathbb{R}_+^2 \times \Sigma_0 \times \mathcal{U} \to \Sigma_0$ by

$$\phi_0(t,t_0,x_0,u(\cdot)) \triangleq E(\phi(t,t_0,x',u(\cdot))) \quad , \qquad (5.1.6.1)$$

where x' is any element of $E^{-1}(x_0)$. Note that the definition of $\phi_0(t, t_0,x_0,u(\cdot))$ is independent of the choice of $x' \in E^{-1}(x_0)$. This is because if $x_a(\cdot)$ and $x_b(\cdot)$ are any two state trajectories of S generated by $u(\cdot)$ with $E(x_a(0)) = E(x_b(0))$, then a simple contradiction argument shows that $E(x_a(t)) = E(x_b(t))$ for all $t \geq 0$. Now define $g_0: \Sigma_0 \times U \to Y$ by

$$g_0(x_0,u) \triangleq g(x',u) \qquad (5.1.6.2)$$

where x' is any element of $E^{-1}(x_0)$. Once again, a simple contradiction argument shows that $g(x',u) = g(x'',u)$ for all $x',x'' \in E^{-1}(x_0)$ and all $u \in U$; so the definition of $g(x_0,u)$ is independent of the choice of $x' \in E^{-1}(x_0)$.

It is straightforward to verify that $S_0 \triangleq \{U,\mathcal{U},\Sigma_0,\phi_0(\cdot,\cdot,\cdot,\cdot),Y, g_0(\cdot,\cdot),\omega(\cdot,\cdot)\}$ is a valid time-invariant dynamical system; by construction, S_0 is observable. Q.E.D.

Proof of Lemma 5.1.8

If N is lossless, then Def. 5.1.5 and Lemma 5.1.3 show that all input-distinguishable time-invariant dynamical system representations for N are lossless. To prove the converse, first note that the set I of all input-distinguishable time-invariant dynamical system representations for N is nonempty by Assumption 2.2.4. Suppose that every element of I is lossless, and let $S \in I$. Let S_0 denote the canonical observable dynamical system equivalent to S (Def. 5.1.7). Then $S_0 \in I$; hence, S_0 is lossless by assumption. It follows that N is lossless (Def. 5.1.5). Q.E.D.

Proof of Lemma 5.1.12

Assertion (a) follows immediately from Def. 5.1.11 and Lemma 5.1.4. To prove assertion (b), suppose that S^* is observable. Then from assertion (a), S is narrow-sense lossless if and only if S^* is lossless; but this is the condition for wide-sense losslessness given in Def. 5.1.9. Q.E.D.

Proof of Lemma 5.1.13

(a) If S_0^* is lossless, then S^* is lossless by Lemma 5.1.3.

(b) If S is time-invariant, then S is equivalent to S_0^*. If S_0^* is lossless, then S is lossless by Lemma 5.1.3. This shows that narrow-sense losslessness implies losslessness. The fact that losslessness implies wide-sense losslessness is obvious and has been remarked upon in the text. Q.E.D.

Proof of Lemma 5.1.14

(Necessity) Suppose that N is lossless. Let S denote an input-distinguishable time-invariant dynamical system representation for N (at least one such S exists by Assumption 2.2.4). By Lemma 5.1.8, S_0^* is lossless; thus S is narrow-sense lossless.

(Sufficiency) Suppose that N has an input-distinguishable time-invariant dynamical system representation S which is narrow-sense lossless. Since S is time-invariant, it is equivalent to S_0^*. Thus S_0^* is a lossless, observable, time-invariant dynamical system representation for N; by definition, N is lossless. Q.E.D.

Proof of Lemma 5.1.16

First note that the set of all input-distinguishable dynamical system representations for N is nonempty by Assumption 2.2.4. The "if" part then follows immediately from Def. 5.1.15. To prove the "only if" part, suppose that N is lossless. Let S denote an input-distinguishable narrow-sense lossless dynamical system representation for N (such an S exists by Def. 5.1.15), and let $\overline{S}$ denote any other input-distinguishable dynamical system representation for N. Note that $\overline{S}_0^*$ is equivalent to $\overline{S}^*$, $\overline{S}^*$ is equivalent to S^*, and S^* is equivalent to S_0^*; hence, $\overline{S}_0^*$ is equivalent to S_0^*. By assumption, S_0^* is lossless: it follows from Lemma 5.1.4 that $\overline{S}_0^*$ is lossless, i.e., $\overline{S}$ is narrow-sense lossless (Def. 5.1.11). Q.E.D.

Proof of Lemma 5.1.20

Suppose that S satisfies (Suff. 5.1.20). If $\{u(\cdot), x(\cdot)\}|[0,T]$ is any input-trajectory pair of S, then in analogy with the proof of Lemma 4.1.4 we obtain

$$\begin{aligned}\int_0^T p(x(t),u(t))dt &= \int_0^T \langle \nabla\phi(x(t)),f(x(t),u(t))\rangle\, dt \\ &= \int_0^T \frac{d}{dt}(\phi(x(t)))dt \\ &= \phi(x(T)) - \phi(x(0)) . \qquad (5.1.20.2)\end{aligned}$$

Therefore S is lossless. Q.E.D.

Proof of Lemma 5.1.23

Let $(x_0,u_0) \in \Sigma \times U$, and let $x(\cdot)|[0,\infty)$ denote the state trajectory of S with $x(0) = x_0$ which is generated by the constant input $u(t) \equiv u_0$. We have

$$\begin{aligned}\langle \nabla\psi(x_0),f(x_0,u_0)\rangle &= \left.\frac{d\psi(x(t))}{dt}\right|_{t=0} \\ &= \lim_{t\to 0^+} \frac{\psi(x(t))-\psi(x(0))}{t} \\ &= \lim_{t\to 0^+} \frac{\phi(x(t))-\phi(x(0))}{t} \\ &= \lim_{t\to 0^+} \frac{1}{t}\int_0^t p(x(\tau),u_0)d\tau \\ &= p(x_0,u_0) . \qquad (5.1.23.2)\end{aligned}$$

The last step in (5.1.23.2) is simply an application of the Fundamental Theorem of Calculus, which is justified because the integrand is continuous. Q.E.D.

Proof of Lemma 5.2.1

The Monotonicity Lemma 4.3.6 shows that the mapping $t \to \int_{t_0}^{t} h(\tau)d\tau - \phi(\gamma(t))$ is monotone increasing; but it also shows that $t \to -[\int_{t_0}^{t} h(\tau)d\tau - \phi(\gamma(t))]$ is monotone increasing. Therefore $\beta(\cdot)$ is both monotone increasing and monotone decreasing, i.e., $\beta(\cdot)$ is constant. Q.E.D.

Proof of Theorem 5.3.1

Assume that S is lossless. Let $h(\cdot,\cdot)$ be the function in Def. 3.1.2, and let $(x_0,u_0) \in \Sigma \times U$.

Suppose first that $f(x_0,u_0) = 0$. Define $\{\hat{u}(t), \hat{x}(t)\} \triangleq \{u_0,x_0\}$ for $t \in \mathbb{R}$. Then for all $T \geqslant 0$, $\{\hat{u}(\cdot), \hat{x}(\cdot)\}|[0, T]$ is a valid input-trajectory pair of S from x_0 to x_0. The energy consumed by this input-trajectory pair is

$$\int_0^T p(\hat{x}(t),\hat{u}(t))dt = \int_0^T p(x_0,u_0)dt = p(x_0,u_0)T \quad ;$$

but this quantity must be independent of T, since S is lossless. Thus $p(x_0,u_0) = 0$. This proves that (5.3.1.1) holds for all $(x_0,u_0) \in \Sigma \times U$ such that $f(x_0,u_0) = 0$.

Now suppose that $f(x_0,u_0) \neq 0$. Let $\gamma_0(\cdot)$ denote the state trajectory of S with $\gamma_0(0) = x_0$ which is generated by the constant input $\hat{u}(t) \equiv u_0$. Choose $T_0' > 0$ such that $\dot{\gamma}_0(t) \neq 0$ for all $t \in [0, T_0']$, and let $T_0 \in [0, T_0']$. Refer now to Lemma 3.1.7. Let $\{u(\cdot), x(\cdot)\}|[0, T]$ be any input-trajectory pair of S with $x(\cdot)|[0, T] \in \mathcal{R}[\gamma_0(\cdot)|[0, T_0]]$. Then, since S is lossless,

$$\int_0^T p(x(t),u(t))dt = \int_0^{T_0} p(\gamma_0(t),u_0)dt.$$

This shows that

$$\inf_{T \geq 0} \{ \int_0^T p(x(t),u(t))dt: \ x(\cdot)|[0, T] \in \mathcal{R}[\gamma_0(\cdot)|[0, T_0]] \}$$

$$= \int_0^{T_0} p(\gamma_0(t),u_0)dt. \tag{5.3.1.2}$$

From (5.3.1.2) and assertion (b) of Lemma 3.1.7, we have

$$\int_0^{T_0} p(\gamma_0(t),u_0)dt = \int_0^{T_0} h\left(\gamma_0(t), \frac{\dot{\gamma}_0(t)}{\|\dot{\gamma}_0(t)\|}\right)\|\dot{\gamma}_0(t)\|dt. \tag{5.3.1.3}$$

Noting that $\dot{\gamma}_0(t) = f(\gamma_0(t),u_0)$ for all $t \in \mathbb{R}^+$, (5.3.1.3) can be rewritten

$$\int_0^{T_0} [p(\gamma_0(t),u_0) - h\left(\gamma_0(t), \frac{f(\gamma_0(t),u_0)}{\|f(\gamma_0(t),u_0)\|}\right)\|f(\gamma_0(t),u_0)\|]dt = 0. \tag{5.3.1.4}$$

Since (5.3.1.4) holds for all $T_0 \in [0, T_0']$, we conclude that

$$p(\gamma_0(t),u_0) - h\left(\gamma_0(t), \frac{f(\gamma_0(t),u_0)}{\|f(\gamma_0(t),u_0)\|}\right)\|f(\gamma_0(t),u_0)\| = 0$$

$$\text{for a.a.} \ t \in [0, T_0']. \tag{5.3.1.5}$$

Now let $\varepsilon > 0$, and choose $\delta > 0$ such that $\|\gamma_0(t) - x_0\| < \varepsilon$ for all $t \in [0, \delta]$. Since (5.3.1.5) holds almost everywhere on $[0, T_0']$, there exists $t_1 \in [0, \delta]$ such that (5.3.1.5) holds at $t = t_1$. Define $x_1 \triangleq \gamma_0(t_1)$. Then by the choice of t_1,

$$p(x_1,u_0) = h\left(x_1, \frac{f(x_1,u_0)}{\|f(x_1,u_0)\|}\right)\|f(x_1,u_0)\|. \tag{5.3.1.6}$$

Thus we have shown that given $(x_0,u_0) \in \Sigma \times U$ and given $\varepsilon > 0$, there exists $x_1 \in \Sigma$ such that $\|x_0 - x_1\| < \varepsilon$ and (5.3.1.1) holds at $(x,u) = (x_1,u_0)$. Q.E.D.

Proof of Theorem 5.4.2

Before proving necessity and sufficiency, we shall prove that a function $h: \Sigma \to \mathbb{R}$ which satisfies $p(x,u) = h(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$ is continuous at each nonsingular state. If a state x_0 is nonsingular, then there exists an input value u_0 and (by continuity) a neighborhood $N(x_0)$ of x_0 such that $f(x,u_0) \neq 0$ for all $x \in N(x_0)$. Thus $h(x) = p(x,u_0)/f(x,u_0)$ for all $x \in N(x_0)$, which shows that $h(\cdot)$ is continuous at x_0.

(Necessity). Suppose that S is lossless. Define $D \triangleq \{(x,u) \in \Sigma \times U: \ f(x,u) \neq 0\}$, and define $h: D \to \mathbb{R}$ by

$$\hat{h}(x,u) \triangleq \frac{p(x,u)}{f(x,u)} . \tag{5.4.2.1}$$

We begin by proving that $\hat{h}(x,u)$ depends only on the first variable x. To obtain a contradiction, suppose that there exist (x_0,u_1), $(x_0,u_2) \in D$ such that $\hat{h}(x_0,u_1) \neq \hat{h}(x_0,u_2)$. Then two cases arise.

Case 1: $f(x_0,u_1)$ and $f(x_0,u_2)$ have the same sign. Assume that $f(x_0,u_1) > 0$ and $f(x_0,u_2) > 0$ (similar arguments apply in the other situation). By continuity, there exists $\delta > 0$ such that

$$f(x,u_1) > 0 \quad \text{for all} \quad x \in [x_0,\ x_0 + \delta] \tag{5.4.2.2}$$

$$f(x,u_2) > 0 \quad \text{for all} \quad x \in [x_0,\ x_0 + \delta] \tag{5.4.2.3}$$

$$\hat{h}(x,u_1) \neq \hat{h}(x,u_2) \quad \text{for all} \quad x \in [x_0,\ x_0 + \delta] . \tag{5.4.2.4}$$

By (5.4.2.2), the constant input $u(t) \equiv u_1$ will generate (for some finite $T_1 > 0$) a state trajectory $x_1(\cdot)|[0,\ T_1]$ of S from x_0 to $x_0 + \delta$. Now

$$\int_0^{T_1} p(x_1(t),u_1)dt = \int_0^{T_1} \frac{p(x_1(t),u_1)}{f(x_1(t),u_1)} \dot{x}_1(t)dt$$

$$= \int_0^{T_1} \hat{h}(x_1(t),u_1)\dot{x}_1(t)dt = \int_{x_0}^{x_0+\delta} \hat{h}(x,u_1)dx. \quad (5.4.2.5)$$

The use of the Change of Variables formula in the last step of (5.4.2.5) is justified because x_1: $[0, T_1] \to \mathbb{R}$ is C^1 and the mapping $x \to \hat{h}(x,u_1)$ is defined and continuous on $x_1([0, T_1])$[22, p. 234, Theorem 30.12]. Similarly, (5.4.2.3) shows that the constant input $u(t) \equiv u_2$ will generate (for some finite $T_2 > 0$) a state trajectory $x_2(\cdot)|[0, T_2]$ of S from x_0 to $x_0 + \delta$; moreover,

$$\int_0^{T_2} p(x_2(t),u_2)dt = \int_{x_0}^{x_0+\delta} \hat{h}(x,u_2)dx. \quad (5.4.2.6)$$

Since the integrands of the integrals on the right-hand sides of (5.4.2.5) and (5.4.2.6) are continuous and unequal at each point of the interval $[x_0,x_0+\delta]$, it follows that the integrals themselves are not equal. This contradicts the assumption of losslessness.

<u>Case 2</u>: $f(x_0,u_1)$ and $f(x_0,u_2)$ have opposite signs. For definiteness, assume that $f(x_0,u_1) > 0$ and $f(x_0,u_2) < 0$. By continuity, there exists $\delta > 0$ such that

$$f(x,u_1) > 0 \quad \text{for all} \quad x \in [x_0, x_0 + \delta] \quad (5.4.2.7)$$

$$f(x,u_2) < 0 \quad \text{for all} \quad x \in [x_0, x_0 + \delta] \quad (5.4.2.8)$$

$$\hat{h}(x,u_1) \neq \hat{h}(x,u_2) \quad \text{for all} \quad x \in [x_0, x_0 + \delta]. \quad (5.4.2.9)$$

Eqs. (5.4.2.7) and (5.4.2.8) show that there exists a finite $T_2 > 0$ and an input-trajectory pair $\{u(\cdot),x(\cdot)\}|[0, T_2]$ of S from x_0 to x_0 with the following property: there exists $T_1 \in (0, T_2)$ such that $u(t) = u_1$ for $t \in [0, T_1]$, $u(t) = u_2$ for $t \in (T_1. T_2]$, and $x(T_1) = x_0 + \delta$. Thus

$$\int_0^{T_2} p(x(t),u(t))dt = \int_0^{T_1} p(x(t),u_1)dt + \int_{T_1}^{T_2} p(x(t),u_2)dt$$
$$= \int_{x_0}^{x_0+\delta} \hat{h}(x,u_1)dx + \int_{x_0+\delta}^{x_0} \hat{h}(x,u_2)dx$$
$$= \int_{x_0}^{x_0+\delta} [\hat{h}(x,u_1) - \hat{h}(x,u_2)]dx. \qquad (5.4.2.10)$$

Since the integrand of the integral on the right-hand side of (5.4.2.10) is continuous and nonzero at every point of the interval $[x_0, x_0 + \delta]$, it follows that the integral itself is nonzero. This contradicts the assumption of losslessness, since $\{u(\cdot),x(\cdot)\}|[0, 0]$ is a valid input-trajectory pair of S from x_0 to x_0 which (unlike $\{u(\cdot),x(\cdot)\}|[0, T_2]$) consumes zero energy.

Thus $\hat{h}(x,u)$ depends only on x. If $pr_\Sigma(D)$ denotes the projection of D onto Σ (i.e., $pr_\Sigma(D) = \{x \in \Sigma: \exists\, u \in U$ such that $f(x,u) \neq 0\}$), then there exists a function $h: pr_\Sigma(D) \to \mathbb{R}$ such that

$$h(x) = \hat{h}(x,u) = \frac{p(x,u)}{f(x,u)} \text{ for all } (x,u) \in D. \qquad (5.4.2.11)$$

Note that $pr_\Sigma(D)$ is precisely the set of all nonsingular states of S. We shall define $h(\cdot)$ arbitrarily at singular states. From Theorem 5.3.1, we know that $p(x,u) = 0$ at all points $(x,u) \in \Sigma \times U$ such that $f(x,u) = 0$; hence, $p(x,u) = h(x)f(x,u)$ at <u>all</u> $(x,u) \in \Sigma \times U$.

(Sufficiency) Suppose that there exists a function $h: \Sigma \to \mathbb{R}$ such that $p(x,u) = h(x)f(x,u)$ for all $(x,u) \in \Sigma \times U$. Let $\{u_1(\cdot),x_1(\cdot)\}|[0, T_1]$ and $\{u_2(\cdot),x_2(\cdot)\}|[0, T_2]$ be any input-trajectory pairs of S for which $x_1(0) = x_2(0) \triangleq a$ and $x_1(T_1) =$

$x_2(T_2) \triangleq b$. We will show that S is lossless by showing that the energy consumed by $\{u_1(\cdot), x_1(\cdot)\}|[0, T_1]$ equals the energy consumed by $\{u_2(\cdot), x_2(\cdot)\}|[0, T_2]$. There are three cases which arise.

Case 1: a is singular. Then $a = b$ and both state trajectories are constant. We have

$$\int_0^{T_1} p(x_1(t), u_1(t))dt = \int_0^{T_1} h(a)\dot{x}_1(t)dt = 0 \qquad (5.4.2.12)$$

and

$$\int_0^{T_2} p(x_2(t), u_2(t))dt = \int_0^{T_2} h(a)\dot{x}_2(t)dt = 0, \qquad (5.4.2.13)$$

since $\dot{x}_1(t) \equiv \dot{x}_2(t) \equiv 0$.

Case 2: a and b are nonsingular. It follows that $x_1(t)$ is nonsingular for all $t \in [0, T_1]$ (otherwise, the condition $x_1(T_1) = b$ would be impossible). Thus

$$\int_0^{T_1} p(x_1(t), u_1(t))dt = \int_0^{T_1} h(x_1(t))\dot{x}_1(t)dt$$
$$= \int_a^b h(x)dx. \qquad (5.4.2.14)$$

The use of the Change of Variables formula in (5.4.2.14) is justified because $x_1(\cdot)$ is absolutely continuous and $h(\cdot)$ is continuous on $x_1([0, T_1])$[15, pp. 95-96, Theorem I.4.42]. Likewise,

$$\int_0^{T_2} p(x_2(t), u_2(t))dt = \int_a^b h(x)dx. \qquad (5.4.2.15)$$

Case 3: a is nonsingular but b is singular. Assume that $b > a$ (similar arguments apply when $b < a$). Define $T^* \triangleq \inf\{t \in [0, T_1]: x_1(t) = b\}$. Since b is singular and $x_1(\cdot)$ is continuous, $x_1(t) = b$ for $t \in [T^*, T_1]$; moreover, $x_1(t) < b$ for $t \in [0, T^*)$. Now

$$\int_0^{T_1} p(x_1(t),u_1(t))dt = \int_0^{T_1} h(x_1(t))\dot{x}_1(t)dt$$
$$= \int_0^{T^*} h(x_1(t))\dot{x}_1(t)dt$$
$$= \lim_{\substack{T\to T^* \\ T<T^*}} \int_0^{T} h(x_1(t))\dot{x}_1(t)dt$$
$$= \lim_{\substack{T\to T^* \\ T<T^*}} \int_a^{x_1(T)} h(x)dx$$
$$= \lim_{\substack{z\to b \\ z<b}} \int_a^{z} h(x)dx. \qquad (5.4.2.16)$$

The second step in (5.4.2.16) follows since $\dot{x}_1(t) = 0$ for $t \in (T^*, T_1)$, and the third step is justified because the integrand is bounded on $[0, T^*]$. The fourth step is a consequence of Case 2, which applies because $x_1(T)$ is nonsingular for $T \in [0, T^*)$, while the final step follows since $x_1(T) \to b$ as $T \to T^*$. Similarly,

$$\int_0^{T_2} p(x_2(t),u_2(t))dt = \lim_{\substack{z\to b \\ z<b}} \int_a^{z} h(x)dx. \qquad (5.4.2.17)$$

In all three cases we have shown that

$$\int_0^{T1} p(x_1(t),u_1(t))dt = \int_0^{T_2} p(x_2(t),u_2(t))dt; \qquad (5.4.2.18)$$

therefore, S is lossless. Q.E.D.