ANALYSIS OF THE POWER FLOW EQUATION

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ABSTRACT

The qualitative properties of the power flow equation for a transmission network are studied in terms of global and local aspects, as well as stability. Global aspects include estimates of the number of solutions and topological properties of the stable region. Local aspects are examined through bifurcations of the flow equation. Examples are given to counter some intuitive conjectures and other conjectures are made.

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1. Introduction

The load flow or power flow equation is a system of nonlinear simultaneous equations expressing the equality between power demand and supply at each node or bus of a transmission network in steady state synchronous operation. Understanding of the power flow equation is essential since it is responsible for the most significant nonlinearities encountered in transient and dynamic stability analysis, economic dispatch studies, and in security assessment and planning excercises.

Because of its central significance much effort has been devoted to the development of numerical methods for solving the power flow equation (see [10] for a review). By comparison scant attention has been given to the analytical investigation of the power flow equation, the references [2,4,6,11,12] and some additional work cited there include, we believe, all reported studies. This low level of effort is unfortunate since, as Galiana [4] has pointed out, analytical developments have sometimes suggested faster and more efficient algorithms, and can provide qualitative insights which escape the more numerically oriented approaches.

In this paper we present the results of our study of the power flow equation. The study is limited to lossless transmission networks consisting of PV buses only. Under this assumption, the power flow function f expresses the vector p of net real power injections at the various buses in terms of the vector θ of bus voltage angles, giving the power flow equation

$$p = f(\theta).$$

The problem is to solve this equation for θ when the value for p is specified, in other words, we wish to "invert" the function f. This inversion is difficult since the nonlinear nature of f is such that for a given p either there is no solution θ of the power flow equation, or there exist several solutions.

If we attempt to understand these difficulties from a mathematical viewpoint, then we are led to differentiate between the study of f in its global and local aspects. The former are relevant to questions about the size and shape of the range of f. The latter deal with the behavior of f in a small neighborhood. As an example of a local property, suppose $f(\theta_0) = p_0$; we know from the Inverse Function Theorem that if the Jacobian $\partial f(\theta_0)/\partial\theta$ is a nonsingular matrix, then as p varies continuously starting at p_0, there is a unique solution $\theta = \theta(p)$ which varies continuously starting at θ_0. (Indeed the success of numerical methods depends upon this fact.) Now if $\partial f(\theta_0)/\partial\theta$ is singular, then the solution $\theta(p)$ may not exist, or may not be unique. If one could tell in advance which of these different behaviors occurs, then such information could be used to design better numerical techniques.

The differentiation between local and global properties of the solutions of the power flow equation stems from differences in mathematical techniques. Consideration of generator dynamics show that certain vectors θ correspond to stable voltage configurations and so we want to distinguish between stable and unstable solutions.

The results presented below are concerned with global and local properties of the stable solutions of the power flow equation. In section 2 we obtain a convenient representation of the function f and its derivatives, define the region of stable solutions, and describe the global properties. Also examples are given to counter what seem to be plausible conjectures. Section 3 examines local properties through a study of fold and cusp bifurcations of the flow equation. Section 4 lists some conjectures. In a companion paper [15] the complete local and global behavior of a 3-node network is given.

2. Global Properties

This section begins with a convenient representation of the power flow function, then defines the stable region and obtains some global properties.

2.1 The power flow function

Consider a transmission network consisting of n + 1 buses or nodes. Taking the (n + 1)st bus voltage angle as reference, denote the voltage phasor at bus i by

$$V_i \exp(j\theta_i), \qquad i = 1,\ldots,n+1,$$

where, by definition,

$$\theta_{n+1} \equiv 0.$$

Then the real power injected into the network at bus i is given by (see [2,3,11])

$$p_i = \sum_{j=1}^{n+1} V_i V_j Y_{ij} \sin(\theta_i - \theta_j), \qquad i = 1,\ldots,n+1, \tag{2.1}$$

where $Y_{ji} = Y_{ij} \geq 0$ is the admittance (susceptance) of the lossless transmission line joining buses i and j. Since the magnitudes V_i are assumed fixed, without losing generality suppose that

$$V_i = 1, \qquad i = 1,\ldots,n+1.$$

Finally, observe that in (2.1) at most n of the p_i can be independently assigned since $p_1 + \ldots + p_{n+1} \equiv 0$. Using these facts we define the <u>power flow function</u> $f = (f_1, \ldots, f_n): R^n \to R^n$ by

$$p_i = f_i(\theta_1,\ldots,\theta_n) := \sum_{j=1}^{n+1} Y_{ij}\,\sin(\theta_i-\theta_j), \qquad i = 1,\ldots,n \tag{2.2}$$

where $\theta_{n+1} \equiv 0$. Denote the vectors $\theta = (\theta_1, \ldots, \theta_n)$ and $p = (p_1, \ldots, p_n)$. The function f is periodic with period 2π i.e., if θ, ϕ are such that $\theta_i - \phi_i = 0 (\mathrm{mod}\ 2\pi)$ for any i, then $f(\theta) = f(\phi)$. Henceforth we restrict the domain of f to the set

$$T^n := [-\pi,\pi]^n$$

with π and $-\pi$ identified. In particular when we say the $f(\theta) = p$ has a certain number of solutions θ we only consider $\theta \in T^n$.

It will be convenient to derive an alternative expression for f. The graph of the transmission network consists of n + 1 nodes and there is a branch joining nodes i and j if and only if $Y_{ij} > 0$. Suppose there are b branches indexed $\ell = 1, \ldots, b$. Let A denote the n x b reduced incidence matrix of the graph taking node n + 1 as reference, and Y denote the b x b diagonal matrix with the ℓth entry Y_{ij} if branch ℓ joins nodes i and j. Then, as can be verified directly, an alternative expression for the power flow function is

$$f(\theta) = AYs(A^T\theta), \tag{2.3}$$

where for $\psi := A^T\theta \in R^b$, $s(\psi) \in R^b$ is defined as

$$s(\psi) := (\sin\psi_1,\ldots,\sin\psi_b). \tag{2.4}$$

It is assumed throughout that the network is connected so that A has rank n.

<u>Notation</u> For $\psi \in R^b$ let $S(\psi)$, respectively $C(\psi)$, denote the b x b diagonal matrix whose ℓth entry is $\sin\psi_\ell$, respectively $\cos\psi_\ell$.

The usefulnes of this notation and the representation (2.3) comes from the next result which is obtained by direct calculation.

<u>Lemma 2.1</u> Fix $\theta \in T^n$ and $\delta \in R^n$. Let $\theta_t := \theta + t\delta (\text{mod } 2\pi)$. The directional derivatives $f^k(\delta) := \partial^k f(\theta_t)/\partial t^k|_{t=0}$, $k \geq 1$, are given by

$$f^1(\delta) = AY[C(A^T\theta)]A^T\delta,$$

$$f^2(\delta) = -AY[S(A^T\theta)](A^T\delta)^2$$

$$f^3(\delta) = -AY[C(A^T\theta)](A^T\delta)^3,$$

$$f^4(\delta) = AY[S(A^T\theta)](A^T\delta)^4, \text{ etc.}$$

Here if $\psi := A^T\delta$, then $(A^T\delta)^k := (\psi_1^k, \ldots, \psi_b^k)$.

From the lemma it follows that the Jacobian of f at θ is the n x n symmetric matrix

$$F(\theta) := \frac{\partial f}{\partial \theta} = AYC(A^T\theta)A^T. \tag{2.5}$$

From this expression we see that $F(\theta)$ can be interpreted as the node admittance matrix of a linear resistive network with conductance of branch ℓ connecting nodes i and j equal to $Y_{ij} \cos(\theta_i - \theta_j)$ (see [1]).

2.2 <u>The stable region</u>

We write $F(\theta) > 0$ or $F(\theta) \geq 0$, according as it is positive definite or positive semi-definite. $\theta \in T^n$ is said to be <u>stable</u> if $F(\theta) > 0$, and let Θ_s denote the stable region. If $\theta \notin \Theta_s$ we call it unstable. We briefly indicate the reason for the term 'stable'. Suppose to each bus $i = 1, \ldots, n$ is attached a generator and suppose bus $n + 1$ is an infinite bus. Then the motion of the generators is governed by the so-called "swing" equations [2,9],

$$M_i\ddot{\theta}_i+D_i\dot{\theta}_i = p_i-f_i(\theta), \qquad i = 1,\ldots,n. \tag{2.6}$$

Here M_i, respectively D_i, is the moment of inertia, respectively damping constant, of the ith generator, and p_i is the mechanical power input to the generator minus the electrical load at the ith bus. Evidently then, the state $(\dot{\theta},\theta)$ is an equilibrium if and only if $\dot{\theta} = 0$ and $f(\theta) = p$. If the equation (2.6) is linearized around an equilibrium $(\dot{\theta} = 0,\theta)$, then it can be directly verified that the eigenvalues of the linearized system lie in the open left-half plane (which implies that the equilibrium is Lyapunov stable) if and only if $F(\theta) > 0$. A more delicate argument can be used to arrive at the same conclusion even when some of the buses are load buses (see [9]).

2.3 Geometric properties of Θ_s

The preceding remark implies that in the steady state the transmission system must operate at a stable equilibrium, and so the shape of the stable region is of interest. Certain "positive" properties of Θ_s are obtained by comparing it with various polytopes of T^n; the main "negative' and possibly surprising conclusion is that Θ_s may be disconnected.

__Definition__ Θ_* is the set of all θ in T^n such that (i) $|\theta_i - \theta_j| \leq \frac{\pi}{2}(\mathrm{mod}\ 2\pi)$ whenever $Y_{ij} > 0$ and (ii) the set of all branches (i,j) such that $Y_{ij} > 0$ and $|\theta_i - \theta_j| = \frac{\pi}{2}(\mathrm{mod}\ 2\pi)$ do not form a cut set of the network graph.

The next result is known [11]. We give a different proof.

__Lemma 2.2__ $\Theta_* \subset \Theta_s$.

__Proof__ $F(\theta) = AYC(A^T\theta)A^T$ and if $\theta \in \Theta_*$ then the diagonal matrix $YC(A^T\theta)$ has non-negative entries and so $F(\theta) \geq 0$. Hence $F(\theta) > 0$ if and only if $\det F(\theta) > 0$. Now $F(\theta)$ is a node-admittance matrix of a linear network and so by the Seshu-Reed result [1],

$$\det F(\theta) = \sum_{\tau} \prod_{(i,j)\in\tau} Y_{ij} \cos(\theta_i - \theta_j)$$

where the sum is over all trees τ of the graph. Since each conductance is positive, $\det F(\theta) = 0$ only if in each tree τ there is a branch (i,j) with $\cos(\theta_i - \theta_j) = 0$, i.e. there is a cutset of branches with $|\theta_i - \theta_j| = \frac{\pi}{2} (\text{mod } 2\pi)$. ¤

<u>Lemma 2.3</u> Let θ be stable and let $\phi = (\phi_1, \ldots, \phi_n) \neq 0$ be such that for each $i, \phi_i = 0$ or $\phi_i = \pi$. Then $(\theta + \phi)$ is unstable.

<u>Proof</u> Suppose for some $0 < k \leq n$, $\phi_i = \pi$ for $i \leq k$ and $\phi_i = 0$ for $k < i \leq n$. Let $x = (x_1, \ldots, x_n)$ be such that $x_i = 1$ for $i \leq k$ and $x_i = 0$ for $k < i \leq n$. From (2.5),

$$x^T F(\theta) x = \sum_{i,j=1}^{n+1} (x_i - x_j)^2 \, Y_{ij} \cos(\theta_i - \theta_j)$$

$$= 2 \sum_{i=1}^{k} \sum_{j=k+1}^{n+1} Y_{ij} \cos(\theta_i - \theta_j) > 0,$$

since θ is stable. (Here and below $x_{n+1} = 0$, $\theta_{n+1} = 0$.) On the other hand,

$$x^T F(\theta+\phi) x = 2 \sum_{i=1}^{k} \sum_{j=k+1}^{n+1} Y_{ij} \cos(\theta_i + \pi - \theta_j) = -x^T F(\theta) x < 0,$$

so $\theta + \phi$ is unstable.

Call $\theta + \phi$ a <u>π-translate</u> of θ if the two vectors are related as in the statement of the lemma. Each stable θ has $2^n - 1$ distinct π-translates so that we can say that Θ_s occupies approximately the fraction $\frac{1}{2^n}$ of the volume of T^n.

Definition [11] The principal polytope Θ_p is the subset of all θ in Θ_* such that $|\theta_i - \theta_j| \leq \frac{\pi}{2}$ whenever $Y_{ij} > 0$.

The power transmitted over a line is $|Y_{ij} \sin(\theta_i - \theta_j)|$ and so its maximum value is Y_{ij}. This value usually exceeds the thermal capacity of the line and so under normal operating conditions $|\sin(\theta_i - \theta_j)| < 1$. In turn this usually implies $|\theta_i - \theta_j| < \frac{\pi}{2}$ i.e. $\theta \in \Theta_p$. Θ_p is convex by definition and stable by Lemma 2.2. Another attractive property is that f is one-to-one on Θ_p. This is a corollary of the next result.

Lemma 2.4 [2] For $\theta^k \in T^n$, k = 1, 2 let $\psi^k = (\psi_1^k, \ldots, \psi_b^k) := A^T\theta^k$. Suppose for each ℓ, $|\psi_\ell^k| \leq \frac{\pi}{2}$ and $\psi_\ell^2 \in (-\pi - \psi_\ell^1, \pi - \psi_\ell^1)$. Then $f(\theta^1) \neq f(\theta^2)$ if $\theta^1 \neq \theta^2$.

Proof From (2.3),

$$
\begin{aligned}
(f(\theta^1)-f(\theta^2))^T(\theta^1-\theta^2) &= (AYs(\psi^1)-AYs(\psi^2))^T(\theta^1-\theta^2) \\
&= (Ys(\psi^1)-Ys(\psi^2))^T(\psi^1-\psi^2) \\
&= \sum Y_\ell(\sin\psi_\ell^1-\sin\psi_\ell^2)(\psi_\ell^1-\psi_\ell^2).
\end{aligned}
$$

Suppose $|\psi_\ell^1| \leq \frac{\pi}{2}$. Then $\sin\psi_\ell^1 < \sin\psi_\ell^2$ if $\psi_\ell^1 < \psi_\ell^2 < \pi - \psi_\ell^1$, and $\sin\psi_\ell^1 > \sin\psi_\ell^2$ if $\psi_\ell^1 > \psi_\ell^2 > -\pi - \psi_\ell^1$; hence $\sum Y_\ell(\sin\psi_\ell^1 - \sin\psi_\ell^2)(\psi_\ell^1 - \psi_\ell^2) > 0$ unless $\psi^1 = \psi^2$. Since A has rank n, $\psi^1 = \psi^2$ implies $\theta^1 = \theta^2$, and the assertion follows. ¤

Corollary 2.1 The power flow function f is one-to-one on Θ_p.

Proof If $\theta^k \in \Theta_p$, k = 1, 2 then by definition $|\psi_\ell^k| \leq \frac{\pi}{2}$ and so the lemma above applies.

For a specified power demand vector p therefore, the power flow equation $f(\theta) = p$ has at most one solution in Θ_p. It might be conjectured that there is at most one solution in the entire stable region Θ_s. An example due to Korsak [6] shows that to be false. We analyze this example further to show that <u>the stable region can be disconnected</u>.

<u>Example 2.1</u> Consider the 5-node loop network of Figure 1 in which all $Y_{ij} = 1$. Then

$$f(\theta) = \begin{bmatrix} \sin\theta_1+\sin(\theta_1-\theta_2) \\ \sin(\theta_2-\theta_1)+\sin(\theta_2-\theta_3) \\ \sin(\theta_3-\theta_2)+\sin(\theta_3-\theta_4) \\ \sin(\theta_4-\theta_3)+\sin\theta_4 \end{bmatrix}$$

$$F(\theta) = \begin{bmatrix} \cos\theta_1+\cos(\theta_1-\theta_2) & -\cos(\theta_1-\theta_2) & 0 & 0 \\ -\cos(\theta_1-\theta_2) & \cos(\theta_2-\theta_1)+\cos(\theta_2-\theta_3) & -\cos(\theta_2-\theta_3) & 0 \\ 0 & -\cos(\theta_2-\theta_3) & \cos(\theta_2-\theta_3)+\cos(\theta_3-\theta_4) & -\cos(\theta_3-\theta_4) \\ 0 & 0 & -\cos(\theta_3-\theta_4) & \cos(\theta_3-\theta_4)+\cos\theta_4 \end{bmatrix}$$

Let $\theta^2 := 0$ and $\theta^1 := (\frac{2}{5}\pi, \frac{4}{5}\pi, -\frac{4}{5}\pi, -\frac{2}{5}\pi, 0)$. Direct verification shows that $f(\theta^1) = f(\theta^2) = 0$ i.e. both solutions give zero net demand. However θ^1 yields a "circulating" power. Direct verification also shows that $F(\theta^1) > 0$, $F(\theta^2) > 0$ so that both solutions are stable.

Consider now any continuous curve $\theta(t)$, $0 \le t \le 1$ such that $\theta(0) = \theta^1$ and $\theta(1) = \theta^2 \bmod(2\pi)$. It will be shown that for some $t, \theta(t)$ must be unstable. The proof is based on the observation that if $F(\theta) > 0$ then none of the angle differences $|\theta_i - \theta_{i+1}|$ can equal $\pi(\bmod 2\pi)$ (here $\theta_5 \equiv 0$), because otherwise one of the diagonal entries of $F(\theta)$ will become zero or negative. Since $\theta(0) = \theta^1$, the observation implies that $\theta(t)$ is stable for all t only if

$$-\pi < \theta_1(t) < \pi, \tag{2.7a}$$

$$-\pi < \theta_2(t)-\theta_1(t) < \pi, \tag{2.7b}$$

$$\pi < \theta_2(t)-\theta_3(t) < 3\pi, \tag{2.7c}$$

$$-\pi < \theta_3(t)-\theta_4(t) < \pi, \tag{2.7d}$$

$$-\pi < \theta_4(t) < \pi, \tag{2.7e}$$

for all t. Since $\theta(1) = 0(\text{mod } 2\pi)$, therefore, (2.7a), (2.7b) imply $\theta_1(1) = \theta_2(1) = 0$, and (2.7d), (2.7e) imply $\theta_3(1) = \theta_4(1) = 0$, and so (2.7c) cannot be satisfied. It is thus impossible to control the system in such a way as to eliminate the circulating power (corresponding to θ^1) without going through the unstable region.

The example shows that in general the stable region is a union of disjoint connected components. Under normal conditions θ lies in the connected component containing the origin which following [11] we call the <u>principal component</u> and denote it by Θ_c.

Evidently $\Theta_p \subset \Theta_c$. It has been conjectured [11] that Θ_c is convex. A partial result in this direction is given next. Define the polytope

$$\Theta_\pi := \{\theta \in T^n |\ |\theta_i-\theta_j| \leq \pi \text{ whenever } Y_{ij}>0\}.$$

<u>Lemma 2.5</u> Let $\theta \in \Theta_\pi \cap \bar{\Theta}_s$ (the closure of Θ_s). Then for $0 \leq \varepsilon < 1$, $\varepsilon\theta \in \Theta_s$.

<u>Proof</u> From (2.5), for any vector $x \in R^n$

$$x^T F(\varepsilon\theta)x = \sum_{i,j=1}^{n+1} (x_i-x_j)^2 Y_{ij} \cos(\varepsilon\theta_i-\varepsilon\theta_j),$$

where, as usual, $x_{n+1} = 0$ and $\theta_{n+1} = 0$. Suppose $x \neq 0$. Since $|\theta_i - \theta_j| \leq \pi$, therefore $\cos(\varepsilon(\theta_i - \theta_j)) > \cos(\theta_i - \theta_j)$ for $0 \leq \varepsilon < 1$ and so

$$x^T F(\varepsilon\theta)x > x^T F(\theta)x \geq 0$$

where the last inequality holds since $\theta \in \bar{\Theta}_s$ implies $F(\theta) \geq 0$. ¤

2.4 Topological properties of Θ_s

In this section the study of Θ_s is based on the derivatives of the flow function. The main result shows the intuitive property that the farther a stable θ is from the boundary of Θ_s the larger its margin of stability - the margin of stability being measured by the smallest eigenvalue of $F(\theta)$.

The next lemma is preliminary.

Lemma 2.6 For $1 \leq k \leq n$ let $e^k := (0, \ldots, 1, 0, \ldots 0)^T$ be the kth unit vector. Let x, y in R^n and θ in T^n be arbitrary. Consider the function

$$g_k(t) := y^T F(\theta + te^k)x.$$

Suppose $g_k^{(1)}(0) := \left.\frac{\partial g_k}{\partial t}\right|_{t=0} = 0$. Then

$$g_k(t) = g_k(0) + g_k^{(2)}(0)\cos t, \qquad \text{for all } t.$$

Moreover, $\sum_{k=1}^{n} g_k^{(2)}(0) = -g_k(0) = -y^T F(\theta)x$.

Proof Setting $x_{n+1} = y_{n+1} = \theta_{n+1} = 0$ as usual, and using Lemma 2.1, we obtain

$$g_k(0) = \sum_{i,j=1}^{n+1} (x_i - x_j)(y_i - y_j)Y_{ij} \cos(\theta_i - \theta_j),$$

$$g_k^{(2r+1)}(0) = \sum_{i,j}(-1)^{r+1}(x_i-x_j)(y_i-y_j)(e_i^k-e_j^k)^{2r+1}\gamma_{ij}\ \sin(\theta_i-\theta_j),$$

$$g_k^{(2r)}(0) = \sum_{i,j}(-1)^{r}(x_i-x_j)(y_i-y_j)(e_i^k-e_j^k)^{2r}\gamma_{ij}\ \cos(\theta_i-\theta_j).$$

Since $(e_i^k - e_j^k)^{2r+1} = e_i^k - e_j^k$, $r \geq 0$, and $(e_i^k - e_i^k)^{2r} = (e_i^k - e_j^k)^2$, $r \geq 1$, the preceding expressions simplify as

$$g_k^{(2r)}(0) = (-1)^r g_k^{(2)}(0), \qquad r \geq 1,$$

$$g_k^{(2r+1)}(0) = (-1)^r g_k^{(1)}(0) = 0, \qquad \text{by hypothesis,}$$

and so

$$g_k(t) = g_k(0)+g_k^{(2)}(0)\sum_{r=0}^{\infty}\frac{(-1)^r t^r}{2r!} = g_k(0)+g_k^{(2)}(0)\ \cos t.$$

Finally, since

$$\sum_{k=1}^{n}(e_i^k-e_j^k)^2 = 1, \qquad \text{for all } i,j \text{ except } i = j,$$

therefore

$$\sum_{k=1}^{n} g_k^{(2)}(0) = -\sum_{i,j}(x_i-x_j)(y_i-y_j)\gamma_{ij}\ \cos(\theta_i-\theta_j) = -g_k(0). \qquad ¤$$

The next lemma is key to the succeeding results.

<u>Lemma 2.7</u> Let $U \subset T^n$ be an open set. Let $M(\theta)$ $[m(\theta)]$ denote the maximum [minimum] eigenvalue of $F(\theta)$. Suppose $M(\theta)[m(\theta)]$ achieves a maximum [minimum] value $\overline{M}$ $[\overline{m}]$ in U. Then $\overline{M} > 0$ $[\overline{m} < 0]$.

<u>Proof</u> We only prove $\overline{M} > 0$, the proof that $\overline{m} < 0$ is similar. Let $\theta \in U$ be such that $M(\theta) = \overline{M}$, and let $x \neq 0$ be the corresponding eigenvector of $F(\theta)$. Then for t so small that $(\theta + te^k) \in U$,

$$0 \geq g_k(t)-\overline{M}|x|^2 := x^T F(\theta+te^k)x-\overline{M}|x|^2 ,$$

$$0 = g_k(0)-\overline{M}|x|^2,$$

where $|x|^2 := x^T x$. Therefore, $g_k(t)$ achieves a local maximum at $t = 0$, so

$$g_k^{(1)}(0) = 0$$

and

$$g_k^{(2)}(0) \leq 0$$

By Lemma 2.6

$$0 \geq \sum_k g_k^{(2)}(0) = -g_k(0) = -M|x|^2,$$

which gives

$$\overline{M} \geq 0.$$

Suppose, contrary to the assertion, that $\overline{M} = 0$. Then $g_k(0) = 0$, $g_k^{(2)}(0) = 0$, and so by Lemma 2.6 $g_k(t) = 0$. Moreover $M(\theta) \leq \overline{M} = 0$ implies $F(\theta)$ is negative semi-definite and so $g_k(t) \equiv 0$ implies

$$F(\theta+te^k)x = 0, \qquad \text{for all } k \text{ and } t.$$

Now consider any $\theta_u^k := (\theta + ue^k) \in U$. Applying the same argument as above with θ_u^k replacing θ leads to the conclusion that

$$F(\theta_u^k+te^j)x = 0, \qquad \text{for all } j \text{ and } t$$

Proceeding in this way one finds

$$F(\phi)x = 0$$

for all ϕ in a neighborhood of θ. Since $F(\phi)x$ is an analytic function of ϕ this implies $F(\phi)x \equiv 0$ for all ϕ. But we know from Lemma 2.2 that $F(0) > 0$ and, so $F(0)x \neq 0$. Hence $\overline{M} > 0$ as asserted. ¤

If θ is stable, it is reasonable to interpret the smallest eigenvalue $m(\theta)$ of $F(\theta)$ as the <u>margin of stability</u> at θ. As a corollary of Lemma 2.7 we have the following intuitive result.

<u>Corollary 2.2</u> Let $U \subset \Theta_s$ be an open stable set, $\mathcal{B}$ its boundary and $\overline{U} = U \cup \mathcal{B}$ its closure. Then the minimum of $m(\theta)$ as θ varies over $\overline{U}$ is achieved only on $\mathcal{B}$.

<u>Proof</u> If the minimum is achieved at $\theta \in U$, then by Lemma 2.7 $m(\theta) < 0$. But θ is stable and so $m(\theta) > 0$. ¤

The preceeding corollary would be more appealing if we know that Θ_s is a 'solid' open set i.e. there are no lower dimensional unstable sets 'inside' Θ_s. Mathematically this means that Θ_s equals the interior of its closure. The proof of this needs an intermediate step. Let Θ_0 denote the collection of all θ such that $F(\theta)$ is positive semi-definite but <u>not</u> positive definite.

<u>Corollary 2.3</u> Every neighborhood of $\theta \in \Theta_0$ contains θ^1 and θ^2 with $M(\theta^1) > 0$ and $m(\theta^2) < 0$.

<u>Proof</u> Immediate from Lemma 2.7. ¤

<u>Corollary 2.4</u> Θ_s is the interior of $\overline{\Theta}_s$.

<u>Proof</u> Observe that $\overline{\Theta}_s \subset \Theta_s \cup \Theta_0$ and then use Corollary 2.3. ¤

We close this section with a couple of results dealing with the boundary of Θ_s. Call $\mathcal{B}_s$ the boundary of the stable region. Evidently

$B_s \subset \Theta_0$, and it is reasonable to conjecture equality. The next example shows this to be false.

Example 2.2 Consider the 4-node loop network of Figure 2 in which all $Y_{ij} = 1$. Then

$$f(\theta) = \begin{bmatrix} \sin\theta_1 + \sin(\theta_1-\theta_2) \\ \sin(\theta_2-\theta_1)+\sin(\theta_2-\theta_3) \\ \sin(\theta_3-\theta_2)+\sin\theta_3 \end{bmatrix}$$

$$F(\theta) = \begin{bmatrix} \cos\theta_1+\cos(\theta_1-\theta_2) & -\cos(\theta_1-\theta_2) & 0 \\ -\cos(\theta_1-\theta_2) & \cos(\theta_1-\theta_2)+\cos(\theta_2-\theta_3) & -\cos(\theta_2-\theta_3) \\ 0 & -\cos(\theta_2-\theta_3) & \cos(\theta_2-\theta_3)+\cos\theta_3 \end{bmatrix}$$

It can be verified directly that $F(\theta)$ vanishes at $\theta = \phi := (\frac{\pi}{2}, \pi, -\frac{\pi}{2})$. Hence $\phi \in \Theta_0$. Then at $\theta = \phi + \delta$ one gets

$$F(\phi+\delta) = \begin{bmatrix} -\sin\delta_1+\sin(\delta_1-\delta_2) & -\sin(\delta_1-\delta_2) & 0 \\ -\sin(\delta_1-\delta_2) & \sin(\delta_1-\delta_2)+\sin(\delta_2-\delta_3) & -\sin(\delta_2-\delta_3) \\ 0 & -\sin(\delta_2-\delta_3) & \sin(\delta_2-\delta_3)+\sin\delta_3 \end{bmatrix}$$

If $F(\phi + \delta) > 0$ then its diagonal entries must be positive. Suppose δ is sufficiently small, and consider two cases.

Case 1 $\delta_1 \geq 0$. Then $-\sin\delta_1 + \sin(\delta_1 - \delta_2) > 0$ requires $\delta_2 < 0$. But then $\sin(\delta_2 - \delta_3) + \sin\delta_3 < 0$ for all $|\delta_3| < |\delta_2|$. Hence $(\phi + \delta)$ is unstable.

Case 2 $\delta_1 < 0$. Then $-\sin\delta_1 + \sin(\delta_1 - \delta_2) > 0$ also requires $\delta_2 < 0$, and so once again $(\phi + \delta)$ is unstable.

This proves that $\phi \notin \mathcal{B}_s$.

This example shows that Θ_0 may be strictly larger than $\mathcal{B}_s$. This suggests that it may not be possible to analytically describe the boundary $\mathcal{B}_s$. However, we can get a partial result as an immediate corollary of Lemma 2.5.

<u>Corollary 2.5</u> $\Theta_0 \cap \Theta_\pi = \mathcal{B}_s \cap \Theta_\pi$.

The next example shows that the boundary $\mathcal{B}_s$ may have "corners".

<u>Example 2.3</u> Consider the 3-node series network of Figure 3 in which the $Y_{ij} = 1$. It is straightforward to check that $\Theta_s = \{\theta \mid |\theta_1 - \theta_2| < \frac{\pi}{2}$ and $|\theta_2| < \frac{\pi}{2}\}$, and that

$$\mathcal{B}_s = \Theta_0 = \{\theta \mid |\theta_1 - \theta_2| = \tfrac{\pi}{2}, |\theta_2| < \tfrac{\pi}{2}\} \cup \{\theta \mid |\theta_1 - \theta_2| \le \tfrac{\pi}{2}, |\theta_2| = \tfrac{\pi}{2}\}.$$

This boundary has four corners at $\theta_1 = 0$, $\theta_2 = \pm\frac{\pi}{2}$ and $\theta_1 = \theta_2 = \pm\frac{\pi}{2}$. Observe that at each of these corner points $F(\theta)$ vanishes i.e. $F(\theta)$ has more than one zero eigenvalue. The last result of this section clarifies this further.

<u>Lemma 2.8</u> Let $\theta \in \Theta_0$ be such that $F(\theta)$ has exactly one zero eigenvalue. Then there is an open set U containing θ such that $U \cap \Theta_0$ is an analytic set.

<u>Proof</u> Let $d(\theta) := \det F(\theta)$. We claim that there is an open set $\underline{U}$ containing θ such that $\Theta_0 \cap U = \{\theta \mid d(\theta) = 0\} \cap U$. If this is false then there is a sequence $\theta^k \to \theta$ with $\theta^k \notin \Theta_0$ and $d(\theta^k) = 0$. The eigenvalues of $F(\theta^k)$ can be arranged as $\lambda_1^k < \ldots < \lambda_r^k \le \lambda_{r+1}^k = 0 \le \ldots \le \lambda_n^k$ and $r \ge 1$ since $\theta^k \notin \Theta_0$. As $k \to \infty$ we must have $\lambda_1^k \to 0$ and so $F(\theta^0)$ has at least $r + 1 \ge 2$ zero eigenvalues, contradicting the hypothesis. The assertion follows since $d(\theta)$ is an analytic function. ¤

2.5 Solutions of $f(\theta) = p$

It is shown first that the number of solutions to the power flow equation is 'usually' even, and that the number of unstable solutions exceeds the number of stable solutions. Then an example is given in which the equation has <u>no</u> stable solution even when it does have an unstable solution.

For any power demand vector p, let $\Theta(p)$ be the set of solutions θ in T^n of the power flow equation $f(\theta) = p$. Say that p is <u>regular</u> if $\det F(\theta) \neq 0$ for all θ in $\Theta(p)$. It is easy to see that if p is regular then $\Theta(p)$ is finite and so we can define the <u>degree</u> of f

$$d(f;p) = \sum_{\theta \in \Theta(p)} \text{sign } F(\theta),$$

where sign $F(\theta)$ equals +1 or -1 according as $\det F(\theta) > 0$ or < 0. It is known [13] that for T^n the degree is zero independently of f. This gives the next result.

<u>Lemma 2.9</u> If p is regular then $\Theta(p)$ contains an even number of solutions. Also the number of unstable solutions is at least as large as the number of stable solutions.

<u>Proof</u> The first assertion is immediate since $d(f;p) = 0$. The second assertion follows from this and the fact that sign $F(\theta) = +1$ if θ is stable.

<u>Example 2.4</u> Consider the 6-node loop network of Figure 4 in which the $Y_{ij} = 1$. Then $f_i(\theta) = \sin(\theta_i - \theta_{i+1}) + \sin(\theta_i - \theta_{i-1})$, $1 \leq i \leq 5$, where $\theta_0 = \theta_6 \equiv 0$. Let

$$\theta^0 = (\tfrac{\pi}{2}, \tfrac{\pi}{4}, 0, \tfrac{\pi}{2}, \tfrac{\pi}{4})^T$$

corresponding to which is the power demand vector

$$p^0 = f(\theta^0) = (1+\sin\tfrac{\pi}{4},\ 0,\ -1+\sin\tfrac{\pi}{4},\ 1+\sin\tfrac{\pi}{4},\ 0)^T.$$

Observe that

$$\theta(\varepsilon) := (\tfrac{\pi}{2}-\varepsilon,\ \tfrac{\pi}{4},\ 0,\ \tfrac{\pi}{2}-\varepsilon,\ \tfrac{\pi}{4})^T$$

is in the principal polytope for $0 < \varepsilon < \frac{\pi}{2}$, hence it is stable. It follows that θ^0 is on the boundary of the stable region.

Let $p^t := p^0 + t(1, 0, 0, 0, 0)^T$. It is shown in section 3.3 that there is a neighborhood N_θ of θ^0 such that for $t > 0$ sufficiently small, the only solution in N_θ of $f(\theta) = p^t$ is unstable. Let $t_n > 0$ be a sequence decreasing to zero and let $f(\theta^n) = p^{t_n}$. Suppose θ^n converges to θ, and suppose θ is stable. Then $\theta \neq \theta^0$ (since $\theta^n \notin N_\theta$ for all n), and $f(\theta) = p^0$, $F(\theta) \geq 0$. It can be shown by a direct, but lengthy calculation which is omitted here that this is impossible. We state this as a lemma.

<u>Lemma 2.10</u> There exists $\bar{t} > 0$ such that for $0 < t < \bar{t}$, $f(\theta) = p^t$ has no stable solution.

Thus <u>there exist power demand vectors which can be met by unstable solutions but by no stable solution</u>.

3. Local Properties

This section is devoted to a study of the qualitative changes in the solution of the flow equation $f(\theta) = p$ as the demand vector p is varied. It is believed that this study will lead to better understanding of the behavior of numerical algorithms for solving the flow equation. These algorithms are often based on continuation methods in which one seeks to obtain in a sequence of steps a solution $\bar{\theta}$ of $f(\theta) = \bar{p}$ starting with a known solution θ^1 of $f(\theta) = p^1$. At the ith step, $i = 1, 2, \ldots,$ one finds a solution θ^{i+1} of $f(\theta) = p^{i+1}$ starting with θ^i and using a locally convergent algorithm such as Newton or Gauss-Seidel. To guarantee success of such a continuation method the "step size" $|p^{i+1} - p^i|$ must be small and, of course, (p^i, θ^i) must converge to $\bar{p}$, $\bar{\theta}$.

Let (θ, p) satisfy $f(\theta) - p = 0$. Suppose $F(\theta)$ is nonsingular. By the Inverse Function Theorem there are neighborhoods N_θ of θ and N_p of p and a function $g: N_p \to N$ such that $f(\theta') - p'$ and (θ', p') is in $N_\theta \times N_p$ if and only if $\theta' = g(p')$. In other words if $F(\theta)$ is nonsingular, there is locally a unique continuation starting at (θ, p). Suppose however that $F(\theta)$ is singular. Then there may still be a unique continuation so that the local behavior is similar to when $F(\theta)$ is nonsingular; but it is also possible that there is no continuation or that it is not unique. In the latter case the qualitative behavior has changed - one then says that p is a bifurcation point. It may be worth noting that computationally, a bifurcation may reveal itself in numerical instability as the surface $\det F(\theta) = 0$ is approached.

It is clear that the study of bifurcations of the flow equation requires a study of the second and higher derivatives of the flow function. We begin by recalling from Hale [5] (see also [7]) the necessary results from bifurcation theory, and then apply them to the flow function. The

6-node network of Figure 4 is then examined as an illustration.

3.1 Some results about bifurcations

Let x, λ denote vectors in R^n and let $g\colon R^n \to R^n$ be twice differentiable. Let $G(x) := \frac{\partial g}{\partial x}(x)$ denote the Jacobian of g. Consider the equation

$$g(x)-\lambda = 0, \tag{3.1}$$

and suppose (x^0, λ^0) satisfies (3.1).

Suppose $G(x^0)$ is a singular matrix of rank $n - 1$. Then one can find nonsingular $n \times n$ matrices $[P \,\vdots\, w]$ and $[Q \,\vdots\, z]$ with w and z in R^n such that

$$Q^T G(x^0) P \text{ is nonsingular,}$$

$$z^T G(x^0) = 0,$$

$$G(x^0) w = 0.$$

Equation (3.1) can be "decomposed" as

$$Q^T g(x) - Q^T\lambda = 0, \qquad z^T g(x) - z^T\lambda = 0. \tag{3.2}$$

Define

$$\mu := Q^T\lambda \in R^{n-1}, \qquad \mu^0 := Q^T\lambda^0,$$

$$\rho := z^T\lambda \in R \quad , \qquad \rho^0 := z^T\lambda^0.$$

We also represent x in R^n by

$$x := Py+uw$$

where $y \in R^{n-1}$, $u \in R$. With this choice of coordinates (3.2) can be rewritten as

$$Q^T g(Py+uw)-\mu = 0, \tag{3.3}$$

$$z^T g(Py+uw)-\rho = 0. \tag{3.4}$$

Let y^0, u^0 (necessarily unique) be such that $x^0 = Py^0 + u^0w$. Equation (3.3) is well behaved, since by the Implicit Function Theorem there is a neighborhood N of (y^0, u^0, μ^0) and a function $y^*(u, \mu)$ such that

$$y^0 = y^*(u^0,\mu^0),$$

$$Q^T g(Py+uw)-\mu = 0 \text{ in } N \Leftrightarrow y = y^*(u,\mu) \tag{3.5}$$

Next, rewrite (3.4) as

$$z^T g(Py^*(u,\mu)+uw)-\rho = 0.$$

Define

$$h(u,\mu) := z^T g(Py^*(u,\mu)+uw),$$

$$x^*(u,\mu) := Py^*(u,\mu)+uw;$$

$$x^*(u) := x^*(u,\mu^0)$$

Then $x^*(u^0, \mu^0) = x^0$. Equation (3.4) is now written as

$$h(u,\mu)-\rho = 0 \tag{3.6}$$

Note that $h(u^0, \mu^0) - \rho^0 = 0$. Also

$$\frac{\partial h}{\partial u}(u,\mu) = z^T G(x^*(u,\mu))\left\{P\frac{\partial y^*}{\partial u}(u,\mu)+w\right\}, \tag{3.7}$$

and so

$$\frac{\partial h}{\partial u}(u^0,\mu^0) = 0,$$

since

$$z^T G(x^0) = 0.$$

Further differentiation of (3.7) yields

$$\frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) = z^T\frac{\partial}{\partial u}[G(x^*(u))]\bigg|_{u^0}[P\frac{\partial y^*}{\partial u}(u^0,\mu^0)+w] \tag{3.8}$$

Next, from (3.5),

$$Q^Tg(Py^*(u,\mu)+uw)-\mu \equiv 0,$$

so, differentiation with respect to u gives

$$Q^TG(x^*(u,\mu))\left\{P\frac{\partial y^*}{\partial u}(u,\mu)+w\right\} = 0.$$

Since $G(x^0)\ w = 0$ and $Q^T\ G(x^0)\ P$ is nonsingular, this implies

$$\frac{\partial y^*}{\partial u}(u^0,\mu^0) = 0. \tag{3.9}$$

Using this in (3.8) gives

$$\frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) = z^T\frac{\partial}{\partial u}[G(x^*(u))]\bigg|_{u^0}w. \tag{3.10}$$

Denote

$$\begin{aligned}\psi^T &:= \frac{\partial}{\partial u}[z^TG(x^*(u))](u^0)\\ &= \lim_{u\to u^0}\frac{1}{u-u^0}[z^TG(x^*(u))-z^TG(x^*(u^0))]\\ &= \lim_{u\to u^0}\frac{1}{u-u^0}\ z^TG(x^*(u)),\end{aligned}$$

since

$$z^TG(x^0) = 0.$$

Write

$$x^*(u) = x^0+(u-u^0)x^1+o(u-u^0).$$

Then

$$\psi^T = \lim_{u\to 0} z^T G(x^0+ux^1). \tag{3.11}$$

Also, by definition

$$\begin{aligned} x^*(u) &= Py^*(u,\mu^0)+uw \\ &= x^0+(u-u^0)\{P\frac{\partial y^*}{\partial u}(u^0,\mu^0)+w\}+o(u-u^0). \end{aligned}$$

Hence,

$$x^1 = P\frac{\partial y^*}{\partial u}(u^0,\mu^0)+w = w, \qquad \text{using (3.9).}$$

Substituting this in (3.11), and the latter in (3.10), gives the desired formula,

$$\frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) = \lim_{u\to 0}\left\{\frac{1}{u}\, z^T G(x^0+uw)w\right\} \tag{3.12}$$

The preceding manipulation which is known as the <u>Liapunov-Schmidt procedure</u> has accomplished the following. We start with a system of n equations (3.1) in the n 'variables' x and n 'parameters' λ. These equations are decomposed into a system of (n - 1) equations (3.3) and a single equation (3.4); the parameter λ is correspondingly split into $\mu \in R^{n-1}$ and $\rho \in R$; and the variables x into (n-1) variables y and a single variable u. The equation (3.4) is well-behaved and we can 'solve' for y as in (3.5), substitute it into (3.4) giving a <u>single</u> equation (3.6) in a <u>single</u> variable u. Since both h and $\frac{\partial h}{\partial u}$ vanish at (u^0, μ^0), h is known as the <u>bifurcation function.</u> If we can evaluate $\frac{\partial^2 h}{\partial u^2}$, which is given by (3.12), then considerable information can be obtained as Theorem 3.1 indicates.

Suppose

$$\frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) \neq 0. \tag{3.13}$$

Then in a sufficiently small neighborhood of (u^0, μ^0) there is a unique $u^*(\mu)$, with $u^0 = u^*(\mu^0)$ such that $\frac{\partial h}{\partial u}(u, \mu)=0$ at $u = u^*(\mu)$; that is $h(u,\mu)$ considered as a function of u has a maximum or minimum at $u^*(\mu)$ according as the sign in (3.13) is negative or positive. Let

$$\eta(\mu) := h(u^*(\mu),\mu)$$

and note that $\eta(\mu^0) - \rho^0 = 0$. (In the statement below it is assumed that $\eta(\mu)$ is the minimum value. If it is a maximum replace $\eta - \rho$ by $-\eta + \rho$.)

<u>Theorem 3.1</u>[5] Suppose condition (3.13) is satisfied. Then there is neighborhood N of $(x^0, \lambda^0) \sim (x^0, \mu^0, \rho^0)$ and a continuously differentiable function $\eta(\mu)$, $\eta(\mu^0) - \rho^0 = 0$, such that the following conclusions are satisfied in N:

(i) Equation (3.1) has no solution if $\eta(\mu) - \rho > 0$,

(ii) Equation (3.1) has exactly one solution if $\eta(\mu) - \rho = 0$,

(iii) Equation (3.1) has exactly two solutions if $\eta(\mu) - \rho < 0$.

The parameter vector λ^0 or the pair (x^0, λ^0) where the equation (3.1) behaves in the manner described above is called a <u>fold</u> bifurcation point. The behavior of (3.1) in a heighborhood of a fold is illustrated in Figure 5. For $\lambda < \lambda^0$, (3.1) gives two well-behaved solutions, which coincide at $\lambda = \lambda^0$, and for $\lambda > \lambda^0$ there is no solution. Theorem 3.1 may be seen as a generalization to the vector case of the behavior of the parabolic equation $x^2 - \lambda = 0$ near (0, 0).

Suppose now that

$$\frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) = 0 \qquad \text{and} \qquad \frac{\partial^3 h}{\partial u^3}(u^0,\mu^0) < 0 \tag{3.14}$$

Then $\frac{\partial h}{\partial u}(u, \mu)$ has a unique maximum value, call it $\gamma_0(\mu)$, in a neighborhood of (u^0, μ^0). If $\gamma_0(\mu) > 0$, then $h(u, \mu)$ regarded as a function of u has a unique local maximum, say $\gamma_1(\mu)$, and a unique local minimum $\gamma_2(\mu)$ in a neighborhood of (u^0, μ^0). Let $\gamma(\mu, \rho) := (\gamma_1(\mu) - \rho)\,(\gamma_2(\mu) - \rho)$. [With obvious changes the result below holds when $\partial^3 h/\partial u^3(u^0, \mu^0) > 0$.]

<u>Theorem 3.2</u>[5] Suppose condition (3.14) is satisfied. Then there is a neighborhood N of $(x^0, \lambda^0) \sim (x^0, \mu^0, \rho^0)$ and two continuously differentiable functions $\gamma_0(\mu)$, $\gamma(\mu, \rho)$, $\gamma_0(\mu^0) = \gamma(\mu^0, \rho^0) = 0$, such that the following conclusion is satisfied:

(i) If $\gamma_0(\mu) \leq 0$ there is a unique solution of equation (3.1),

(ii) If $\gamma_0(\mu) > 0$ then $\gamma(\mu, \rho)$ is defined and

- (a) $\gamma(\mu, \rho) > 0$ implies one simple solution of equation (3.1),
- (b) $\gamma(\mu, \rho) = 0$ implies one simple and one double solution of equation (3.1),
- (c) $\gamma(\mu, \rho) < 0$ implies three simple solutions of (3.1).

If $\gamma_0(\mu) \leq 0$ for all μ near μ^0, then by (i) above there is no bifurcation. Otherwise λ^0 is said to be a <u>cusp</u> bifurcation. The behavior of (3.1) near a cusp is illustrated in Figure 6.

Observe that if $\gamma_0(\mu) \leq 0$ for μ near μ^0 then $\frac{\partial h}{\partial u}(u, \mu) \leq 0$ for all (u, μ) near (u^0, μ^0). This gives the next result which will be used in Section 3.3.

<u>Corollary 3.1</u> Suppose condition (3.14) is satisfied and suppose there are points (u, μ) arbitrarily close to (u^0, μ^0) where $\frac{\partial h}{\partial u}(u, \mu) > 0$. Then λ^0 is a cusp.

Theorem 3.2 can be seen as a generalization to the vector case of the behavior of the equation $x^3 + \mu x + \rho = 0$ near $(x, \mu, \rho) = (0, 0, 0)$. It is known [14] that the cubic equation has 3 simple real solutions if the discriminant $4\mu^3 + 27\rho^2 < 0$, one simple and one double real solution if $4\mu^3 + 27\rho^2 = 0$, and one simple real solution if $4\mu^3 + 27\rho^2 > 0$.

At the beginning of this discussion it was assumed that $G(x^0)$ has rank n - 1 i.e. it has a single zero eigenvalue. If two or more eigenvalues vanish simultaneously, more complex bifurcations may occur. More complex behavior also occurs even when the rank is n - 1 but where the first non-vanishing derivative of the function h above is of order four or larger. We do not pursue this further since we are unable as yet to evaluate these derivatives for the power flow function.

In conclusion here it may be worth mentioning that since bifurcations are "local" phenomena one may be tempted to ignore them considering "global" behavior. This is not the case; as we see in [15], the global behavior of the power flow equation is essentially structured by the bifurcations which do occur.

3.2 Fold bifurcation of power flow equation

Let θ^0, p^0 satisfy

$$f(\theta)-p = 0, \tag{3.15}$$

and suppose that $F(\theta^0) = \frac{\partial f}{\partial \theta}(\theta^0)$ has rank n - 1. Let $w \neq 0$ be such that $F(\theta^0)\, w = 0$. Let P be a matrix such that $[\, P \,\vdots\, w]$ is nonsingular and $P^T w = 0$. Since $F(\theta^0)$ is symmetric, therefore $w^T F(\theta^0) = 0$ also. Hence, in terms of the notation of Section 3.1, we can take $w = z$ and $P = Q$.

To study the behavior of (3.15) near θ^0, p^0 introduce the coordinates μ, ρ, y and u such that

$$\mu = P^T p, \qquad \rho = w^T p, \qquad \theta = Py+uw.$$

Then there is a neighborhood N of (y^0, u^0, μ^0) and $y^*(u, \mu)$ so that

$$P^T f(Py+uw)-\mu = 0 \quad \Leftrightarrow \quad y = y^*(u,\mu).$$

The bifurcation function is

$$h(u,\mu) = w^T f(Py^*(u,\mu)+uw). \tag{3.16}$$

Then $h(u^0, \mu^0) = \rho^0$, $\frac{\partial h}{\partial u}(u^0, \mu^0) = 0$, and from (3.12)

$$\begin{aligned} \frac{\partial^2 h}{\partial u^2}(u^0,\mu^0) &= \lim_{u\to 0} \frac{1}{u} w^T F(\theta^0+uw)w \\ &= -w^T AY[S(A^T\theta^0)](A^T w)^2, \qquad \text{by Lemma 2.1,} \\ &= -\sum_{i,j=1}^{n+1} (w_i-w_j)^3 Y_{ij} \sin(\theta_i^0-\theta_j^0), \\ &=: \alpha(\theta^0), \text{ say.} \end{aligned} \tag{3.17}$$

Here, as usual, $w_{n+1} = 0$, $\theta_{n+1}^0 = 0$. An application of Theorem 3.1 gives the next result.

<u>Lemma 3.1</u> Suppose $\alpha(\theta^0) \neq 0$. Then p^0 is a fold bifurcation i.e. there is a neighborhood $N = N_\theta \times N_p$ of $(\theta^0, p^0) \sim (\theta^0, \mu^0, \rho^0)$ and a function $\eta(\mu)$, $\eta(\mu^0) = \rho^0$, such that the following conclusions hold in N: the power flow equation has

(i) no solution if $\eta(\mu) > \rho$,

(ii) exactly one solution if $\eta(\mu) = \rho$,

(iii) exactly two solutions if $\eta(\mu) < \rho$.

Let S_0, respectively S_+, S_-, denote the set $N_p \cap \{\eta(P^T p) - w^T p = 0$, respectively > 0, $< 0\}$. Then S_0 is an $(n - 1)$-dimensional surface containing p^0, while S_+ and S_- are open sets lying on opposite sides of S_0. Since $\eta(\mu)$ is the maximum or minimum value of $h(u, \mu)$, therefore the eigenvector w is the normal to S_0 at p^0.

We apply Lemma 3.1 to the practically significant case when μ^0 is in the boundary $\mathcal{B}_s$ of the stable region. Choose the neighborhood N_θ so small that for θ in N_θ, $F(\theta)$ has at most one negative eigenvalue. Now if $p \in S_+$, then by Lemma 3.1 (i) $H(p) \cap N_\theta$ is empty, which implies [13] that the local degree of f at p,

$$\sum_{\mathcal{H}(p)\cap N_\theta} \text{sign } F(\theta) = 0$$

for all p in N_p sufficiently small. Hence if $p \in S_0$, and so by Lemma 3.1 (ii) there is exactly one θ in $\mathcal{H}(p) \cap N_\theta$, it must be that $\det F(\theta) = 0$ i.e. $\theta \in \mathcal{B}_s$. Finally, if $p \in S_-$, then by Lemma 3.1 (iii) there are exactly two solutions, say θ^1 and θ^2, in $\mathcal{H}(p) \cap N_\theta$. Since $\det F(\theta^1) + \det F(\theta^2) = 0$ we must have the determinant positive for one solution say θ^1, and negative for the other. Since $F(\theta^1)$ and $F(\theta^2)$ have at most one negative eigenvalue, therefore $F(\theta^1)$ must be positive definite i.e. θ^1 is stable, and θ^2 must be unstable. We summarize the conclusion as a corollary.

Corollary 3.2 Suppose $\theta^0 \in \mathcal{B}_s$. Then in the neighborhood N_θ, the power flow equation has

(i) no solution, if $p \in S_+$

(ii) exactly one solution which is in $\mathcal{B}_s$, if $p \in S_0$

(iii) exactly two solutions, one of which is stable and the other unstable, if $p \in S_-$.

3.3 Cusp bifurcation of power flow equation

Starting with (3.16), some further manipulations lead to

$$\frac{\partial^3 h}{\partial u^3}(u^0,\mu^0) = w^T\left\{\frac{\partial^2}{\partial u^2}F(\theta(u))-2\frac{\partial}{\partial u}F(\theta(u))P[P^TF(\theta^0)P]^{-1}P^T\frac{\partial}{\partial u}F(\theta(u))\right\}w\Big|_{u^0}$$

$$=: \beta(\theta^0) \qquad \text{say,} \tag{3.18}$$

where $\theta(u) := Py^*(u, \mu^0) + uw$. Lemma 2.1 gives the following formulas for the derivatives in (3.18),

$$\frac{\partial^2}{\partial u^2}F(\theta(u))\Big|_{u^0} = -AY[C(A^T\theta^0)][A^Tw]^2A^T, \tag{3.19}$$

$$w^T\frac{\partial^2}{\partial u^2}F(\theta(u))w\Big|_{u^0} = -\sum_{i,j=1}^{n+1}(w_i-w_j)^4Y_{ij}\cos(\theta_i^0-\theta_j^0), \tag{3.20}$$

$$\frac{\partial}{\partial u}F(\theta(u))w\Big|_{u^0} = -AY[S(A^T\theta)][A^Tw]A^T, \tag{3.21}$$

$$\frac{\partial}{\partial u}F(\theta(u))w\Big|_{u^0} = -AY[S(A^T\theta)](A^Tw)^2. \tag{3.22}$$

In (3.19) and (3.21), $[A^Tw]$ is the diagonal matrix whose entries are the components of the vector A^Tw.

Suppose $\alpha(\theta^0) = 0$ in (3.17) and $\beta(\theta^0) < 0$ in (3.18). Suppose moreover that there are points (u, μ) arbitrarily close to (u^0, μ^0) where

$$\frac{\partial h}{\partial u}(u,\mu) > 0. \tag{3.23}$$

Then p^0 is a cusp by Corollary 3.1. Let $N = N_\theta \times N_p$ be the neighborhood in Theorem 3.2. Let N_p^-, respectively N_p^+ denote the open set $\{p \in N_p | \gamma_0(\mu) = \gamma_0(P^Tp) < 0$, respectively $> 0\}$. N_p^+ is non-empty and p^0 is in its boundary. N_p^+ can be further partitioned into three sets S_0, S_+, S_- defined by $N_p^+ \cap \{\gamma(\mu, \rho) = \gamma(P^Tp, w^Tp) = 0$, respectively > 0, $< 0\}$.

<u>Lemma 3.2</u> Under the conditions above, and in the neighborhood N, the

power flow equation has

(i) a unique solution if $p \in N_p^-$ (in fact if $p \notin N_p^+$),

(ii) (a) one simple solution if $p \in S_+$,

(b) one simple solution and one double solution if $p \in S_0$,

(c) three simple solutions if $p \in S_-$.

We examine the remarkable implications of these results when θ^0 is in the boundary $\mathcal{B}_s$ of the stable region. We assume again that N_θ is so small that $F(\theta)$ has at most one negative eigenvalue. First observe using (3.16) that

$$\frac{\partial h}{\partial u}(u,\mu) = -w^T F(\theta^*(u,\mu))P[P^T F(\theta^*(u,\mu))P]^{-1}P^T F(\theta^*(u,\mu))w + w^T F(\theta^*(u,\mu))w. \tag{3.24}$$

where $\theta^*(u, \mu) := Py^*(u, \mu) + uw$. Since $\theta^0 \in \mathcal{B}_s$, therefore there exist (u, μ) arbitrarily close to (u^0, μ^0) where $F(\theta^*(u, \mu)) > 0$. By, Lemma 3.4 below, this implies $\frac{\partial h}{\partial u}(u, \mu) > 0$. Hence (3.23) holds and so the conditions of Lemma 3.2 are satisfied. Second, suppose that $p \in N_p^-$ and let $\theta \in N_\theta$ be the unique solution to $f(\theta) = p$. Now since $p \in N_p^-$, therefore $\gamma_0(\mu) = \gamma_0(P^T p) < 0$, and since $\gamma_0(\mu)$ is the maximum value of $\frac{\partial h}{\partial u}(u, \mu)$, this implies that $\frac{\partial h}{\partial u}(u, \mu) < 0$. Hence θ must be unstable; and so the local degree of f at p

$$d(f,p) := \sum_{\theta \in \Theta(p) \cap N_\theta} \operatorname{sign} F(\theta) = -1. \tag{3.25}$$

<u>Lemma 3.3</u> Suppose $\theta^0 \in \mathcal{B}_s$, $\alpha(\theta^0) = 0$, $\beta(\theta^0) < 0$. Then p^0 is a cusp bifurcation, and in the neighborhood N, the power flow equation has

(i) a unique unstable solution if $p \in N_p^-$,

(ii) (a) one simple unstable solution if $p \in S_+$

(b) one simple unstable solution and one double solution in $\mathcal{B}_s$ if

$p \in S_0$,

(c) two simple unstable solutions and one simple stable solution if $p \in S_-$.

<u>Proof</u> (i) has already been proved, (ii) now follows from Lemma 3.2 and (3.25). ¤

Observe that conclusion (i) implies that if a cusp occurs at p^0 with θ^0 in $\mathcal{B}_s$, then there is always a perturbation in the power demand which can be met by a small change in bus angles only at <u>unstable</u> solutions. We see this in more detail in the next example.

<u>Lemma 3.4</u> Let $F = F^T > 0$, and let $[P \,\vdots\, w]$ be nonsingular with $P^T w = 0$. Then

$$w^T R w := w^T\{F - FP[P^T FP]^{-1} P^T F\} w > 0.$$

<u>Proof</u> First observe that $R = \lim_{\varepsilon \to 0} R_\varepsilon$, where

$$R_\varepsilon := F - FP[P^T FP + \varepsilon I]^{-1} P^T F.$$

By a well-known matrix identity (see e.g. [8]), R_ε can be written as

$$R_\varepsilon = [F^{-1} + \tfrac{1}{\varepsilon} P^T P]^{-1} > 0.$$

Hence $R \geq 0$ and so $w^T R w \geq 0$. Therefore, $w^T R w = 0$ only if

$$0 = Rw = Fw - FP[P^T FP]^{-1} P^T Fw,$$

and since F is nonsingular, this implies

$$w = P[P^T FP]^{-1} P^T Fw.$$

So,

$$w^T w = w^T P[P^T F P]^{-1} P^T F,$$

which is a contradiction, since $w^T w > 0$ and $w^T P = 0$. ¤

Example 2.4 (continued) Consider again the 6-node network of Figure 4 in which $Y_{ij} = 1$. Take θ^0 and p^0 as in Section 2.5 where it was shown that $\theta^0 \in \mathcal{B}_s$. It can be verified directly that $w := (1, 1, 1, 0, 0)^T$ satisfies $F(\theta^0)\, w = 0$. Substituting this w in the formulas (3.17) and (3.18) - (3.22) we get $\alpha(\theta^0) = 0$, $\beta(\theta^0) < 0$. By Lemma 3.3, p^0 is a cusp. It can be directly verified that near (θ^0, p^0) there is a unique solution θ^t of $f(\theta) = p^t$ where $p^t := p^0 + t(1, 0, 0, 0, 0)^T$ for $t > 0$ small. By Lemma 3.3, θ^t must be unstable.

4. Conclusions

The most interesting global properties of the power flow equation concern the disconnectedness of the stable region, and the fact that $F(\theta)$ becomes "more" positive definite as one moves towards the interior of the stable region. The interesting local properties are that in the boundary of the stable region, a fold bifurcation occurs by the coming together of a stable and an unstable solution, whereas a cusp occurs by the coming together of two unstable and one stable solution.

The examples given in the paper have dispelled some conjectures which had appeared in the literature, or which at first seemed plausible to us. The most surprising discovery is the existence of power demands which can only be met by unstable solutions.

Finally, our study has led us to formulate these conjectures:

(i) The number of components of the stable region is equal to the number of stable solutions of $f(\theta) = 0$,

(ii) The principal component of the stable region is not generally convex,

(iii) The flow function $f(\theta)$ is one-one in each component of the stable region,

(iv) If $f(\theta) = p$ has a stable solution, then it has a stable solution in the principal component.

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FIGURE CAPTIONS

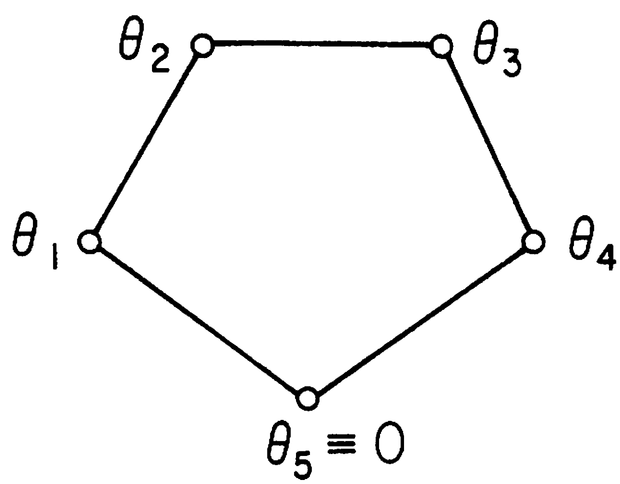

Figure 1

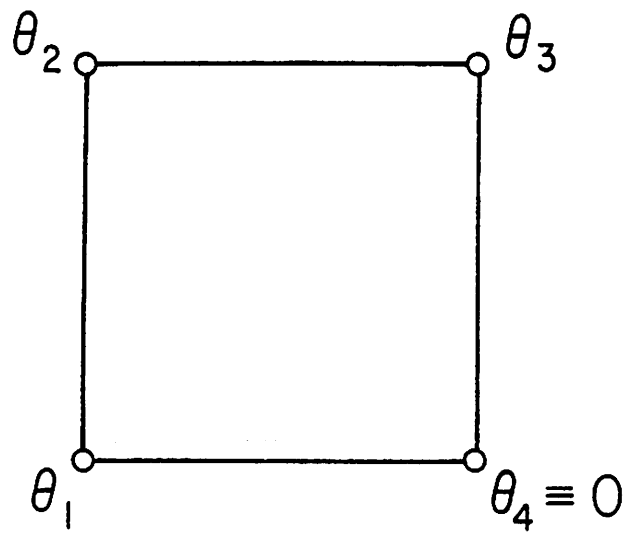

Figure 2

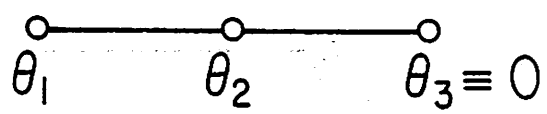

Figure 3

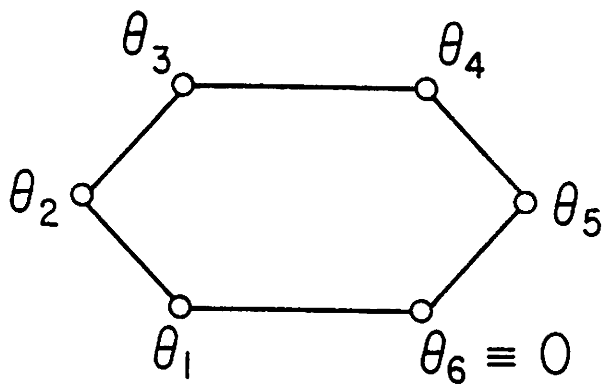

Figure 4

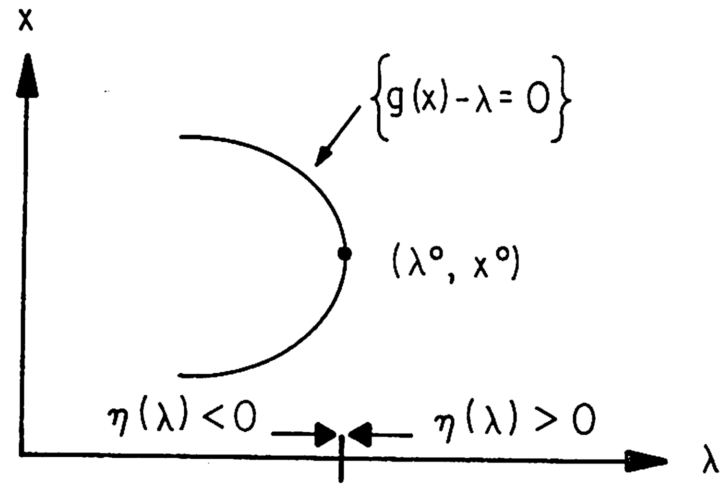
x
$\{g(x)-\lambda=0\}$
(λ°, x°)
$\eta(\lambda)<0$
$\eta(\lambda)>0$
λ

Figure 5

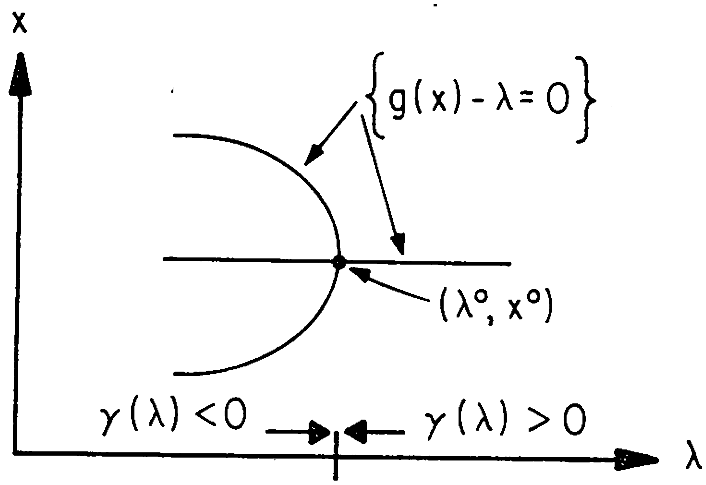

Figure 6